On the Green function of an orthotropic clamped plate in a half-plane

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Received: 11 February 2021 / Accepted: 5 May 2021 / Published online: 19 May 2021
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Abstract

First, we calculate, in a heuristic manner, the Green function of an orthotropic plate in a half-plane which is clamped along the boundary. We then justify the solution and generalize our approach to operators of the form $(Q(\partial') - a^2 \partial_n^2)(Q(\partial') - b^2 \partial_n^2)$ (where $\partial' = (\partial_1, \dots, \partial_{n-1})$ and $a > 0, b > 0, a \neq b$) with respect to Dirichlet boundary conditions at $x_n = 0$. The Green function G_ξ is represented by a linear combination of fundamental solutions E^c of $Q(\partial')(Q(\partial') - c^2 \partial_n^2)$, $c \in \{a, b\}$, that are shifted to the source point ξ , to the mirror point $-\xi$, and to the two additional points $-\frac{a}{b}\xi$ and $-\frac{b}{a}\xi$, respectively.

Keywords Green function · Orthotropic plate · Dirichlet problem · Cauchy–Dirichlet problem

Mathematics Subject Classification 35C05 · 35J40 · 35L35 · 35E05 · 35K35

1 Introduction and notation

We will derive in this study an explicit formula for the deflection $G_\xi(x)$ of a semi-infinite orthotropic plate in $H = \{x \in \mathbb{R}^2; x_2 > 0\}$, which is clamped along the boundary $x_2 = 0$ and loaded by a unit point force at $\xi = (0, \xi_2)$ with $\xi_2 > 0$.

Hence, G_ξ is the Green function of the differential operator

$$\begin{aligned} P(\partial) &= \partial_1^4 + 2(1 - 2\epsilon^2)\partial_1^2\partial_2^2 + \partial_2^4 \\ &= (\partial_1^2 + 2\epsilon\partial_1\partial_2 + \partial_2^2)(\partial_1^2 - 2\epsilon\partial_1\partial_2 + \partial_2^2), \quad 0 < \epsilon < 1, \end{aligned} \quad (1.1)$$

with respect to Dirichlet boundary conditions at $x_2 = 0$, i.e., G_ξ is the unique solution of $P(\partial)G_\xi = \delta(x - \xi)$ in H satisfying $\lim_{\epsilon \searrow 0} \partial_2^k G_\xi|_{x_2=\epsilon} = 0$, $k = 0, 1$, and some growth condition for $x_2 \rightarrow \infty$. The parameter ϵ characterizes the orthotropy of the plate.

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If $\epsilon = 0$, then $P(\partial)$ coincides with the bi-harmonic operator Δ_2^2 , which is the operator of isotropic plates. The Green function G_ξ^0 of isotropic plates was derived in 1901 by J. H. Michell, see [10, p. 225, last line], [6, Equ. (633), p. 233]. In our notation, it is given by $G_\xi^0(\xi) = |\xi|^2/(4\pi)$ and

$$G_\xi^0(x) = \frac{1}{16\pi} |x - \xi|^2 \log \left(\frac{|x - \xi|^2}{|x + \xi|^2} \right) + \frac{x\xi}{4\pi}, \quad x_2 > 0, x \neq \xi. \quad (1.2)$$

A fundamental solution (also called singularity function) E of $P(\partial)$ in (1.1) is known since 1959 at least, see [16, Equ. (B9), p. 11], [17, p. 44], [13, Ex. 5.2.4, p. 351]. It reads

$$\begin{aligned} E(x) = & \frac{1}{32\pi\sqrt{1-\epsilon^2}} \left[(x_1^2 + \frac{2}{\epsilon}x_1x_2 + x_2^2) \log(x_1^2 + 2\epsilon x_1x_2 + x_2^2) + \right. \\ & \left. + (x_1^2 - \frac{2}{\epsilon}x_1x_2 + x_2^2) \log(x_1^2 - 2\epsilon x_1x_2 + x_2^2) \right] \\ & + \frac{x_1^2 - x_2^2}{16\pi\epsilon} \arctan \left(\frac{\epsilon}{\sqrt{1-\epsilon^2}} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \right), \quad 0 < \epsilon < 1, \end{aligned} \quad (1.3)$$

and the limit $\epsilon \searrow 0$ yields, up to the bi-harmonic polynomial $\frac{|x|^2}{16\pi}$, the well-known fundamental solution E^0 of the isotropic plate operator Δ_2^2 , i.e.,

$$E^0(x) = \frac{|x|^2}{16\pi} \log |x|^2. \quad (1.4)$$

Let us note, in parentheses, that formula (1.3) also furnishes, by linear transformations, a fundamental solution E of the fourth-order operator $\nabla^T A \nabla \cdot \nabla^T B \nabla$ in $\mathbb{R}^2$, if $A, B \in \mathbb{R}^{2 \times 2}$ are linearly independent symmetric positive definite matrices. The result is

$$\begin{aligned} E = & \frac{1}{8\pi\alpha\sqrt{\det A}} \cdot x^T [\text{tr}(BA^{\text{ad}}) \cdot A^{\text{ad}} - 2(\det A) \cdot B^{\text{ad}}] x \cdot \log(x^T A^{\text{ad}} x) \\ & + \frac{1}{8\pi\alpha\sqrt{\det B}} \cdot x^T [\text{tr}(AB^{\text{ad}}) \cdot B^{\text{ad}} - 2(\det B) \cdot A^{\text{ad}}] x \cdot \log(x^T B^{\text{ad}} x) \\ & + \frac{\beta(x)}{2\pi\alpha} \cdot \arctan \left(\frac{\beta(x)}{\sqrt{\det A} \cdot x^T B^{\text{ad}} x + \sqrt{\det B} \cdot x^T A^{\text{ad}} x} \right) \end{aligned} \quad (1.5)$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x^T = (x_1, x_2)$, $A^{\text{ad}} = (\det A) \cdot A^{-1}$ and $\alpha = \text{tr}^2(BA^{\text{ad}}) - 4 \det A \det B$ is positive since B is not a multiple of A and

$$\beta(x) = \sqrt{\text{tr}(BA^{\text{ad}})x^T A^{\text{ad}} x \cdot x^T B^{\text{ad}} x - \det B \cdot (x^T A^{\text{ad}} x)^2 - \det A \cdot (x^T B^{\text{ad}} x)^2}.$$

Let us observe, furthermore, that we can derive the Green function G_ξ^0 in formula (1.2) from the fundamental solution E^0 in (1.4) by the ansatz

$$G_\xi^0 = \frac{|x - \xi|^2}{16\pi} [\log(|x - \xi|^2) - \log(|x + \xi|^2)] + Z$$

where Z fulfills the conditions $\Delta_2^2 Z = 0$ and $Z|_{x_2=0} = 0$, $\partial_2 Z|_{x_2=0} = \frac{\xi_2}{4\pi}$. This yields $Z = x\xi/(4\pi)$.

