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A multiplicity result for a (p, q)-Schrödinger–Kirchhoff type equation

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Abstract
In this paper, we study a class of (p, q)-Schrödinger–Kirchhoff type equations involving a continuous positive potential satisfying del Pino–Felmer type conditions and a continuous nonlinearity with subcritical growth at infinity. By applying variational methods, penalization techniques and Lusternik–Schnirelman category theory, we relate the number of positive solutions with the topology of the set where the potential attains its minimum values.

Keywords $(p \cdot q)$-Laplacian problem · Penalization technique · Lusternik–Schnirelman category theory

Mathematics Subject Classification 35A15 · 35J62 · 35Q55 · 55M30

1 Introduction

1.1 Background and motivations

In this paper, we consider the following class of (p, q)-Schrödinger–Kirchhoff type problems:

$$\begin{cases} -\left(1+a\displaystyle\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x\right)\Delta_p u-\left(1+b\displaystyle\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\right)\Delta_q u+V(\varepsilon x)(u^{p-1}+u^{q-1})=f(u) \text{ in } \mathbb{R}^N, \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a small parameter, $a, b > 0$, $1 < p < q < N < 2q$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, with $r \in \{p, q\}$, is the r-Laplacian operator, the potential $V : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ are continuous functions.

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When $a = b = 0$, the equation in (1) becomes a (p, q)-Schrödinger equation of the form

$$-\Delta_p u - \Delta_q u + V(x)(u^{p-1} + u^{q-1}) = f(x, u) \quad \text{in } \mathbb{R}^N, \tag{2}$$

whose study is motivated by the general reaction diffusion system

$$u_t = \operatorname{div}[D(u)\nabla u] + c(x, u)$$

where $D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related sciences such as biophysics, plasma physics and chemical reaction design. In these applications, the function u describes a concentration; the first term on the right-hand side of (2) corresponds to the diffusion with a diffusion coefficient Du, whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration u; see [14] for more details about the applications.

We point out that several existence, multiplicity and regularity results for (2) have been established in these last years by several authors. In [28], the authors established some regularity results for (2). By combining concentration-compactness principle and mountain pass theorem, an existence result has been obtained in [29]. The author in [22] proved the existence of a ground-state positive solution for a (p, q)-Schrödinger equation with critical growth. The multiplicity and concentration of nontrivial solutions for (2) have been established in [3]. The authors in [39] combined refined variational methods based on critical point theory with Morse theory and truncation techniques to obtain a multiplicity result for a (p, q)-Laplacian problem in bounded domains. For other interesting results, one can consult [2, 7, 8, 34, 40, 44] and references therein.

When $p = q$, (2) boils down a p-Schrödinger equation of the type

$$-\Delta_p u + V(x)u^{p-1} = f(x, u) \quad \text{in } \mathbb{R}^N, \tag{3}$$

for which different and interesting results have been obtained in the literature; see for instance [3, 4, 19, 23, 52]. Note that, in the case $p = q = 2$, Eq. (3) reduces to the well-known Schrödinger equation which has been widely studied in the last three decades; see for example [18, 25, 43, 48, 50].

When $a = b \neq 0$, $p = q = 2$ and $N = 3$, problem (1) becomes a Kirchhoff equation of the form

$$-\left(1 + a\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^3. \tag{4}$$

Equation (4) is related to the stationary analog of the Kirchhoff equation [32]

$$u_{tt} - \left(\alpha + \beta\int_{\Omega} |\nabla u|^2\,dx\right)\Delta u = f(x, u) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\alpha > 0$, $\beta \geq 0$, and u satisfies some boundary conditions, which was proposed by Kirchhoff in 1883 as a nonlinear extension of D'Alembert's wave equation for free vibration of elastic strings

$$\rho u_{tt} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2 dx\right)u_{xx} = 0. \tag{5}$$

Here, $u = u(x,t)$ is the transverse string displacement at the space coordinate x and the time t, L is the length of the string, h is the area of the cross section, E is the Young's modulus of the material, ρ is the mass density and p_0 is the initial axial tension.

From a purely mathematical point of view, it is important to mention that the early studies dedicated to the Kirchhoff equation (5) were given by Bernstein [10] and Pohozaev [42]. However, the Kirchhoff equation (5) began to attract the attention of more researchers only after the paper by Lions [36], in which a functional analysis approach was proposed to attack it. For some interesting results on Kirchhoff problems, we refer to [13, 24, 30, 41, 49].

Finally, if $a = b \neq 0$, $p = q > 1$ and $N = 3$ in (1), we have the following p-Laplacian Kirchhoff type equation

$$-\left(1 + a\int_{\mathbb{R}^3} |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u + V(x)|u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^3, \tag{6}$$

which has been investigated in several works; see for instance [15, 16, 27, 35, 51].

Due to the interest shared by the mathematical community toward quasilinear problems and Kirchhoff type equations, in [12, 31], the authors studied Kirchhoff type equations involving the (p,q)-Laplacian operator with $p \neq q$, in a bounded domain and in the whole of $\mathbb{R}^3$, respectively.

Motivated by the above works, the purpose of this paper is to study the multiplicity and the concentration of solutions for (1).

1.2 Assumptions and main result

For simplicity, we assume that $a = b = 1$ in (1). Let us now introduce the hypotheses on the potential V and the nonlinearity f that we are going to consider throughout the paper.

Let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous function that satisfies the following assumptions due to del Pino–Felmer [18]:

(V_1) there exists $V_0 > 0$ such that $V_0 = \inf_{x\in\mathbb{R}^N} V(x)$;

(V_2) there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad 0 \in M = \{x \in \Lambda : V(x) = V_0\}.$$

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(t) = 0$ for $t \leq 0$ and fulfills the following hypotheses:

(f_1) $\displaystyle\lim_{|t|\to 0} \frac{|f(t)|}{|t|^{2p-1}} = 0$;

(f_2) there exists $\nu \in (2q, q^*)$ such that $\displaystyle\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{\nu-1}} = 0$, where $q^* = \frac{Nq}{N-q}$;

(f_3) there exists $\mu \in (2q, \nu)$ such that $0 < \mu F(t) = \mu \displaystyle\int_0^t f(\tau)\,\mathrm{d}\tau \leq tf(t)$ for all $t > 0$;

(f_4) the map $t \mapsto \dfrac{f(t)}{t^{2q-1}}$ is increasing for $t > 0$.

In order to give the precise statement of our main theorem, let us recall that, for any closed subset Y of a topological space X, the Lusternik–Schnirelman category of Y in

X, $cat_X(Y)$, stands for the least number of closed and contractible sets in X which cover Y; see [50]. Then, we can state the following result.

Theorem 1 *Let V satisfy (V_1) and (V_2), and let f be a continuous function such that the hypotheses (f_0)–(f_4) hold. Then, for any $\delta > 0$ such that*

$$M_\delta = \left\{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \le \delta\right\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1) has at least $cat_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^N$ is a global maximum point of u_ε, then

$$\lim_{\varepsilon \to 0} V(\varepsilon x_\varepsilon) = V_0.$$

1.3 Main difficulties and ideas

Due to the lack of information about the behavior of the potential V at infinity and the fact that our problem is set in an unbounded domain, we adapt the local mountain pass argument introduced by del Pino and Felmer [18]. It consists in making a suitable modification on f, solving a modified problem, whose corresponding energy functional has a nice geometric structure, and then checking that, for $\varepsilon > 0$ small enough, the solutions of the new problem are indeed solutions of the original one. We note that, because of the presence of the (p, q)-Laplacian operators and Kirchhoff terms, even for the corresponding modified energy functional, it is hard to obtain compactness, and an accurate analysis will be done to prove a first existence result for the modified problem; see Lemmas 5, 6 and 7. Secondly, we make use of a technique given by Benci and Cerami [9] to establish a relationship between the category of the set M and the number of solutions for the modified problem. We underline that, since f is merely continuous, standard $\mathcal{C}^1$-Nehari manifold arguments as in [2–5, 30, 50] do not work in our setting, and so we take advantage of some abstract results due to Szulkin and Weth [46]. Note that, this type of approach has been also used in [24] where a Schrödinger–Kirchhoff elliptic equation was considered. Clearly, with respect to [24], a more careful analysis will be needed and some refined estimates will be used to overcome some technical difficulties. Finally, to obtain a uniform L^∞-estimate for an appropriate translated sequence of solutions to the modified problem, we do not use the classical Moser iteration argument [38] as in [3, 19, 23, 24, 30], because such technique does not seem to work well in our situation, but we follow some arguments found in [2, 21, 26, 33] which are inspired by the well-known method pioneered by De Giorgi [17]; see Lemma 15.

As far as we know, all results presented in this work are new in the literature. Moreover, we believe that the ideas developed here can be applied in other situation to study (p, q)-Schrödinger–Kirchhoff type problems involving potentials satisfying local conditions and continuous nonlinearities.

The outline of the paper is the following. In Sect. 2, we introduce the modified problem. Section 3 is devoted to the study of the autonomous problem associated with (1). In Sect. 4, we prove a multiplicity result for the modified problem. The proof of Theorem 1 is given in Sect. 5.

2 The modified problem

2.1 Notations and preliminary results

In order to simplify the presentation, we denote by C a generic positive constant, which may change from line to line, but does not depend on crucial quantities. Let A be a measurable subset of $\mathbb{R}^N$. By A^c, we denote the complement of A. Let $1 \leq r \leq \infty$. We will use the notation $|\cdot|_{L^r(A)}$ for the norm in $L^r(A)$, and when $A = \mathbb{R}^N$, we simply write $|\cdot|_r$. By $\mathcal{B}_r(x_0)$, we indicate the open ball in $\mathbb{R}^N$ centered at $x_0 \in \mathbb{R}^N$ and radius $r > 0$. In the case $x_0 = 0$, we simply write $\mathcal{B}_r$.

Let $1 < r < \infty$ and $N > r$. By $\mathcal{D}^{1,r}(\mathbb{R}^N)$, we mean the closure of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ functions with respect to the norm

$$|\nabla u|_r^r = \int_{\mathbb{R}^N} |\nabla u|^r \mathrm{d}x.$$

By $W^{1,r}(\mathbb{R}^N)$, we denote the Sobolev space equipped with the norm

$$\|u\|_{W^{1,r}(\mathbb{R}^N)} = \left(|u|_r^r + |\nabla u|_r^r\right)^{\frac{1}{r}}.$$

The following embeddings are well known.

Theorem 2 [1] *Let $p \in (1, \infty)$ and $N > p$. Then, there exists a constant $S_* > 0$ such that, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,*

$$|u|_{p^*}^p \leq S_*^{-1} |\nabla u|_p^p.$$

Moreover, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, p^*]$ and compactly in $L^t(\mathcal{B}_R)$, for all $R > 0$ and any $t \in [1, p^*)$.

For the reader's convenience, we also recall the following vanishing lemma.

Lemma 1 [37] *Let $p \in (1, \infty)$, $N > p$ and $r \in [p, p^*)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathcal{B}_R(y)} |u_n|^r \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p^*)$.

Let $p, q \in (1, \infty)$ and set

$$W^{p,q} = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$$

endowed with the norm

$$\|u\|_{W^{p,q}} = \|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{W^{1,q}(\mathbb{R}^N)}.$$

For any $\varepsilon > 0$, we introduce the space

$$\mathbb{W}_\varepsilon = \left\{ u \in W^{p,q} \,:\, \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^p + |u|^q)\,\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_{\mathbb{W}_\varepsilon} = \|u\|_{V_\varepsilon,p} + \|u\|_{V_\varepsilon,q},$$

where

$$\|u\|_{V_\varepsilon,r} = \left(|\nabla u|_r^r + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^r\,\mathrm{d}x \right)^{\frac{1}{r}} \quad \text{for all } r \in (1,\infty).$$

Finally, we recall the following well-known elementary inequalities [45] which will be used in the sequel: for any $\xi, \eta \in \mathbb{R}^N$, we have

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta)\cdot(\xi - \eta) \geq c_1|\xi - \eta|^r \quad \text{for } r \geq 2, \tag{7}$$

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta)\cdot(\xi - \eta) \geq c_2 \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-r}} \quad \text{for } 1 < r < 2, \tag{8}$$

for some $c_1, c_2 > 0$ constants. In particular,

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta)\cdot(\xi - \eta) > 0 \quad \text{for all } \xi \neq \eta. \tag{9}$$

Note that, when $1 < r < 2$ using (8) and the following elementary inequality

$$(|\xi| + |\eta|)^r \leq 2^{r-1}(|\xi|^r + |\eta|^r) \quad \text{for all } \xi, \eta \in \mathbb{R}^N,$$

we deduce that there exists $c_3 > 0$ such that for any $\xi, \eta \in \mathbb{R}^N$, the following relation satisfies

$$[(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta)\cdot(\xi - \eta)]^{\frac{r}{2}} \geq c_3 \frac{|\xi - \eta|^r}{(|\xi|^r + |\eta|^r)^{\frac{2-r}{2}}} \quad \text{for } 1 < r < 2. \tag{10}$$

2.2 The penalization approach

To deal with (1), we use a del Pino–Felmer penalization type approach [18]. Firstly, we note that the map $t \mapsto \frac{f(t)}{t^{p-1}+t^{q-1}}$ is increasing in $(0,\infty)$. Indeed, once we write

$$\frac{f(t)}{t^{p-1} + t^{q-1}} = \frac{f(t)}{t^{2q-1}} \frac{t^{2q-1}}{t^{p-1} + t^{q-1}}$$

then, by (f_4), we know that $t \mapsto \frac{f(t)}{t^{2q-1}}$ is increasing in $(0,\infty)$, and since $2q > p$, we deduce that $t \mapsto \frac{t^{2q-1}}{t^{p-1}+t^{q-1}}$ is increasing in $(0,\infty)$.

Take

$$K > \frac{q}{p}\left(\frac{\mu - p}{\mu - q}\right) > 1,$$

and let $a > 0$ be such that

$$f(a) = \frac{V_0}{K}(a^{p-1} + a^{q-1}).$$

Consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \le a, \\ \dfrac{V_0}{K}(t^{p-1} + t^{q-1}) & \text{if } t > a. \end{cases}$$

Denote by χ_A the characteristic function of $A \subset \mathbb{R}^N$, and define the function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as

$$g(x,t) = \begin{cases} \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t) & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

Using the hypotheses on f, we infer that g is a Carathéodory function such that

- (g_1) $\displaystyle\lim_{t\to 0} \frac{g(x,t)}{t^{2p-1}} = 0$ uniformly with respect to $x \in \mathbb{R}^N$,
- (g_2) $g(x,t) \le f(t)$ for all $x \in \mathbb{R}^N$ and $t > 0$,
- (g_3) (i) $0 < \mu G(x,t) \le g(x,t)t$ for all $x \in \Lambda$ and $t > 0$, where $G(x,t) = \displaystyle\int_0^t g(x,\tau)\,\mathrm{d}\tau$,

  (ii) $0 \le pG(x,t) \le g(x,t)t \le \dfrac{V_0}{\mathrm{K}}(t^p + t^q)$ for all $x \in \Lambda^c$ and $t > 0$,
- (g_4) for each $x \in \Lambda$, the function $t \mapsto \dfrac{g(x,t)}{(t^{p-1} + t^{q-1})}$ is increasing in $(0,\infty)$, and for each $x \in \Lambda^c$, the function $t \mapsto \dfrac{g(x,t)}{(t^{p-1} + t^{q-1})}$ is increasing in $(0,a)$.

We point that, from (g_1), (g_2), (f_1) and (f_2), for any $\zeta > 0$, there exists $C_\zeta > 0$ such that

$$|g(x,t)| \le \zeta |t|^{p-1} + C_\zeta |t|^{\nu-1} \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{11}$$

Let us introduce the following auxiliary problem:

$$\begin{cases} -\left(1 + \displaystyle\int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x\right)\Delta_p u - \left(1 + \displaystyle\int_{\mathbb{R}^N} |\nabla u|^q\,\mathrm{d}x\right)\Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = g(\varepsilon x, u) \text{ in } \mathbb{R}^N, \\ u \in W^{p,q}, \quad u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{12}$$

Define the set $\Lambda_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$. We underline that if u_ε is a solution to (12) satisfying $u_\varepsilon(x) \le a$ for all $x \in \Lambda_\varepsilon^c$, then u_ε is also a solution to (1).

