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Multiple solutions for coupled gradient-type quasilinear elliptic systems with supercritical growth

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Abstract

In this paper, we consider the following coupled gradient-type quasilinear elliptic system

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = G_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(b(x,v,\nabla v)) + B_t(x,v,\nabla v) = G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 2$. We suppose that some $\mathcal{C}^1$–Carathéodory functions $A, B : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ exist such that $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$, $A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi)$, $b(x,t,\xi) = \nabla_\xi B(x,t,\xi)$, $B_t(x,t,\xi) = \frac{\partial B}{\partial t}(x,t,\xi)$, and that $G_u(x,u,v)$, $G_v(x,u,v)$ are the partial derivatives of a $\mathcal{C}^1$–Carathéodory nonlinearity $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Roughly speaking, we assume that $A(x,t,\xi)$ grows at least as $(1+|t|^{s_1 p_1})|\xi|^{p_1}$, $p_1 > 1$, $s_1 \geq 0$, while $B(x,t,\xi)$ grows as $(1+|t|^{s_2 p_2})|\xi|^{p_2}$, $p_2 > 1$, $s_2 \geq 0$, and that $G(x, u, v)$ can also have a supercritical growth related to s_1 and s_2. Since the coefficients depend on the solution and its gradient themselves, the study of the interaction of two different norms in a suitable Banach space is needed. In spite of these difficulties, a variational approach is used to show that the system admits a nontrivial weak bounded solution and, under hypotheses of symmetry, infinitely many ones.

Keywords Coupled gradient-type quasilinear elliptic system · p-Laplacian-type operator · Supercritical growth · Weak Cerami–Palais–Smale condition · Ambrosetti–Rabinowitz condition · Mountain Pass theorem · Critical Sobolev exponent · Nontrivial weak bounded solution · Pseudo-eigenvalue

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1 Introduction

The study of partial differential equations involving nonlinearities with critical or supercritical growths is a very complex matter, and for many critical and supercritical problems, some basic issues are mostly unknown or undiscovered. For example, let us consider the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^N, N \geq 2$, and $1 < p < N$. In spite of the simple looking structure of the problem, if we ask q to be critical or supercritical from the viewpoint of the Sobolev embedding theorem, namely $q \geq p^* = \frac{Np}{N-p}$, some significant difficulties arise. Among other problems, in general, the lack of compactness which occurs does not guarantee even the existence of solutions, which has been derived only in few cases and frequently under assumptions on the shape of the domain Ω (for the classical nonexistence result due to the Pohožaev's identity see [36], or also [37, Theorem III.1.3]).

The existence of positive solutions of (1.1) has been successfully addressed either by adding some lower-order term to the critical nonlinearity (see [9]) or by considering domains which are not starshaped (see, e.g., [6, 22]) if $p = 2$, $q = 2^*$ and $\lambda = 0$, while the existence of sign-changing solutions of (1.1) has been obtained if $p = 2$, $q = 2^*$ but $\lambda \neq 0$ (see, e.g., [3, 20]). To our knowledge, all the results carried over so far are built on the key assertion that the functional associated with the critical problem (1.1) satisfies the Palais–Smale condition even if only in certain ranges of energy.

On the other hand, taking $p \neq 2$, due to the hardship in handling a quasilinear operator, very few results of existence have been derived so far, not even under assumptions of symmetry on the domain (we refer to [31] for a wider discussion). We limit ourselves to point out that as derived in [34], a Pohožaev-type nonexistence result is not yet available for sign-changing solutions of (1.1), as the unique continuation principle for the p-Laplacian is not known, while it has been proved for nonnegative solutions (see [23]). However, the existence of a positive solution in a domain with a sufficiently small hole has been shown for $\frac{2N}{N+2} \leq p \leq 2$, as well as an existence and multiplicity result have been proved under further assumptions of symmetry (see [21, 31, 32, 34, 35] and references therein).

In spite of the mentioned difficulties, in recent years, there has been a marked increase in research in critical and supercritical problems. The interest in these problems is related to their similarity to some variational problems which arise in Geometry and Physics where the lack of compactness also occurs. In this sense, one of the best known challenges is the so-called Yamabe's problem, but also some examples related to the existence of extremal functions for isoperimetric inequalities, Hardy–Littlewood–Sobolev inequalities and trace inequalities can be addressed (see, e.g., [25, 27, 29]). However, in general, also in the "simplest" cases, some problems are still open, and some classical variational tools, largely used in the subcritical case, do not work in the critical and supercritical ones.

Anyway, recently, quasilinear problems which generalize

$$\begin{cases} -\mathrm{div}((1+A(x)|u|^{sp})|\nabla u|^{p-2}\nabla u) + sA(x)|u|^{sp-2}u|\nabla u|^p = |u|^{\mu-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

have been studied and, by means of a suitable variational setting, the existence of infinitely many weak bounded solutions is proved also if the nonlinear term has a supercritical growth such as $2 < 1+p < p(s+1) < \mu < p^*(s+1)$, when $A \in L^\infty(\Omega)$ is such that $A(x) \geq \alpha_0 > 0$ for a.e. $x \in \Omega$ (see [14] and also, for other approaches, [4, 30]). One of the most remarkable feature of this work is that unlike the results mentioned above, both an existence and a multiplicity result have been provided in the supercritical case for a more general problem without taking any symmetry assumption on the domain Ω.

Here, following the ideas introduced in [14], we look for solutions of the family of coupled gradient-type quasilinear elliptic systems

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = G_u(x,u,v) & \text{in } \Omega, \\ -\mathrm{div}(b(x,v,\nabla v)) + B_t(x,v,\nabla v) = G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, and $A, B : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are given functions with partial derivatives

$$A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi), \qquad a(x,t,\xi) = \left(\frac{\partial A}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial A}{\partial \xi_N}(x,t,\xi)\right), \tag{1.3}$$

$$B_t(x,t,\xi) = \frac{\partial B}{\partial t}(x,t,\xi), \qquad b(x,t,\xi) = \left(\frac{\partial B}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial B}{\partial \xi_N}(x,t,\xi)\right), \tag{1.4}$$

for a.e. $x \in \Omega$, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Moreover, a nonlinear function $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ exists so that

$$G_u(x,u,v) = \frac{\partial G}{\partial u}(x,u,v),\ G_v(x,u,v) = \frac{\partial G}{\partial v}(x,u,v) \ \text{ for a.e. } x \in \Omega, \text{ all } (u,v) \in \mathbb{R}^2. \tag{1.5}$$

Roughly speaking, here we assume that $A(x,u,\nabla u)$ grows at least as $(1+|u|^{s_1p_1})|\nabla u|^{p_1}$, $p_1 > 1$, $s_1 \geq 0$, while $B(x,v,\nabla v)$ grows at least as $(1+|v|^{s_2p_2})|\nabla v|^{p_2}$, $p_2 > 1$, $s_2 \geq 0$ (see Remark 3.2 and assumption (h_7)), and that $G(x, u, v)$ can also have a supercritical growth depending on s_1 and s_2 (see hypothesis (g_2)).

While subcritical quasilinear systems have been handled through several techniques (see, e.g., [5, 8, 10, 16, 18]), as far as we know very few existence results have been determined for supercritical quasilinear elliptic systems (see, for example, [19, 24] and references therein), even though no result occurs for supercritical systems with coefficients depending on the solution and its gradient themselves, as that one in (1.2). Moreover, as in [14], even if a supercritical growth occurs, the domain Ω is only open and bounded as we consider homogeneous Dirichlet boundary condition, and the solutions we are looking for are weak.

Thus, following the same approach used in [16] and [18], but carefully adapting the ideas in [14] to our supercritical setting, we give some sufficient conditions for recognizing the variational structure of problem (1.2), so that investigating solutions of (1.2) reduces to find critical points of functional

$$\mathcal{J}(u,v) = \int_\Omega A(x,u,\nabla u)\,dx + \int_\Omega B(x,v,\nabla v)\,dx - \int_\Omega G(x,u,v)\,dx \tag{1.6}$$

in the product Banach space $X = X_1 \times X_2$, with $X_i = W_0^{1,p_i}(\Omega) \cap L^\infty(\Omega)$ if $i \in \{1,2\}$.

Moreover, since in the Banach space X our functional $\mathcal{J}$ does not satisfy the Palais–Smale condition, or one of its standard variants, we are not allowed to use directly existence and multiplicity results as the classical Ambrosetti–Rabinowitz theorems stated in [2] or in [7]. Hence, we have to submit a weaker definition of the Cerami's variant of Palais–Smale condition, the so-called weak Cerami–Palais–Smale condition (see Definition 2.1). We believe that the use of this definition, introduced in the pioneering paper [11] and employed in the framework of a quasilinear supercritical system, represents another major improvement of the work in this field. In fact, here Definition 2.1 is used for stating an extended Mountain Pass theorem and also its symmetric version of which we avail to gain our existence and multiplicity results (see Theorems 2.2 and 2.3), but we do not exclude the chance that this feature may be also employed to recover other kind of problems (see, e.g., [33]). In fact, we highlight that this technique has been adapted to address problems placed over unbounded domains both in radial and in non-radial setting (see [15], respectively [17]) but so far only in subcritical growth assumptions (in [1], the existence of solutions for some critical and supercritical problems has been proved by using a different (radial) approach, which is not applicable for non-autonomous equations).

On the other hand, this enhancement imposes to *pay the price* that some technical assumptions on the involved functions are needed. Namely, if we just assume the Carathéodory functions $A(x,t,\xi)$, $B(x,t,\xi)$, $G(x, u, v)$ and their partial derivatives fit some proper polynomial growths to show the $\mathcal{C}^1$ regularity of the functional $\mathcal{J}$ in (1.6), on the other hand, the proof of the weak Cerami–Palais–Smale condition passes through some fine requirements on the involved functions (see Sect. 3) and a very remarkable result (see Lemma 3.7) which has interest own self and can be employed regardless of this scenario to fix a problem of common trouble in this field.

Now, in order to draw the attention to the enhancement of our main results, we state them here in a "streamlined" version but we refer the reader to Sect. 4 for all the needed hypotheses on the involved functions and the precise statement of the results (see Theorems 4.1 and 4.2).

Theorem 1.1 *Suppose that $A(x,t,\xi)$ grows at least as $(1+|t|^{s_1p_1})|\xi|^{p_1}$, with $p_1 > 1$, $s_1 \ge 0$, while $B(x,t,\xi)$ grows at least as $(1+|t|^{s_2p_2})|\xi|^{p_2}$, with $p_2 > 1$, $s_2 \ge 0$. Moreover, assume that the $\mathcal{C}^1$–Carathéodory function $A(x,t,\xi)$, respectively $B(x,t,\xi)$, and its partial derivatives fits some suitable interaction properties among themselves, while the $\mathcal{C}^1$–Carathéodory nonlinear term $G(x, u, v)$ satisfies the Ambrosetti–Rabinowitz condition for systems with coefficients θ_1, $\theta_2 > 0$ such that $\theta_i < \frac{1}{p_i}$, $i \in \{1,2\}$, and has a proper polynomial growth which can also be supercritical depending on s_1 and s_2. If*

$$\limsup_{(u,v)\to(0,0)} \frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} < \alpha_2 \min\{\lambda_{1,1}, \lambda_{2,1}\} \quad \text{uniformly a.e. in } \Omega,$$

with $\lambda_{i,1}$ first eigenvalue of $-\Delta_{p_i}$ in $W_0^{p_i}(\Omega)$, $i \in \{1,2\}$, then problem (1.2) admits a nontrivial weak bounded solution.

Theorem 1.2 *In the same hypotheses of Theorem 1.1, assume that $A(x,\cdot,\cdot)$ and $B(x,\cdot,\cdot)$ are even in $\mathbb{R} \times \mathbb{R}^N$ while $G(x,\cdot,\cdot)$ is even in $\mathbb{R}^2$ for a.e. $x \in \Omega$. Then, if*

$$\liminf_{|(u,v)|\to+\infty} \frac{G(x,u,v)}{|u|^{\frac{1}{\theta_1}} + |v|^{\frac{1}{\theta_2}}} > 0 \quad \text{uniformly a.e. in } \Omega,$$

problem (1.2) *admits infinitely many distinct weak bounded solutions*.