In the case of the orthotropic plate operator, the deduction of G_g from E in (1.3) is more complicated. In Section 2, we shall derive the Green functions of the operators $(\partial_1^2 + a^2 \partial_2^2)(\partial_1^2 + b^2 \partial_2^2)$, $a > 0, b > 0, a \neq b$, and $\partial_1^4 + 2(1 - 2\epsilon^2)\partial_1^2 \partial_2^2 + \partial_2^4$, $0 < \epsilon < 1$, in a heuristic manner by partial Fourier transform with respect to x_1 . The correctness and the uniqueness of the Green functions under appropriate conditions will be investigated in Sections 3 and 4 more generally for operators of the form $P(\partial) = (Q(\partial') - a^2 \partial_n^2)(Q(\partial') - b^2 \partial_n^2)$. Therein, we shall also provide further examples.

Let us introduce some notation. $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and of nonnegative integers, respectively. We consider as differentiation symbols

$$\partial_t := \frac{\partial}{\partial t}, \quad \partial_j := \frac{\partial}{\partial x_j}, \quad \partial := (\partial_1, \dots, \partial_n), \quad \Delta_n := \partial_1^2 + \dots + \partial_n^2$$

and we denote by $P(\partial)$ linear partial differential operators $\sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$ with constant coefficients $c_\alpha \in \mathbb{C}$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$. In some examples, we set $\partial = (\partial_t, \partial_1, \dots, \partial_n)$ and $P(\partial)$ is then an operator in the $n + 1$ variables $t, x_1, \dots, x_n$.

We employ the standard notation for the distribution spaces $\mathcal{D}', \mathcal{S}'$, the dual spaces of the spaces $\mathcal{D}, \mathcal{S}$ of “test functions” and of “rapidly decreasing functions,” respectively, see [13, 15]. In order to display the active variable in a distribution, say $x \in \mathbb{R}^n$, we use notation as $T(x)$ or $T \in \mathcal{D}'(\mathbb{R}_x^n)$. For the evaluation of a distribution T on a test function ϕ , we use angle brackets, i.e., $\langle \phi, T \rangle$.

The Heaviside function is denoted by Y , see [15, p. 36], and we set

$$\chi^z(t) = \frac{Y(t)t^z}{\Gamma(z+1)} \in L_{\text{loc}}^1(\mathbb{R}_t^1) \quad \text{for } z \in \mathbb{C} \text{ with } \operatorname{Re} z > -1. \quad (1.6)$$

The function $z \mapsto \chi^z$ can be analytically continued in $\mathcal{S}'(\mathbb{R}^1)$ and thus yields an entire function

$$\chi : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^1) : z \longmapsto \chi^z,$$

see [4, Eqs. (3.1), (3.2), pp. 314, 315], [8, (3.2.17), p. 73]. Note that $(\chi^z)' = \chi^{z-1}$, $z \in \mathbb{C}$, and $\chi^{-m} = \delta^{(m-1)}$, $m \in \mathbb{N}$. We write δ for the delta distribution with support in 0 i.e., $\langle \phi, \delta \rangle = \phi(0)$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution T in one variable t with respect to a submersive C^∞ function $h : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is defined as in [5, Equ. (7.2.4/5), p. 82] or in [13, Def. 1.2.12, p. 19], i.e.,

$$\langle \phi, h^*T \rangle = \left\langle \frac{d}{dt} \left(\int_{\Omega} Y(t - h(x)) \phi(x) dx \right), T \right\rangle, \quad \phi \in \mathcal{D}(\Omega).$$

We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\phi)(\xi) := \int_{\mathbb{R}^n} e^{-i\xi x} \phi(x) dx, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

this being extended to $\mathcal{S}'$ by continuity. (Herein and also elsewhere, the Euclidean inner product $(\xi, x) \mapsto \xi x$ is simply expressed by juxtaposition.) For the partial Fourier transforms of a distribution $T \in \mathcal{S}'(\mathbb{R}_{xy}^{m+n})$ with respect to $x \in \mathbb{R}^m$ or $y \in \mathbb{R}^n$, respectively, we use the notation $\mathcal{F}_x T$ and $\mathcal{F}_y T$, respectively.

The restriction of a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ to an open set $H \subset \mathbb{R}^n$ will be denoted by $T|_H$. Similarly, we write $T|_{x_n=\epsilon}$ for the restriction of T to the hyperplane $\{x \in \mathbb{R}^n; x_n = \epsilon\}$, $\epsilon > 0$, if the distribution T continuously depends on x_n , i.e., if it belongs to the subspace of $\mathcal{D}'(H)$, $H = \{x \in \mathbb{R}^n; x_n > 0\}$, constituted by the continuous mappings

$$(0, \infty) \longrightarrow \mathcal{D}'(\mathbb{R}^{n-1}) : x_n \longmapsto T_{x_n}(x').$$

According to [8, Thm. 4.4.8, p. 115], T continuously depends on x_n if it solves a linear partial differential equation of order m with constant coefficients and with a non-vanishing coefficient of ∂_n^m .

2 Green functions of $(\partial_1^2 + a^2 \partial_2^2)(\partial_1^2 + b^2 \partial_2^2)$ and of $\partial_1^4 + 2(1 - 2\epsilon^2)\partial_1^2 \partial_2^2 + \partial_2^4$

We shall first determine the Green function $g_\xi(x) = g_\xi^\lambda(x)$, $\xi > 0$, $x > 0$, of the ordinary differential operator

$$p\left(\frac{d}{dx}\right) = \left(\lambda - a^2 \frac{d^2}{dx^2}\right)\left(\lambda - b^2 \frac{d^2}{dx^2}\right), \quad a > 0, b > 0, \lambda > 0, a \neq b, \quad (2.1)$$

with respect to Dirichlet boundary conditions at $x = 0$. Hence, g_ξ fulfills

$$\begin{aligned} \text{(i)} \quad & p\left(\frac{d}{dx}\right)g_\xi = \delta(x - \xi) \text{ for } x > 0, \\ \text{(ii)} \quad & \exists T_\xi \in \mathcal{S}'(\mathbb{R}) : g_\xi = T_\xi|_{(0, \infty)}, \\ \text{(iii)} \quad & \lim_{\epsilon \searrow 0} g_\xi^{(k)}(\epsilon) = 0 \text{ for } k = 0, 1. \end{aligned} \quad (2.2)$$