Let us introduce the functional $L_\varepsilon : \mathbb{W}_\varepsilon \to \mathbb{R}$ associated with (12), that is

$$L_\varepsilon(u) = \frac{1}{p}\|u\|_{V_\varepsilon,p}^p + \frac{1}{2p}|\nabla u|_p^{2p} + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q + \frac{1}{2q}|\nabla u|_q^{2q} - \int_{\mathbb{R}^N} G(\varepsilon x, u)\,\mathrm{d}x.$$

We note that $L_\varepsilon \in \mathcal{C}^1(\mathbb{W}_\varepsilon, \mathbb{R})$ and

$$\begin{aligned}\langle L_\varepsilon'(u), \varphi\rangle &= (1 + |\nabla u|_p^p)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,\mathrm{d}x + (1 + |\nabla u|_q^q)\int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla\varphi\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}u\,\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{q-2}u\,\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} g(\varepsilon x, u)\varphi\,\mathrm{d}x\end{aligned}$$

for any $u, \varphi \in \mathbb{W}_\varepsilon$.

The Nehari manifold associated with L_ε is given by

$$\mathcal{N}_\varepsilon = \big\{u \in \mathbb{W}_\varepsilon : \langle L_\varepsilon'(u), u\rangle = 0\big\},$$

and let

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} L_\varepsilon(u).$$

Let $\mathbb{S}_\varepsilon = \big\{u \in \mathbb{W}_\varepsilon : \|u\|_{\mathbb{W}_\varepsilon} = 1\big\}$ be the unit sphere in $\mathbb{W}_\varepsilon$, and set $\mathbb{S}_\varepsilon^+ = \mathbb{S}_\varepsilon \cap \mathbb{W}_\varepsilon^+$, where $\mathbb{W}_\varepsilon^+$ stands for the open set

$$\mathbb{W}_\varepsilon^+ = \{u \in \mathbb{W}_\varepsilon : |\mathrm{supp}(u^+) \cap \Lambda_\varepsilon| > 0\}.$$

Note that, $\mathbb{S}_\varepsilon^+$ is an incomplete $\mathcal{C}^{1,1}$-manifold of codimension one. Hence, for all $u \in \mathbb{S}_\varepsilon^+$, $\mathbb{W}_\varepsilon = T_u\mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$, where

$$\begin{aligned}T_u\mathbb{S}_\varepsilon^+ = \Bigg\{v \in \mathbb{W}_\varepsilon :\ & (1 + |\nabla u|_p^p)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x \\ & + (1 + |\nabla u|_q^q)\int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla v\,\mathrm{d}x \\ & + \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^{p-2}uv + |u|^{q-2}uv)\,\mathrm{d}x = 0\Bigg\}.\end{aligned}$$

First, we show that L_ε has a mountain pass geometry [6].

Lemma 2 *The functional L_ε has the following properties:*

(i) *There exist $\alpha, \rho > 0$ such that $L_\varepsilon(u) \ge \alpha$ for $\|u\|_{\mathbb{W}_\varepsilon} = \rho$.*
(ii) *There exists $e \in \mathbb{W}_\varepsilon$ with $\|e\|_{\mathbb{W}_\varepsilon} > \rho$ and $L_\varepsilon(e) < 0$.*

Proof (i) Fix $\zeta \in (0, V_0)$. From (11), we have

$$L_\varepsilon(u) \geq C_1 \|u\|_{V_\varepsilon,p}^p + \frac{1}{q}\|u\|_{V_\varepsilon,q}^q - \frac{C_\zeta}{\nu}|u|_\nu^\nu.$$

Choosing $\|u\|_{\mathbb{W}_\varepsilon} = \rho \in (0,1)$ and using $1 < p < q$, we have $\|u\|_{V_\varepsilon,p} < 1$ and thus $\|u\|_{V_\varepsilon,p}^p \geq \|u\|_{V_\varepsilon,p}^q$. Recalling that

$$a^t + b^t \geq C_t(a+b)^t \quad \text{for all } a, b \geq 0, t > 1,$$

and using Theorem 2, we find

$$L_\varepsilon(u) \geq C_2\|u\|_{\mathbb{W}_\varepsilon}^q - \frac{C_\zeta}{\nu}|u|_\nu^\nu \geq C_2\|u\|_{\mathbb{W}_\varepsilon}^q - C_3\|u\|_{\mathbb{W}_\varepsilon}^\nu.$$

Since $\nu > q$, there exists $\alpha > 0$ such that $L_\varepsilon(u) \geq \alpha$ for $\|u\|_{\mathbb{W}_\varepsilon} = \rho$.

(ii) By (f_3), we deduce that

$$F(t) \geq At^\mu - B \quad \text{for all } t > 0.$$

Then, for all $u \in \mathbb{W}_\varepsilon^+$ and $t > 0$, we have

$$\begin{aligned} L_\varepsilon(tu) \leq\ & \frac{t^p}{p}\|u\|_{\varepsilon,p}^p + \frac{t^{2p}}{2p}|\nabla u|_p^{2p} + \frac{t^q}{q}\|u\|_{\varepsilon,q}^q + \frac{t^{2q}}{2q}|\nabla u|_q^{2q} \\ & - At^\mu \int_{\Lambda_\varepsilon} (u^+)^\mu \, \mathrm{d}x + B|\mathrm{supp}(u^+) \cap \Lambda_\varepsilon|, \end{aligned}$$

and observing that $\mu > 2q > 2p$ we deduce that $L_\varepsilon(tu) \to -\infty$ as $t \to \infty$. □

In order to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}_\varepsilon^+$, we prove the following results.

Lemma 3 *Under the assumptions (V_1)–(V_2) and (f_1)–(f_4), the following properties hold:*

(i) *For each $u \in \mathbb{W}_\varepsilon^+$, there exists a unique $t_u > 0$ such that if $\alpha_u(t) = L_\varepsilon(tu)$, then $\alpha_u'(t) > 0$ for $0 < t < t_u$ and $\alpha_u'(t) < 0$ for $t > t_u$.*

(ii) *There exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}_\varepsilon^+$ and for each compact set $\mathcal{W} \subset \mathbb{S}_\varepsilon^+$, there is a positive constant $C_{\mathcal{W}}$ such that $t_u \leq C_{\mathcal{W}}$ for any $u \in \mathcal{W}$.*

(iii) *The map $\hat{m}_\varepsilon : \mathbb{W}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\hat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon = \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ is a homeomorphism between $\mathbb{S}_\varepsilon^+$ and $\mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{W}_\varepsilon}}$.*

(iv) *If there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ such that $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$, then $\|m_\varepsilon(u_n)\|_{\mathbb{W}_\varepsilon} \to \infty$ and $L_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

Proof (i) Similar to the proof of Lemma 2, we can see that $\alpha_u(0) = 0$, $\alpha_u(t) > 0$ for $t > 0$ small enough and $\alpha_u(t) < 0$ for $t > 0$ sufficiently large. Then, there exists a global maximum point $t_u > 0$ for α_u in $[0, \infty)$ such that $\alpha_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. We claim that $t_u > 0$

is unique. We argue by contradiction and suppose that there exist $t_1 > t_2 > 0$ such that $\alpha_u'(t_1) = \alpha_u'(t_2) = 0$. Therefore,

$$\frac{\|u\|^p_{V_\varepsilon,p}}{t_1^{2q-p}} + \frac{\|u\|^q_{V_\varepsilon,q}}{t_1^q} + \frac{|\nabla u|_p^{2p}}{t_1^{2q-2p}} + |\nabla u|_q^{2q} = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} u^{2q} \mathrm{d}x,$$

and

$$\frac{\|u\|^p_{V_\varepsilon,p}}{t_2^{2q-p}} + \frac{\|u\|^q_{V_\varepsilon,q}}{t_2^q} + \frac{|\nabla u|_p^{2p}}{t_2^{2q-2p}} + |\nabla u|_q^{2q} = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}} u^{2q} \mathrm{d}x.$$

From the definition of g, (g_4) and (f_4), we get

$$\begin{aligned}
&\left(\frac{1}{t_1^{2q-p}} - \frac{1}{t_2^{2q-p}}\right)\|u\|^p_{V_\varepsilon,p} + \left(\frac{1}{t_1^q} - \frac{1}{t_2^q}\right)\|u\|^q_{V_\varepsilon,q} + \left(\frac{1}{t_1^{2q-2p}} - \frac{1}{t_2^{2q-2p}}\right)|\nabla u|_p^{2p} \\
&\quad = \int_{\mathbb{R}^N} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}}\right] u^{2q} \mathrm{d}x \\
&\quad \geq \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}}\right] u^{2q} \mathrm{d}x \\
&\qquad + \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}}\right] u^{2q} \mathrm{d}x \\
&\qquad + \int_{\Lambda_\varepsilon^c \cap \{t_1 u < a\}} \left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{2q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{2q-1}}\right] u^{2q} \mathrm{d}x \\
&\quad \geq \frac{V_0}{K} \int_{\Lambda_\varepsilon^c \cap \{t_2 u > a\}} \left[\left(\frac{1}{(t_1 u)^{2q-p}} - \frac{1}{(t_2 u)^{2q-p}}\right) + \left(\frac{1}{(t_1 u)^q} - \frac{1}{(t_2 u)^q}\right)\right] u^{2q} \mathrm{d}x \\
&\qquad + \int_{\Lambda_\varepsilon^c \cap \{t_2 u \leq a < t_1 u\}} \left[\frac{V_0}{K}\left(\frac{1}{(t_1 u)^{2q-p}} + \frac{1}{(t_1 u)^q}\right) - \frac{f(t_2 u)}{(t_2 u)^{2q-1}}\right] u^{2q} \mathrm{d}x.
\end{aligned}$$

Multiplying both sides by $\frac{(t_1 t_2)^{2q-p}}{t_2^{2q-p} - t_1^{2q-p}} < 0$ (since $t_1 > t_2$), we have

$$
\begin{aligned}
&\|u\|_{V_\varepsilon,p}^p + \frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q - t_1^q)\|u\|_{V_\varepsilon,q}^q\\
&= \|u\|_{V_\varepsilon,p}^p + \frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\frac{t_2^q - t_1^q}{(t_1t_2)^q}\|u\|_{V_\varepsilon,q}^q\\
&\le \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^p \mathrm{d}x + \frac{V_0}{K}\frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\frac{t_2^q - t_1^q}{(t_2t_1)^q}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^q \mathrm{d}x\\
&\quad + \frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}}\left[\frac{V_0}{K}\left(\frac{1}{(t_1u)^{2q-p}}+\frac{1}{(t_1u)^q}\right) - \frac{f(t_2u)}{(t_2u)^{2q-1}}\right]u^{2q}\mathrm{d}x\\
&\le \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^p \mathrm{d}x + \frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q - t_1^q)\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^q \mathrm{d}x\\
&\quad + \frac{V_0}{K}\frac{t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^p\,\mathrm{d}x + \frac{V_0}{K}\frac{t_1^{q-p}t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^q\,\mathrm{d}x\\
&\quad - \frac{(t_1t_2)^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}}\frac{f(t_2u)}{(t_2u)^{2q-1}}u^{2q}\mathrm{d}x\\
&\le \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^p \mathrm{d}x + \frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q - t_1^q)\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^q \mathrm{d}x\\
&\quad + \frac{V_0}{K}\frac{t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^p\,\mathrm{d}x + \frac{V_0}{K}\frac{t_1^{q-p}t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^q\,\mathrm{d}x\\
&\quad - \frac{V_0}{K}\frac{t_1^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^p\,\mathrm{d}x - \frac{V_0}{K}\frac{t_1^{2q-p}t_2^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^q\,\mathrm{d}x\\
&= \frac{V_0}{K}\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^p \mathrm{d}x + \frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q - t_1^q)\int_{\Lambda_\varepsilon^c\cap\{t_2u>a\}} u^q \mathrm{d}x\\
&\quad + \frac{V_0}{K}\frac{t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^p\,\mathrm{d}x + \frac{V_0}{K}\frac{t_1^{q-p}t_2^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^q\,\mathrm{d}x\\
&\quad - \frac{V_0}{K}\frac{t_1^{2q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^p\,\mathrm{d}x - \frac{V_0}{K}\frac{t_1^{2q-p}t_2^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}\int_{\Lambda_\varepsilon^c\cap\{t_2u\le a<t_1u\}} u^q\,\mathrm{d}x\\
&\le \frac{V_0}{K}\int_{\Lambda_\varepsilon^c} u^p\,\mathrm{d}x + \frac{V_0}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q - t_1^q)\int_{\Lambda_\varepsilon^c} u^q \mathrm{d}x\\
&\le \frac{1}{K}\|u\|_{V_\varepsilon,p}^p + \frac{1}{K}\frac{(t_1t_2)^{q-p}}{t_2^{2q-p}-t_1^{2q-p}}(t_2^q - t_1^q)\|u\|_{V_\varepsilon,q}^q,
\end{aligned}
$$

where we used the fact that (f_4) and our choice of the constant a give

$$
\begin{aligned}
\frac{f(t_2u)}{(t_2u)^{2q-1}} &= \frac{f(t_2u)}{(t_2u)^{p-1}+(t_2u)^{q-1}}\,\frac{(t_2u)^{p-1}+(t_2u)^{q-1}}{(t_2u)^{2q-1}}\\
&\le \frac{f(a)}{a^{p-1}+a^{q-1}}\,\frac{(t_2u)^{p-1}+(t_2u)^{q-1}}{(t_2u)^{2q-1}}\\
&= \frac{V_0}{K}\left(\frac{1}{(t_2u)^{2q-p}}+\frac{1}{(t_2u)^{q}}\right) \quad \text{in } \Lambda_\varepsilon^c \cap \{t_2u \le a < t_1u\}.
\end{aligned}
$$

Since $u \neq 0$ and $K > 1$, we get a contradiction.

(ii) Let $u \in \mathbb{S}_\varepsilon^+$. Using (i), we can find $t_u > 0$ such that $\alpha_u'(t_u) = 0$, that is

$$
t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q + t_u^{2p-1}|\nabla u|_p^{2p} + t_u^{2q-1}|\nabla u|_q^{2q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_u u)\, u \,\mathrm{d}x.
$$

Fix $\zeta > 0$. By (11) and Theorem 2, we have

$$
t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q \le \int_{\mathbb{R}^3} g(\varepsilon x, t_u u)\, u \,\mathrm{d}x \le \zeta t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + C_\zeta t_u^{\nu-1}\|u\|_{V_\varepsilon,q}^\nu.
$$

Taking $\zeta > 0$ sufficiently small, we find

$$
Ct_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q \le Ct_u^{\nu-1}\|u\|_{V_\varepsilon,q}^\nu \le Ct_u^{\nu-1}.
$$

Now, if $t_u \le 1$, then $t_u^{q-1} \le t_u^{p-1}$, and using the facts that $1 = \|u\|_{\mathbb{W}_\varepsilon} \ge \|u\|_{V_\varepsilon,p}$ and $q > p$ imply that $\|u\|_{V_\varepsilon,p}^p \ge \|u\|_{V_\varepsilon,p}^q$, we can see that

$$
Ct_u^{q-1} = Ct_u^{q-1}\|u\|_{\mathbb{W}_\varepsilon}^q \le t_u^{q-1}(C\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \le t_u^{q-1}(C\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \le Ct_u^{\nu-1}.
$$

Thanks to $\nu > q$, we can find $\tau > 0$, independent of u, such that $t_u \ge \tau$.

When $t_u > 1$, then $t_u^{q-1} > t_u^{p-1}$, and noting that $1 = \|u\|_{\mathbb{W}_\varepsilon} \ge \|u\|_{V_\varepsilon,p}$ and $q > p$ imply $\|u\|_{V_\varepsilon,p}^p \ge \|u\|_{V_\varepsilon,p}^q$, we obtain

$$
Ct_u^{p-1} = Ct_u^{p-1}\|u\|_{\mathbb{W}_\varepsilon}^q \le t_u^{p-1}(C\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \le t_u^{p-1}(C\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \le Ct_u^{\nu-1}.
$$

Since $\nu > q > p$, there exists $\tau > 0$, independent of u, such that $t_u \ge \tau$.