Finally, in order to better explain the required hypotheses, we consider the particular setting

$$A(x,t,\xi) = \frac{1}{p_1}(1+|t|^{s_1p_1})|\xi|^{p_1}, \quad B(x,t,\xi) = \frac{1}{p_2}(1+|t|^{s_2p_2})|\xi|^{p_2}, \tag{1.7}$$

and

$$G(x,u,v) = \frac{1}{q_1}|u|^{q_1} + \frac{1}{q_2}|v|^{q_2} + c_*|u|^{\gamma_1}|v|^{\gamma_2}, \tag{1.8}$$

with $c_* \geq 0$ and some positive exponents p_i, s_i, q_i, γ_i for each $i \in \{1,2\}$. So, $\mathcal{J}$ in (1.6) reduces to the functional $\mathcal{J}_0 : X \to \mathbb{R}$ such that

$$\begin{aligned}\mathcal{J}_0(u,v) = &\frac{1}{p_1}\int_\Omega (1+|u|^{s_1p_1})|\nabla u|^{p_1}dx + \frac{1}{p_2}\int_\Omega (1+|v|^{s_2p_2})|\nabla v|^{p_2}dx \\ &- \int_\Omega \left(\frac{1}{q_1}|u|^{q_1} + \frac{1}{q_2}|v|^{q_2} + c_*|u|^{\gamma_1}|v|^{\gamma_2}\right)dx,\end{aligned}$$

and, in a suitable set of assumptions, system (1.2) turns into the model problem

$$\begin{cases} -\mathrm{div}((1+|u|^{s_1p_1})|\nabla u|^{p_1-2}\nabla u) + s_1|u|^{s_1p_1-2}u\,|\nabla u|^{p_1} & \\ \qquad\qquad = |u|^{q_1-2}u + \gamma_1 c_*|u|^{\gamma_1-2}u|v|^{\gamma_2} & \text{in } \Omega, \\ -\mathrm{div}((1+|v|^{s_2p_2})|\nabla v|^{p_2-2}\nabla v) + s_2|v|^{s_2p_2-2}v\,|\nabla v|^{p_2} & \\ \qquad\qquad = \gamma_2 c_*|u|^{\gamma_1}|v|^{\gamma_2-2}v + |v|^{q_2-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Hence, the previous results can be reworded in this way.

Theorem 1.3 *Let $A(x,t,\xi)$, $B(x,t,\xi)$ and $G(x,u,v)$ be as in* (1.7) *and* (1.8) *with $p_i > 1$, $i \in \{1,2\}$, $c_* \geq 0$ and either $p_1 < N$ or $p_2 < N$. Assume that θ_1, θ_2 exist such that*

$$2 < 1 + p_i < p_i(s_i+1) < \frac{1}{\theta_i} \leq q_i < p_i^*(s_i+1) \quad \textit{for } i \in \{1,2\}, \tag{1.10}$$

where p_1^*, p_2^* are the critical Sobolev exponents, and also

$$1 < \gamma_1 < q_1, \quad 1 < \gamma_2 < q_2 \quad \textit{are such that} \quad \gamma_1\theta_1 + \gamma_2\theta_2 \geq 1. \tag{1.11}$$

Then, if

$$\gamma_j \frac{q_i - 1}{q_i - \gamma_i} < \frac{p_i}{N}\left(1 - \frac{1}{p_i^*(s_i+1)}\right)p_j^*(s_j+1) \quad \textit{for } i,j \in \{1,2\}, i \neq j, \tag{1.12}$$

problem (1.9) *admits infinitely many weak bounded distinct solutions*.

Our paper is organized as follows. In Sect. 2, we introduce the abstract setting needed to recognize the variational structure of our problem (1.2), as well as some extended versions of the Mountain Pass theorems are shown up. Furthermore, a regularity result for the functional $\mathcal{J}$ in (1.6) is provided, too. Then, in Sect. 3, some further assumptions on $A(x,t,\xi)$, $B(x,t,\xi)$ and $G(x, u, v)$ are addressed in order to show that the functional $\mathcal{J}$ verifies the weak Cerami–Palais–Smale condition. Lastly, in Sect. 4, our main results are stated and proved.

2 Abstract tools and variational setting

We denote $\mathbb{N} = \{1, 2, \dots\}$ and, as long as we introduce our abstract setting, we employ the following notations:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$,
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e., $X \subset W$ and a constant $\sigma_0 > 0$ exists such that

$$\|y\|_W \;\leq\; \sigma_0\, \|y\|_X \qquad \text{for all } y \in X,$$

- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in \mathcal{C}^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$.

In order to avoid any ambiguity and simplify, when possible, the notation, from now on by X we denote the space equipped with its given norm $\|\cdot\|_X$ while if the norm $\|\cdot\|_W$ is involved, we write it explicitly.

Now, taking $\beta \in \mathbb{R}$, we say that a sequence $(y_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*-sequence*, if

$$\lim_{n\to+\infty} J(y_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ\big(y_n\big)\|_{X'}(1 + \|y_n\|_X) = 0.$$

As pointed out in [13, Example 4.3], a $(CPS)_\beta$ sequence can be constructed so that it is unbounded in $\|\cdot\|_X$ but converges with respect to $\|\cdot\|_W$. Thus, as in [14], we introduce the following definition.

Definition 2.1 The functional J satisfies the weak Cerami-Palais-Smale condition at level β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(CPS)_\beta$-sequence $(y_n)_n$, a point $y \in X$ exists, such that

(i) $\lim_{n\to+\infty} \|y_n - y\|_W = 0$ (up to subsequences),
(ii) $J(y) = \beta$, $dJ(y) = 0$.

We say that J satisfies the $(wCPS)$ condition in I, I real interval, if J satisfies the $(wCPS)_\beta$ condition in X at each level $\beta \in I$.

Anyway, even if we deal with a weaker version of the Cerami's variant of the Palais–Smale condition, some classical abstract results can be extended so to fit to our purposes. Actually, as in [14, Lemma 2.2] (see also [12, Lemma 2.3]), a Deformation Lemma can be stated which provides the following extended version of the Mountain Pass theorem given in [2] (see [14, Theorem 2.3] for a detailed proof).

Theorem 2.2 *Let $J \in \mathcal{C}^1(X, \mathbb{R})$ be such that $J(0) = 0$ and the (wCPS) condition holds in $\mathbb{R}_+$. Moreover, assume that there exist a continuous map $\ell : X \to \mathbb{R}$, some constants $r_0, \varrho_0 > 0$, and $e \in X$ such that*

(*i*) $\ell(0) = 0$ *and* $\ell(y) \geq \|y\|_W$ *for all* $y \in X$;
(*ii*) $y \in X, \quad \ell(y) = r_0 \implies J(y) \geq \varrho_0$;
(*iii*) $\|e\|_W > r_0$ *and* $J(e) < \varrho_0$.

Then, J has a Mountain Pass critical point $y^* \in X$ such that $J(y^*) \geq \varrho_0$.

If, in addition, we require that J is symmetric, then a more general version of the Symmetric Mountain Pass Theorem in [2] can be stated, too (for the proof, see [14, Theorem 2.4]).

Theorem 2.3 *Let $J \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$ and the (wCPS) condition holds in $\mathbb{R}_+$. Moreover, assume that $\varrho > 0$ exists so that:*

($\mathcal{H}_\varrho$) *three closed subsets V_ϱ, Z_ϱ and $\mathcal{M}_\varrho$ of X and a constant $R_\varrho > 0$ exist which satisfy the following conditions:*

(*i*) *V_ϱ and Z_ϱ are subspaces of X such that*

$$V_\varrho + Z_\varrho = X, \qquad \operatorname{codim} Z_\varrho < \dim V_\varrho < +\infty;$$

(*ii*) *$\mathcal{M}_\varrho = \partial\mathcal{N}$, where $\mathcal{N} \subset X$ is a neighborhood of the origin which is symmetric and bounded with respect to* $\|\cdot\|_W$;
(*iii*) $y \in \mathcal{M}_\varrho \cap Z_\varrho \implies J(y) \geq \varrho$;
(*iv*) $y \in V_\varrho, \quad \|y\|_X \geq R_\varrho \implies J(y) \leq 0$.

Then, if we put

$$\beta_\varrho = \inf_{\gamma \in \Gamma_\varrho} \sup_{y \in V_\varrho} J(\gamma(y)),$$

with

$$\Gamma_\varrho = \{\gamma : X \to X : \ \gamma \text{ odd homeomorphism, } \gamma(y) = y \text{ if } y \in V_\varrho \text{ with } \|y\|_X \geq R_\varrho\},$$

the functional J possesses at least a pair of symmetric critical points in X with corresponding critical level β_ϱ which belongs to $[\varrho, \varrho_1]$, where $\varrho_1 \geq \sup_{y \in V_\varrho} J(y) > \varrho$.

Remark 2.4 Since the vector space V_ϱ in Theorem 2.3 has finite dimension, then condition $(\mathcal{H}_\varrho)(iv)$ implies that $\sup_{y \in V_\varrho} \mathcal{J}(y) < +\infty$. Moreover, such hypothesis still holds if we replace $\|\cdot\|_X$ with $\|\cdot\|_W$.

Finally, if we can apply Theorem 2.3 infinitely many times, then the following multiplicity abstract result can be stated, too.

Corollary 2.5 *Let $J \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$, the (wCPS) condition holds in $\mathbb{R}_+$ and assumption $(\mathcal{H}_\varrho)$ holds for all $\varrho > 0$. Then, the functional J possesses a sequence of critical points $(y_n)_n \subset X$ such that $J(y_n) \nearrow +\infty$ as $n \nearrow +\infty$.*

Now, we proceed introducing the notations related to our specific setting. If $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $N \geq 2$, we denote by:

- $L^q(\Omega)$ the Lebesgue space with norm $|y|_q = \left(\int_\Omega |y|^q dx\right)^{1/q}$ if $1 \leq q < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue-measurable and essentially bounded functions $y : \Omega \to \mathbb{R}$ with norm $|y|_\infty = \operatorname{ess\,sup}_\Omega |y|$;
- $W_0^{1,p}(\Omega)$ the Sobolev space equipped with the norm $\|y\|_{W_0^{1,p}} = |\nabla y|_p$ if $1 \leq p < +\infty$;
- meas (D) the usual Lebesgue measure of a measurable set D in $\mathbb{R}^N$;
- $|\cdot|$ the standard norm on any Euclidean space, as the dimension of the vector taken into account is clear and no ambiguity occurs.

Moreover, for any $m \in \mathbb{N}$, we say that $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a $\mathcal{C}^k$–Carathéodory function, $k \in \mathbb{N} \cup \{0\}$, if $h(\cdot, v)$ is measurable in Ω for all $v \in \mathbb{R}^m$ while $h(x, \cdot)$ is $\mathcal{C}^k$ in $\mathbb{R}^m$ for a.e. $x \in \Omega$.