The uniquely determined temperate fundamental solution of $\lambda(\lambda - a^2 d^2/dx^2)$ is given by

$$E^a(x) = E^{a,\lambda}(x) = \frac{1}{2a\lambda^{3/2}} e^{-\sqrt{\lambda}|x|/a}, \quad x \in \mathbb{R}, \quad (2.3)$$

and this easily yields that the linear combination $E = (a^2 E^a - b^2 E^b)/(a^2 - b^2)$ coincides with the temperate fundamental solution of $p(\frac{d}{dx})$. In fact,

$$p\left(\frac{d}{dx}\right)E = \frac{1}{a^2 - b^2} \left[\frac{a^2}{\lambda} \left(\lambda - b^2 \frac{d^2}{dx^2}\right)\delta - \frac{b^2}{\lambda} \left(\lambda - a^2 \frac{d^2}{dx^2}\right)\delta \right] = \delta.$$

If we use the ansatz

$$g_\xi(x) = E(x - \xi) + c_1 E^a(x + \xi) + c_2 E^b(x + \xi) + c_3 E^a(x + \frac{a\xi}{b}) + c_4 E^b(x + \frac{b\xi}{a}),$$

then conditions (i) and (ii) in (2.2) are clearly fulfilled. Now note that $E^a(\frac{a\xi}{b}) = \frac{b}{a} E^b(\xi)$ and $E^{a'}(\frac{a\xi}{b}) = \frac{b^2}{a^2} E^{b'}(\xi)$. Therefore, the remaining condition (iii) in (2.2) yields

$$\begin{aligned} 0 = g_\xi(0) &= \frac{a^2}{a^2 - b^2} E^a(\xi) - \frac{b^2}{a^2 - b^2} E^b(\xi) \\ &\quad + c_1 E^a(\xi) + c_2 E^b(\xi) + c_3 \frac{b}{a} E^b(\xi) + c_4 \frac{a}{b} E^a(\xi), \end{aligned}$$

i.e.,

$$c_1 + \frac{a}{b}c_4 = \frac{a^2}{b^2 - a^2}, \quad c_2 + \frac{b}{a}c_3 = \frac{b^2}{a^2 - b^2},$$

and

$$0 = g'_\xi(0) = -\frac{a^2}{a^2 - b^2}E^{a'}(\xi) + \frac{b^2}{a^2 - b^2}E^{b'}(\xi) \\ + c_1E^{a'}(\xi) + c_2E^{b'}(\xi) + c_3\frac{b^2}{a^2}E^{b'}(\xi) + c_4\frac{a^2}{b^2}E^{a'}(\xi),$$

i.e.,

$$c_1 + \frac{a^2}{b^2}c_4 = \frac{a^2}{a^2 - b^2}, \quad c_2 + \frac{b^2}{a^2}c_3 = \frac{b^2}{b^2 - a^2}.$$

These linear equations for $c_1, \dots, c_4$ have the solutions

$$c_1 = -\frac{a^2}{(a-b)^2} = \frac{a^2}{b^2}c_2, \quad c_3 = \frac{2a^2b}{(a-b)^2(a+b)} = \frac{a}{b}c_4,$$

and hence,

$$g_\xi(x) = g_\xi^\lambda(x) = F_\xi^{a,b,\lambda}(x) + F_\xi^{b,a,\lambda}(x), \quad x > 0, \\ F_\xi^{a,b,\lambda}(x) = \frac{a^2}{a^2 - b^2}E^{a,\lambda}(x - \xi) - \frac{a^2}{(a-b)^2}E^{a,\lambda}(x + \xi) \\ + \frac{2a^2b}{(a-b)^2(a+b)}E^{a,\lambda}\left(x + \frac{a\xi}{b}\right). \quad (2.4)$$

We finally observe that the expansion into a power series with respect to $\sqrt{\lambda}$ shows that g_ξ^λ continuously depends on λ also near $\lambda = 0$ and that

$$\lim_{\lambda \searrow 0} g_\xi^\lambda(x) = \frac{1}{12a^2b^2} [|x - \xi|^3 - x^3 + 3x^2\xi + 3x\xi^2 - \xi^3]. \quad (2.5)$$

In the next step, we consider the operator

$$P(\partial) = (\partial_1^2 + a^2\partial_2^2)(\partial_1^2 + b^2\partial_2^2), \quad a > 0, b > 0, a \neq b.$$

We are going to derive the Green function $G_\xi(x)$, $\xi = (0, \xi_2)$, $\xi_2 > 0$, in the half-space $H = \{x \in \mathbb{R}^2; x_2 > 0\}$ subject to Dirichlet boundary conditions at the border line $x_2 = 0$. Hence, G_ξ fulfills

$$(i) \quad P(\partial)G_\xi = \delta(x - \xi) \text{ for } x_2 > 0, \\ (ii) \quad \exists T_\xi \in \mathcal{S}'(\mathbb{R}^2) : G_\xi = T_\xi|_H, \\ (iii) \quad \lim_{\epsilon \searrow 0} \partial_2^k G_\xi|_{x_2=\epsilon} = 0 \text{ in } \mathcal{S}'(\mathbb{R}_{x_1}^1) \text{ for } k = 0, 1. \quad (2.6)$$

Note that (2.6) does not determine G_ξ uniquely since, e.g., for each solution G_ξ , the distributions $G_\xi + cx_2^2$, $c \in \mathbb{C}$, also fulfill (2.6). As we shall show in Section 4, G_ξ becomes uniquely determined if we add to (2.6) the growth condition

$$(iv) \quad \lim_{N \rightarrow \infty} N^{-2}G_\xi|_{x_2=N} = 0 \text{ in } \mathcal{S}'(\mathbb{R}_{x_1}^1). \quad (2.7)$$

Upon a partial Fourier transform with respect to x_1 , we obtain the ordinary differential operator in (2.1) as an operator in d/dx_2 with λ being the square of the transformed variable of x_1 . Hence, we conclude, at least heuristically, that $G_\xi(x) = \mathcal{F}_{x_1}^{-1} g_{\xi_2}^{x_1^2}(x_2)$.