Now, let $\mathcal{W} \subset \mathbb{S}_\varepsilon^+$ be a compact set and assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{W}$ such that $t_n = t_{u_n} \to \infty$. Then, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in $\mathbb{W}_\varepsilon$. From (ii) of Lemma 2, we have that

$$
L_\varepsilon(t_n u_n) \to -\infty. \tag{13}
$$

On the other hand, if $v \in \mathcal{N}_\varepsilon$, by $\langle L_\varepsilon'(v), v\rangle = 0$ and (g_3), we have that

$$
L_\varepsilon(v) = L_\varepsilon(v) - \frac{1}{\mu}\langle L_\varepsilon'(v), v\rangle \ge \tilde{C}(\|v\|_{V_\varepsilon,p}^p + \|v\|_{V_\varepsilon,q}^q).
$$

Taking $v_n = t_{u_n} u_n \in \mathcal{N}_\varepsilon$ in the above inequality, we find

$$
L_\varepsilon(t_n u_n) \ge \tilde{C}(\|v_n\|_{V_\varepsilon,p}^p + \|v_n\|_{V_\varepsilon,q}^q).
$$

Since $\|v_n\|_{\mathbb{W}_\varepsilon} = t_n \to \infty$ and $\|v_n\|_{\mathbb{W}_\varepsilon} = \|v_n\|_{\varepsilon,p} + \|v_n\|_{\varepsilon,q}$, we can use (13) to get a contradiction.

(iii) Let us observe that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (i), for each $u \in \mathbb{W}_\varepsilon^+$, there is a unique $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$, then $u \in \mathbb{W}_\varepsilon^+$. Otherwise, we have

$$|\operatorname{supp}(u^+) \cap \Lambda_\varepsilon| = 0,$$

and by (g_3)-(ii), we deduce that

$$\begin{aligned}
\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q &\le \int_{\mathbb{R}^N} g(\varepsilon x, u)\, u \,\mathrm{d}x = \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u)\, u \,\mathrm{d}x + \int_{\Lambda_\varepsilon} g(\varepsilon x, u)\, u \,\mathrm{d}x \\
&= \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u^+)\, u^+ \,\mathrm{d}x \\
&\le \frac{1}{K} \int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u|^p + |u|^q)\mathrm{d}x \\
&\le \frac{1}{K}\left(\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q \right)
\end{aligned}$$

which is impossible due to $K > 1$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{W}_\varepsilon}} \in \mathbb{S}_\varepsilon^+$ is well defined and continuous. From

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{\mathbb{W}_\varepsilon}} = \frac{u}{\|u\|_{\mathbb{W}_\varepsilon}} = u \quad \text{for all } u \in \mathbb{S}_\varepsilon^+,$$

we infer that m_ε is a bijection. To prove that $\hat{m}_\varepsilon : \mathbb{W}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{W}_\varepsilon^+$ and $u \in \mathbb{W}_\varepsilon^+$ be such that $u_n \to u$ in $\mathbb{W}_\varepsilon$. Since $\hat{m}(tu) = \hat{m}(u)$ for all $t > 0$, we may assume that $\|u_n\|_{\mathbb{W}_\varepsilon} = \|u\|_{\mathbb{W}_\varepsilon} = 1$ for all $n \in \mathbb{N}$. By (ii), there exists $t_0 > 0$ such that $t_n = t_{u_n} \to t_0$. Since $t_n u_n \in \mathcal{N}_\varepsilon$,

$$t_n^p \|u_n\|_{V_\varepsilon,p}^p + t_n^q \|u_n\|_{V_\varepsilon,q}^q + t_n^{2p} |\nabla u_n|_p^{2p} + t_n^{2q} |\nabla u_n|_q^{2q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_n u_n)\, t_n u_n \,\mathrm{d}x,$$

and passing to the limit as $n \to \infty$, we obtain

$$t_0^p \|u\|_{V_\varepsilon,p}^p + t_0^q \|u\|_{V_\varepsilon,q}^q + t_0^{2p} |\nabla u|_p^{2p} + t_0^{2q} |\nabla u|_q^{2q} = \int_{\mathbb{R}^N} g(\varepsilon x, t_0 u)\, t_0 u \,\mathrm{d}x,$$

which yields $t_0 u \in \mathcal{N}_\varepsilon$. From (i), $t_u = t_0$, and this means that $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in $\mathbb{W}_\varepsilon^+$. Thus, $\hat{m}_\varepsilon$ and m_ε are continuous functions.

(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be a sequence such that $\operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Then, for each $v \in \partial\mathbb{S}_\varepsilon^+$ and $n \in \mathbb{N}$, we have $u_n^+ \le |u_n - v|$ a.e. in Λ_ε. Therefore, by (V_1), (V_2) and Theorem 2, we can see that for each $r \in [p, q_s^*]$, there exists $C_r > 0$ such that

$$|u_n^+|_{L^r(\Lambda_\varepsilon)} \le \inf_{v\in\partial\mathbb{S}_\varepsilon^+} |u_n - v|_{L^r(\Lambda_\varepsilon)} \le C_r \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_{\mathbb{W}_\varepsilon} \quad \text{for all } n \in \mathbb{N}.$$

By virtue of (g_1), (g_2), (g_3)-(ii) and $q > p$, we get, for all $t > 0$,

$$
\begin{aligned}
\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\,\mathrm{d}x &= \int_{\Lambda_\varepsilon^c} G(\varepsilon x, tu_n)\,\mathrm{d}x + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)\,\mathrm{d}x \\
&\le \frac{V_0}{Kp}\int_{\Lambda_\varepsilon^c} (t^p|u_n|^p + t^q|u_n|^q)\mathrm{d}x + \int_{\Lambda_\varepsilon} F(tu_n)\,\mathrm{d}x \\
&\le \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,\mathrm{d}x + \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,\mathrm{d}x \\
&\quad + C_1 t^p \int_{\Lambda_\varepsilon} (u_n^+)^p \mathrm{d}x + C_2 t^\nu \int_{\Lambda_\varepsilon} (u_n^+)^\nu \mathrm{d}x \\
&\le \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,\mathrm{d}x + \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,\mathrm{d}x \\
&\quad + C_p' t^p \mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^p + C_\nu' t^\nu \mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^\nu.
\end{aligned}
$$

Therefore,

$$
\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\,\mathrm{d}x \le \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,\mathrm{d}x + \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,\mathrm{d}x + o_n(1). \tag{14}
$$

Now, we note that $K > 1$, and that $1 = \|u_n\|_{\mathbb{W}_\varepsilon} \ge \|u_n\|_{V_\varepsilon,p}$ implies that $\|u_n\|_{V_\varepsilon,p}^p \ge \|u_n\|_{V_\varepsilon,p}^q$. Then, for all $t > 1$, we obtain that

$$
\begin{aligned}
&\frac{t^p}{p}\|u_n\|_{V_\varepsilon,p}^p + \frac{t^q}{q}\|u_n\|_{V_\varepsilon,q}^q - \frac{t^p}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,\mathrm{d}x - \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,\mathrm{d}x \\
&= \frac{t^p}{p}|\nabla u_n|_p^p + t^p\left(\frac{1}{p} - \frac{1}{Kp}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\,\mathrm{d}x + \frac{t^q}{q}|\nabla u_n|_q^q \\
&\quad + t^q\left(\frac{1}{q} - \frac{1}{Kp}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,\mathrm{d}x \\
&\ge C_1 t^p\|u_n\|_{V_\varepsilon,p}^p + C_2 t^q\|u_n\|_{V_\varepsilon,q}^q \\
&\ge C_1 t^p\|u_n\|_{V_\varepsilon,p}^q + C_2 t^q\|u_n\|_{V_\varepsilon,q}^q \\
&\ge C_1 t^p\|u_n\|_{V_\varepsilon,p}^q + C_2 t^p\|u_n\|_{V_\varepsilon,q}^q \\
&\ge C_3 t^p(\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^q = C_3 t^p.
\end{aligned} \tag{15}
$$

Bearing in mind the definition of $m_\varepsilon(u_n)$ and using (14), (15), we find

$$
\begin{aligned}
&\liminf_{n\to\infty} L_\varepsilon(m_\varepsilon(u_n)) \ge \liminf_{n\to\infty} L_\varepsilon(tu_n) \\
&\ge \liminf_{n\to\infty}\left[\frac{t^p}{p}\|u_n\|_{V_\varepsilon,p}^p + \frac{t^q}{q}\|u_n\|_{V_\varepsilon,q}^q - \int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\,\mathrm{d}x\right] \ge C_3 t^p \quad \text{for all } t > 1.
\end{aligned}
$$

By sending $t \to \infty$, we get $L_\varepsilon(m_\varepsilon(u_n)) \to \infty$ as $n \to \infty$. On the other hand, by the definition of L_ε, we see that for all $n \in \mathbb{N}$

$$\frac{1}{p}\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p}(1+\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p})+\frac{1}{q}\|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q}(1+\|m_\varepsilon(u_n)\|^q_{V_{\varepsilon,q}})$$
$$\geq \frac{1}{p}\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p}+\frac{1}{2p}|\nabla m_\varepsilon(u_n)|^{2p}_p+\frac{1}{q}\|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q}+\frac{1}{2q}|\nabla m_\varepsilon(u_n)|^{2q}_q \geq L_\varepsilon(m_\varepsilon(u_n))$$

which implies that $\|m_\varepsilon(u_n)\|_{\mathbb{W}_\varepsilon} \to \infty$ as $n \to \infty$. □

Remark 1 If $u \in \mathcal{N}_\varepsilon$, it follows from (11) and Theorem 2 that

$$\|u\|^p_{V_\varepsilon,p}+\|u\|^q_{V_\varepsilon,q} \leq \int_{\mathbb{R}^N} g(\varepsilon x,u)\,u\,\mathrm{d}x \leq \zeta|u|^p_p + C_\zeta|u|^{q^*}_{q^*} \leq \frac{\zeta}{V_0}\|u\|^p_{V_\varepsilon,p}+C'_\zeta\|u\|^{q^*}_{V_\varepsilon,q}.$$

Choosing $\zeta \in (0,V_0)$ we find $\|u\|_{V_\varepsilon,q} \geq \kappa = (C'_\zeta)^{-\frac{1}{q^*-q}}$ which implies that $\|u\|_\varepsilon \geq \|u\|_{V_\varepsilon,q} \geq \kappa$.

Define the maps

$$\hat{\psi}_\varepsilon : \mathbb{W}^+_\varepsilon \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}^+_\varepsilon \to \mathbb{R},$$

by $\hat{\psi}_\varepsilon(u) = L_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon = \hat{\psi}_\varepsilon|_{\mathbb{S}^+_\varepsilon}$. From Lemma 3 and arguing as in the proofs of Proposition 9 and Corollary 10 in [46], we may obtain the following result.

Proposition 1 *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *are satisfied. Then,*

(a) $\hat{\psi}_\varepsilon \in \mathcal{C}^1(\mathbb{W}^+_\varepsilon,\mathbb{R})$ *and*

$$\langle \hat{\psi}'_\varepsilon(u),v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_{\mathbb{W}_\varepsilon}}{\|u\|_{\mathbb{W}_\varepsilon}}\langle L'_\varepsilon(\hat{m}_\varepsilon(u)),v\rangle \quad \text{for all } u \in \mathbb{W}^+_\varepsilon, v \in \mathbb{W}_\varepsilon.$$

(b) $\psi_\varepsilon \in \mathcal{C}^1(\mathbb{S}^+_\varepsilon,\mathbb{R})$ *and*

$$\langle \psi'_\varepsilon(u),v\rangle = \|m_\varepsilon(u)\|_{\mathbb{W}_\varepsilon}\langle L'_\varepsilon(m_\varepsilon(u)),v\rangle, \quad \text{for all } v \in T_u\mathbb{S}^+_\varepsilon.$$

(c) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(\mathrm{PS})_c$ *sequence for* ψ_ε, *then* $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ *is a* $(\mathrm{PS})_c$ *sequence for* L_ε. *If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(\mathrm{PS})_c$ *sequence for* L_ε, *then* $\{m^{-1}_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ *is a* $(\mathrm{PS})_c$ *sequence for* ψ_ε.

(d) *u is a critical point of ψ_ε if, and only if, $m_\varepsilon(u)$ is a critical point for L_ε. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}^+_\varepsilon}\psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon}L_\varepsilon(u).$$

Remark 2 As in [46], we have the following variational characterization of the infimum of L_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon}L_\varepsilon(u) = \inf_{u\in\mathbb{W}^+_\varepsilon}\max_{t>0}L_\varepsilon(tu) = \inf_{u\in\mathbb{S}^+_\varepsilon}\max_{t>0}L_\varepsilon(tu).$$

Next, we claim that L_ε satisfies the Palais–Smale condition. First of all, we have the following result.

Lemma 4 *Let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{W}_\varepsilon$ *be a* $(\mathrm{PS})_c$ *sequence for* L_ε. *Then,* $\{u_n\}_{n\in\mathbb{N}}$ *is bounded in* $\mathbb{W}_\varepsilon$.

Proof From (g_3), $q > p$ and $\mu > 2q$, we have that

$$\begin{aligned}
C(1+\|u_n\|_\varepsilon) &\geq L_\varepsilon(u_n) - \frac{1}{\mu}\langle L'_\varepsilon(u_n), u_n\rangle \\
&= \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|^p_{V_\varepsilon,p} + \left(\frac{1}{2p}-\frac{1}{\mu}\right)|\nabla u_n|_p^{2p} + \left(\frac{1}{q}-\frac{1}{\mu}\right)\|u_n\|^q_{V_\varepsilon,q} \\
&\quad + \left(\frac{1}{2q}-\frac{1}{\mu}\right)|\nabla u_n|_q^{2q} \\
&\quad + \frac{1}{\mu}\int_{\Lambda_\varepsilon^c}[g(\varepsilon x, u_n)u_n - \mu G(\varepsilon x, u_n)]\,\mathrm{d}x + \frac{1}{\mu}\int_{\Lambda_\varepsilon}[g(\varepsilon x, u_n)u_n - \mu G(\varepsilon x, u_n)]\,\mathrm{d}x \\
&\geq \left(\frac{1}{q}-\frac{1}{\mu}\right)[\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}] - \left(\frac{1}{p}-\frac{1}{\mu}\right)\frac{1}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u_n|^p + |u_n|^q)\,\mathrm{d}x \\
&\geq \left[\left(\frac{1}{q}-\frac{1}{\mu}\right) - \left(\frac{1}{p}-\frac{1}{\mu}\right)\frac{1}{K}\right](\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) \\
&= \tilde{C}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}),
\end{aligned}$$

where $\tilde{C} = \left(\frac{1}{q}-\frac{1}{\mu}\right) - \left(\frac{1}{p}-\frac{1}{\mu}\right)\frac{1}{K} > 0$ since $K > \left(\frac{\mu-p}{\mu-q}\right)\frac{q}{p}$.

Now, we assume by contradiction that $\|u_n\|_{\mathbb{W}_\varepsilon} \to \infty$ and consider the following cases:

(1) $\|u_n\|_{V_\varepsilon,p} \to \infty$ and $\|u_n\|_{V_\varepsilon,q} \to \infty$;
(2) $\|u_n\|_{V_\varepsilon,p} \to \infty$ and $\|u_n\|_{V_\varepsilon,q}$ is bounded;
(3) $\|u_n\|_{V_\varepsilon,q} \to \infty$ and $\|u_n\|_{V_\varepsilon,p}$ is bounded.