Let $A,\ B : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be such that the following conditions hold:

(h_0) $A(x, t, \xi)$ and $B(x, t, \xi)$ are $\mathcal{C}^1$–Carathéodory functions with partial derivatives as in (1.3), respectively (1.4);

(h_1) two exponents $p_1 > 1$, $p_2 > 1$, and some positive functions $\Phi_i, \phi_i, \Psi_i, \psi_i \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$, if $i \in \{0, 1, 2\}$, exist such that

$$|A(x,t,\xi)| \leq \Phi_0(t) + \phi_0(t)|\xi|^{p_1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{2.1}$$

$$\begin{aligned} |A_t(x,t,\xi)| &\leq \Phi_1(t) + \phi_1(t)|\xi|^{p_1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\ |a(x,t,\xi)| &\leq \Phi_2(t) + \phi_2(t)|\xi|^{p_1-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \end{aligned} \tag{2.2}$$

and

$$|B(x,t,\xi)| \leq \Psi_0(t) + \psi_0(t)|\xi|^{p_2} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{2.3}$$

$$\begin{aligned} |B_t(x,t,\xi)| &\leq \Psi_1(t) + \psi_1(t)|\xi|^{p_2} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\ |b(x,t,\xi)| &\leq \Psi_2(t) + \psi_2(t)|\xi|^{p_2-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N. \end{aligned} \tag{2.4}$$

Furthermore, let $G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a map which satisfies the following hypotheses:

(g_0) $G(x, u, v)$ is a $\mathcal{C}^1$–Caratheodory function with partial derivatives as in (1.5), such that

$$G(\cdot, 0, 0) \in L^\infty(\Omega) \quad \text{and} \quad G_u(x, 0, 0) = G_v(x, 0, 0) = 0 \quad \text{for a.e. } x \in \Omega;$$

(g_1) a constant $\sigma > 0$ and some exponents $q_i \geq 1$, $t_i \geq 0$, if $i \in \{1, 2\}$, exist such that

$$\begin{aligned} |G_u(x,u,v)| &\le \sigma(1+|u|^{q_1-1}+|v|^{t_1}) \quad \text{for a.e. } x\in\Omega, \text{ for all } (u,v)\in\mathbb{R}^2,\\ |G_v(x,u,v)| &\le \sigma(1+|u|^{t_2}+|v|^{q_2-1}) \quad \text{for a.e. } x\in\Omega, \text{ for all } (u,v)\in\mathbb{R}^2. \end{aligned} \tag{2.5}$$

Remark 2.6 Hypotheses (g_0)–(g_1), the mean value theorem and direct computations ensure the existence of a positive constant $\sigma_1 > 0$ such that

$$|G(x,u,v)| \le \sigma_1\left(1+|u|^{q_1}+|v|^{t_1}|u|+|u|^{t_2}|v|+|v|^{q_2}\right) \text{ for a.e. } x\in\Omega, \text{ for all } (u,v)\in\mathbb{R}^2. \tag{2.6}$$

Now, taking any couple of real numbers $t_3, t_5 > 1$, from Young inequality, we obtain

$$|v|^{t_1}|u| \le |u|^{t_3}+|v|^{t_4}, \qquad |u|^{t_2}|v| \le |u|^{t_6}+|v|^{t_5} \quad \text{for all } (u,v)\in\mathbb{R}^2, \tag{2.7}$$

where for simplicity, we set

$$t_4 := \frac{t_1 t_3}{t_3-1} \ge t_1 \quad \text{and} \quad t_6 := \frac{t_2 t_5}{t_5-1} \ge t_2. \tag{2.8}$$

Thus, from (2.6) and (2.7), we infer that

$$|G(x,u,v)| \le \sigma_2(1+|u|^{\overline{q}_1}+|v|^{\overline{q}_2}) \quad \text{for a.e. } x\in\Omega, \text{ for all } (u,v)\in\mathbb{R}^2, \tag{2.9}$$

with

$$\overline{q}_1 := \max\{q_1,t_3,t_6\} \qquad \text{and} \qquad \overline{q}_2 := \max\{q_2,t_4,t_5\}, \tag{2.10}$$

for a suitable constant $\sigma_2 > 0$.

In order to recall some features shared by the subcritical systems in [16] and [18], if needed, here we introduce similar notations.

For each $i\in\{1,2\}$ let $p_i > 1$ be as in assumption (h_1) and let us consider the related Sobolev space

$$W_i = W_0^{1,p_i}(\Omega) \quad \text{with norm } \|\cdot\|_{W_i} = \|\cdot\|_{W_0^{1,p_i}}.$$

From the Sobolev embedding theorem, for any $r\in[1,p_i^*]$ with $p_i^* = \frac{Np_i}{N-p_i}$ if $N > p_i$, or any $r\in[1,+\infty[$ if $p_i \ge N$, W_i is continuously embedded in $L^r(\Omega)$, i.e., $\tau_{i,r} > 0$ exists such that

$$|y|_r \le \tau_{i,r}\|y\|_{W_i} \quad \text{for all } y\in W_i. \tag{2.11}$$

Furthermore, if $p_i \ge N$, we place

$$p_i^* = +\infty \quad \text{and} \quad \frac{1}{p_i^*} = 0.$$

Here, the notation $(W,\|\cdot\|_W)$, introduced for the abstract setting at the beginning of this section, is referred to our problem with

$$W = W_1\times W_2 \qquad \text{and} \qquad \|(u,v)\|_W = \|u\|_{W_1}+\|v\|_{W_2} \quad \text{if } (u,v)\in W. \tag{2.12}$$

Since $(W_i,\|\cdot\|_{W_i})$ is a reflexive Banach space for both $i\in\{1,2\}$, so is $\left(W,\|\cdot\|_W\right)$ in (2.12).

Moreover, we consider the Banach space $(X,\|\cdot\|_X)$ defined as

$$X = X_1 \times X_2 \quad \text{with} \quad \|(u,v)\|_X = \|u\|_{X_1} + \|v\|_{X_2} \quad \text{if } (u,v) \in X, \tag{2.13}$$

where

$$X_1 := W_1 \cap L^\infty(\Omega) \quad \text{and} \quad X_2 := W_2 \cap L^\infty(\Omega) \tag{2.14}$$

are endowed with the norms

$$\|u\|_{X_1} = \|u\|_{W_1} + |u|_\infty \ \text{ if } u \in X_1 \quad \text{and} \quad \|v\|_{X_2} = \|v\|_{W_2} + |v|_\infty \ \text{ if } v \in X_2.$$

Setting

$$L := L^\infty(\Omega) \times L^\infty(\Omega) \quad \text{with} \quad \|(u,v)\|_L = |u|_\infty + |v|_\infty,$$

we have that X in (2.13) can also be written as

$$X = W \cap L \tag{2.15}$$

and its norm is such that

$$\|(u,v)\|_X = \|(u,v)\|_W + \|(u,v)\|_L.$$

Clearly, from (2.14), for both $i \in \{1,2\}$ we have that the continuous embeddings $X_i \hookrightarrow W_i$ and $X_i \hookrightarrow L^\infty(\Omega)$ hold.

Remark 2.7 If $p_i > N$ for both $i \in \{1,2\}$, then the embedding $W_i \hookrightarrow L^\infty(\Omega)$ means that $X_i = W_i$. Thus, $X = W$ and the classical Mountain Pass theorems in [2] may be applied.

Firstly, we note that if conditions (h_0)–(h_1), (g_0)–(g_1) hold, then direct computations imply that $\mathcal{J}(u,v)$ in (1.6) is well-defined for all $(u,v) \in X$. Moreover, taking any (u, v), $(w,z) \in X$, the Gâteaux differential of functional $\mathcal{J}$ in (u, v) along the direction (w, z) is given by

$$\begin{aligned} d\mathcal{J}(u,v)[(w,z)] = & \int_\Omega a(x,u,\nabla u) \cdot \nabla w \, dx + \int_\Omega A_u(x,u,\nabla u) w \, dx \\ & + \int_\Omega b(x,v,\nabla v) \cdot \nabla z \, dx + \int_\Omega B_v(x,v,\nabla v) z \, dx \\ & - \int_\Omega G_u(x,u,v) w \, dx - \int_\Omega G_v(x,u,v) z \, dx. \end{aligned} \tag{2.16}$$

For simplicity, we set

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial u}(u,v) &: w \in X_1 \mapsto \frac{\partial \mathcal{J}}{\partial u}(u,v)[w] = d\mathcal{J}(u,v)[(w,0)] \in \mathbb{R}, \\ \frac{\partial \mathcal{J}}{\partial v}(u,v) &: z \in X_2 \mapsto \frac{\partial \mathcal{J}}{\partial v}(u,v)[z] = d\mathcal{J}(u,v)[(0,z)] \in \mathbb{R}; \end{aligned}$$

hence, from (2.16), it follows that

$$\frac{\partial \mathcal{J}}{\partial u}(u,v)[w] = \int_\Omega a(x,u,\nabla u) \cdot \nabla w \, dx + \int_\Omega A_u(x,u,\nabla u) w \, dx - \int_\Omega G_u(x,u,v) w \, dx \tag{2.17}$$

and

$$\frac{\partial \mathcal{J}}{\partial v}(u,v)[z] = \int_\Omega b(x,v,\nabla v)\cdot \nabla z\,dx + \int_\Omega B_v(x,v,\nabla v)z\,dx - \int_\Omega G_v(x,u,v)z\,dx. \tag{2.18}$$

Taking $(u, v) \in X$, since $d\mathcal{J}(u, v) \in X'$, then

$$\frac{\partial \mathcal{J}}{\partial u}(u,v) \in X_1', \qquad \frac{\partial \mathcal{J}}{\partial v}(u,v) \in X_2'$$

and

$$d\mathcal{J}(u,v)[(w,z)] = \frac{\partial \mathcal{J}}{\partial u}(u,v)[w] + \frac{\partial \mathcal{J}}{\partial v}(u,v)[z] \quad \text{ for all } (w,z) \in X. \tag{2.19}$$

Furthermore, above remarks and direct computations give not only the estimates

$$\left\|\frac{\partial \mathcal{J}}{\partial u}(u,v)\right\|_{X_1'} \le \|d\mathcal{J}(u,v)\|_{X'} \quad \text{ and } \quad \left\|\frac{\partial \mathcal{J}}{\partial v}(u,v)\right\|_{X_2'} \le \|d\mathcal{J}(u,v)\|_{X'}, \tag{2.20}$$

but also

$$\|d\mathcal{J}(u,v)\|_{X'} \le \left\|\frac{\partial \mathcal{J}}{\partial u}(u,v)\right\|_{X_1'} + \left\|\frac{\partial \mathcal{J}}{\partial v}(u,v)\right\|_{X_2'}.$$

At last, from (2.19), we infer that

$$d\mathcal{J}(u,v) = 0 \text{ in } X \quad \iff \quad \frac{\partial \mathcal{J}}{\partial u}(u,v) = 0 \text{ in } X_1 \text{ and } \frac{\partial \mathcal{J}}{\partial v}(u,v) = 0 \text{ in } X_2.$$

Finally, we can state the regularity of functional $\mathcal{J}$ defined in (1.6) (for the proof, see [18, Proposition 3.5]).

Proposition 2.8 *Assume that conditions* (h_0)–(h_1), (g_0)–(g_1) *hold. Let* $((u_n, v_n))_n \subset X$ *and* $(u, v) \in X$ *be such that*

$$(u_n, v_n) \to (u,v) \text{ in } W \quad \text{ and } \quad (u_n, v_n) \to (u,v) \text{ a.e. in } \Omega \quad \text{ if } n \to +\infty.$$

If $M > 0$ *exists such that*

$$|u_n|_\infty \le M \quad \text{ and } \quad |v_n|_\infty \le M \quad \text{ for all } n \in \mathbb{N},$$

then

$$\mathcal{J}(u_n, v_n) \to \mathcal{J}(u,v) \quad \text{ and } \quad \|d\mathcal{J}(u_n,v_n) - d\mathcal{J}(u,v)\|_{X'} \to 0 \text{ as } n \to +\infty.$$

Hence, $\mathcal{J}$ *is a* $\mathcal{C}^1$ *functional on* X *with Fréchet differential defined as in* (2.16).