We next observe that

$$E^{a, x_1^2}(x_2) = \frac{1}{2a|x_1|^3} e^{-|x_1 x_2|/a}$$

is, at first sight, not well defined at $x_1 = 0$, but that the linear combination of these functions in $g_{\xi_2}^{x_1^2}(x_2)$ is continuous at $x_1 = 0$ due to Eq. (2.5). We can therefore evaluate the inverse Fourier transform of $g_{\xi_2}^{x_1^2}(x_2)$ by replacing $E^{a, x_1^2}(x_2)$ by the finite part at $z = -3$ of the meromorphic distribution-valued function

$$z \mapsto \frac{1}{2a} |x_1|^z e^{-|x_1 x_2|/a} = S_z^a(x) \in \mathcal{S}'(\mathbb{R}^2),$$

which has simple poles for $z \in -\mathbb{N}$.

For $\operatorname{Re} z > -1$ and fixed $x_2 \neq 0$, $S_z^a(x)$ is an integrable function of x_1 . Hence, we obtain, for $\operatorname{Re} z > -1$ and $x_2 \neq 0$, the following:

$$\begin{aligned} \mathcal{F}_{x_1}^{-1} S_z^a &= \frac{1}{4\pi a} \int_{-\infty}^{\infty} e^{itx_1 - |x_2 t|/a} |t|^z dt \\ &= \frac{\Gamma(z+1)}{4\pi a} \left[\left(\frac{|x_2|}{a} + ix_1 \right)^{-z-1} + \left(\frac{|x_2|}{a} - ix_1 \right)^{-z-1} \right], \end{aligned} \quad (2.8)$$

see [7, Equ. 3.381.4], [9, p. 103]. Furthermore, $\mathbb{R} \rightarrow \mathcal{S}'(\mathbb{R}_{x_1}^1) : x_2 \mapsto S_z^a(x)$ is continuous if $\operatorname{Re} z > -1$. Therefore, the result in (2.8) represents $\mathcal{F}_{x_1}^{-1} S_z^a$ as a locally integrable function in $\mathbb{R}^2$ if $|\operatorname{Re} z| < 1$. By analytic continuation, this generally holds for $\operatorname{Re} z < 1$ outside the poles of $\Gamma(z+1)$, i.e., if $z \notin -\mathbb{N}$. Thus, we obtain

$$\mathcal{F}_{x_1}^{-1} \left(\operatorname{Pf}_{z=-3} S_z^a \right) = \operatorname{Pf}_{z=-3} (\mathcal{F}_{x_1}^{-1} S_z^a) = \frac{1}{2\pi a} \operatorname{Re} \left[\operatorname{Pf}_{z=-3} \Gamma(z+1) \left(\frac{|x_2|}{a} + ix_1 \right)^{-z-1} \right].$$

We now use [12, Prop. 1.6.3, p. 28], and $\operatorname{Res}_{z=-2} \Gamma(z) = \frac{1}{2}$, $\operatorname{Pf}_{z=-2} \Gamma(z) = \frac{1}{2} \psi(3)$ in order to conclude that

$$\begin{aligned} \mathcal{F}_{x_1}^{-1} \left(\operatorname{Pf}_{z=-3} S_z^a \right) &= \frac{1}{2\pi a} \operatorname{Re} \left[\operatorname{Pf}_{z=-3} \Gamma(z+1) \cdot \left(\frac{|x_2|}{a} + ix_1 \right)^2 \right. \\ &\quad \left. + \operatorname{Res}_{z=-3} \Gamma(z+1) \cdot \frac{d}{dz} \left(\frac{|x_2|}{a} + ix_1 \right)^{-z-1} \Big|_{z=-3} \right] \\ &= \frac{1}{2\pi a} \operatorname{Re} \left[\frac{1}{2} \psi(3) \left(\frac{|x_2|}{a} + ix_1 \right)^2 - \frac{1}{2} \left(\frac{|x_2|}{a} + ix_1 \right)^2 \log \left(\frac{|x_2|}{a} + ix_1 \right) \right] \\ &= -\frac{\psi(3)}{4\pi a} \left(x_1^2 - \frac{x_2^2}{a^2} \right) + \frac{1}{8\pi a} \left(x_1^2 - \frac{x_2^2}{a^2} \right) \log \left(x_1^2 + \frac{x_2^2}{a^2} \right) + \frac{x_1 x_2}{2\pi a^2} \arctan \left(\frac{ax_1}{x_2} \right). \end{aligned}$$

Upon summing up the six terms which constitute $g_{\xi_2}^{x_1^2}(x_2)$, the second-order polynomials with the factor $\psi(3)$ cancel out. Furthermore, for $\operatorname{Re} z > 0$, we have $x_1^2(x_1^2 - a^2\partial_2^2)S_z^a = |x_1|^{z+3}\delta(x_2)$, and this implies, by analytic continuation, that $E^a = \mathcal{F}_{x_1}^{-1}(\operatorname{Pf}_{z=-3}S_z^a)$ is a fundamental solution of $\partial_1^2(\partial_1^2 + a^2\partial_2^2)$, also compare [13, Equ. (3.1.15), p. 194]. Let us eventually observe that, here and similarly in the following sections, dilation by the factor a implies $E^a = a^{-1}E^1(x_1, x_2/a)$. Thus, we arrive at the following proposition.

Proposition 2.1 *For $c > 0$, let E^c denote the following fundamental solution of $\partial_1^2(\partial_1^2 + c^2\partial_2^2)$:*

$$E^c(x) = \frac{1}{8\pi c} \left(x_1^2 - \frac{x_2^2}{c^2} \right) \log \left(x_1^2 + \frac{x_2^2}{c^2} \right) + \frac{x_1 x_2}{2\pi c^2} \arctan \left(\frac{c x_1}{x_2} \right). \quad (2.9)$$

Let $G_\xi(x)$, $x_2 > 0$, $\xi = (0, \xi_2)$, $\xi_2 > 0$ be the Green function of the operator $P(\partial) = (\partial_1^2 + a^2\partial_2^2)(\partial_1^2 + b^2\partial_2^2)$, $a > 0, b > 0, a \neq b$, with respect to Dirichlet boundary conditions at $x_2 = 0$, i.e., let G_ξ be determined by the conditions (2.6) and (2.7).