In case (1), for n large, we have $\|u_n\|^{q-p}_{V_\varepsilon,q} \geq 1$, that is $\|u_n\|^q_{V_\varepsilon,q} \geq \|u_n\|^p_{V_\varepsilon,q}$. Therefore,

$$C_0(1+\|u_n\|_{\mathbb{W}_\varepsilon}) \geq \tilde{C}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^p_{V_\varepsilon,q}) \geq C_1(\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^p = C_1\|u_n\|^p_{\mathbb{W}_\varepsilon}$$

that is an absurd. In case (2), we have

$$C_0(1+\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q}) = C_0(1+\|u_n\|_{\mathbb{W}_\varepsilon}) \geq \tilde{C}\|u_n\|^p_{V_\varepsilon,p}$$

and consequently

$$C_0\left(\frac{1}{\|u_n\|^p_{V_\varepsilon,p}} + \frac{1}{\|u_n\|^{p-1}_{V_\varepsilon,p}} + \frac{\|u_n\|_{V_\varepsilon,q}}{\|u_n\|^p_{V_\varepsilon,p}}\right) \geq \tilde{C}.$$

Since $p > 1$ and passing to the limit as $n \to \infty$, we obtain $0 < \tilde{C} \leq 0$ which is impossible. The last case is similar to the case (2), so we omit the details. Consequently, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{W}_\varepsilon$. □

Lemma 5 *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{W}_\varepsilon$ be a $(\mathrm{PS})_c$ sequence for L_ε. Then, for any $\eta > 0$, there exists $R = R(\eta) > 0$ such that*

$$\limsup_{n\to\infty}\int_{\mathcal{B}_R^c} |\nabla u_n|^p + |\nabla u_n|^q + V(\varepsilon x)(|u_n|^p + |u_n|^q)\,\mathrm{d}x < \eta. \tag{16}$$

Proof For $R > 0$, let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $0 \le \psi_R \le 1$, $\psi_R = 0$ in $\mathcal{B}_{\frac{R}{2}}$, $\psi_R = 1$ in $\mathcal{B}_R^c$, and $|\nabla \psi_R| \le \frac{C}{R}$, for some constant $C > 0$ independent of R. From the boundedness of $\{\psi_R u_n\}_{n\in\mathbb{N}}$ in $\mathbb{W}_\varepsilon$, it follows that $\langle L'_\varepsilon(u_n), \psi_R u_n\rangle = o_n(1)$, namely

$$
\begin{aligned}
&(1 + |\nabla u_n|_p^p) \int_{\mathbb{R}^N} |\nabla u_n|^p \psi_R \, \mathrm{d}x + (1 + |\nabla u_n|_q^q) \int_{\mathbb{R}^N} |\nabla u_n|^q \psi_R \, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p \psi_R \, \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q \psi_R \, \mathrm{d}x \\
&= o_n(1) + \int_{\mathbb{R}^N} g(\varepsilon x, u_n)\psi_R u_n \, \mathrm{d}x - (1 + |\nabla u_n|_p^p) \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_R u_n \, \mathrm{d}x \\
&\quad - (1 + |\nabla u_n|_q^q) \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla \psi_R u_n \, \mathrm{d}x.
\end{aligned}
$$

Take $R > 0$ such that $\Lambda_\varepsilon \subset \mathcal{B}_{\frac{R}{2}}$. From the definition of ψ_R and (g_3)-(ii), we see that

$$
\begin{aligned}
&\int_{\mathcal{B}_R^c} |\nabla u_n|^p \, \mathrm{d}x + \int_{\mathcal{B}_R^c} |\nabla u_n|^q \, \mathrm{d}x + \left(1 - \frac{1}{K}\right) \int_{\mathcal{B}_R^c} V(\varepsilon x)(|u_n|^p + |u_n|^q) \, \mathrm{d}x \\
&\le o_n(1) - (1 + |\nabla u_n|_p^p) \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_R u_n \, \mathrm{d}x \\
&\quad - (1 + |\nabla u_n|_q^q) \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla \psi_R u_n \, \mathrm{d}x.
\end{aligned} \tag{17}
$$

Now, using the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{W}_\varepsilon$, we have, for $t \in \{p, q\}$,

$$
\left| \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla \psi_R u_n \, \mathrm{d}x \right| \le \frac{C}{R} |\nabla u_n|_t^{t-1} |u_n|_t \le \frac{C}{R}
$$

which implies that

$$
\lim_{R\to\infty} \limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla \psi_R u_n \, \mathrm{d}x \right| = 0. \tag{18}
$$

Thanks to (17) and (18), we deduce that (16) holds true. □

Due to the presence of the Kirchhoff terms, the following lemma plays a crucial role to get the strong convergence of bounded Palais–Smale sequences.

Lemma 6 *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{W}_\varepsilon$ be a $(\mathrm{PS})_c$ sequence for L_ε, and let $R > 0$. Then,*

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathcal{B}_R} |\nabla u_n|^p + |\nabla u_n|^q + V(\varepsilon x)(|u_n|^p + |u_n|^q) \, \mathrm{d}x \\
&= \int_{\mathcal{B}_R} |\nabla u|^p + |\nabla u|^q + V(\varepsilon x)(|u|^p + |u|^q) \, \mathrm{d}x.
\end{aligned} \tag{19}
$$

Proof Take $\eta_\rho \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that

$$\eta_\rho(x) = \begin{cases} 1 & \text{for } x \in \mathcal{B}_\rho, \\ 0 & \text{for } x \notin \mathcal{B}_{2\rho}, \end{cases}$$

with $0 \le \eta_\rho(x) \le 1$ and $|\nabla \eta_\rho| \le \frac{2}{\rho}$. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{W}_\varepsilon$ (by Lemma 4), we may assume that

$$|\nabla u_n|_p^p \to T_p \quad \text{and} \quad |\nabla u_n|_q^q \to T_q \quad \text{as } n \to \infty. \tag{20}$$

Fix $R > 0$ and take $\rho > R$. For $t \in \{p, q\}$ and $n \in \mathbb{N}$, define

$$\begin{aligned} A_n^t &= \left(1 + |\nabla u_n|_t^t\right) \int_{\mathcal{B}_R} \left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u\right) \cdot \left(\nabla u_n - \nabla u\right) \mathrm{d}x \\ &\quad + \int_{\mathcal{B}_R} V(\varepsilon x)\left(|u_n|^{t-2}u_n - |u|^{t-2}u\right)\left(u_n - u\right) \mathrm{d}x. \end{aligned}$$

By (9), we note that $A_n^t \ge 0$. Moreover, we see that

$$\begin{aligned} 0 \le A_n^t &= \left(1 + |\nabla u_n|_t^t\right) \int_{\mathcal{B}_R} \left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u\right) \cdot \left(\nabla u_n - \nabla u\right)\eta_\rho \, \mathrm{d}x \\ &\quad + \int_{\mathcal{B}_R} V(\varepsilon x)\left(|u_n|^{t-2}u_n - |u|^{t-2}u\right)\left(u_n - u\right)\eta_\rho \, \mathrm{d}x \\ &\le \left(1 + |\nabla u_n|_t^t\right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u\right) \cdot \left(\nabla u_n - \nabla u\right)\eta_\rho \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon x)\left(|u_n|^{t-2}u_n - |u|^{t-2}u\right)\left(u_n - u\right)\eta_\rho \, \mathrm{d}x \\ &= \left(1 + |\nabla u_n|_t^t\right) \int_{\mathbb{R}^N} |\nabla u_n|^t \eta_\rho \, \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^t \eta_\rho \, \mathrm{d}x \\ &\quad + \left(1 + |\nabla u_n|_t^t\right) \int_{\mathbb{R}^N} |\nabla u|^t \eta_\rho \, \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t \eta_\rho \, \mathrm{d}x \\ &\quad - \left[\left(1 + |\nabla u_n|_t^t\right) \int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla u \eta_\rho \, \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{t-2}u_n u \eta_\rho \, \mathrm{d}x\right] \\ &\quad - \left[\left(1 + |\nabla u_n|_t^t\right) \int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla u_n \eta_\rho \, \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{t-2}u u_n \eta_\rho \, \mathrm{d}x\right]. \end{aligned}$$

Set

$$
\begin{aligned}
I_{n,\rho}^1 &= \left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u_n|^p\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u_n|^p\eta_\rho\,\mathrm{d}x\\
&\quad+\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u_n|^q\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u_n|^q\eta_\rho\,\mathrm{d}x-\int_{\mathbb{R}^N}g(\varepsilon x,u_n)u_n\eta_\rho\,\mathrm{d}x,\\
I_{n,\rho}^2 &= \left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u|^p\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^p\eta_\rho\,\mathrm{d}x\\
&\quad-\left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla u_n\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^{p-2}uu_n\eta_\rho\,\mathrm{d}x\\
&\quad+\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u|^q\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^q\eta_\rho\,\mathrm{d}x\\
&\quad-\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\cdot\nabla u_n\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^{q-2}uu_n\eta_\rho\,\mathrm{d}x,\\
I_{n,\rho}^3 &= \left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u_n|^{p-2}u_nu\eta_\rho\,\mathrm{d}x\\
&\quad+\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla u\eta_\rho\,\mathrm{d}x+\int_{\mathbb{R}^N}V(\varepsilon x)|u_n|^{q-2}u_nu\eta_\rho\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N}g(\varepsilon x,u_n)u\eta_\rho\,\mathrm{d}x
\end{aligned}
$$

and

$$
I_{n,\rho}^4=\int_{\mathbb{R}^N}g(\varepsilon x,u_n)(u_n-u)\eta_\rho\,\mathrm{d}x.
$$

Then, we have

$$
0\le A_n^p+A_n^q\le |I_{n,\rho}^1|+|I_{n,\rho}^2|+|I_{n,\rho}^3|+|I_{n,\rho}^4|. \tag{21}
$$

We note that

$$
\begin{aligned}
I_{n,\rho}^1 =&\langle L_\varepsilon'(u_n),u_n\eta_\rho\rangle-\Bigg[\left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_nu_n\nabla\eta_\rho\,\mathrm{d}x\\
&+\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_nu_n\nabla\eta_\rho\,\mathrm{d}x\Bigg]
\end{aligned}
$$

and since $\{u_n\eta_\rho\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{W}_\varepsilon$, we have $\langle L_\varepsilon'(u_n),u_n\eta_\rho\rangle=o_n(1)$. A direct computation and (20) yield

$$
\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}\left|\left(1+|\nabla u_n|_t^t\right)\int_{\mathbb{R}^N}|\nabla u_n|^{t-2}\nabla u_nu_n\nabla\eta_\rho\,\mathrm{d}x\right|\right]=0\quad\text{for } t\in\{p,q\}, \tag{22}
$$

and so

$$
\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}\left|I_{n,\rho}^1\right|\right]=0. \tag{23}
$$

On the other hand, the weak convergence and (20) imply

$$\begin{aligned}\lim_{n\to\infty}|I^2_{n,\rho}| = \lim_{n\to\infty}\Bigg|&\left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot(\nabla u_n-\nabla u)\eta_\rho\,\mathrm{d}x\\&+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^{p-2}u(u_n-u)\eta_\rho\,\mathrm{d}x\\&+\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\cdot(\nabla u_n-\nabla u)\eta_\rho\,\mathrm{d}x\\&+\int_{\mathbb{R}^N}V(\varepsilon x)|u|^{q-2}u(u_n-u)\eta_\rho\,\mathrm{d}x\Bigg| = 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}I^3_{n,\rho} =&\langle L'_\varepsilon(u_n), u\eta_\rho\rangle - \Bigg[\left(1+|\nabla u_n|_p^p\right)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n u\nabla\eta_\rho\,\mathrm{d}x\\&+\left(1+|\nabla u_n|_q^q\right)\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n u\nabla\eta_\rho\,\mathrm{d}x\Bigg],\end{aligned}$$

and by (20), (22) and $\langle L'_\varepsilon(u_n), u\eta_\rho\rangle = o_n(1)$, we deduce

$$\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}|I^3_{n,\rho}|\right] = 0.$$

Finally, from the growth assumptions on g and Theorem 2, we see that

$$\lim_{n\to\infty}|I^4_{n,\rho}| = 0 \quad \text{for any } \rho > R. \tag{24}$$

Combining (21), (23), (24), we get

$$0 \le \limsup_{n\to\infty}\left(A_n^p + A_n^q\right) \le 0.$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty}\Bigg[&\int_{\mathcal{B}_R}\left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u\right)\cdot\left(\nabla u_n-\nabla u\right)\mathrm{d}x\\&+\int_{\mathcal{B}_R}V(\varepsilon x)\left(|u_n|^{t-2}u_n-|u|^{t-2}u\right)\left(u_n-u\right)\mathrm{d}x\Bigg] = 0 \quad \text{for } t\in\{p,q\}.\end{aligned}$$

In particular, if $t \ge 2$, from (7), we have

$$\int_{\mathcal{B}_R}|\nabla u_n-\nabla u|^t\,\mathrm{d}x \le C\int_{\mathcal{B}_R}\left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u\right)\cdot\left(\nabla u_n-\nabla u\right)\mathrm{d}x \to 0.$$

When $1 < t < 2$, by (10) and Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla u_n - \nabla u|^t \, dx \\
&\quad \le C \left[\int_{\mathcal{B}_R} \left(|\nabla u_n|^{t-2} \nabla u_n - |\nabla u|^{t-2} \nabla u \right) \cdot \left(\nabla u_n - \nabla u \right) dx \right]^{\frac{t}{2}} \left[|\nabla u_n|_t^t + |\nabla u|_t^t \right]^{\frac{2-t}{2}} \\
&\quad \le C \left[\int_{\mathcal{B}_R} \left(|\nabla u_n|^{t-2} \nabla u_n - |\nabla u|^{t-2} \nabla u \right) \cdot \left(\nabla u_n - \nabla u \right) dx \right]^{\frac{t}{2}} \to 0.
\end{aligned}
$$

Arguing as before, we deduce that for $t \in \{p, q\}$

$$
\int_{\mathcal{B}_R} V(\varepsilon x) |u_n - u|^t \, dx \to 0 \text{ as } n \to \infty.
$$

Accordingly, for $t \in \{p, q\}$, we get

$$
\lim_{n \to \infty} \int_{\mathcal{B}_R} \left[|\nabla u_n|^t + V(\varepsilon x) |u_n|^t \right] dx = \int_{\mathcal{B}_R} \left[|\nabla u|^t + V(\varepsilon x) |u|^t \right] dx,
$$

which gives (19). □

Now, we show that L_ε verifies the Palais–Smale compactness condition.

Lemma 7 *L_ε satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.*

Proof Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{W}_\varepsilon$ be a $(PS)_c$ sequence for L_ε. From Lemma 4, we know that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{W}_\varepsilon$. Up to a subsequence, we may assume that $u_n \rightharpoonup u$ in $\mathbb{W}_\varepsilon$ and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q^*)$. By Lemma 5, for each $\eta > 0$, there exists $R = R(\eta) > \frac{C}{\eta}$, with $C > 0$ independent of η, such that (19) is satisfied. This together with Lemma 6 implies that

$$
\begin{aligned}
\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q &\le \liminf_{n \to \infty} (\|u_n\|_{V_\varepsilon,p}^p + \|u_n\|_{V_\varepsilon,q}^q) \\
&\le \limsup_{n \to \infty} (\|u_n\|_{V_\varepsilon,p}^p + \|u_n\|_{V_\varepsilon,q}^q) \\
&= \limsup_{n \to \infty} \left[\int_{\mathcal{B}_R} |\nabla u_n|^p + |\nabla u_n|^q + V(\varepsilon x)(|u_n|^p + |u_n|^q) \, dx \right. \\
&\qquad \left. + \int_{\mathcal{B}_R^c} |\nabla u_n|^p + |\nabla u_n|^q + V(\varepsilon x)(|u_n|^p + |u_n|^q) \, dx \right] \\
&= \int_{\mathcal{B}_R} |\nabla u|^p + |\nabla u|^q + V(\varepsilon x)(|u|^p + |u|^q) \, dx \\
&\qquad + \limsup_{n \to \infty} \left[\int_{\mathcal{B}_R^c} |\nabla u_n|^p + |\nabla u_n|^q + V(\varepsilon x)(|u_n|^p + |u_n|^q) \, dx \right] \\
&< \int_{\mathcal{B}_R} |\nabla u|^p + |\nabla u|^q + V(\varepsilon x)(|u|^p + |u|^q) \, dx + \eta.
\end{aligned}
$$

Since $R \to \infty$ when $\eta \to 0$, it follows that

$$\begin{aligned}\|u\|^p_{V_\varepsilon,p}+\|u\|^q_{V_\varepsilon,q} &\le \liminf_{n\to\infty}(\|u_n\|^p_{V_\varepsilon,p}+\|u_n\|^q_{V_\varepsilon,q})\\ &\le \limsup_{n\to\infty}(\|u_n\|^p_{V_\varepsilon,p}+\|u_n\|^q_{V_\varepsilon,q})\\ &\le \|u\|^p_{V_\varepsilon,p}+\|u\|^q_{V_\varepsilon,q}\end{aligned}$$

and thus

$$\|u_n\|^p_{V_\varepsilon,p}+\|u_n\|^q_{V_\varepsilon,q}=\|u\|^p_{V_\varepsilon,p}+\|u\|^q_{V_\varepsilon,q}+o_n(1).$$

From the Brezis–Lieb lemma [11], we have

$$\|u_n-u\|^p_{V_\varepsilon,p}=\|u_n\|^p_{V_\varepsilon,p}-\|u\|^p_{V_\varepsilon,p}+o_n(1)$$

and

$$\|u_n-u\|^q_{V_\varepsilon,q}=\|u_n\|^q_{V_\varepsilon,q}-\|u\|^q_{V_\varepsilon,q}+o_n(1).$$

Therefore,

$$\|u_n-u\|^p_{V_\varepsilon,p}+\|u_n-u\|^q_{V_\varepsilon,q}=o_n(1)$$

which yields $u_n\to u$ in $\mathbb{W}_\varepsilon$ as $n\to\infty$. □

Remark 3 We can assume that any (PS) sequence $\{u_n\}_{n\in\mathbb{N}}$ of L_ε is nonnegative. To see this, from $\langle L'_\varepsilon(u_n),u_n^-\rangle=o_n(1)$ and $g(\varepsilon x,t)=0$ for $t\le 0$, we have

$$\begin{aligned}&(1+|\nabla u_n|_p^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u_n^-\,\mathrm{d}x+(1+|\nabla u_n|_q^q)\int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla u_n^-\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}V(\varepsilon x)(|u_n|^{p-2}u_n+|u_n|^{q-2}u_n)\,u_n^-\,\mathrm{d}x=o_n(1),\end{aligned}$$

where $u_n^-=\min\{u_n,0\}$, and then

$$\|u_n^-\|^p_{V_\varepsilon,p}+\|u_n^-\|^q_{V_\varepsilon,q}=o_n(1),$$

namely $u_n^-\to 0$ in $\mathbb{W}_\varepsilon$. In particular, $\{u_n^+\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{W}_\varepsilon$. Since $|\nabla u_n|_t^t=|\nabla u_n^+|_t^t+o_n(1)$ and $\|u_n\|_{V_\varepsilon,t}=\|u_n^+\|_{V_\varepsilon,t}+o_n(1)$ for $t\in\{p,q\}$, we deduce that $L_\varepsilon(u_n)=L_\varepsilon(u_n^+)+o_n(1)$ and $L'_\varepsilon(u_n)=L'_\varepsilon(u_n^+)+o_n(1)$. Hence, $L_\varepsilon(u_n^+)\to c$ and $L'_\varepsilon(u_n^+)\to 0$ as $n\to\infty$.