3 The set up for the weak Cerami–Palais–Smale condition

In order to prove some more properties of functional $\mathcal{J}$ in (1.6), let $p_1 > 1$ and $p_2 > 1$ as in the earlier hypothesis (h_1). Then, assume that $R \ge 1$ exists such that the following conditions hold:

(h_2) some constants $\eta_1, \eta_2 > 0$ exist such that

$$A(x,t,\xi) \leq \eta_1 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R, \tag{3.1}$$

$$B(x,t,\xi) \leq \eta_1 b(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R, \tag{3.2}$$

and

$$\sup_{|(t,\xi)|\leq R} |A(x,t,\xi)| \leq \eta_2, \qquad \sup_{|(t,\xi)|\leq R} |B(x,t,\xi)| \leq \eta_2 \quad \text{a.e. in } \Omega; \tag{3.3}$$

(h_3) some exponents $s_1, s_2 \geq 0$ and a constant $\mu_0 > 0$ exist so that

$$a(x,t,\xi)\cdot\xi \geq \mu_0(1+|t|^{s_1p_1})|\xi|^{p_1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N,$$
$$b(x,t,\xi)\cdot\xi \geq \mu_0(1+|t|^{s_2p_2})|\xi|^{p_2} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$

(h_4) a constant $\mu_1 > 0$ exists such that

$$a(x,t,\xi)\cdot\xi + A_t(x,t,\xi)t \geq \mu_1 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R,$$
$$b(x,t,\xi)\cdot\xi + B_t(x,t,\xi)t \geq \mu_1 b(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R;$$

(h_5) some constants $\theta_1, \theta_2, \mu_2 > 0$ exist such that

$$\theta_1 < \frac{1}{p_1}, \qquad \theta_2 < \frac{1}{p_2}, \tag{3.4}$$

and

$$A(x,t,\xi) - \theta_1 a(x,t,\xi)\cdot\xi - \theta_1 A_t(x,t,\xi)t \geq \mu_2 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R,$$
$$B(x,t,\xi) - \theta_2 b(x,t,\xi)\cdot\xi - \theta_2 B_t(x,t,\xi)t \geq \mu_2 b(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R;$$

(h_6) for all $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$, it is

$$[a(x,t,\xi) - a(x,t,\xi')]\cdot[\xi-\xi'] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t\in\mathbb{R},$$
$$[b(x,t,\xi) - b(x,t,\xi')]\cdot[\xi-\xi'] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t\in\mathbb{R};$$

(g_2) for $i \in \{1,2\}$, taking p_i as in hypothesis (h_1), q_i, t_i as in assumption (g_1) and s_i as in (h_3), we assume that

$$1 \leq q_1 < p_1^*(s_1+1), \qquad 1 \leq q_2 < p_2^*(s_2+1), \tag{3.5}$$

and

$$0 \leq t_1 < \frac{p_1}{N}\left(1 - \frac{1}{p_1^*(s_1+1)}\right)p_2^*(s_2+1), \; 0 \leq t_2 < \frac{p_2}{N}\left(1 - \frac{1}{p_2^*(s_2+1)}\right)p_1^*(s_1+1); \tag{3.6}$$

(g_3) taking θ_1, θ_2 as in (h_5), we assume that

$$0 < G(x,u,v) \leq \theta_1 G_u(x,u,v)u + \theta_2 G_v(x,u,v)v \quad \text{a.e. in } \Omega, \text{ if } |(u,v)| \geq R.$$

Remark 3.1 Assumption (3.5) shows up the supercritical nature of our problem which vanishes if $s_1 = s_2 = 0$ as it reduces exactly to the subcritical condition (g_1) in [16, 18].

However, in general, if one or both $s_1 > 0$, $s_2 > 0$ hold, then a supercritical growth on the nonlinear term $G(x, u, v)$ is allowed. Moreover, we emphasize that the growth hypothesis (g_2) is needed to prove that the functional $\mathcal{J}$ satisfies the $(wCPS)$ condition, but has not been required for the variational principle stated in Proposition 2.8.

Remark 3.2 If we consider hypothesis (h_4) with $t = 0$ and $|\xi| \geq R$, then assumption (h_3) gives $\mu_1 \leq 1$. Moreover, we note that (h_4) and (h_5) yield

$$A(x,t,\xi) \geq (\theta_1\mu_1 + \mu_2)\, a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R, \tag{3.7}$$

$$B(x,t,\xi) \geq (\theta_2\mu_1 + \mu_2)\, b(x,t,\xi)\cdot\xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R \tag{3.8}$$

which, together with condition (h_3), imply that

$$A(x,t,\xi) \geq \mu_0(\theta_1\mu_1 + \mu_2)(1 + |t|^{s_1p_1})|\xi|^{p_1} \geq 0 \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R, \tag{3.9}$$

$$B(x,t,\xi) \geq \mu_0(\theta_2\mu_1 + \mu_2)(1 + |t|^{s_2p_2})|\xi|^{p_2} \geq 0 \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R. \tag{3.10}$$

Hence, from (3.3) and (3.9), respectively (3.10), and direct computations, we have that

$$A(x,t,\xi) \geq \mu_0(\theta_1\mu_1 + \mu_2)(1 + |t|^{s_1p_1})|\xi|^{p_1} - \eta_3 \quad \text{a.e. in } \Omega \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.11}$$

$$B(x,t,\xi) \geq \mu_0(\theta_2\mu_1 + \mu_2)(1 + |t|^{s_2p_2})|\xi|^{p_2} - \eta_3 \quad \text{a.e. in } \Omega \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.12}$$

for a suitable constant $\eta_3 > 0$. On the other hand, from (h_2) and (2.2), respectively (2.4), direct computations imply that

$$A(x,t,\xi) \leq \eta_1\Phi_2(t) + \eta_1(\Phi_2(t) + \phi_2(t))|\xi|^{p_1} + \eta_4 \quad \text{a.e. in } \Omega \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.13}$$

$$B(x,t,\xi) \leq \eta_1\Psi_2(t) + \eta_1(\Psi_2(t) + \psi_2(t))|\xi|^{p_2} + \eta_4 \quad \text{a.e. in } \Omega \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.14}$$

for a suitable constant $\eta_4 > 0$. Then, if hypotheses (h_2)–(h_5) hold, the growth conditions on $A(x,t,\xi)$ and $B(x,t,\xi)$ stated in (2.1) and (2.3) are a direct consequence of (2.2), respectively (2.4), as (2.1) follows from (3.11) and (3.13), while (2.3) follows from (3.12) and (3.14). Hence, even if (2.1) and (2.3) are part of assumption (h_1), they can be ruled-out from the hypotheses if (h_2)–(h_5) hold, too.

Remark 3.3 In the set of hypotheses (h_2) and (h_5) a more precise growth condition on both the functions $A(x,t,\xi)$ and $B(x,t,\xi)$ can be pointed out. In fact, (3.1), respectively (3.2), (h_5) and direct calculations imply that

$$\left(\frac{\eta_1 - \theta_1 - \mu_2}{\eta_1\theta_1}\right) A(x,t,\xi) \geq A_t(x,t,\xi)t \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R, \tag{3.15}$$

$$\left(\frac{\eta_1 - \theta_2 - \mu_2}{\eta_1\theta_2}\right) B(x,t,\xi) \geq B_t(x,t,\xi)t \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R. \tag{3.16}$$

Now, taking $t = 0$ and $|\xi| \geq R$ in both (h_2) and (h_5), without loss of generality we can choose μ_2 small enough so that

$$\eta_1 - \theta_1 - \mu_2 > 0 \quad \text{and} \quad \eta_1 - \theta_2 - \mu_2 > 0.$$

Thus, from (2.1), (3.9), (3.15), respectively (2.3), (3.10), (3.16), and direct computations, we obtain that

$$A(x,t,\xi) \le \eta_5 |t|^{\frac{\eta_1-\theta_1-\mu_2}{\eta_1\theta_1}} |\xi|^{p_1} \quad \text{a.e. in } \Omega \text{ if } |t| \ge 1 \text{ and } |\xi| \ge R,$$
$$B(x,t,\xi) \le \eta_5 |t|^{\frac{\eta_1-\theta_2-\mu_2}{\eta_1\theta_2}} |\xi|^{p_2} \quad \text{a.e. in } \Omega \text{ if } |t| \ge 1 \text{ and } |\xi| \ge R,$$

for a suitable $\eta_5 > 0$, and then from (3.7), respectively (3.8), we have that

$$a(x,t,\xi)\cdot\xi \le \frac{\eta_5}{\theta_1\mu_1+\mu_2} |t|^{\frac{\eta_1-\theta_1-\mu_2}{\eta_1\theta_1}} |\xi|^{p_1} \quad \text{a.e. in } \Omega \text{ if } |t| \ge 1 \text{ and } |\xi| \ge R, \tag{3.17}$$

$$b(x,t,\xi)\cdot\xi \le \frac{\eta_5}{\theta_2\mu_1+\mu_2} |t|^{\frac{\eta_1-\theta_2-\mu_2}{\eta_1\theta_2}} |\xi|^{p_2} \quad \text{a.e. in } \Omega \text{ if } |t| \ge 1 \text{ and } |\xi| \ge R. \tag{3.18}$$

Finally, from (3.17), (3.18) and assumption (h_3), we infer that

$$0 \le p_1 s_1 \le \frac{1}{\theta_1} - \frac{\theta_1+\mu_2}{\eta_1\theta_1} \quad \text{and} \quad 0 \le p_2 s_2 \le \frac{1}{\theta_2} - \frac{\theta_2+\mu_2}{\eta_1\theta_2}. \tag{3.19}$$

We note that if

$$0 \le s_1 < \frac{1}{\theta_1 p_1} \quad \text{and} \quad 0 \le s_2 < \frac{1}{\theta_2 p_2}, \tag{3.20}$$

then we can always choose η_1 in (h_2) large enough so that (3.19) is satisfied.

Remark 3.4 Conditions in (3.20) not only relate the exponents s_1, s_2 provided in assumption (h_3) with the powers $p_1, p_2 > 1$ used in the growth conditions (h_1) and θ_1, θ_2 claimed in (3.4), but also they tell us how far we can take it. In particular, it implies that in our set of hypotheses, a supercritical growth is allowed as long as s_1, s_2 cover the whole range stated in (3.20).

Remark 3.5 Assumptions (g_0)–(g_1), (g_3) and direct calculations imply that for each $i \in \{1,2\}$ a function $h_i \in L^\infty(\Omega)$, $h_i(x) > 0$ for a.e. $x \in \Omega$, exists such that

$$G(x,u,0) \ge h_1(x)|u|^{\frac{1}{\theta_1}} \quad \text{for a.e. } x \in \Omega, \text{ if } |u| \ge R,$$
$$G(x,0,v) \ge h_2(x)|v|^{\frac{1}{\theta_2}} \quad \text{for a.e. } x \in \Omega, \text{ if } |v| \ge R.$$

Thus, from (2.6), we obtain that

$$\begin{aligned} h_1(x)|u|^{\frac{1}{\theta_1}} - \sigma_3 \le G(x,u,0) \le \sigma_1(1+|u|^{q_1}) \quad \text{for a.e. } x \in \Omega, \text{ for all } u \in \mathbb{R}, \\ h_2(x)|v|^{\frac{1}{\theta_2}} - \sigma_3 \le G(x,0,v) \le \sigma_1(1+|v|^{q_2}) \quad \text{for a.e. } x \in \Omega, \text{ for all } v \in \mathbb{R} \end{aligned} \tag{3.21}$$

for a positive constant $\sigma_3 > 0$. Then, (3.20) and (3.21) imply that

$$p_1 s_1 < \frac{1}{\theta_1} \leq q_1, \qquad p_2 s_2 < \frac{1}{\theta_2} \leq q_2,$$

while from (3.4), it is

$$p_1 < \frac{1}{\theta_1}, \qquad p_2 < \frac{1}{\theta_2}.$$

So, if condition (3.5) holds, without loss of generality, we can take q_1, q_2 in (g_1) large enough so that

$$p_1(s_1+1) < q_1 < p_1^*(s_1+1) \quad \text{and} \quad p_2(s_2+1) < q_2 < p_2^*(s_2+1). \tag{3.22}$$

In order to show that the $(wCPS)$ condition holds also in our supercritical setting, we need some preliminary results.

Firstly, we note that taking $p > 1$ and $s \geq 0$, then straightforward computations give

$$|\nabla(|y|^s y)|^p = (s+1)^p |y|^{sp} |\nabla y|^p \quad \text{a.e. in } \Omega, \ \text{ for all } y \in W_0^{1,p}(\Omega). \tag{3.23}$$

Such an equality allows us to prove the following Rellich-type embedding theorem (for the proof, see [14, Lemma 3.8]).