Then $G_\xi(x) = F_\xi^{a,b}(x) + F_\xi^{b,a}(x)$, $x_1 \in \mathbb{R}$, $x_2 > 0$, where

$$\begin{aligned} F_\xi^{a,b}(x) &= \frac{a^2}{a^2 - b^2} E^a(x - \xi) - \frac{a^2}{(a - b)^2} E^a(x + \xi) \\ &\quad + \frac{2a^2 b}{(a - b)^2(a + b)} E^a \left(x + \frac{a\xi}{b} \right). \end{aligned}$$

In order to derive the Green function (of the Dirichlet problem) for the orthotropic plate operator in (1.1), let us represent G_ξ slightly more explicitly in the case $b = \frac{1}{a}$. We assume that $a > 1$ and we set $\mu = \frac{1}{2}(a - b)$. This implies $a^2 + b^2 = 2 + 4\mu^2$, $\left. \begin{smallmatrix} a \\ b \end{smallmatrix} \right\} = \pm\mu + \sqrt{1 + \mu^2}$ and $P(\partial) = \partial_1^4 + 2(1 + 2\mu^2)\partial_1^2\partial_2^2 + \partial_2^4$. Then, the fundamental solution $E = (a^2E^a - b^2E^b)/(a^2 - b^2)$ of $P(\partial)$ takes the form

$$\begin{aligned} E &= \frac{1}{8\pi(a^2 - b^2)} \left[(ax_1^2 - bx_2^2) \log(x_1^2 + b^2x_2^2) - (bx_1^2 - ax_2^2) \log(x_1^2 + a^2x_2^2) \right. \\ &\quad \left. + 4x_1x_2 \arctan \left(\frac{ax_1}{x_2} \right) - 4x_1x_2 \arctan \left(\frac{bx_1}{x_2} \right) \right] \\ &= \frac{1}{8\pi(a^2 - b^2)} \left[\frac{a - b}{2} |x|^2 \log P(x) + \frac{a + b}{2} (x_1^2 - x_2^2) \log \left(\frac{x_1^2 + b^2x_2^2}{x_1^2 + a^2x_2^2} \right) \right. \\ &\quad \left. + 4x_1x_2 \arctan \left(\frac{(a - b)x_1x_2}{|x|^2} \right) \right]. \end{aligned}$$

Hence, we obtain

$$E = \frac{|x|^2}{32\pi\sqrt{1+\mu^2}} \log P(x) + \frac{x_1^2 - x_2^2}{32\pi\mu} \log \left(\frac{x_1^2 + b^2 x_2^2}{x_1^2 + a^2 x_2^2} \right) + \frac{x_1 x_2}{8\pi\mu\sqrt{1+\mu^2}} \arctan \left(\frac{2\mu x_1 x_2}{|x|^2} \right). \quad (2.10)$$

Similarly, the term $H = (a^2 E^a + b^2 E^b)/(a-b)^2$ yields

$$H = \frac{\sqrt{1+\mu^2}(x_1^2 - x_2^2)}{32\pi\mu^2} \log P(x) + \frac{|x|^2}{32\pi\mu} \log \left(\frac{x_1^2 + b^2 x_2^2}{x_1^2 + a^2 x_2^2} \right) + \frac{x_1 x_2}{8\pi\mu^2} \left[\arctan \left(\frac{2\sqrt{1+\mu^2} x_1 x_2}{x_2^2 - x_1^2} \right) + \pi Y(x_1^2 - x_2^2) \operatorname{sign}(x_1 x_2) \right]. \quad (2.11)$$

Of course, the last part of G_ξ , i.e.,

$$J = \frac{2ab}{(a-b)^2(a+b)} \left[aE^a \left(x + \frac{a\xi}{b} \right) + bE^b \left(x + \frac{b\xi}{a} \right) \right],$$

is the most laborious one. It gives

$$J = \frac{x_1^2 - (x_2 + \xi_2)^2 - 2\mu^2(x_2^2 + \xi_2^2)}{32\pi\mu^2\sqrt{1+\mu^2}} \times \log \left[(x_1^2 + (bx_2 + a\xi_2)^2)(x_1^2 + (ax_2 + b\xi_2)^2) \right] + \frac{x_2^2 - \xi_2^2}{16\pi\mu} \log \left(\frac{x_1^2 + (bx_2 + a\xi_2)^2}{x_1^2 + (ax_2 + b\xi_2)^2} \right) + \frac{x_1(x_2 + \xi_2)}{8\pi\mu^2} \left[\arctan \left(\frac{2\sqrt{1+\mu^2} x_1(x_2 + \xi_2)}{(x_2 + \xi_2)^2 + 4\mu^2 x_2 \xi_2 - x_1^2} \right) + \pi Y(x_1^2 - (x_2 + \xi_2)^2 - 4\mu^2 x_2 \xi_2) \operatorname{sign}(x_1(x_2 + \xi_2)) \right] + \frac{x_1(\xi_2 - x_2)}{8\pi\mu\sqrt{1+\mu^2}} \arctan \left(\frac{2\mu x_1(x_2 - \xi_2)}{(x_2 + \xi_2)^2 + 4\mu^2 x_2 \xi_2 + x_1^2} \right). \quad (2.12)$$

We now obtain the Green function of the orthotropic plate operator $P(\partial)$ in (1.1) by continuing $G_\xi(x) = E(x - \xi) - H(x + \xi) + J(x, \xi)$ analytically with respect to μ , i.e., we set $\mu = i\epsilon$, $0 < \epsilon < 1$. The result is the following.

Proposition 2.2 *The Green function $G_\xi(x)$, $x_2 > 0$, $\xi = (0, \xi_2)$, $\xi_2 > 0$, of the operator of the orthotropic plate $P(\partial) = \partial_1^4 + 2(1 - 2\epsilon^2)\partial_1^2\partial_2^2 + \partial_2^4$, $0 < \epsilon < 1$, with respect to Dirichlet boundary conditions at $x_2 = 0$, see (2.6) and (2.7), is given by $G_\xi(x) = \tilde{G}_\xi(x) + \tilde{G}_\xi(-x_1, x_2)$ where $\tilde{G}_\xi(x) = f_1(x - \xi) + f_2(x + \xi) + f_3(x, \xi)$ and*

$$\begin{aligned}
f_1(x) &= \frac{\epsilon|x|^2 + 2x_1x_2}{32\pi\epsilon\sqrt{1-\epsilon^2}} \log(|x|^2 + 2\epsilon x_1x_2) + \frac{x_2^2 - x_1^2}{16\pi\epsilon} \arctan\left(\frac{\epsilon|x_2| + x_1}{|x_2|\sqrt{1-\epsilon^2}}\right), \\
f_2(x) &= \frac{\sqrt{1-\epsilon^2}(x_1^2 - x_2^2)}{32\pi\epsilon^2} \log(|x|^2 + 2\epsilon x_1x_2) + \\
&\quad + \frac{\epsilon|x|^2 + 2x_1x_2}{16\pi\epsilon^2} \arctan\left(\frac{\epsilon x_2 + x_1}{x_2\sqrt{1-\epsilon^2}}\right), \\
f_3(x, \xi) &= \frac{(x_2 + \xi_2)^2 - 2\epsilon^2(x_2^2 + \xi_2^2) - x_1^2 - 2\epsilon x_1(x_2 - \xi_2)}{32\pi\epsilon^2\sqrt{1-\epsilon^2}} \times \\
&\quad \times \log[x_1^2 + (x_2 + \xi_2)^2 - 4\epsilon^2 x_2 \xi_2 + 2\epsilon x_1(x_2 - \xi_2)] \\
&\quad - \frac{\epsilon(x_2^2 - \xi_2^2) + x_1(x_2 + \xi_2)}{8\pi\epsilon^2} \arctan\left(\frac{\epsilon(x_2 - \xi_2) + x_1}{(x_2 + \xi_2)\sqrt{1-\epsilon^2}}\right).
\end{aligned}$$