Corollary 1 *The functional ψ_ε satisfies the Palais–Smale condition on $\mathbb{S}_\varepsilon^+$ at any level $c\in\mathbb{R}$.*

Proof Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathbb{S}_\varepsilon^+$ be a Palais–Smale sequence for ψ_ε at the level c. Then,

$$\psi_\varepsilon(u_n)\to c \quad\text{and}\quad \psi'_\varepsilon(u_n)\to 0 \text{ in } (T_{u_n}\mathbb{S}_\varepsilon^+)'.$$

By Proposition 1-(c), we see that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}\subset\mathbb{W}_\varepsilon$ is a Palais–Smale sequence for L_ε at the level c. From Lemma 7, we deduce that L_ε satisfies the $(\mathrm{PS})_c$ condition in $\mathbb{W}_\varepsilon$. Then, up to a subsequence, we can find $u\in\mathbb{S}_\varepsilon^+$ such that

$$m_\varepsilon(u_n) \to m_\varepsilon(u) \text{ in } \mathbb{W}_\varepsilon.$$

In view of Lemma 3-(iii), we conclude that $u_n \to u$ in $\mathbb{S}_\varepsilon^+$. □

3 The autonomous problem

In this section, we consider the following autonomous problem related to (1):

$$\begin{cases} -\left(1+\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u - \left(1+\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x\right)\Delta_q u + V_0(u^{p-1}+u^{q-1}) = f(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \quad u>0 \text{ in } \mathbb{R}^N. \end{cases} \tag{25}$$

Set $\mathbb{Y}_{V_0} = W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{\mathbb{Y}_{V_0}} = \|u\|_{1,p} + \|u\|_{1,q},$$

where

$$\|u\|_{1,t} = \left(|\nabla u|_t^t + V_0|u|_t^t\right)^{\frac{1}{t}} \quad \text{for all } t\in\{p,q\}.$$

Let $\mathcal{L}_{V_0} : \mathbb{Y}_{V_0} \to \mathbb{R}$ be the energy functional associated with (25), then

$$\mathcal{L}_{V_0}(u) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{2p}|\nabla u|_p^{2p} + \frac{1}{q}\|u\|_{1,q}^q + \frac{1}{2q}|\nabla u|_q^{2q} - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

It is easy to check that $\mathcal{L}_{V_0} \in \mathcal{C}^1(\mathbb{Y}_{V_0},\mathbb{R})$ and that

$$\begin{aligned}\langle \mathcal{L}_{V_0}'(u),\varphi\rangle &= (1+|\nabla u|_p^p)\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x + (1+|\nabla u|_q^q)\int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x \\ &\quad + V_0\left[\int_{\mathbb{R}^N}|u|^{p-2}u\,\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N}|u|^{q-2}u\,\varphi\,\mathrm{d}x\right] - \int_{\mathbb{R}^N} f(u)\varphi\,\mathrm{d}x\end{aligned}$$

for any $u,\varphi \in \mathbb{Y}_{V_0}$.

The Nehari manifold $\mathcal{M}_{V_0}$ associated with $\mathcal{L}_{V_0}$ is

$$\mathcal{M}_{V_0} = \left\{u\in\mathbb{Y}_{V_0}\setminus\{0\} : \langle\mathcal{L}_{V_0}'(u),u\rangle = 0\right\},$$

and we set

$$d_{V_0} = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u).$$

Denote by $\mathbb{S}_{V_0}$ the unit sphere of $\mathbb{Y}_{V_0}$ and set $\mathbb{S}_{V_0}^+ = \mathbb{S}_{V_0}\cap\mathbb{Y}_{V_0}^+$, where

$$\mathbb{Y}_{V_0}^+ = \left\{u\in\mathbb{Y}_{V_0} : |\operatorname{supp}(u^+)|>0\right\}.$$

Note that, $\mathbb{S}_{V_0}^+$ is an incomplete $\mathcal{C}^{1,1}$-manifold of codimension one contained in $\mathbb{Y}_{V_0}^+$. Thus, $\mathbb{Y}_{V_0} = T_u\mathbb{S}_{V_0}^+ \oplus \mathbb{R}u$ for each $u\in\mathbb{S}_{V_0}^+$, where

$$T_u\mathbb{S}^+_{V_0} = \left\{ v \in \mathbb{Y}_{V_0} : (1+|\nabla u|_p^p) \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + (1+|\nabla u|_q^q) \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla v \, dx \right.$$
$$\left. +V_0 \int_{\mathbb{R}^N} (|u|^{p-2}u + |u|^{q-2}u)v \, dx = 0 \right\}.$$

Arguing as in Sect. 2, we can see that the following results hold.

Lemma 8 *Under the assumptions* (f_1)-(f_4), *the following properties hold:*

(i) *For each* $u \in \mathbb{Y}^+_{V_0}$, *there exists a unique* $t_u > 0$ *such that if* $\alpha_u(t) = \mathcal{L}_{V_0}(tu)$, *then* $\alpha'_u(t) > 0$ *for* $0 < t < t_u$ *and* $\alpha'_u(t) < 0$ *for* $t > t_u$.
(ii) *There exists* $\tau > 0$ *independent of* u *such that* $t_u \geq \tau$ *for any* $u \in \mathbb{S}^+_{V_0}$ *and for each compact set* $\mathcal{W} \subset \mathbb{S}^+_{V_0}$, *there is a positive constant* $C_{\mathcal{W}}$ such that $t_u \leq C_{\mathcal{W}}$ *for any* $u \in \mathcal{W}$.
(iii) *The map* $\hat{m}_{V_0} : \mathbb{Y}^+_{V_0} \to \mathcal{M}_{V_0}$ *given by* $\hat{m}_{V_0}(u) = t_u u$ *is continuous and* $m_{V_0} = \hat{m}_{V_0}|_{\mathbb{S}^+_{V_0}}$ *is a homeomorphism between* $\mathbb{S}^+_{V_0}$ and $\mathcal{M}_{V_0}$. *Moreover,* $m^{-1}_{V_0}(u) = \frac{u}{\|u\|_{\mathbb{Y}_{V_0}}}$.
(iv) *If there is a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^+_{V_0}$ *such that* $\mathrm{dist}(u_n, \partial\mathbb{S}^+_{V_0}) \to 0$, *then* $\|m_{V_0}(u_n)\|_{\mathbb{Y}_{V_0}} \to \infty$ *and* $\mathcal{L}_{V_0}(m_{V_0}(u_n)) \to \infty$.

Let us consider the maps

$$\hat{\psi}_{V_0} : \mathbb{Y}^+_{V_0} \to \mathbb{R} \quad \text{and} \quad \psi_{V_0} : \mathbb{S}^+_{V_0} \to \mathbb{R},$$

defined as $\hat{\psi}_{V_0}(u) = \mathcal{L}_{V_0}(\hat{m}_{V_0}(u))$ and $\psi_{V_0} = \hat{\psi}_{V_0}|_{\mathbb{S}^+_{V_0}}$.

Proposition 2 *Assume that* (f_1)-(f_4) *are satisfied. Then,*

(a) $\hat{\psi}_{V_0} \in \mathcal{C}^1(\mathbb{Y}^+_{V_0}, \mathbb{R})$ *and*
$$\langle \hat{\psi}'_{V_0}(u), v\rangle = \frac{\|\hat{m}_{V_0}(u)\|_{\mathbb{Y}_{V_0}}}{\|u\|_{\mathbb{Y}_{V_0}}} \langle \mathcal{L}'_{V_0}(\hat{m}_{V_0}(u)), v\rangle \quad \forall u \in \mathbb{Y}^+_{V_0}, \forall v \in \mathbb{Y}_{V_0}.$$
(b) $\psi_{V_0} \in \mathcal{C}^1(\mathbb{S}^+_{V_0}, \mathbb{R})$ *and*
$$\langle \psi'_{V_0}(u), v\rangle = \|m_{V_0}(u)\|_{\mathbb{Y}_{V_0}} \langle \mathcal{L}'_{V_0}(m_{V_0}(u)), v\rangle, \quad \forall v \in T_u\mathbb{S}^+_{V_0}.$$
(c) If $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_d$ *sequence for* ψ_{V_0}, *then* $\{m_{V_0}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* $\mathcal{L}_{V_0}$. *If* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ *is a bounded* $(PS)_d$ *sequence for* $\mathcal{L}_{V_0}$, *then* $\{m^{-1}_{V_0}(u_n)\}_{n\in\mathbb{N}}$ *is a* $(PS)_d$ *sequence for* ψ_{V_0}.
(d) u *is a critical point of* ψ_{V_0} *if, and only if,* $m_{V_0}(u)$ *is a nontrivial critical point for* $\mathcal{L}_{V_0}$. *Moreover, the corresponding critical values coincide and*
$$\inf_{u\in\mathbb{S}^+_{V_0}} \psi_{V_0}(u) = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u).$$

Remark 4 As in [46], we have the following characterization of the infimum of $\mathcal{L}_{V_0}$ over $\mathcal{M}_{V_0}$:

$$0 < d_{V_0} = \inf_{u \in \mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u) = \inf_{u \in \mathbb{Y}_{V_0}^+} \max_{t>0} \mathcal{L}_{V_0}(tu) = \inf_{u \in \mathbb{S}_{V_0}^+} \max_{t>0} \mathcal{L}_{V_0}(tu).$$

The next lemma allows us to assume that the weak limit of a $(\mathrm{PS})_{d_{V_0}}$ sequence of $\mathcal{L}_{V_0}$ is nontrivial.

Lemma 9 *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ be a $(\mathrm{PS})_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$ such that $u_n \rightharpoonup 0$ in $\mathbb{Y}_{V_0}$. Then, one and only one of the following alternatives occurs:*

(a) $u_n \to 0$ *in* $\mathbb{Y}_{V_0}$, *or*
(b) *there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q \,\mathrm{d}x \geq \beta.$$

Proof Assume that (b) does not hold. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, we can apply Lemma 1 to see that

$$u_n \to 0 \quad \text{in } L^r(\mathbb{R}^N) \text{ for all } r \in (p, q^*).$$

In particular, by (f_1)–(f_2), it follows that

$$\int_{\mathbb{R}^N} f(u_n)u_n \,\mathrm{d}x = o_n(1) \quad \text{as } n \to \infty.$$

Recalling that $\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle = o_n(1)$, we have

$$\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q \leq \int_{\mathbb{R}^N} f(u_n)u_n \,\mathrm{d}x = o_n(1),$$

that is $\|u_n\|_{\mathbb{Y}_{V_0}} \to 0$ as $n \to \infty$ and the item (a) holds true. □

Remark 5 From the above result, we deduce that if u is the weak limit of a $(\mathrm{PS})_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$, then we can assume $u \neq 0$. In fact, if $u_n \rightharpoonup 0$ in $\mathbb{Y}_{V_0}$ and, if $u_n \not\to 0$ in $\mathbb{Y}_{V_0}$, by Lemma 9, we can find $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(y_n)} |u_n|^q \,\mathrm{d}x \geq \beta.$$

Set $v_n(x) = u_n(x + y_n)$. Then, using the invariance of $\mathbb{R}^N$ by translation, we see that $\{v_n\}_{n\in\mathbb{N}}$ is a bounded $(\mathrm{PS})_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$ such that $v_n \rightharpoonup v$ in $\mathbb{Y}_{V_0}$ for some $v \neq 0$.

In what follows, we prove the existence of a positive ground-state solution for (25).

Theorem 3 *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ be a Palais–Smale sequence of $\mathcal{L}_{V_0}$ at the level d_{V_0}. Then, there exists $u \in \mathbb{Y}_{V_0} \setminus \{0\}$ with $u \geq 0$ such that, up to a subsequence, $u_n \to u$ in $\mathbb{Y}_{V_0}$. Moreover, u is a positive ground-state solution to* (25).

Proof As in the proof of Lemma 7, we can see that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{Y}_{V_0}$ so, going if necessary to a subsequence, we may assume that

$$
\begin{aligned}
&u_n \rightharpoonup u \quad \text{in } \mathbb{Y}_{V_0}, \\
&u_n \to u \quad \text{in } L^r_{loc}(\mathbb{R}^N) \text{ for all } r \in [1, p^*).
\end{aligned} \tag{26}
$$

From Remark 5, we may suppose that $u \neq 0$. Moreover, we may assume that

$$
|\nabla u_n|_p^p \to t_1 \quad \text{and} \quad |\nabla u_n|_q^q \to t_2.
$$

Step 1 $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$.

Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since $u_n \rightharpoonup u$ in $\mathbb{Y}_{V_0}$ and $\mathcal{L}'_{V_0}(u_n) \to 0$, we have that $\langle \mathcal{L}'_{V_0}(u_n) - \mathcal{L}'_{V_0}(u), (u_n - u)\varphi \rangle = o_n(1)$. Therefore,

$$
\begin{aligned}
o_n(1) &= \langle \mathcal{L}'_{V_0}(u_n) - \mathcal{L}'_{V_0}(u), (u_n - u)\varphi \rangle \\
&= (1 + |\nabla u_n|_p^p) \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + (1 + |\nabla u_n|_q^q) \int_{\mathbb{R}^N} (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u) \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + [(1 + |\nabla u_n|_p^p) - (1 + |\nabla u|_p^p)] \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + [(1 + |\nabla u_n|_q^q) - (1 + |\nabla u|_q^q)] \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} V_0 |u_n|^{p-2} u_n (u_n - u)\varphi\, \mathrm{d}x + \int_{\mathbb{R}^N} V_0 |u_n|^{q-2} u_n (u_n - u)\varphi\, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} f(u_n)(u_n - u)\varphi\, \mathrm{d}x \\
&\quad + (1 + |\nabla u|_p^p) \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + (1 + |\nabla u|_q^q) \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x + o_n(1) \\
&= (1 + |\nabla u_n|_p^p) \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + (1 + |\nabla u_n|_q^q) \int_{\mathbb{R}^N} (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u) \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + (1 + |\nabla u_n|_p^p) \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + (1 + |\nabla u_n|_q^q) \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla[(u_n - u)\varphi]\, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} V_0 |u_n|^{p-2} u_n (u_n - u)\varphi\, \mathrm{d}x + \int_{\mathbb{R}^N} V_0 |u_n|^{q-2} u_n (u_n - u)\varphi\, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} f(u_n)(u_n - u)\varphi\, \mathrm{d}x + o_n(1) \\
&= \mathcal{A}_n^p + \mathcal{A}_n^q + \mathcal{B}_n^p + \mathcal{B}_n^q + \mathcal{C}_n^p + \mathcal{C}_n^q - \mathcal{D}_n + o_n(1).
\end{aligned} \tag{27}
$$

By using (26) and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{Y}_{V_0}$, it is easy to check that, for $t \in \{p, q\}$,

$$\int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot (\nabla u_n - \nabla u)\varphi \,\mathrm{d}x \to 0, \tag{28}$$

$$\int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla\varphi(u_n - u)\,\mathrm{d}x \to 0, \tag{29}$$

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u) \cdot \nabla\varphi(u_n - u)\,\mathrm{d}x \to 0, \tag{30}$$

$$\mathcal{C}_n^t = \int_{\mathbb{R}^N} V_0|u_n|^{t-2}u_n(u_n - u)\varphi \,\mathrm{d}x \to 0, \tag{31}$$

$$\mathcal{D}_n = \int_{\mathbb{R}^N} f(u_n)(u_n - u)\varphi \,\mathrm{d}x \to 0. \tag{32}$$

In particular, from the boundedness of $\{|\nabla u_n|_t\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ together with (28) and (29), we deduce that

$$\begin{aligned}
&(1 + |\nabla u_n|_t^t)\int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot (\nabla[(u_n - u)\varphi])\,\mathrm{d}x \\
&\quad = (1 + |\nabla u_n|_t^t)\left[\int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot (\nabla u_n - \nabla u)\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla\varphi(u_n - u)\,\mathrm{d}x\right] \to 0.
\end{aligned} \tag{33}$$

Hence, by combining (27)–(33), we have

$$o_n(1) = \mathcal{A}_n^p + \mathcal{A}_n^q + o_n(1),$$

from which

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u) \cdot (\nabla u_n - \nabla u)\varphi \,\mathrm{d}x \to 0$$

for $t \in \{p, q\}$. Now, if $t \geq 2$, from (7), we get

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^t \varphi \,\mathrm{d}x \leq C\int_{\mathbb{R}^N} (|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u) \cdot (\nabla u_n - \nabla u)\varphi \,\mathrm{d}x \to 0.$$

When $1 < t < 2$, from (7), we obtain

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^t \varphi \,\mathrm{d}x\right)^{\frac{2}{t}} &\leq \left(\int_{\mathbb{R}^N} \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-t}}\varphi \,\mathrm{d}x\right)\left(\int_{\mathbb{R}^N} (|\nabla u_n| + |\nabla u|^t)\varphi \,\mathrm{d}x\right)^{\frac{2-t}{t}} \\
&\leq C\int_{\mathbb{R}^N} \left(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u|^{t-2}\nabla u\right) \cdot \left(\nabla u_n - \nabla u\right)\varphi \,\mathrm{d}x \to 0.
\end{aligned}$$

Therefore, for $t \in \{p, q\}$, we have

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^t \varphi \,\mathrm{d}x \to 0.$$

Then, for some subsequence of $\{u_n\}_{n\in\mathbb{N}}$, we have $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$.

Step 2 $|\nabla u_n|_t \to |\nabla u|_t$ for $t \in \{p, q\}$.

By Step 1 and Fatou's lemma, we know that $|\nabla u|_p^p \le t_1$ and $|\nabla u|_q^q \le t_2$. Now, we show that

$$|\nabla u|_p^p = t_1 \quad \text{and} \quad |\nabla u|_q^q = t_2.$$

Assume by contradiction that $|\nabla u|_p^p < t_1$ and $|\nabla u|_q^q \le t_2$. Since $\langle \mathcal{L}'_{V_0}(u_n), \varphi\rangle \to 0$ for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and $\mathcal{C}_c^\infty(\mathbb{R}^N)$ is dense in $\mathbb{Y}_{V_0}$, we see that

$$(1+t_1)|\nabla u|_p^p + (1+t_2)|\nabla u|_q^q + V_0(|u|_p^p + |u|_q^q) = \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x.$$

Hence,

$$\begin{aligned}
&(1+|\nabla u|_p^p)|\nabla u|_p^p + (1+|\nabla u|_q^q)|\nabla u|_q^q + V_0(|u|_p^p + |u|_q^q) - \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x \\
&\quad < (1+t_1)|\nabla u|_p^p + (1+t_2)|\nabla u|_q^q + V_0(|u|_p^p + |u|_q^q) - \int_{\mathbb{R}^N} f(u)u\,\mathrm{d}x = 0,
\end{aligned}$$

which implies that $\langle \mathcal{L}'_{V_0}(u), u\rangle < 0$. Using (f_1) and (f_2), we see that $\langle \mathcal{L}'_{V_0}(t_0u), t_0u\rangle > 0$ for some $0 < t_0 \ll 1$. Then, we can find $\tau \in (t_0, 1)$ such that $\langle \mathcal{L}'_{V_0}(\tau u), \tau u\rangle = 0$. This fact together with the characterization of d_{V_0}, $t \mapsto \frac{1}{2q} f(t)t - F(t)$ is increasing (by (f_3) and (f_4)), the Fatou's lemma gives

$$\begin{aligned}
d_{V_0} \le \mathcal{L}_{V_0}(\tau u) &= \mathcal{L}_{V_0}(\tau u) - \frac{1}{2q}\langle \mathcal{L}'_{V_0}(\tau u), \tau u\rangle \\
&< \mathcal{L}_{V_0}(u) - \frac{1}{2q}\langle \mathcal{L}'_{V_0}(u), u\rangle \\
&\le \liminf_{n\to\infty} \left[\mathcal{L}_{V_0}(u_n) - \frac{1}{2q}\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle\right] = d_{V_0},
\end{aligned}$$

and this is an absurd. Consequently, $|\nabla u_n|_t \to |\nabla u|_t$ for $t \in \{p, q\}$ and we have $\mathcal{L}'_{V_0}(u) = 0$.

Step 3 u is positive.

Using $\langle \mathcal{L}'_{V_0}(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$, and $f(t) = 0$ for $t \le 0$, we have

$$\|u^-\|_{1,p}^p + \|u^-\|_{1,q}^q \le 0$$

which gives $u^- = 0$, that is $u \ge 0$ in $\mathbb{R}^N$. Therefore, $u \ge 0$ and $u \not\equiv 0$ in $\mathbb{R}^N$. Arguing as in [28], we deduce that $u \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^N)$ and $|u(x)| \to 0$ as $|x| \to \infty$. By means of the Harnack inequality [47], we conclude that $u > 0$ in $\mathbb{R}^N$. □

The next compactness result will be used in the sequel.

Lemma 10 *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ be a sequence such that $\mathcal{L}_{V_0}(u_n) \to d_{V_0}$. Then, $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}_{V_0}$.*

Proof By Lemma 8-(iii), Proposition 2-(d) and the definition of d_{V_0} we have that

$$v_n = m_{V_0}^{-1}(u_n) = \frac{u_n}{\|u_n\|_{\mathbb{Y}_{V_0}}} \in \mathbb{S}_{V_0}^+ \quad \text{for all } n \in \mathbb{N}$$

and

$$\psi_{V_0}(v_n) = \mathcal{L}_{V_0}(u_n) \to d_{V_0} = \inf_{v \in \mathbb{S}_{V_0}^+} \psi_{V_0}(v).$$

Note that, $(\overline{\mathbb{S}}_{V_0}^+, \delta_{V_0})$, where $\delta_{V_0}(u,v) = \|u - v\|_{\mathbb{Y}_{V_0}}$, is a complete metric space. Consider the map $\mathcal{G} : \overline{\mathbb{S}}_{V_0}^+ \to \mathbb{R} \cup \{\infty\}$ given by

$$\mathcal{G}(u) = \begin{cases} \psi_{V_0}(u) & \text{if } u \in \mathbb{S}_{V_0}^+, \\ \infty & \text{if } u \in \partial \mathbb{S}_{V_0}^+. \end{cases}$$

By Lemma 8-(iv), $\mathcal{G} \in \mathcal{C}(\overline{\mathbb{S}}_{V_0}^+, \mathbb{R} \cup \{\infty\})$, and by Proposition 2-($d$), we have that $\mathcal{G}$ is bounded below. Then, we can apply Ekeland's variational principle [20] to deduce that there exists a sequence $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_{V_0}^+$ such that $\{\hat{v}_n\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_{d_{V_0}}$ sequence for ψ_{V_0} at the level d_{V_0} and $\|\hat{v}_n - v_n\|_{\mathbb{Y}_{V_0}} = o_n(1)$. At this point, the proof follows from Proposition 2, Theorem 3, and arguing as in the proof of Corollary 1. □

For the minimax levels c_ε and d_{V_0}, we have the following relation.

Lemma 11 $\lim_{\varepsilon \to 0} c_\varepsilon = d_{V_0}$.

Proof By Theorem 3, we know that there exists a positive ground-state ω of (25). Define $\omega_\varepsilon(x) = \psi_\varepsilon(x)\omega(x)$, where $\psi_\varepsilon(x) = \psi(\varepsilon x)$ with $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$. We also assume that $\mathrm{supp}(\psi) \subset \mathcal{B}_2 \subset \Lambda$. By the dominated convergence theorem, it follows that

$$\omega_\varepsilon \to \omega \text{ in } W^{p,q} \quad \text{and} \quad \mathcal{L}_{V_0}(\omega_\varepsilon) \to \mathcal{L}_{V_0}(\omega) = d_{V_0} \quad \text{as } \varepsilon \to 0. \tag{34}$$

Now, for each $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$L_\varepsilon(t_\varepsilon \omega_\varepsilon) = \max_{t \ge 0} L_\varepsilon(t\omega_\varepsilon).$$

Consequently, $\langle L_\varepsilon'(t_\varepsilon \omega_\varepsilon), \omega_\varepsilon \rangle = 0$, that is

$$\begin{aligned} &t_\varepsilon^p |\nabla \omega_\varepsilon|_p^p + t_\varepsilon^{2p} |\nabla \omega_\varepsilon|_p^{2p} + t_\varepsilon^q |\nabla \omega_\varepsilon|_q^q + t_\varepsilon^{2q} |\nabla \omega_\varepsilon|_q^{2q} \\ &\quad + t_\varepsilon^p \int_{\mathbb{R}^N} V(\varepsilon x)\omega_\varepsilon^p \,\mathrm{d}x + t_\varepsilon^q \int_{\mathbb{R}^N} V(\varepsilon x)\omega_\varepsilon^q \,\mathrm{d}x = \int_{\mathbb{R}^N} f(t_\varepsilon \omega_\varepsilon) t_\varepsilon \omega_\varepsilon \,\mathrm{d}x. \end{aligned}$$

Let us prove that $t_\varepsilon \to t_0 \in (0, \infty)$. Assume by contradiction that $t_\varepsilon \to \infty$. Since

$$\begin{aligned} &t_\varepsilon^{p-2q} |\nabla \omega_\varepsilon|_p^p + t_\varepsilon^{2p-2q} |\nabla \omega_\varepsilon|_p^{2p} + t^{-q} |\nabla \omega_\varepsilon|_q^q + |\nabla \omega_\varepsilon|_q^{2q} \\ &\quad + t_\varepsilon^{p-2q} \int_{\mathbb{R}^N} V(\varepsilon x)\omega_\varepsilon^p \,\mathrm{d}x + t^{-q} \int_{\mathbb{R}^N} V(\varepsilon x)\omega_\varepsilon^q \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(t_\varepsilon \omega_\varepsilon)}{(t_\varepsilon \omega_\varepsilon)^{2q-1}} \omega_\varepsilon^{2q} \,\mathrm{d}x, \end{aligned} \tag{35}$$

using (34), $p < 2q$ and (f_3), we deduce that $|\nabla \omega|_q^{2q} = \infty$ which is impossible. Hence, $t_0 \in [0, \infty)$. If by contradiction $t_0 = 0$, from (f_1) and (f_2), we see that, for $\zeta \in (0, V_0)$ fixed, we have

$$\left(1 - \frac{\zeta}{V_0}\right) \|\omega_\varepsilon\|_{V_{\varepsilon,p}}^p + t_\varepsilon^{q-p} \|\omega_\varepsilon\|_{V_{\varepsilon,q}}^q \le C_\zeta t_\varepsilon^{q-p} \|\omega_\varepsilon\|_{V_{\varepsilon,q}}^{q^*}.$$

This combined with $q > p$ gives $\|\omega\|_{1,p}^p = 0$, that is a contradiction.

Now, letting $\varepsilon \to 0$ in (35), we obtain

$$\begin{aligned} &t_0^{p-2q} |\nabla \omega|_p^p + t_0^{2p-2q} |\nabla \omega|_p^{2p} + t_0^{-q} |\nabla \omega|_q^q + |\nabla \omega|_q^{2q} \\ &\quad + t_0^{p-2q} \int_{\mathbb{R}^N} V_0 \omega^p \,\mathrm{d}x + t_0^{-q} \int_{\mathbb{R}^N} V_0 \omega^q \,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{f(t_0 \omega)}{(t_0 \omega)^{2q-1}} \omega^{2q} \,\mathrm{d}x. \end{aligned}$$

Using $2q > q > p$, (f_4) and $\omega \in \mathcal{M}_{V_0}$, we conclude that $t_0 = 1$.

Finally, we observe that

$$\begin{aligned} c_\varepsilon \le \max_{t \ge 0} L_\varepsilon(t\omega_\varepsilon) =& L_\varepsilon(t_\varepsilon \omega_\varepsilon) = \mathcal{L}_{V_0}(t_\varepsilon \omega_\varepsilon) + \frac{t_\varepsilon^p}{p} \int_{\mathbb{R}^N} (V_\varepsilon(x) - V_0) \omega_\varepsilon^p \,\mathrm{d}x \\ &+ \frac{t_\varepsilon^q}{q} \int_{\mathbb{R}^N} (V_\varepsilon(x) - V_0) \omega_\varepsilon^q \,\mathrm{d}x. \end{aligned}$$

Since $V(\varepsilon \cdot)$ is bounded on the support of ω_ε, we use the dominated convergence theorem, (34) and the above inequality to see that $\limsup_{\varepsilon \to 0} c_\varepsilon \le d_{V_0}$. On the other hand, it follows from (V_1) that $\liminf_{\varepsilon \to 0} c_\varepsilon \ge d_{V_0}$. In conclusion, $\lim_{\varepsilon \to 0} c_\varepsilon = d_{V_0}$. □

4 Multiplicity of solutions to (12)

In this section, we collect some technical results which will be used to implement the barycenter machinery below. Take $\delta > 0$ such that

$$M_\delta = \left\{ x \in \mathbb{R}^N : \mathrm{dist}(x, M) \le \delta \right\} \subset \Lambda,$$

and choose a non-increasing function $\eta \in \mathcal{C}^\infty([0, \infty), [0, 1])$ such that $\eta(t) = 1$ if $0 \le t \le \frac{\delta}{2}$, $\eta(t) = 0$ if $t \ge \delta$ and $|\eta'(t)| \le c$ for some $c > 0$. For any $y \in M$, we define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right),$$

and take $t_\varepsilon > 0$ satisfying

$$\max_{t \ge 0} L_\varepsilon(t \Psi_{\varepsilon,y}) = L_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}),$$

where $w \in \mathbb{Y}_{V_0}$ is a positive ground-state solution to (25) whose existence is guaranteed by Theorem 3.