Lemma 3.6 *Taking $1 < p < N$ and $s > 0$, let $(y_n)_n \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a sequence such that*

$$\left(\int_\Omega (1 + |y_n|^{sp}) |\nabla y_n|^p dx \right)_n \quad \text{is bounded.}$$

Then, $y \in W_0^{1,p}(\Omega)$ exists such that $|y|^s y \in W_0^{1,p}(\Omega)$, too, and, up to subsequences, we have that

$$\begin{aligned} & y_n \rightharpoonup y \quad \text{weakly in } W_0^{1,p}(\Omega),\\ & |y_n|^s y_n \rightharpoonup |y|^s y \quad \text{weakly in } W_0^{1,p}(\Omega),\\ & y_n \to y \quad \text{strongly in } L^r(\Omega) \ \text{ for each } r \in [1, p^*(s+1)[,\\ & y_n \to y \quad \text{a.e. in } \Omega. \end{aligned}$$

Furthermore, we state the following boundedness result (for the proof, see [26, Theorem II.5.1]).

Lemma 3.7 *Let Ω be an open bounded subset of $\mathbb{R}^N$ and consider $y \in W_0^{1,p}(\Omega)$ with $p \leq N$. Suppose that $\gamma > 0$ and $k_0 \in \mathbb{N}$ exist such that*

$$\int_{\Omega_k^+} |\nabla y|^p dx \leq \gamma \left(\int_{\Omega_k^+} (y-k)^r dx \right)^{\frac{p}{r}} + \gamma \sum_{j=1}^m k^{\alpha_j} [\text{meas}(\Omega_k^+)]^{1-\frac{p}{N}+\varepsilon_j} \quad \text{for all } k \geq k_0,$$

with $\Omega_k^+ = \{x \in \Omega : y(x) > k\}$ and r, m, α_j, ε_j positive constants such that

$$1 \leq r < p^*, \qquad \varepsilon_j > 0, \qquad p \leq \alpha_j < \varepsilon_j p^* + p.$$

Then, $\operatorname{ess\,sup}_\Omega y$ *is bounded from above by a positive constant which can be chosen so that it depends only on* $\operatorname{meas}(\Omega)$, N, p, γ, k_0, r, m, ε_j, α_j, $|y|_{p^*}$ *(eventually,* $|y|_l$ *for some* $l > r$ *if* $p^* = +\infty$).

As pointed out in Remark 3.1, the upper bounds in (g_2) were not required so far, but will be essential in the incoming results. Therefore, some consequences of the estimates in (3.5) and (3.6) are needed.

Remark 3.8 Suppose $1 < p_1 < N$, $1 < p_2 < N$ and take t_1, t_2 as in (3.6). Following the ideas in Remark 2.6, we can choose t_3 and t_5 in (2.7) so that

$$\begin{aligned} &1 < \frac{p_1 p_2^*(s_2+1)}{p_1 p_2^*(s_2+1) - N t_1} < t_3 < p_1^*(s_1+1), \\ &1 < \frac{p_2 p_1^*(s_1+1)}{p_2 p_1^*(s_1+1) - N t_2} < t_5 < p_2^*(s_2+1), \end{aligned} \tag{3.24}$$

as (3.6) implies that

$$p_1 p_2^*(s_2+1) - N t_1 > 0 \qquad \text{and} \qquad 1 < \frac{p_1 p_2^*(s_2+1)}{p_1 p_2^*(s_2+1) - N t_1} < p_1^*(s_1+1),$$

and also

$$p_2 p_1^*(s_1+1) - N t_2 > 0 \qquad \text{and} \qquad 1 < \frac{p_2 p_1^*(s_1+1)}{p_2 p_1^*(s_1+1) - N t_2} < p_2^*(s_2+1).$$

Then, t_4 and t_6 in (2.8) are such that

$$t_1 \le t_4 < \frac{p_1}{N}\, p_2^*(s_2+1) \le p_2^*(s_2+1),\ \ t_2 \le t_6 < \frac{p_2}{N}\, p_1^*(s_1+1) \le p_1^*(s_1+1). \tag{3.25}$$

Clearly, (3.24) and (3.25) still hold if $p_1 = N$ and/or $p_2 = N$.

Remark 3.9 In the set of hypotheses (g_0)–(g_3), by reasoning as in Remark 2.6, we can consider estimate (2.9) with $\overline{q}_1$ and $\overline{q}_2$ as in (2.10) but taking t_j, $j \in \{3,4,5,6\}$, as in Remark 3.8. Hence, from (3.5), (3.22), (3.24) and (3.25), we infer that

$$1 < p_1 < \frac{\overline{q}_1}{s_1+1} < p_1^* \quad \text{and} \quad 1 < p_2 < \frac{\overline{q}_2}{s_2+1} < p_2^*. \tag{3.26}$$

Finally, we are able to prove that the weak Cerami–Palais–Smale condition holds.

Proposition 3.10 *Under assumptions* (h_0)–(h_6) *and* (g_0)–(g_3) *functional* $\mathcal{J} : X \to \mathbb{R}$, *defined as in* (1.6), *satisfies* (*wCPS*) *condition in* $\mathbb{R}$.

Proof Taking any $\beta \in \mathbb{R}$, let $((u_n, v_n))_n \subset X$ be a sequence such that

$$\mathcal{J}(u_n, v_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(u_n, v_n)\|_{X'}(1 + \|(u_n, v_n)\|_X) \to 0 \quad \text{as } n \to +\infty. \tag{3.27}$$

We want to prove that a couple $(u, v) \in X$ exists such that

(i) $(u_n, v_n) \to (u, v)$ in W (up to subsequences),
(ii) $\mathcal{J}(u, v) = \beta$, $d\mathcal{J}(u, v) = 0$.

To this aim, for simplicity, we organize our proof in the following steps:

1. Both the sequences

$$\left(\int_\Omega (1 + |u_n|^{s_1 p_1})|\nabla u_n|^{p_1} dx\right)_n \text{ and } \left(\int_\Omega (1 + |v_n|^{s_2 p_2})|v_n|^{p_2} dx\right)_n \text{ are bounded,} \tag{3.28}$$

so, by applying Lemma 3.6, a couple $(u, v) \in W$ exists such that also $(|u|^{s_1}u, |v|^{s_2}v) \in W$ and, up to subsequences, we have:

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in } W, \tag{3.29}$$

$$(|u_n|^{s_1}u_n, |v_n|^{s_2}v_n) \rightharpoonup (|u|^{s_1}u, |v|^{s_2}v) \quad \text{weakly in } W, \tag{3.30}$$

$$(u_n, v_n) \to (u, v) \text{ in } L^{r_1}(\Omega) \times L^{r_2}(\Omega) \text{ if } 1 \le r_i < p_i^*(s_i + 1),\ i \in \{1, 2\}, \tag{3.31}$$

$$(u_n, v_n) \to (u, v) \quad \text{a.e. in } \Omega; \tag{3.32}$$

2. $(u, v) \in L^\infty(\Omega) \times L^\infty(\Omega)$;
3. for any $k > 0$, define $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$T_k t := \begin{cases} t & \text{if } |t| \le k \\ k\frac{t}{|t|} & \text{if } |t| > k \end{cases}$$

and

$$\mathbb{T}_k : (y_1, y_2) \in \mathbb{R}^2 \mapsto \mathbb{T}_k(y_1, y_2) = (T_k y_1, T_k y_2) \in \mathbb{R}^2,$$

then, if $k \ge \max\{\|(u, v)\|_L, R\} + 1$ (with $R \ge 1$ as in our set of hypotheses), it is

$$\|d\mathcal{J}(\mathbb{T}_k(u_n, v_n))\|_{X'} \to 0 \quad \text{and} \quad \mathcal{J}(\mathbb{T}_k(u_n, v_n)) \to \beta;$$

4. $\|\mathbb{T}_k(u_n, v_n) - (u, v)\|_W \to 0$ and then (i) holds;
5. (ii) is satisfied.

For simplicity, here and in the following, we will use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $((u_n, v_n))_n$. Moreover, we denote by c_i every positive constant which arises during our computations.

Step 1. Firstly, we note that (2.20) and (3.27) imply

$$\frac{\partial \mathcal{J}}{\partial u}(u_n, v_n)[u_n] = \varepsilon_n \quad \text{and} \quad \frac{\partial \mathcal{J}}{\partial v}(u_n, v_n)[v_n] = \varepsilon_n. \tag{3.33}$$

Thus, if we take θ_1, θ_2 as in (h_5), (g_3), and s_1, s_2 as in (h_3), by reasoning as in [18Step 1 in Proposition 4.8], from (1.6), (2.17), (2.18), (3.27), (3.33), assumptions (h_1), (h_3), (h_5), (g_3) together with estimate (2.9), and direct computations it follows that

$$\beta + \varepsilon_n = \mathcal{J}(u_n, v_n) - \theta_1 \frac{\partial \mathcal{J}}{\partial u}(u_n, v_n)[u_n] - \theta_2 \frac{\partial \mathcal{J}}{\partial v}(u_n, v_n)[v_n]$$
$$\geq \mu_0 \mu_2 \int_\Omega (1 + |u_n|^{s_1 p_1}) |\nabla u_n|^{p_1} dx + \mu_0 \mu_2 \int_\Omega (1 + |v_n|^{s_2 p_2}) |\nabla v_n|^{p_2} dx - c_1,$$

which implies that (3.28) is satisfied and, up to subsequences, $(u, v) \in W$ exists such that (3.29)–(3.32) hold.

Step 2. Due to the Sobolev Embedding Theorem, this step requires a proof only if either $p_1 \leq N$ or $p_2 \leq N$. So, if $p_1 < N$ (when $p_1 = N$ the arguments can be simplified), we want to prove that $u \in L^\infty(\Omega)$. Arguing by contradiction, we assume that $u \notin L^\infty(\Omega)$ as either

$$\text{ess sup}_\Omega u = +\infty \tag{3.34}$$

or

$$\text{ess sup}_\Omega (-u) = +\infty. \tag{3.35}$$

If (3.34) holds, then for any $k \in \mathbb{N}$, we have that

$$\text{meas}(\Omega^+_{u,k}) > 0 \qquad \text{with} \quad \Omega^+_{u,k} = \{x \in \Omega : \; u(x) > k\}, \tag{3.36}$$

and for an integer $\tilde{k} > 0$, we consider the function $R^+_{\tilde{k}} : t \in \mathbb{R} \mapsto R^+_{\tilde{k}} t \in \mathbb{R}$ defined as

$$R^+_{\tilde{k}} t = \begin{cases} 0 & \text{if } t \leq \tilde{k} \\ t - \tilde{k} & \text{if } t > \tilde{k} \end{cases}. \tag{3.37}$$

Now, we consider condition (3.36) for a fixed integer $k > R$ (with $R \geq 1$ as in our setting of hypotheses) and, taking $\tilde{k} = k^{s_1+1}$, for simplicity, we put

$$w_n = |u_n|^{s_1} u_n, \qquad w = |u|^{s_1} u, \tag{3.38}$$

and, as $|t|^{s_1} t > \tilde{k} \iff t > k$, we have that

$$\Omega^+_{u,k} = \{x \in \Omega : \; w(x) > \tilde{k}\}. \tag{3.39}$$

Thus, from condition (3.30) and the sequentially weakly lower semicontinuity of $\|\cdot\|_{W_1}$, we have that

$$\|R^+_{\tilde{k}} w\|_{W_1} \leq \liminf_{n \to +\infty} \|R^+_{\tilde{k}} w_n\|_{W_1},$$

i.e.,

$$\int_{\Omega^+_{u,k}} |\nabla w|^{p_1} dx \leq \liminf_{n \to +\infty} \int_{\Omega^+_{n,k}} |\nabla w_n|^{p_1} dx, \tag{3.40}$$

with

$$\Omega^+_{n,k} = \{x \in \Omega : \; u_n(x) > k\} = \{x \in \Omega : \; w_n(x) > \tilde{k}\}.$$

On the other hand, by definition (3.37) with $\tilde{k}$ replaced with k, it is $\|R^+_k u_n\|_{X_1} \leq \|u_n\|_{X_1}$, so from (2.20), (3.27) and (3.36), an integer $n_k \in \mathbb{N}$ exists such that

$$\frac{\partial \mathcal{J}}{\partial u}(u_n, v_n)\left[R_k^+ u_n\right] < \text{meas}(\Omega_{u,k}^+) \quad \text{for all } n \geq n_k. \tag{3.41}$$