Proof If we replace μ by $i\epsilon$, then $\frac{a}{b} = \pm i\epsilon + \sqrt{1-\epsilon^2}$ are conjugate complex numbers of modulus one. Hence, in the second term of E in formula (2.10), the logarithm is purely imaginary and given by

$$\begin{aligned}
\log\left(\frac{x_1^2 + b^2x_2^2}{x_1^2 + a^2x_2^2}\right) &= 2i \left[\arctan\left(\frac{\operatorname{Im}(b^2)x_2^2}{x_1^2 + \operatorname{Re}(b^2)x_2^2}\right) - \pi Y(-x_1^2 - \operatorname{Re}(b^2)x_2^2) \right] \\
&= -2i \left[\arctan\left(\frac{2\epsilon\sqrt{1-\epsilon^2}x_2^2}{|x|^2 - 2\epsilon^2x_2^2}\right) + \pi Y(2\epsilon^2x_2^2 - |x|^2) \right] \\
&= -2i \left[\arctan\left(\frac{\epsilon|x_2| + x_1}{|x_2|\sqrt{1-\epsilon^2}}\right) + \arctan\left(\frac{\epsilon|x_2| - x_1}{|x_2|\sqrt{1-\epsilon^2}}\right) \right].
\end{aligned}$$

Furthermore, the arctangent in Eq. (2.10) yields

$$\arctan\left(\frac{2i\epsilon x_1x_2}{|x|^2}\right) = \frac{i}{2} \log\left(\frac{|x|^2 + 2\epsilon x_1x_2}{|x|^2 - 2\epsilon x_1x_2}\right).$$

The analytic continuation of H in formula (2.11) is similar.

Let us yet explain the analytic continuation of formula (2.12). The polynomial

$$Q(x) = (x_1^2 + (bx_2 + a\xi_2)^2)(x_1^2 + (ax_2 + b\xi_2)^2)$$

can be factored as $Q(x) = q(x)q(-x_1, x_2)$ where

$$\begin{aligned}
q(x) &= (x_1 + i(bx_2 + a\xi_2))(x_1 - i(ax_2 + b\xi_2)) \\
&= x_1^2 + (x_2 + \xi_2)^2 - 4\epsilon^2 x_2 \xi_2 + 2\epsilon x_1(x_2 - \xi_2).
\end{aligned}$$

This settles the first term in formula (2.12). For the second one, observe that

$$\begin{aligned}
\log\left(\frac{x_1^2 + (bx_2 + a\xi_2)^2}{x_1^2 + (ax_2 + b\xi_2)^2}\right) &= 2i\left[\arctan\left(\frac{\operatorname{Im}((bx_2 + a\xi_2)^2)}{x_1^2 + \operatorname{Re}((bx_2 + a\xi_2)^2)}\right) + \begin{Bmatrix} 0 \\ \pm\pi \end{Bmatrix}\right] \\
&= -2i\left[\arctan\left(\frac{2\epsilon\sqrt{1-\epsilon^2}(x_2^2 - \xi_2^2)}{x_1^2 + (x_2 + \xi_2)^2 - 2\epsilon^2(x_2^2 + \xi_2^2)}\right) + \begin{Bmatrix} 0 \\ \mp\pi \end{Bmatrix}\right] \\
&= -2i\left[\arctan\left(\frac{\epsilon(x_2 - \xi_2) + x_1}{(x_2 + \xi_2)\sqrt{1-\epsilon^2}}\right) + \arctan\left(\frac{\epsilon(x_2 - \xi_2) - x_1}{(x_2 + \xi_2)\sqrt{1-\epsilon^2}}\right)\right],
\end{aligned}$$

where π or $-\pi$ is added in the first two formula lines if the complex number $x_1^2 + (bx_2 + a\xi_2)^2$ belongs to the second or to the third quadrant, respectively. The remaining two terms in Eq. (2.12) are treated similarly. We therefore obtain for $G_\varepsilon(x)$ the representation which is enunciated in the proposition.

The three conditions in (2.6) then hold automatically by analytic continuation. Condition (2.7) can be checked directly. Finally, the uniqueness of $G_\varepsilon(x)$ under these conditions follows from a reasoning which is similar to that in the proof of Proposition 4.1 below. This completes the proof. $\square$

Remark 2.3 If f_1 is as in Proposition 2.2, then $f_1(x) + f_1(-x_1, x_2)$ coincides with the fundamental solution $E(x)$ in Eq. (1.3) of the orthotropic plate operator (1.1) up to the second-order polynomial $\frac{\arcsin \epsilon}{16\pi\epsilon}(x_1^2 - x_2^2)$.

3 Quasi-hyperbolic and non-vanishing symbol operators

Let us generalize now the context of Section 2 and consider operators of the form

$$P(\partial) = (Q(\partial') - a^2\partial_n^2)(Q(\partial') - b^2\partial_n^2), \quad a > 0, b > 0, a \neq b, \quad (3.1)$$

where $Q(\partial')$ is an operator in the $n-1$ variables $x' = (x_1, \dots, x_{n-1})$ and $P(\partial)$ either is quasi-hyperbolic or has a non-vanishing symbol $P(ix)$.

Let us first recall the notion of “quasi-hyperbolicity”, see [13, Def. and Prop. 2.4.13, p. 162].