Let $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ be given by

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 12 *The functional Φ_ε verifies the following limit:*

$$\lim_{\varepsilon \to 0} L_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y \in M.$$

Proof Suppose that the thesis of the lemma is false. Then, we can find $\delta_0 > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ such that

$$|L_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - d_{V_0}| \geq \delta_0. \tag{36}$$

Now, for each $n \in \mathbb{N}$ and for all $z \in \mathcal{B}_{\frac{\delta}{\varepsilon_n}}$, we have $\varepsilon_n z \in \mathcal{B}_\delta$, and so

$$\varepsilon_n z + y_n \in \mathcal{B}_\delta(y_n) \subset M_\delta \subset \Lambda.$$

Using the definition of $\Phi_{\varepsilon_n}(y_n)$, that $G = F$ in $\Lambda \times \mathbb{R}$ and taking the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we have

$$\begin{aligned}
L_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \frac{t_{\varepsilon_n}^{2p}}{2p}|\nabla\Psi_{\varepsilon_n,y_n}|_p^{2p} + \frac{t_{\varepsilon_n}^q}{q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q + \frac{t_{\varepsilon_n}^{2p}}{2q}|\nabla\Psi_{\varepsilon_n,y_n}|_q^{2q} \\
&\quad - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})\,\mathrm{d}x \\
&= \frac{t_{\varepsilon_n}^p}{p}\left(|\nabla(\eta(|\varepsilon_n\cdot|)w)|_p^p + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^p\,\mathrm{d}z\right) \\
&\quad + \frac{t_{\varepsilon_n}^{2p}}{2p}|\nabla(\eta(|\varepsilon_n\cdot|)w)|_p^{2p} \\
&\quad + \frac{t_{\varepsilon_n}^q}{q}\left(|\nabla(\eta(|\varepsilon_n\cdot|)w)|_q^q + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^q\,\mathrm{d}z\right) \\
&\quad + \frac{t_{\varepsilon_n}^{2q}}{2q}|\nabla(\eta(|\varepsilon_n\cdot|)w)|_q^{2q} \\
&\quad - \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))\,\mathrm{d}z.
\end{aligned} \tag{37}$$

Our purpose is to show that $t_{\varepsilon_n} \to 1$ as $n \to \infty$. First, we prove that $t_{\varepsilon_n} \to t_0 \in [0,\infty)$. Note that, $\langle L'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$ and $g = f$ on $\Lambda \times \mathbb{R}$ yield

$$\begin{aligned}
&\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^p + \frac{1}{t_\varepsilon^{2q-2p}}|\nabla\Psi_{\varepsilon_n,y_n}|_p^{2p} + \frac{1}{t_\varepsilon^q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^q + |\nabla\Psi_{\varepsilon_n,y_n}|_q^{2q} \\
&= \int_{\mathbb{R}^N}\left[\frac{f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))}{(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^{2q-1}}\right](\eta(|\varepsilon_n z|)w(z))^{2q}\,\mathrm{d}z.
\end{aligned} \tag{38}$$

Since $\eta(|x|) = 1$ for $x \in \mathcal{B}_{\frac{\delta}{2}}$ and $\mathcal{B}_{\frac{\delta}{2}} \subset \mathcal{B}_{\frac{\delta}{\varepsilon_n}}$ for all n large enough, from (38) we deduce that

$$\begin{aligned}&\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|^p_{V_{\varepsilon_n},p}+\frac{1}{t_{\varepsilon}^{2q-2p}}|\nabla\Psi_{\varepsilon_n,y_n}|_p^{2p}+\frac{1}{t_{\varepsilon}^{q}}\|\Psi_{\varepsilon_n,y_n}\|^q_{V_{\varepsilon_n},q}+|\nabla\Psi_{\varepsilon_n,y_n}|_q^{2q}\\&\quad\geq\int_{\mathcal{B}_{\frac{\delta}{2}}}\left[\frac{f(t_{\varepsilon_n}w(z))}{(t_{\varepsilon_n}w(z))^{2q-1}}\right]|w(z)|^{2q}\,\mathrm{d}z.\end{aligned}$$

Then, using (f_4), we obtain

$$\begin{aligned}&\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|^p_{V_{\varepsilon_n},p}+\frac{1}{t_{\varepsilon}^{2q-2p}}|\nabla\Psi_{\varepsilon_n,y_n}|_p^{2p}+\frac{1}{t_{\varepsilon}^{q}}\|\Psi_{\varepsilon_n,y_n}\|^q_{V_{\varepsilon_n},q}+|\nabla\Psi_{\varepsilon_n,y_n}|_q^{2q}\\&\quad\geq\left[\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{2q-1}}\right]|w(\hat{z})|^{2q}|\mathcal{B}_{\frac{\delta}{2}}|,\end{aligned}\tag{39}$$

where $w(\hat{z})=\min_{z\in\overline{\mathcal{B}}_{\frac{\delta}{2}}}w(z)>0$ (w is continuous and positive in $\mathbb{R}^N$). If, by contradiction, $t_{\varepsilon_n}\to\infty$, using $2q>q>p$ and the dominated convergence theorem, we have

$$\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},r}\to\|w\|_{1,r}\in(0,\infty)\quad\forall r\in\{p,q\},\tag{40}$$

and

$$\frac{1}{t_{\varepsilon_n}^{2q-p}}\|\Psi_{\varepsilon_n,y_n}\|^p_{V_{\varepsilon_n},p}+\frac{1}{t_{\varepsilon}^{2q-2p}}|\nabla\Psi_{\varepsilon_n,y_n}|_p^{2p}+\frac{1}{t_{\varepsilon}^{q}}\|\Psi_{\varepsilon_n,y_n}\|^q_{V_{\varepsilon_n},q}+|\nabla\Psi_{\varepsilon_n,y_n}|_q^{2q}\to|\nabla\omega|_q^{2q}.\tag{41}$$

On the other hand, condition (f_3) gives

$$\lim_{n\to\infty}\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{2q-1}}=\infty.\tag{42}$$

In view of (39), (41) and (42), we obtain an absurd. Therefore, $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded, and we may suppose that $t_{\varepsilon_n}\to t_0$ for some $t_0\geq 0$. Taking into account (38), (40), (f_1)–(f_2), we deduce that $t_0\in(0,\infty)$. Now, letting $n\to\infty$ in (38), and using (40) and the dominated convergence theorem, we obtain that

$$t_0^{p-2q}\|w\|_{1,p}^p+t_0^{2p-2q}|\nabla w|_p^{2p}+t_0^{-q}\|w\|_{1,q}^q+|\nabla w|_q^{2q}=\int_{\mathbb{R}^N}\frac{f(t_0w)}{(t_0w)^{2q-1}}\,w^{2q}\,\mathrm{d}x.$$

Since $w\in\mathcal{M}_{V_0}$, then

$$\|w\|_{1,p}^p+|\nabla w|_p^{2p}+\|w\|_{1,q}^q+|\nabla w|_q^{2q}=\int_{\mathbb{R}^N}f(w)w\,\mathrm{d}x,$$

Combining the above identities, we find

$$\begin{aligned}&(t_0^{p-2q}-1)\|w\|_{1,p}^p+(t_0^{2p-2q}-1)|\nabla w|_p^{2p}+(t_0^{-q}-1)|\nabla w|_q^{2q}\\&\quad=\int_{\mathbb{R}^N}\left[\frac{f(t_0w)}{(t_0w)^{2q-1}}-\frac{f(w)}{w^{2q-1}}\right]w^{2q}\,\mathrm{d}x,\end{aligned}$$

and using $2q>q>p$ and (f_4), we deduce that $t_0=1$ and the claim is proved.

By sending $n\to\infty$ in (37), we have that

$$\lim_{n\to\infty} L_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = \mathcal{L}_{V_0}(w) = d_{V_0},$$

which is in contrast with (36). This completes the proof of the lemma. □

Let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset \mathcal{B}_\rho$. Define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ as $\Upsilon(x) = x$ for $|x| < \rho$ and $\Upsilon(x) = \frac{\rho x}{|x|}$ for $|x| > \rho$. Finally, we introduce the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)(|u(x)|^p + |u(x)|^q)\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^p + |u(x)|^q\,\mathrm{d}x}.$$

Since $M \subset \mathcal{B}_\rho$, by the definition of Υ and applying the dominated convergence theorem, we conclude that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in M. \tag{43}$$

The next compactness result is fundamental for showing that the solutions of (12) are solutions of (1).

Lemma 13 *Let $\varepsilon_n \to 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$ be such that $L_{\varepsilon_n}(u_n) \to d_{V_0}$. Then, there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $\mathbb{Y}_{V_0}$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}} = \{\varepsilon_n \tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n \to y_0 \in M$.*

Proof As in the proof of Lemma 7, we can prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$. In view of $d_{V_0} > 0$, we have that $\|u_n\|_{\mathbb{W}_{\varepsilon_n}} \nrightarrow 0$. Arguing as in the proof of Lemma 9 and Remark 5, we can find $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{\mathcal{B}_R(\tilde{y}_n)} |u_n|^q \mathrm{d}x \geq \beta.$$

Putting $v_n(x) = u_n(x + \tilde{y}_n)$, we see that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, and, up to a subsequence, we may suppose that $v_n \rightharpoonup v \not\equiv 0$ in $\mathbb{Y}_{V_0}$. Take $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{M}_{V_0}$ and set $y_n = \varepsilon_n \tilde{y}_n$. Using $u_n \in \mathcal{N}_{\varepsilon_n}$ and (g_2), we get

$$\begin{aligned}
d_{V_0} &\leq \mathcal{L}_{V_0}(\tilde{v}_n)\\
&\leq \frac{1}{p}|\nabla \tilde{v}_n|_p^p + \frac{1}{2p}|\nabla \tilde{v}_n|_p^{2p} + \frac{1}{q}|\nabla \tilde{v}_n|_q^q + \frac{1}{2q}|\nabla \tilde{v}_n|_q^{2q} + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right)\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} F(\tilde{v}_n)\,\mathrm{d}x\\
&\leq \frac{t_n^p}{p}|\nabla u_n|_p^p + \frac{t_n^{2p}}{2p}|\nabla u_n|_p^{2p} + \frac{t_n^q}{q}|\nabla u_n|_q^q + \frac{t_n^{2q}}{2q}|\nabla u_n|_q^{2q} + \int_{\mathbb{R}^N} V(\varepsilon_n x)\left(\frac{t_n^p}{p}|u_n|^p + \frac{t_n^q}{q}|u_n|^q\right)\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_n u_n)\,\mathrm{d}x\\
&= L_{\varepsilon_n}(t_n u_n) \leq L_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1).
\end{aligned}$$

Hence,

$$\mathcal{L}_{V_0}(\tilde{v}_n) \to d_{V_0} \text{ and } \{\tilde{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}. \tag{44}$$

Moreover, $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$ and we may suppose that $\tilde{v}_n \rightharpoonup \tilde{v}$. We may assume that $t_n \to t_0 \in (0, \infty)$. From the uniqueness of the weak limit, we have that $\tilde{v} = t_0 v \not\equiv 0$. By (44) and Lemma 10, $\tilde{v}_n \to \tilde{v}$ in $\mathbb{Y}_{V_0}$ and thus $v_n \to v$ in $\mathbb{Y}_{V_0}$. In particular,

$$\mathcal{L}_{V_0}(\tilde{v}) = d_{V_0} \text{ and } \langle \mathcal{L}'_{V_0}(\tilde{v}), \tilde{v} \rangle = 0.$$

Next, we prove that $\{y_n\}_{n\in\mathbb{N}}$ admits a bounded subsequence. Assume, by contradiction, that there exists a subsequence of $\{y_n\}_{n\in\mathbb{N}}$, still denoted by itself, such that $|y_n| \to \infty$. Let $R > 0$ be such that $\Lambda \subset \mathcal{B}_R$. Then, for n large enough, we have $|y_n| > 2R$, and for each $x \in \mathcal{B}_{\frac{R}{\varepsilon_n}}$ we obtain

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R.$$

Then, taking into account that $v_n \to v$ in $\mathbb{Y}_{V_0}$, the definition of g, and the dominated convergence theorem, we deduce that

$$\begin{aligned}
\|v_n\|_{1,p}^p + \|v_n\|_{1,q}^q &\leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, v_n) v_n \,\mathrm{d}x \\
&\leq \int_{\mathcal{B}_{\frac{R}{\varepsilon_n}}} \tilde{f}(v_n) v_n \,\mathrm{d}x + \int_{\mathcal{B}^c_{\frac{R}{\varepsilon_n}}} f(v_n) v_n \,\mathrm{d}x \\
&\leq \frac{1}{K} \int_{\mathcal{B}_{\frac{R}{\varepsilon_n}}} V_0(|v_n|^p + |v_n|^q) \,\mathrm{d}x + o_n(1)
\end{aligned}$$

which implies that

$$\left(1 - \frac{1}{K}\right)\left(\|v_n\|_{1,p}^p + \|v_n\|_{1,q}^q\right) \leq o_n(1),$$

and this gives a contradiction because of $v_n \to v \not\equiv 0$ in $\mathbb{Y}_{V_0}$. Therefore, $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$ and, up to a subsequence, we may assume that $y_n \to y_0$. If $y_0 \notin \overline{\Lambda}$, we can proceed as above to conclude $v_n \to 0$ in $\mathbb{Y}_{V_0}$. Hence, $y \in \overline{\Lambda}$. In order to prove that $V(y_0) = V_0$, we suppose, by contradiction, that $V(y_0) > V_0$. Then, using $\tilde{v}_n \to \tilde{v}$ in $\mathbb{Y}_{V_0}$, Fatou's lemma and the invariance of $\mathbb{R}^N$ by translations, we deduce that

$$\begin{aligned}
d_{V_0} &= \mathcal{L}_{V_0}(\tilde{v}) \\
&< \liminf_{n\to\infty} \left[\frac{1}{p}|\nabla \tilde{v}_n|_p^p + \frac{1}{2p}|\nabla \tilde{v}_n|_p^{2p} + \frac{1}{q}|\nabla \tilde{v}_n|_q^q + \frac{1}{2q}|\nabla \tilde{v}_n|_q^{2q} \right. \\
&\quad \left. + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(\tilde{v}_n) \,\mathrm{d}x\right] \\
&\leq \liminf_{n\to\infty} L_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} L_{\varepsilon_n}(u_n) = d_{V_0}
\end{aligned}$$

which yields a contradiction. Thus, $V(y_0) = V_0$ and $y_0 \in \overline{M}$. From (V_2), we get that $y_0 \notin \partial M$ and so $y_0 \in M$. $\square$

We now define the following subset of the Nehari manifold

$$\widetilde{\mathcal{N}}_\varepsilon = \left\{u \in \mathcal{N}_\varepsilon \,:\, L_\varepsilon(u) \leq d_{V_0} + \pi(\varepsilon)\right\},$$

where $\pi(\varepsilon) = \sup_{y \in M} |L_\varepsilon(\Phi_\varepsilon(y)) - d_{V_0}|$. By Lemma 12, we deduce that $\pi(\varepsilon) \to 0$ as $\varepsilon \to 0$. By the definition of $\pi(\varepsilon)$, we have that, for all $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$.

In what follows, we provide an interesting relation between $\widetilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 14 *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon \to 0} \sup_{u \in \widetilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof Let $\varepsilon_n \to 0$ as $n \to \infty$. By definition, there exists $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ be such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n), M_\delta) = \sup_{u \in \widetilde{\mathcal{N}}_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u), M_\delta) + o_n(1).$$

Then, it is enough to find a sequence $\{y_n\}_{n\in\mathbb{N}} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Since $\mathcal{L}_{V_0}(tu_n) \leq L_{\varepsilon_n}(tu_n)$ and $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we have that

$$d_{V_0} \leq c_{\varepsilon_n} \leq L_{\varepsilon_n}(u_n) \leq d_{V_0} + h(\varepsilon_n)$$

from which $L_{\varepsilon_n}(u_n) \to d_{V_0}$. Then, we can apply Lemma 13 to find $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n large enough. Hence,

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\displaystyle\int_{\mathbb{R}^N} [\Upsilon(\varepsilon_n z + y_n) - y_n](|u_n(z+\tilde{y}_n)|^p + |u_n(z+\tilde{y}_n)|^q)\,\mathrm{d}z}{\displaystyle\int_{\mathbb{R}^N} (|u_n(z+\tilde{y}_n)|^p + |u_n(z+\tilde{y}_n)|^q)\,\mathrm{d}z}.$$

Since $u_n(\cdot + \tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $\varepsilon_n z + y_n \to y \in M_\delta$ for all $z \in \mathbb{R}^N$, we deduce that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$. Therefore, $\{y_n\}_{n\in\mathbb{N}}$ satisfies the required property and the lemma is proved. □

5 Proof of the main result

In this section, we give the proof of the main result of this work. We start by proving a multiplicity result for (12). Note that, since $\mathbb{S}_\varepsilon^+$ is not a complete metric space, we cannot use directly an abstract result as in [2–5, 23]. However, we can apply the abstract category result in [46] to deduce the following result.

Theorem 4 *Assume that (V_1)–(V_2) and (f_1)–(f_4) hold. Then, for any given $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\bar{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_\delta)$, problem (12) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.*

Proof For each $\varepsilon > 0$, we consider $\gamma_\varepsilon : M \to \mathbb{S}_\varepsilon^+$ given by

$$\gamma_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y)).$$

By Lemma 12, we get

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(\gamma_\varepsilon(y)) = \lim_{\varepsilon\to 0} L_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y \in M.$$

Hence, there exists $\hat{\varepsilon} > 0$ such that

$$\widetilde{\mathcal{S}}_\varepsilon^+ = \{w \in \mathbb{S}_\varepsilon^+ : \psi_\varepsilon(w) \le d_{V_0} + \pi(\varepsilon)\} \neq \emptyset$$

for all $\varepsilon \in (0, \hat{\varepsilon})$, in view of $\psi_\varepsilon(M) \subset \widetilde{\mathcal{S}}_\varepsilon^+$. Here, $\pi(\varepsilon) = \sup_{y\in M} |\psi_\varepsilon(\gamma_\varepsilon(y)) - d_{V_0}| \to 0$ as $\varepsilon \to 0$.