Then, by reasoning as in [18, Step 2 in Proposition 4.8], from (2.17), hypotheses (h_3), (h_4) with $\mu_1 \leq 1$ (see Remark 3.2), equality (3.23) and estimate (3.41) we obtain that

$$\int_{\Omega_{n,k}^+} |\nabla w_n|^{p_1} dx \leq \frac{(s_1+1)^{p_1}}{\mu_0 \mu_1} \text{meas}(\Omega_{u,k}^+) + \int_\Omega G_u(x, u_n, v_n) R_k^+ u_n dx \ \text{ for all } n \geq n_k. \tag{3.42}$$

We claim that

$$\int_\Omega G_u(x, u_n, v_n) R_k^+ u_n dx \ \rightarrow \ \int_\Omega G_u(x, u, v) R_k^+ u \, dx. \tag{3.43}$$

In fact, from (3.32) and (g_0) , we have that

$$G_u(x, u_n, v_n) R_k^+ u_n \to G_u(x, u, v) R_k^+ u \quad \text{ a.e. in } \Omega,$$

while thanks to assumption (g_2), formulae (2.5), (2.7), (3.24), (3.25) and (3.31) ensure the existence of $h \in L^1(\Omega)$ such that

$$|G_u(x, u_n, v_n) R_k^+ u_n| \leq \sigma(|u_n| + |u_n|^{q_1} + |u_n|^{t_3} + |v_n|^{t_4}) \leq h(x) \quad \text{ for a.e. } x \in \Omega,$$

so the dominated convergence theorem implies (3.43). Thus, summing up, via (3.40), (3.42), (3.43) and again (2.5), from definition (3.37) ($\tilde{k}$ replaced with k), we infer that

$$\begin{aligned}\int_{\Omega_{u,k}^+} |\nabla w|^{p_1} dx &\leq c_2 \left(\int_\Omega |R_k^+ u| dx + \int_\Omega |u|^{q_1 - 1} |R_k^+ u| dx + \int_\Omega |v|^{t_1} |R_k^+ u| dx + \text{meas}(\Omega_{u,k}^+) \right) \\ &\leq c_2 \left(\int_{\Omega_{u,k}^+} |u| dx + \int_{\Omega_{u,k}^+} |u|^{q_1} dx + \int_{\Omega_{u,k}^+} |u| |v|^{t_1} dx + \text{meas}(\Omega_{u,k}^+) \right),\end{aligned}$$

or better, from (2.7) but according to the choises in Remark 3.8, by taking $\overline{q}_1 > 1$ as in (2.10) and being $u > 1$ in $\Omega_{u,k}^+$, definition (3.38) implies that

$$\int_{\Omega_{u,k}^+} |\nabla w|^{p_1} dx \leq c_3 \left(\int_{\Omega_{u,k}^+} |w|^{\frac{\overline{q}_1}{s_1+1}} dx + \int_{\Omega_{u,k}^+} |v|^{t_4} dx + \text{meas}(\Omega_{u,k}^+) \right). \tag{3.44}$$

We claim that

$$\int_{\Omega_{u,k}^+} |v|^{t_4} dx \leq (\tau_{2,p_2^*} \| |v|^{s_2} v \|_{W_2})^{\frac{t_4}{s_2+1}} [\text{meas}(\Omega_{u,k}^+)]^{1 - \frac{t_4}{p_2^*(s_2+1)}}. \tag{3.45}$$

In fact, if $t_4 = 0$ then (3.45) reduces to

$$\int_{\Omega_{u,k}^+} |v|^{t_4} dx = \text{meas}(\Omega_{u,k}^+),$$

while if $t_4 > 0$ from Step 1 we have that $z = |v|^{s_2} v \in W_2$, so (3.25) gives $\frac{p_2^*(s_2+1)}{t_4} > 1$ and the Hölder inequality with such an exponent, together with (2.11), implies that

$$\int_{\Omega^+_{u,k}} |v|^{t_4} dx = \int_{\Omega^+_{u,k}} |z|^{\frac{t_4}{s_2+1}} dx \le |z|_{p_2^*}^{\frac{t_4}{s_2+1}} [\mathrm{meas}(\Omega^+_{u,k})]^{1-\frac{t_4}{p_2^*(s_2+1)}}$$
$$\le (\tau_{2,p_2^*} \|z\|_{W_2})^{\frac{t_4}{s_2+1}} [\mathrm{meas}(\Omega^+_{u,k})]^{1-\frac{t_4}{p_2^*(s_2+1)}}.$$

On the other hand, if, for simplicity, we put $r = \frac{\bar{q}_1}{s_1+1}$, from (3.26), direct computations and, again, (2.11) we have that

$$\int_{\Omega^+_{u,k}} |w|^{\frac{\bar{q}_1}{s_1+1}} dx \le 2^{r-1} \left(\int_{\Omega^+_{u,k}} |w - \tilde{k}|^r dx + \tilde{k}^r \mathrm{meas}(\Omega^+_{u,k}) \right)$$
$$\le 2^{r-1} \left((\tau_{1,r} \|w\|_{W_1})^{r-p_1} \left(\int_{\Omega^+_{u,k}} |w - \tilde{k}|^r dx \right)^{\frac{p_1}{r}} + \tilde{k}^r \mathrm{meas}(\Omega^+_{u,k}) \right),$$

which, together with (3.45), allows us to reduce (3.44) to the estimate

$$\int_{\Omega^+_{u,k}} |\nabla w|^{p_1} dx \le c_4 \left(\left(\int_{\Omega^+_{u,k}} |w - \tilde{k}|^r dx \right)^{\frac{p_1}{r}} + \tilde{k}^r \mathrm{meas}(\Omega^+_{u,k}) + [\mathrm{meas}(\Omega^+_{u,k})]^{1-\frac{t_4}{p_2^*(s_2+1)}} \right) \tag{3.46}$$

with $c_4 = c_4(\|w\|_{W_1}, \|z\|_{W_2}) > 0$. At last, as $p_1 < N$, from (3.25), we have that

$$\mathrm{meas}(\Omega^+_{u,k}) = \mathrm{meas}(\Omega^+_{u,k})^{1-\frac{p_1}{N}+\varepsilon_1}, \quad \text{with } \varepsilon_1 = \frac{p_1}{N} > 0,\ \varepsilon_1 p_1^* + p_1 = p_1^*,$$
$$\mathrm{meas}(\Omega^+_{u,k})^{1-\frac{t_4}{p_2^*(s_2+1)}} = \mathrm{meas}(\Omega^+_{u,k})^{1-\frac{p_1}{N}+\varepsilon_2}, \quad \varepsilon_2 = \frac{p_1}{N} - \frac{t_4}{p_2^*(s_2+1)} > 0,$$

so, from (3.26) and (3.39), since (3.46) holds for all $\tilde{k}$ large enough, we have that Lemma 3.7 applies and $\mathrm{ess\,sup}_\Omega w < +\infty$ in contradiction with (3.34). Similar arguments, but modified in a suitable way, ensures that even (3.35) cannot occur, then it has to be $u \in L^\infty(\Omega)$, and also that it has to be $v \in L^\infty(\Omega)$.

Step 3 The proof can be obtained by reasoning as in the proof of [18, *Step 3* in Proposition 4.8] but with $m = 2$ and by replacing the estimates in [18, Remark 4.5] with those ones in Remark 3.8 together with (3.26), and also by using (3.31) at the place of [18, (4.19)].

Steps 4 and 5. The proofs are as in the corresponding steps of [18, Proposition 4.8] (see also [11, Proposition 4.6]). □

4 Existence and multiplicity results

Now, we can state our leading results. To this aim, we refer to the decomposition of X already introduced in [16, Section 5]. For the sake of convenience, here we recall the main issues. For $i \in \{1, 2\}$, the first eigenvalue of $-\Delta_{p_i}$ in W_i is given by

$$\lambda_{i,1} := \inf_{y\in W_i\setminus\{0\}} \frac{\int_\Omega |\nabla y|^{p_i} dx}{\int_\Omega |y|^{p_i} dx}. \tag{4.1}$$

Such an eigenvalue is simple, positive, isolated and has a unique eigenfunction $\varphi_{i,1}$ such that

$$\varphi_{i,1} > 0 \text{ a.e. in } \Omega, \quad \varphi_{i,1} \in L^\infty(\Omega) \quad \text{and} \quad |\varphi_{i,1}|_{p_i} = 1 \tag{4.2}$$

(see, e.g., [28]). Furthermore, a sequence of positive real numbers exists such that

$$0 < \lambda_{i,1} < \lambda_{i,2} \le \dots \le \lambda_{i,m} \le \dots, \qquad \text{with } \lambda_{i,m} \nearrow +\infty \text{ as } m \to +\infty, \tag{4.3}$$

with corresponding pseudo-eigenfunctions $(\psi_{i,m})_m$ which not only generate the whole space W_i, but are in $L^\infty(\Omega)$, too. Thus, $(\psi_{i,m})_m \subset X_i$, and, for any fixed $m \in \mathbb{N}$, we consider

$$V_{i,m} = \operatorname{span}\{\psi_{i,1}, \dots, \psi_{i,m}\}$$

and denote $Y_{i,m}$ its topological complement in W_i so that $W_i = V_{i,m} \oplus Y_{i,m}$ and the inequality

$$\lambda_{i,m+1} \int_\Omega |y|^{p_i} dx \le \int_\Omega |\nabla y|^{p_i} dx \quad \text{for all } y \in Y_{i,m} \tag{4.4}$$

is satisfied (cf. [11, Proposition 5.4]).

Thus, for any $m \in \mathbb{N}$ definition (2.12) implies that

$$W = (V_{1,m} \times V_{2,m}) \oplus (Y_{1,m} \times Y_{2,m}),$$

while from (2.15), it follows that

$$X = (V_{1,m} \times V_{2,m}) \oplus (Y_m^{X_1} \times Y_m^{X_2})$$

where, for $i \in \{1, 2\}$, it is $Y_m^{X_i} = Y_{i,m} \cap L^\infty(\Omega) \subset X_i$ and $X_i = V_{i,m} \oplus Y_m^{X_i}$, with

$$\dim (V_{i,m}) = m \quad \text{and} \quad \operatorname{codim} (Y_m^{X_i}) = m.$$

Now, we are ready to provide our existence and multiplicity results.

Theorem 4.1 *Suppose that $A(x,t,\xi)$, $B(x,t,\xi)$ comply with assumptions (h_0)–(h_6) and that a given function $G(x, u, v)$ satisfies hypotheses (g_0)–(g_3). Furthermore, assume that a constant $\alpha_2 > 0$ exists such that the following conditions hold:*

(h_7) *taking p_1, p_2 as in hypothesis (h_1) and s_1, $s_2 \ge 0$ as in assumption (h_3), we have that*

$$A(x,t,\xi) \ge \alpha_2(1 + |t|^{s_1 p_1})|\xi|^{p_1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N,$$
$$B(x,t,\xi) \ge \alpha_2(1 + |t|^{s_2 p_2})|\xi|^{p_2} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(g_4) *taking $\lambda_{1,1}$ and $\lambda_{2,1}$ as in (4.1), we have that*

$$\limsup_{(u,v)\to(0,0)} \frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} < \alpha_2 \min\{\lambda_{1,1}, \lambda_{2,1}\} \quad \text{uniformly a.e. in } \Omega.$$

Thus, functional $\mathcal{J}$ in (1.6) possesses at least one nontrivial critical point in X; hence, problem (1.2) admits a nontrivial weak bounded solution.