Definition 3.1 The operator $P(\partial) = P(\partial_1, \dots, \partial_n) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$, $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}_0^n$, is called quasi-hyperbolic in the direction $N \in \mathbb{R}^n \setminus \{0\}$ if and only if there exists $\sigma_0 \in \mathbb{R}$ such that

$$\forall \sigma > \sigma_0 : \forall x \in \mathbb{R}^n : P(ix + \sigma N) \neq 0. \quad (3.2)$$

As shown in [13, Def. and Prop. 2.4.13, p. 162], for a quasi-hyperbolic operator $P(\partial)$, there exists a uniquely determined fundamental solution E of $P(\partial)$ fulfilling $e^{-\sigma Nx} E \in \mathcal{S}'(\mathbb{R}^n)$ for some σ larger than the constant σ_0 in (3.2). This fundamental solution is then given by the formula

$$E = e^{\sigma Nx} \mathcal{F}^{-1}(P(ix + \sigma N)^{-1}), \quad \sigma > \sigma_0, \quad (3.3)$$

and it holds $\operatorname{supp} E \subset \{x \in \mathbb{R}^n; x \cdot N \geq 0\}$ and $e^{-\sigma Nx} E \in \mathcal{S}'(\mathbb{R}^n)$ for each $\sigma \geq \sigma_0$. Note that operators $P(\partial)$ whose symbols $P(ix)$ do not vanish are formally contained in the above definition if we allow for $N = 0$.

In the next proposition, we shall state uniqueness conditions and we shall give a formula for the Green function $G_\xi(x)$ of the Dirichlet problem in the half-space $H = \{x \in \mathbb{R}^n; x_n > 0\}$ for quasi-hyperbolic and for non-vanishing symbol operators $P(\partial)$ as in (3.1). We assume that $Q(\partial')$ is an operator in the $n-1$ variables $x' = (x_1, \dots, x_{n-1})$, that $N' \in \mathbb{R}^{n-1}$ and we set $N = (N', 0)$. Then, $P(\partial)$ as in (3.1) is quasi-hyperbolic in direction N (in case $N \neq 0$) or has a non-vanishing symbol $P(ix)$ (in case $N = 0$) if and only if, for some $\sigma_0 \in \mathbb{R}$, the following condition is satisfied:

$$\forall \sigma > \sigma_0 : \forall x = (x', x_n) \in \mathbb{R}^n : Q(ix' + \sigma N') + x_n^2 \neq 0. \quad (3.4)$$

Proposition 3.2 *Let $N' \in \mathbb{R}^{n-1}$, $\sigma_0 \in \mathbb{R}$, $H = \{x \in \mathbb{R}^n; x_n > 0\}$, $\xi = (0, \dots, 0, \xi_n) \in \mathbb{R}^n$, $\xi_n > 0$. Let $Q(\partial') = Q(\partial_1, \dots, \partial_{n-1})$ be a linear partial differential operator with constant coefficients in $n-1$ variables such that condition (3.4) is satisfied. Let $P(\partial)$ be defined as in (3.1). For $c > 0$, let $E^c = c^{-1}E^1(x', x_n/c)$ be the uniquely determined fundamental solution of $Q(\partial')(Q(\partial') - c^2\partial_n^2)$ such that $e^{-\sigma N'x'}E^c$ is temperate for $\sigma \geq \sigma_0$.*

Then, the Green function $G_\xi \in \mathcal{D}'(H)$ of $P(\partial)$ fulfilling

$$\begin{aligned} & \text{(i) } P(\partial)G_\xi = \delta(x - \xi) \text{ in } \mathcal{D}'(H), \\ & \text{(ii) } \exists \sigma > \sigma_0 : \exists T_\xi \in \mathcal{S}'(\mathbb{R}^n) : e^{-\sigma N'x'}G_\xi = T_\xi|_H, \\ & \text{(iii) } \lim_{\epsilon \searrow 0} \partial_n^k T_\xi|_{x_n=\epsilon} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}) \text{ for } T_\xi \text{ as in (b) and } k = 0, 1 \end{aligned} \quad (3.5)$$

is uniquely determined and given by

$$\begin{aligned} G_\xi(x) &= F_\xi^{a,b}(x) + F_\xi^{b,a}(x), \quad x \in H, \\ F_\xi^{a,b}(x) &= \frac{a^2}{a^2 - b^2} E^a(x - \xi) - \frac{a^2}{(a - b)^2} E^a(x + \xi) \\ &\quad + \frac{2a^2b}{(a - b)^2(a + b)} E^a\left(x + \frac{a\xi}{b}\right). \end{aligned} \quad (3.6)$$

[Note that the restrictions of $\partial_n^k T_\xi$ to the subspaces $\{x \in \mathbb{R}^n; x_n = \epsilon\}$, $\epsilon > 0$, are well defined due to (i) and [8, Thm. 4.4.8, p. 115]. Condition (iii) requires that these restrictions belong to $\mathcal{S}'(\mathbb{R}^{n-1})$ and converge therein to 0.]

Proof The statement in the proposition corresponds to Lemma 3.2 and Proposition 3.7 in [14], where the operator $(Q(\partial') - \partial_n^2)^m$, $m \in \mathbb{N}$, is investigated. We therefore only outline the proof here and refer to [14] for details.

Since the values of the polynomial $\lambda(x') = Q(ix' + \sigma N')$ lie in the set $\mathbb{C} \setminus (-\infty, 0]$ due to condition (3.4), we can choose the root $\sqrt{\lambda(x')}$ such the $\operatorname{Re} \sqrt{\lambda(x')} > 0$. If $G_\xi, \tilde{G}_\xi$ are two solutions of (3.5), then we consider the difference $S = G_\xi - \tilde{G}_\xi$ and the partial Fourier transform with respect to x'

$$\tilde{S} = \mathcal{F}_{x'}(e^{-\sigma N'x'} S) = \mathcal{F}_{x'}(T_\xi - \tilde{T}_\xi)|_H \in \mathcal{D}'(H).$$

It fulfills $(\lambda(x') - a^2\partial_n^2)(\lambda(x') - b^2\partial_n^2)\tilde{S} = 0$ in $\mathcal{D}'(H)$ and accordingly we obtain the representation

$$\tilde{S} = \sum_{\epsilon=\pm} A_{\epsilon}(x')e^{\epsilon x_n \sqrt{\lambda(x')}/a} + B_{\epsilon}(x')e^{\epsilon x_n \sqrt{\lambda(x')}/b}$$

with distributions $A_{\epsilon}, B_{\epsilon} \in \mathcal{D}'(\mathbb{R}^{n-1})$, $\epsilon = \pm$. From $\mathcal{F}_{x'}(T_{\xi} - \tilde{T}_{\xi}) \in \mathcal{S}'(\mathbb{R}^n)$, we infer that A_{+} and B_{+} must vanish, and from the boundary condition (iii) in (3.5), we then conclude that also $A_{-} = B_{-} = 0$, i.e., $S = 0$ and $G_{\xi} = \tilde{G}_{\xi}$. Therefore, G_{ξ} is uniquely determined.