From the above considerations, and using Lemma 12, Lemma 3-(iii), Lemma 14 and (43), we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the diagram of continuous mappings below is well defined for $\varepsilon \in (0, \bar{\varepsilon})$:

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \gamma_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta.$$

From (43), we can choose a function $\varsigma(\varepsilon, y)$ with $|\varsigma(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$ and for all $\varepsilon \in (0, \bar{\varepsilon})$, such that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \varsigma(\varepsilon, y)$ for all $y \in M$. Therefore, the map $\mathcal{H} : [0,1] \times M \to M_\delta$ given by $\mathcal{H}(t, y) = y + (1-t)\varsigma(\varepsilon, y)$, with $(t, y) \in [0,1] \times M$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $id : M \to M_\delta$. Consequently,

$$cat_{\gamma_\varepsilon(M)} \gamma_\varepsilon(M) \ge cat_{M_\delta}(M). \tag{45}$$

Applying Corollary 1, Lemma 11, and Theorem 27 in [46], with $c = c_\varepsilon \le d_{V_0} + \pi(\varepsilon) = d$ and $K = \gamma_\varepsilon(M)$, we deduce that Ψ_ε has at least $cat_{\gamma_\varepsilon(M)}\gamma_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Then, by Proposition 1-(d) and (45), we can infer that L_ε admits at least $cat_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

The next result will be crucial to study the behavior of the maximum points of the solutions. The proof is based on some arguments found in [2, 21, 26, 33].

Lemma 15 *Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution to (12). Then, $L_{\varepsilon_n}(u_n) \to d_{V_0}$, and there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N)$ and for some $\bar{C} > 0$ it holds*

$$|v_n|_\infty \le \bar{C} \quad \text{for all } n \in \mathbb{N}.$$

Moreover,

$$v_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N}. \tag{46}$$

Proof Observing that $L_{\varepsilon_n}(u_n) \le d_{V_0} + \pi(\varepsilon_n)$ with $\pi(\varepsilon_n) \to 0$ as $n \to \infty$, we can repeat the same arguments used at the beginning of the proof of Lemma 13 to show that $L_{\varepsilon_n}(u_n) \to d_{V_0}$. Then, applying Lemma 13, there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \to v$ in $\mathbb{Y}_{V_0}$ for some $v \in \mathbb{Y}_{V_0} \setminus \{0\}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

Let $x_0 \in \mathbb{R}^N$, $R_0 > 1$, $0 < t < s < 1 < R_0$ and $\xi \in C_c^\infty(\mathbb{R}^N)$ such that

$$0 \le \xi \le 1, \quad \operatorname{supp} \xi \subset \mathcal{B}_s(x_0), \quad \xi \equiv 1 \text{ on } \mathcal{B}_t(x_0), \quad |\nabla \xi| \le \frac{2}{s-t}.$$

For $\zeta \geq 1$, set $A_{n,\zeta,\rho} = \{x \in \mathcal{B}_\rho(x_0) \ : \ v_n(x) > \zeta\}$ and

$$Q_n = \int_{A_{n,\zeta,s}} \left(|\nabla v_n|^p + |\nabla v_n|^q\right)\xi^q \, \mathrm{d}x.$$

Note that, v_n satisfies

$$\begin{aligned} A_n \int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \eta \, \mathrm{d}x + B_n \int_{\mathbb{R}^N} |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla \eta \, \mathrm{d}x \\ + \int_{\mathbb{R}^N} V_n(x)(v_n^{p-1} + v_n^{q-1})\eta \, \mathrm{d}x = \int_{\mathbb{R}^N} g_n(x, v_n)\eta \, \mathrm{d}x, \end{aligned}$$

for all $\eta \in \mathbb{X}_\varepsilon$, where $A_n = 1 + |\nabla v_n|_p^p$ and $B_n = 1 + |\nabla v_n|_q^q$. Taking $\eta_n = \xi^q(v_n - \zeta)^+$ as test function, we obtain

$$\begin{aligned} A_n &\left[q \int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)^+ |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \xi \, \mathrm{d}x + \int_{A_{n,\zeta,s}} \xi^q |\nabla v_n|^p \, \mathrm{d}x\right] \\ &+ B_n \left[q \int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)^+ |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla \xi \, \mathrm{d}x + \int_{A_{n,\zeta,s}} \xi^q |\nabla v_n|^q \, \mathrm{d}x\right] \\ &+ \int_{A_{n,\zeta,s}} V_n(v_n^{p-1} + v_n^{q-1})\xi^q(v_n - \zeta)^+ \, \mathrm{d}x \\ = &\int_{A_{n,\zeta,s}} g_n(x, v_n)\xi^q(v_n - \zeta)^+ \, \mathrm{d}x. \end{aligned}$$

Let us observe that $v_n \to v \not\equiv 0$ in $\mathbb{Y}_{V_0}$ so that $1 \leq A_n \leq C_1$ and $1 \leq B_n \leq C_2$ for some $C_1, C_2 > 0$. Therefore, by (V_1), we get

$$\begin{aligned} Q_n \leq C &\int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)^+ |\nabla \xi| \left(|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1}\right) \mathrm{d}x \\ &- \int_{A_{n,\zeta,s}} V_0 \xi^{q-1}(v_n - \zeta)^+(v_n^{p-1} + v_n^{q-1}) \, \mathrm{d}x + \int_{A_{n,\zeta,s}} g_n(x, v_n)\xi^q(v_n - \zeta)^+ \, \mathrm{d}x. \end{aligned}$$

Using the growth assumptions on g, for any $\alpha > 0$ there exists $C_\alpha > 0$ such that

$$|g(x,t)| \leq \alpha|t|^{p-1} + C_\alpha|t|^{q^*-1} \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Then, choosing $\alpha > 0$ sufficiently small, we find

$$Q_n \leq C \int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)^+ |\nabla \xi| \left(|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1}\right) \mathrm{d}x + \int_{A_{n,\zeta,s}} v_n^{q^*-1}\xi^q(v_n - \zeta)^+ \, \mathrm{d}x.$$

Proceeding similarly to the proof of Lemma 3.4 in [2], we get

$$Q_n \leq C\left(\int_{A_{n,\zeta,s}} \left|\frac{v_n - \zeta}{s - t}\right|^{q^*} \mathrm{d}x + (\zeta^{q^*} + 1)|A_{n,\zeta,s}|\right).$$

Exploiting the definition of ξ, we can infer that

$$\int_{A_{n,\zeta,t}} |\nabla v_n|^q \, dx \le C\left(\int_{A_{n,\zeta,s}} \left| \frac{v_n - \zeta}{s-t} \right|^{q^*} dx + (\zeta^{q^*} + 1)|A_{n,\zeta,s}| \right)$$

where C does not depend on ζ and $\zeta \ge \zeta_0 \ge 1$, for some constant ζ_0.

Now, fix $R_1 > 0$ and define

$$\begin{aligned}
\sigma_j &= \frac{R_1}{2}\left(1 + \frac{1}{2^j}\right), \\
\bar{\sigma}_j &= \frac{1}{2}\left(\sigma_j + \sigma_{j+1}\right), \\
\zeta_j &= \frac{\zeta_0}{2}\left(1 - \frac{1}{2^{j+1}}\right), \\
Q_{j,n} &= \int_{A_{n,\zeta_j,\sigma_j}} \left((v_n - \zeta_j)^+\right)^{q^*} dx.
\end{aligned}$$

Then, arguing as in Step 1 in Lemma 3.5 in [2], we can see that for each $n \in \mathbb{N}$

$$Q_{j,n} \le C A^\tau Q_{j,n}^{1+\tau} \quad \text{for all } j \in \mathbb{N} \cup \{0\},$$

where $C, \tau > 0$ are independent of n and $A > 1$. Since $v_n \to v$ in $\mathbb{Y}_{V_0}$, we see that

$$\limsup_{\zeta_0 \to \infty} \left(\limsup_{n \to \infty} Q_{0,n} \right) = \limsup_{\zeta_0 \to \infty} \left(\limsup_{n \to \infty} \int_{A_{n,\zeta_0\sigma_0}} \left(\left(v_n - \frac{\zeta_0}{4} \right)^+ \right)^{q^*} dx \right) = 0.$$

Then, there exists $n_0 \in \mathbb{N}$ and $\zeta_0^* > 0$ such that

$$Q_{0,n} \le C^{\frac{1}{\tau}} A^{-\frac{1}{\tau^2}} \quad \text{for } n \ge n_0 \text{ and } \zeta_0 \ge \zeta_0^*.$$

Exploiting Lemma 4.7 in [33], we have that $\lim_{j \to \infty} Q_{j,n} = 0$ for $n \ge n_0$. On the other hand,

$$\lim_{j \to \infty} Q_{j,n} = \lim_{j \to \infty} \int_{A_{n,K_n,\sigma_n}} \left((v_n - \zeta_j)^+\right)^{q^*} dx = \int_{A_{n,\frac{\zeta}{2},\frac{R_1}{2}}} \left(\left(v_n - \frac{\zeta_0}{2} \right)^+ \right)^{q^*} dx.$$

Hence,

$$\int_{A_{n,\frac{\zeta_0}{2},\frac{R_1}{2}}} \left(\left(v_n - \frac{\zeta_0}{2} \right)^+ \right)^{q^*} dx = 0 \quad \text{for all } n \ge n_0,$$

and so, for all $n \ge n_0$,

$$v_n(x) \le \frac{\zeta_0}{2} \quad \text{for a.e. } x \in \mathcal{B}_{\frac{R_1}{2}}(x_0).$$

From the arbitrariness of $x_0 \in \mathbb{R}^N$, we deduce that $v_n(x) \le \frac{\zeta_0}{2}$ for a.e. $x \in \mathbb{R}^N$ and for all $n \ge n_0$, that is

$$|v_n|_\infty \le \frac{\zeta_0}{2}, \quad \text{for all } n \ge n_0.$$

Setting $\bar{C} = \max\left\{\frac{\zeta_0}{2}, |v_1|_\infty, \dots, |v_{n_0-1}|_\infty\right\}$, we find that $|v_n|_\infty \leq \bar{C}$ for all $n \in \mathbb{N}$. Combining this estimate with the regularity results in [28], we can see that $\{v_n\}_{n\in\mathbb{N}} \subset C^{1,\alpha}_{loc}(\mathbb{R}^N)$.

Finally, we show that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. Arguing as before, we can see that for each $\delta > 0$, we have that

$$\limsup_{|x_0|\to\infty}\left(\limsup_{n\to\infty} Q_{0,n}\right) = \limsup_{|x_0|\to\infty}\left(\limsup_{n\to\infty}\int_{A_{n,K_0,\sigma_0}}\left(\left(v_n - \frac{\delta}{4}\right)^+\right)^{q^*} \mathrm{d}x\right) = 0.$$

Therefore, applying lemma Lemma 4.7 in [33], there exist $R_* > 0$ and $n_0 \in \mathbb{N}$ such that

$$\lim_{j\to\infty} Q_{j,n} = 0 \quad \text{if } |x_0| > R_*, \text{ for } n \geq n_0,$$

which yields

$$v_n(x) \leq \frac{\delta}{4} \quad \text{for } x \in \mathcal{B}_{\frac{R_1}{2}}(x_0) \text{ and } |x_0| > R_*, \text{ for all } n \geq n_0.$$

Now, increasing R_* if necessary, it holds

$$v_n(x) \leq \frac{\delta}{4} \quad \text{for } |x| > R_*, \text{ for all } n \geq n_0.$$

This completes the proof of the lemma. □

We are now ready to provide the main result of this section.

Proof of Theorem 1 Fix $\delta > 0$ such that $M_\delta \subset \Lambda$. We first claim that there exists $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (12), it holds

$$|u_\varepsilon|_{L^\infty(\Lambda^c_\varepsilon)} < a. \tag{47}$$

Suppose, by contradiction, that for some sequence $\varepsilon_n \to 0$ we can find $u_n = u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $L'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_n|_{L^\infty(\Lambda^c_{\varepsilon_n})} \geq a. \tag{48}$$

As in Lemma 13, we have that $L_{\varepsilon_n}(u_n) \to d_{V_0}$ and therefore we can use Lemma 13 to find a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \to v$ in $\mathbb{Y}_{V_0}$ and $\varepsilon_n\tilde{y}_n \to y_0 \in M$.

Take $r > 0$ such that $\mathcal{B}_r(y_0) \subset \mathcal{B}_{2r}(y_0) \subset \Lambda$, and so $\mathcal{B}_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Then, for any $y \in \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, it holds

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \quad \text{for } n \text{ sufficiently large.}$$

For these values of n, we have that $\Lambda^c_{\varepsilon_n} \subset \mathcal{B}^c_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$. In view of (46), there exists $R > 0$ such that

$$v_n(x) < a \quad \text{for any } |x| \geq R, n \in \mathbb{N},$$

from which

$$u_n(x) < a \quad \text{for any } x \in \mathcal{B}^c_R(\tilde{y}_n)\,, n \in \mathbb{N}.$$

On the other hand, there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ it holds

$$\Lambda^c_{\varepsilon_n} \subset \mathcal{B}^c_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathcal{B}^c_R(\tilde{y}_n).$$

Consequently, $u_n(x) < a$ for any $x \in \Lambda^c_{\varepsilon_n}$ and $n \geq \nu$, which contradicts (48).

Let $\bar{\varepsilon}_\delta > 0$ be given by Theorem 4 and set $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. Take $\varepsilon \in (0, \varepsilon_\delta)$. By Theorem 4, we obtain at least $cat_{M_\delta}(M)$ positive solutions to (12). If u_ε is one of these solutions, we have that $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$, and we can use (47) and the definition of g to deduce that $g(\varepsilon x, u_\varepsilon) = f(u_\varepsilon)$. This means that u_ε is also a solution of (1). Therefore, (1) has at least $cat_{M_\delta}(M)$ solutions.

Now, we consider $\varepsilon_n \to 0$ and take a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{W}_{\varepsilon_n}$ of solutions to (12) as above. In order to study the behavior of the maximum points of u_n, we first note that, by the definition of g and (g_1), there exists $\sigma \in (0, a)$ sufficiently small such that

$$g(\varepsilon x, t)t \leq \frac{V_0}{K}(t^p + t^q) \quad \text{for any } x \in \mathbb{R}^N, 0 \leq t \leq \sigma. \tag{49}$$

As before, we can take $R > 0$ such that

$$|u_n|_{L^\infty(\mathcal{B}^c_R(\tilde{y}_n))} < \sigma. \tag{50}$$

Up to a subsequence, we may also assume that

$$|u_n|_{L^\infty(\mathcal{B}_R(\tilde{y}_n))} \geq \sigma. \tag{51}$$

Otherwise, if this is not the case, we see that $|u_n|_\infty < \sigma$. Then, it follows from $\langle L'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and (49) that

$$\|u_n\|^p_{V_{\varepsilon_n},p} + \|u_n\|^q_{V_{\varepsilon_n},q} \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n)u_n \, dx \leq \frac{V_0}{K}\int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q)\, dx$$

which implies that $\|u_n\|^p_{V_{\varepsilon_n},p}, \|u_n\|^q_{V_{\varepsilon_n},q} \to 0$, and thus $L_\varepsilon(u_n) \to 0$. This last fact is impossible because $L_\varepsilon(u_n) \to d_{V_0} > 0$. Hence, (51) holds.

By virtue of (50) and (51), we can see that if p_n is a global maximum point of u_n, then $p_n = \tilde{y}_n + q_n$ for some $q_n \in \mathcal{B}_R$. Recalling that $\varepsilon_n \tilde{y}_n \to y_0 \in M$ and using the fact that $\{q_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_R$, we obtain that $\varepsilon_n p_n \to y_0$. Since V is a continuous function, we deduce that

$$\lim_{n\to\infty} V(\varepsilon_n p_n) = V(y_0) = V_0.$$

This completes the proof of Theorem 1. □

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