Theorem 4.2 *Suppose that $A(x,t,\xi)$, $B(x,t,\xi)$ and $G(x, u, v)$ satisfy hypotheses (h_0)–(h_6), (g_0)–(g_3). Moreover, if we assume also that:*

(h_8) *$A(x,\cdot,\cdot)$ and $B(x,\cdot,\cdot)$ are even in $\mathbb{R}\times\mathbb{R}^N$ for a.e. $x\in\Omega$;*
(g_5) *taking θ_1, θ_2 as in hypotheses (h_5) and (g_3), we have that*

$$\liminf_{|(u,v)|\to+\infty} \frac{G(x,u,v)}{|u|^{\frac{1}{\theta_1}}+|v|^{\frac{1}{\theta_2}}} > 0 \quad \text{uniformly a.e. in } \Omega;$$

(g_6) *$G(x,\cdot,\cdot)$ is even in $\mathbb{R}^2$ for a.e. $x\in\Omega$;*

then functional $\mathcal{J}$ in (1.6) possesses an unbounded sequence of critical points $((u_m,v_m))_m\subset X$ such that $\mathcal{J}(u_m,v_m)\nearrow+\infty$; hence, problem (1.2) admits infinitely many distinct weak bounded solutions.

Finally, by reasoning as in [16, Corollary 5.4], we can state this further multiplicity result since the supercritical growth in (3.26) does not affect its proof.

Corollary 4.3 *Let $p_1,p_2>1$ and suppose that the functions $A(x,t,\xi)$, $B(x,t,\xi)$ and $G(x,u,v)$ satisfy assumptions (h_0)–(h_6), (h_8), (g_0)–(g_3) and (g_6). Furthermore, if*

(g_7) $\inf\{G(x,w,z)\ :\ x\in\Omega,\ (w,z)\in\mathbb{R}^2 \text{ such that } |(w,z)|=R\}>0$, *with R as in (g_2);*
(g_8) *$\theta_1=\theta_2$, with θ_1, θ_2 as in (h_5) and (g_3);*

are satisfied too, the even functional $\mathcal{J}$ in (1.6) possesses a sequence of critical points $((u_m,v_m))_m$ in X such that $\mathcal{J}(u_m,v_m)\nearrow+\infty$; hence, problem (1.2) admits infinitely many distinct weak bounded solutions.

Before turning to the proof of our main results, we observe that if assumption (h_3), and then (h_7), holds with $s_1=s_2=0$, then Theorem 4.1 reduces to [18, Theorem 5.1] while Theorem 4.2 reduces to [18, Theorem 5.2] but with $m=2$. Actually, the same holds true if both $p_1\ge N$ and $p_2\ge N$. Thus, in order to improve such previous results, here we assume that either $s_1>0$ or $s_2>0$ and we define

$$\ell_i(y)=\max\{\|y\|_{W_i},\||y|^{s_i}y\|_{W_i}\}\quad \text{if } y\in X_i,\qquad \text{with } i\in\{1,2\}, \tag{4.5}$$

and then

$$\ell(u,v)=\max\{\|(u,v)\|_W,\|(|u|^{s_1}u,|v|^{s_2}v)\|_W\}\quad \text{if } (u,v)\in X. \tag{4.6}$$

From definitions (4.5) and (4.6), we have that

$$[\ell_i(y)]^{p_i}\ \le\ \|y\|_{W_i}^{p_i}+\||y|^{s_i}y\|_{W_i}^{p_i}\quad \text{if } y\in X_i,\qquad \text{with } i\in\{1,2\}, \tag{4.7}$$

and

$$\max\{\ell_1(u),\ell_2(v)\}\ \le\ \ell(u,v)\ \le\ \ell_1(u)+\ell_2(v)\quad \text{for all } (u,v)\in X. \tag{4.8}$$

Moreover, taking $\bar{p} = \min\{p_1, p_2\}$, direct computations imply that

$$[\ell_1(u)]^{p_1} + [\ell_2(v)]^{p_2} \geq \left[\frac{\ell(u,v)}{2}\right]^{\bar{p}} \quad \text{if } (u,v) \in X \text{ is such that } \ell(u,v) \geq 2. \tag{4.9}$$

Remark 4.4 For both $i \in \{1,2\}$ definition (2.14) and identity (3.23) imply that the function $y \mapsto \||y|^{s_i}y\|_{W_i}$ is continuous and well-defined in $(X_i, \|\cdot\|_{X_i})$ and so $\ell_i : X_i \to \mathbb{R}$ is continuous, too. Thus, from (2.15), we have that $(u,v) \mapsto \|(|u|^{s_1}u, |v|^{s_2}v)\|_W$ is continuous and well-defined in $(X, \|\cdot\|_X)$, then also $\ell : X \to \mathbb{R}$ is continuous with respect $\|\cdot\|_X$ and definition (4.6) implies that $\ell(u,v) \geq \|(u,v)\|_W$ for all $(u,v) \in X$ with $\ell(0,0) = 0$.

Throughout the remaining part of this section, for simplicity, we assume that

$$\int_\Omega A(x,0,\mathbf{0}_N)\,dx = 0, \qquad \int_\Omega B(x,0,\mathbf{0}_N)\,dx = 0, \tag{4.10}$$

with $\mathbf{0}_N = (0,\dots,0) \in \mathbb{R}^N$, and

$$\int_\Omega G(x,0,0)\,dx = 0. \tag{4.11}$$

Differently, one can always replace $\mathcal{J}(u,v)$ in (1.6) with the new functional

$$\mathcal{J}^*(u,v) = \mathcal{J}(u,v) - \int_\Omega A(x,0,\mathbf{0}_N)\,dx - \int_\Omega B(x,0,\mathbf{0}_N)\,dx + \int_\Omega G(x,0,0)\,dx,$$

since they share the same differential on X and so the same critical points.

For simplicity, we denote by c_i every positive constant which arises during computations.

Now, we can prove our existence result.

Proof of Theorem 4.1 Firstly, hypothesis (g_4) allows us to take $\bar{\lambda} > 0$ such that

$$\limsup_{(u,v)\to(0,0)} \frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} < \bar{\lambda} < \alpha_2 \min\{\lambda_{1,1}, \lambda_{2,1}\} \quad \text{uniformly a.e. in } \Omega. \tag{4.12}$$

Thus, from (4.12) and direct computations, estimate (2.9) ensures the existence of a constant $\sigma^* > 0$ such that

$$G(x,u,v) \leq \bar{\lambda}(|u|^{p_1} + |v|^{p_2}) + \sigma^*(|u|^{\bar{q}_1} + |v|^{\bar{q}_2}) \text{ for a.e. } x \in \Omega, \text{ for all } (u,v) \in \mathbb{R}^2, \tag{4.13}$$

with $\bar{q}_1, \bar{q}_2$ as in (2.10) so that (3.26) holds. Moreover, taking s_1, s_2 as in our setting of hypotheses and fixing any couple $(u,v) \in X$, from definition (1.6), condition (h_7), estimate (4.13) together with (3.23) and (4.1), it follows that

$$\begin{aligned} \mathcal{J}(u,v) \geq{}& \left(\alpha_2 - \frac{\bar{\lambda}}{\lambda_{1,1}}\right)\|u\|_{W_1}^{p_1} + \frac{\alpha_2}{(s_1+1)^{p_1}}\||u|^{s_1}u\|_{W_1}^{p_1} - \sigma^*|u|_{\bar{q}_1}^{\bar{q}_1} \\ &+ \left(\alpha_2 - \frac{\bar{\lambda}}{\lambda_{2,1}}\right)\|v\|_{W_2}^{p_2} + \frac{\alpha_2}{(s_2+1)^{p_2}}\||v|^{s_2}v\|_{W_2}^{p_2} - \sigma^*|v|_{\bar{q}_2}^{\bar{q}_2}, \end{aligned} \tag{4.14}$$

where from (3.26) and the Sobolev inequality (2.11), we have that

$$\int_\Omega |y|^{\bar{q}_i} dx = \int_\Omega ||y|^{s_i} y|^{\frac{\bar{q}_i}{s_i+1}} dx \leq c_1 \||y|^{s_i} y\|_{W_i}^{\frac{\bar{q}_i}{s_i+1}} \quad \text{for all } y \in X_i, i \in \{1, 2\}, \tag{4.15}$$

for a suitable $c_1 > 0$ independent of i. Then, by using (4.15) in (4.14), from (4.12), a positive constant $c_2 > 0$ exists such that definition (4.5), estimate (4.7) and direct computations imply that

$$\begin{aligned}\mathcal{J}(u,v) &\geq c_2(\|u\|_{W_1}^{p_1} + \||u|^{s_1}u\|_{W_1}^{p_1}) - c_3\||u|^{s_1}u\|_{W_1}^{\frac{\bar{q}_1}{s_1+1}} + c_2(\|v\|_{W_2}^{p_2} + \||v|^{s_2}v\|_{W_2}^{p_2}) - c_3\||v|^{s_2}v\|_{W_2}^{\frac{\bar{q}_2}{s_2+1}}\\ &\geq [\ell_1(u)]^{p_1}\left(c_2 - c_3[\ell_1(u)]^{\frac{\bar{q}_1}{s_1+1}-p_1}\right) + [\ell_2(v)]^{p_2}\left(c_2 - c_3[\ell_2(v)]^{\frac{\bar{q}_2}{s_2+1}-p_2}\right),\end{aligned}$$

for a suitable $c_3 > 0$; hence, from (3.26) and (4.8), we obtain that

$$\mathcal{J}(u,v) \;\geq\; [\ell_1(u)]^{p_1}\left(c_2 - c_3[\ell(u,v)]^{\frac{\bar{q}_1}{s_1+1}-p_1}\right) + [\ell_2(v)]^{p_2}\left(c_2 - c_3[\ell(u,v)]^{\frac{\bar{q}_2}{s_2+1}-p_2}\right). \tag{4.16}$$

We note that again from (3.26), a radius $r_0 > 0$ and a constant ϱ_1 can be found so that

$$c_2 - c_3 r_0^{\frac{\bar{q}_i}{s_i+1}-p_i} \;\geq\; \varrho_1 > 0 \quad \text{for both } i = 1 \text{ and } i = 2,$$

thus, from (4.8) and (4.16), we infer that a constant $\varrho_0 > 0$ exists such that

$$\ell(u,v) = r_0 \quad \implies \quad \mathcal{J}(u,v) \geq \varrho_0. \tag{4.17}$$

On the other hand, from (h_0)–(h_2) and (h_5) we have that [11, Proposition 6.5] implies the existence of some constants $b_1^*, b_2^* > 0$ such that

$$|A(x,t,\xi)| \leq b_1^*\left(1 + |t|^{\frac{1}{\theta_1}\left(1-\frac{\mu_2}{\eta_1}\right)}\right) + b_2^*\left(1 + |t|^{\frac{1}{\theta_1}\left(1-\frac{\mu_2}{\eta_1}\right)-p_1}\right)|\xi|^{p_1}$$

a.e. in Ω and for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$, with η_1, μ_2 as in (h_2), respectively (h_5), and, without loss of generality, we can assume $\frac{1}{\theta_1}\left(1 - \frac{\mu_2}{\eta_1}\right) - p_1 > 0$ (a priori, we can take either μ_2 small enough or η_1 large enough). Thus, taking $\varphi_{1,1} \in X_1$ as in (4.2), from (1.6), (3.21), (4.10) and direct computations, we obtain that

$$\begin{aligned}\mathcal{J}(\tau\varphi_{1,1}, 0) &\leq b_1^*\tau^{\frac{1}{\theta_1}(1-\frac{\mu_2}{\eta_1})}\int_\Omega |\varphi_{1,1}|^{\frac{1}{\theta_1}(1-\frac{\mu_2}{\eta_1})}dx + b_2^*\tau^{p_1}\int_\Omega |\nabla\varphi_{1,1}|^{p_1}dx\\ &\quad + b_2^*\tau^{\frac{1}{\theta_1}(1-\frac{\mu_2}{\eta_1})}\int_\Omega |\varphi_{1,1}|^{\frac{1}{\theta_1}(1-\frac{\mu_2}{\eta_1})-p_1}|\nabla\varphi_{1,1}|^{p_1}dx - \tau^{\frac{1}{\theta_1}}\int_\Omega h_1(x)|\varphi_{1,1}|^{\frac{1}{\theta_1}}dx + c_4,\end{aligned}$$

for a suitable $c_4 > 0$, which implies, from (3.4) that

$$\mathcal{J}(\tau\varphi_{1,1}, 0) \to -\infty \quad \text{as } \tau \to +\infty$$

as (4.2) and Remark 3.5 ensure that $\int_\Omega h_1(x)|\varphi_{1,1}|^{\frac{1}{\theta_1}}dx > 0$. Hence, considering r_0, ϱ_0 so that (4.17) holds, a point $e_1 \in X_1$ can be found so that

$$\|(e_1, 0)\|_W > r_0 \quad \text{and} \quad \mathcal{J}(e_1, 0) < \varrho_0. \tag{4.18}$$

Finally, from (1.6), (4.10) and (4.11), it is $\mathcal{J}(0,0)=0$, which, together with Remark 4.4, (4.17), (4.18) and Propositions 2.8 and 3.10, ensures that Theorem 2.2 applies and a critical point (u, v) exists in X such that $\mathcal{J}(u,v)\geq \varrho_0>0$. $\square$

In order to prove our multiplicity theorem, some geometric conditions are needed. In particular, if assumptions (h_0)–(h_6) and (g_0)–(g_3) hold, we are able to state the following results.

Proposition 4.5 *For any fixed $\varrho\in\mathbb{R}$, an integer $m=m(\varrho)\geq 1$ and a radius $R_m>0$ exist such that*

$$(u,v)\in Y_m^{X_1}\times Y_m^{X_2},\quad \ell(u,v)=R_m \qquad\Longrightarrow\qquad \mathcal{J}(u,v)\geq\varrho.$$

Proof Firstly, we note that (3.23) and (4.7) imply that

$$\int_\Omega (1+|y|^{s_ip_i})|\nabla y|^{p_i}dx\geq \frac{1}{(s_i+1)^{p_i}}\,[\ell_i(y)]^{p_i}\quad \text{if } y\in X_i,\ \text{for each } i\in\{1,2\}. \tag{4.19}$$

Then, taking $(u,v)\in X$, from (1.6), (3.11), (3.12), (4.19), together with (2.9) where $\bar q_1$, $\bar q_2$ satisfy (3.26), we obtain that

$$\begin{aligned}\mathcal{J}(u,v)\ \geq\ & \frac{\mu_0(\mu_1\theta_1+\mu_2)}{(s_1+1)^{p_1}}\,[\ell_1(u)]^{p_1}+\frac{\mu_0(\mu_1\theta_2+\mu_2)}{(s_2+1)^{p_2}}\,[\ell_2(v)]^{p_2}\\ &-\ \sigma_2\int_\Omega |u|^{\bar q_1}dx\ -\ \sigma_2\int_\Omega |v|^{\bar q_2}dx-c_1,\end{aligned} \tag{4.20}$$

for some $c_1>0$. We note that for each $i\in\{1,2\}$ condition (3.26) allows us to take $r_i>0$ so that

$$\frac{r_i}{p_i}+\frac{\bar q_i-r_i}{p_i^*(s_i+1)}=1,$$

then, reasoning as in [14, Proposition 4.5], from classical interpolation arguments, (2.11) and (4.5), we obtain that

$$\int_\Omega |y|^{\bar q_i}dx\ \leq\ c_2[\ell_i(y)]^{\frac{\bar q_i-r_i}{s_i+1}}\left(\int_\Omega |y|^{p_i}dx\right)^{\frac{r_i}{p_i}}\qquad \text{for all } y\in X_i,$$

for a suitable constant $c_2>0$ independent of i.

Thus, fixing any $m\in\mathbb{N}$, from (4.4) and, again, (4.5), it follows that

$$\int_\Omega |y|^{\bar q_i}dx\ \leq\ c_2\lambda_{i,m+1}^{-\frac{r_i}{p_i}}[\ell_i(y)]^{\frac{r_is_i+\bar q_i}{s_i+1}}\qquad \text{for all } y\in Y_m^{X_i}, \tag{4.21}$$

where from (3.26), it is

$$\frac{r_is_i+\bar q_i}{s_i+1}\ >\ p_i. \tag{4.22}$$

Hence, taking any couple $(u,v)\in Y_m^{X_1}\times Y_m^{X_2}$, by using estimate (4.21) in (4.20), we obtain that

$$\mathcal{J}(u,v) \geq c_3\,[\ell_1(u)]^{p_1} - c_4\lambda_{1,m+1}^{-\frac{r_1}{p_1}}[\ell_1(u)]^{\frac{r_1 s_1+\bar{q}_1}{s_1+1}} + c_3[\ell_2(v)]^{p_2} - c_4\lambda_{2,m+1}^{-\frac{r_2}{p_2}}[\ell_2(v)]^{\frac{r_2 s_2+\bar{q}_2}{s_2+1}} - c_1$$

or better, from (4.8) and (4.22), we have that

$$\begin{aligned}\mathcal{J}(u,v) \;&\geq [\ell_1(u)]^{p_1}\left(c_3 - c_4\lambda_{1,m+1}^{-\frac{r_1}{p_1}}\,[\ell(u,v)]^{\frac{r_1 s_1+\bar{q}_1}{s_1+1}-p_1}\right)\\ &\quad + [\ell_2(v)]^{p_2}\left(c_3 - c_4\lambda_{2,m+1}^{-\frac{r_2}{p_2}}\,[\ell(u,v)]^{\frac{r_2 s_2+\bar{q}_2}{s_2+1}-p_2}\right) - c_1.\end{aligned} \tag{4.23}$$

Now, for each $i \in \{1,2\}$, from (4.22), we can define $R_{i,m} > 0$ so that

$$c_4\lambda_{i,m+1}^{-\frac{r_i}{p_i}}\,R_{i,m}^{\frac{r_i s_i+\bar{q}_i}{s_i+1}-p_i} = \frac{c_3}{2} \quad\Longleftrightarrow\quad R_{i,m} = \left(\frac{c_3}{2c_4}\,\lambda_{i,m+1}^{\frac{r_i}{p_i}}\right)^{\frac{s_i+1}{r_i s_i+\bar{q}_i-p_i(s_i+1)}} \tag{4.24}$$

and, since from (4.3), it follows that $R_{i,m} \nearrow +\infty$ as $m \to +\infty$, we have that

$$R_m := \min\{R_{1,m}, R_{2,m}\} \;\to\; +\infty \quad \text{as } m \to +\infty \tag{4.25}$$

which implies $R_m \geq 2$ for all $m \geq m_0$ if $m_0 \in \mathbb{N}$ is large enough. So, for any $m \geq m_0$, taking $(u,v) \in Y_m^{X_1} \times Y_m^{X_2}$ such that $\ell(u,v) = R_m$, from (4.9), we have that

$$[\ell_1(u)]^{p_1} + [\ell_2(v)]^{p_2} \;\geq\; \left(\frac{R_m}{2}\right)^{\bar{p}}, \tag{4.26}$$

while from (4.23), by using (4.22), (4.24) and the definition in (4.25), we obtain

$$\begin{aligned}\mathcal{J}(u,v) &\geq [\ell_1(u)]^{p_1}\left(c_3 - c_4\lambda_{1,m+1}^{-\frac{r_1}{p_1}}R_m^{\frac{r_1 s_1+\bar{q}_1}{s_1+1}-p_1}\right) + [\ell_2(v)]^{p_2}\left(c_3 - c_4\lambda_{2,m+1}^{-\frac{r_2}{p_2}}R_m^{\frac{r_2 s_2+\bar{q}_2}{s_2+1}-p_2}\right) - c_1\\ &\geq [\ell_1(u)]^{p_1}\left(c_3 - c_4\lambda_{1,m+1}^{-\frac{r_1}{p_1}}R_{1,m}^{\frac{r_1 s_1+\bar{q}_1}{s_1+1}-p_1}\right) + [\ell_2(v)]^{p_2}\left(c_3 - c_4\lambda_{2,m+1}^{-\frac{r_2}{p_2}}R_{2,m}^{\frac{r_2 s_2+\bar{q}_2}{s_2+1}-p_2}\right) - c_1\\ &= \frac{c_3}{2}\,\left([\ell_1(u)]^{p_1} + [\ell_2(v)]^{p_2}\right) - c_1.\end{aligned}$$

Thus, for any $m \geq m_0$ estimate (4.26) implies that

$$\mathcal{J}(u,v) \;\geq\; \frac{c_3}{2}\left(\frac{R_m}{2}\right)^{\bar{p}} - c_1 \quad \text{if } (u,v) \in Y_m^{X_1} \times Y_m^{X_2} \text{ is such that } \ell(u,v) = R_m. \tag{4.27}$$

Finally, we note that the proof follows from (4.25) and (4.27). □

At last, by reasoning as in the first part of the proof of [18, Theorem 5.2] (we note that the computations do not involve the supercritical growth of $G(x, u, v)$ but only its lower bound coming from assumption (g_5)), the following result can be stated, too.

Proposition 4.6 *If also hypothesis (g_5) holds, then for any finite-dimensional subspace V of X a suitable radius $R_V > 0$ exists such that*

$$\mathcal{J}(u,v) \leq 0 \quad \text{for all } (u,v) \in V \text{ such that } \|(u,v)\|_X \geq R_V.$$

In particular, the functional $\mathcal{J}$ is bounded form above in V.

Now, we can prove our multiplicity results.

Proof of Theorem 4.2 Firstly, we observe that (1.6), (4.10) and (4.11) give $\mathcal{J}(0,0) = 0$, while assumptions (h_8) and (g_6) imply that the functional $\mathcal{J}$ is even in X. Furthermore, taking any $r > 0$, we set

$$\mathcal{M}_r = \{(u,v) \in X : \ \ell(u,v) = r\}.$$

By definition, $\mathcal{M}_r$ is the boundary of a symmetric neighborhood of the origin which is bounded with respect to $\|\cdot\|_W$. Now, fixing any $\varrho > 0$, from Proposition 4.5, it follows that an integer $m_\varrho \geq 1$ and a radius $r_\varrho = r_\varrho(m_\varrho) > 0$ exist so that

$$(u,v) \in \mathcal{M}_{r_\varrho} \cap (Y^{X_1}_{m_\varrho} \times Y^{X_2}_{m_\varrho}) \quad \Longrightarrow \quad \mathcal{J}(u,v) \geq \varrho,$$

while, by choosing $m > m_\varrho$, the m-dimensional space V_m is such that codim $Y_{m_\varrho} <$ dim V_m, and from Proposition 4.6 a radius $R_{V_m} > 0$ exists so that

$$\mathcal{J}(u,v) \leq 0 \quad \text{for all } (u,v) \in V_m \text{ such that } \|(u,v)\|_X \geq R_{V_m}.$$

Hence, assumption $(\mathcal{H}_\varrho)$ in Theorem 2.3 is verified. Then, the arbitrariness of $\varrho > 0$ so that $(\mathcal{H}_\varrho)$ holds, together with Propositions 2.8 and 3.10, allows us to apply Corollary 2.5 and the existence of a sequence of diverging critical levels for the functional $\mathcal{J}$ in X is provided. □

Proof of Theorem 1.3 Taking $A(x,t,\xi)$ and $B(x,t,\xi)$ as in (1.7), from (1.10), it follows that conditions (h_0)–(h_4) and (h_6) hold. Moreover, if $G(x, u, v)$ is as in (1.8), assumptions (1.10)–(1.12) and Young inequality imply that (g_0)–(g_2) are satisfied with

$$t_1 = \gamma_2 \frac{q_1 - 1}{q_1 - \gamma_1}, \quad t_2 = \gamma_1 \frac{q_2 - 1}{q_2 - \gamma_2}.$$

On the other hand, again from (1.10), direct computations allow us to prove that hypotheses (h_5) and (g_3) are verified, too. At last, also condition (g_5) holds as (1.10) and direct computations allow us to prove that for any $R \geq 2$ it is

$$\frac{G(x,u,v)}{|u|^{\frac{1}{\theta_1}} + |v|^{\frac{1}{\theta_2}}} \geq \frac{1}{2} \min\left\{\frac{1}{q_1}, \frac{1}{q_2}\right\} \quad \text{if } (u,v) \in \mathbb{R}^2 \text{ is such that } |(u,v)| \geq R.$$

Then, since the symmetric assumptions (h_8) and (g_6) are trivially satisfied, the thesis follows from Theorem 4.2. □

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