In order to show that G_{ξ} defined by formula (3.6) satisfies the conditions in (3.5), let E denote the uniquely determined fundamental solution of $P(\partial)$ such that $e^{-\sigma N'x'}E$ is temperate for $\sigma \geq \sigma_0$. Then, E satisfies

$$\begin{aligned} E &= e^{\sigma N'x'} \mathcal{F}^{-1}((\lambda(x') + a^2 x_n^2)^{-1}(\lambda(x') + b^2 x_n^2)^{-1}) \\ &= \frac{e^{\sigma N'x'}}{a^2 - b^2} \mathcal{F}^{-1}\left(\frac{a^2}{\lambda(x')(\lambda(x') + a^2 x_n^2)} - \frac{b^2}{\lambda(x')(\lambda(x') + b^2 x_n^2)}\right) \\ &= \frac{a^2 E^a - b^2 E^b}{a^2 - b^2}. \end{aligned}$$

Furthermore, $P(\partial)E^a = P(\partial)E^b = 0$ in H . This implies that G_{ξ} satisfies conditions (i) and (ii) in (3.5).

Finally, condition (iii) in (3.5) follows from the calculation leading to formula (2.4), which corresponds to the representation (3.6) in the one-dimensional case. We observe that the definition in (2.3) and the ensuing calculations are equally valid for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ if we choose $\sqrt{\lambda}$ with positive real part. $\square$

Example 3.3 Let us calculate the Green function $G_{(0,\xi)}(t, x)$ where $\xi = (0, \dots, 0, \xi_n) \in \mathbb{R}^n$, $\xi_n > 0$, for the product of wave operators

$$P(\partial) = (\partial_t^2 - \Delta_{n-1} - a^2 \partial_n^2)(\partial_t^2 - \Delta_{n-1} - b^2 \partial_n^2), \quad a > 0, b > 0, a \neq b, \quad (3.7)$$

with respect to Dirichlet conditions at the hyperplane $x_n = 0$ and vanishing Cauchy data at $t = 0$. According to Proposition 3.2, we can express $G_{(0,\xi)}$ through a combination of shifted fundamental solutions E^c of $(\partial_t^2 - \Delta_{n-1})(\partial_t^2 - \Delta_{n-1} - c^2 \partial_n^2)$, $c \in \{a, b\}$.

Using the method of parameter integration, see [13, Chap. 3], we have $E^c = c^{-2} \int_0^{c^2} E_{\mu} d\mu$ with E_{μ} being the “forward” fundamental solution of $(\partial_t^2 - \Delta_{n-1} - \mu \partial_n^2)^2$, i.e., the one with support in the half-space $\{(t, x) \in \mathbb{R}^{n+1}; t \geq 0, x \in \mathbb{R}^n\}$.

If χ^z is defined as in (1.6), then the forward fundamental solution E_1 of $(\partial_t^2 - \Delta_n)^2$ can be written as

$$E_1(t, x) = \frac{Y(t) \chi^{2-(n+1)/2}(t^2 - |x|^2)}{8\pi^{(n-1)/2}}$$

according to [4, Lemma 4.2, p. 317], [8, Section 6.2], [13, Eqs. (2.3.11), (2.3.12), pp. 141, 142]. (Note that the function $(t, x) \mapsto t^2 - |x|^2$ is submersive outside the origin and that the composition with χ^z defined in (1.6) thus is meaningful outside the origin. Furthermore, the resulting distribution continuously depends on $t \in \mathbb{R}$, and it can therefore be multiplied by $Y(t)$, see also [12, Prop. 2.4.2, p. 56], and the ensuing remark.)

Hence,

$$E^c = \frac{Y(t)}{8\pi^{(n-1)/2} c^2} \int_0^{c^2} \chi^{2-(n+1)/2}\left(t^2 - |x'|^2 - \frac{x_n^2}{\mu}\right) \frac{d\mu}{\sqrt{\mu}}. \quad (3.8)$$

E.g., let us evaluate the integral in (3.8) in the case of $n = 3$ spatial dimensions. Due to $\chi^0 = Y$ and setting $x' = (x_1, x_2)$, $R = \sqrt{t^2 - |x'|^2}$, $t_+ = \chi^1(t) = Y(t)t$, we obtain

$$E^c(t, x) = \frac{Y(t)}{8\pi c^2} \int_0^{c^2} Y(t^2 - |x'|^2 - x_3^2/\mu) \frac{d\mu}{\sqrt{\mu}} = \frac{Y(t - |x'|)}{4\pi c^2 R} (cR - |x_3|)_+,$$

see also [17, Ex. 5, p. 27]. Therefore, formula (3.6) yields the following Green function for the operator $P(\partial)$ in (3.7) with $n = 3$:

$$G_{(0,\xi)}(t, x) = \frac{Y(t - |x'|)}{4\pi R} \left[\frac{(aR - |x_3 - \xi_3|)_+ - (bR - |x_3 - \xi_3|)_+}{a^2 - b^2} - \frac{(aR - x_3 - \xi_3)_+ + (bR - x_3 - \xi_3)_+}{(a - b)^2} + 2 \frac{(abR - ax_3 - b\xi_3)_+ + (abR - bx_3 - a\xi_3)_+}{(a - b)^2(a + b)} \right].$$

Example 3.4 By means of Proposition 3.2 and using analytic continuation, we can also derive the Green function for the Cauchy–Dirichlet problem of the equation of transverse vibrations of a semi-infinite clamped beam, see [11].

The fundamental solution E^c referring to formula (3.3) for the operator $\partial_t(\partial_t - c^2\partial_x^2)$, $c > 0$, is given by

$$E^c = \frac{Y(t)}{2c\sqrt{\pi}} \int_0^t e^{-x^2/(4c^2\tau)} \frac{d\tau}{\sqrt{\tau}}.$$

Hence, formula (3.6) in Proposition 3.2 yields the following for the Green function $G_{(0,\xi)}$, $\xi > 0$, of the operator

$$\begin{aligned} P(\partial) &= (\partial_t - a^2\partial_x^2)(\partial_t - a^{-2}\partial_x^2) = \partial_t^2 + \partial_x^4 - (a^2 + a^{-2})\partial_t\partial_x^2, \quad a > 0 : \\ G_{(0,\xi)}(t, x) &= \frac{Y(t)}{2\sqrt{\pi}} \int_0^t \left\{ \frac{1}{a^2 - a^{-2}} \left[a e^{-(x-\xi)^2/(4a^2\tau)} - a^{-1} e^{-a^2(x-\xi)^2/(4\tau)} \right] \right. \\ &\quad - \frac{1}{(a - a^{-1})^2} \left[a e^{-(x+\xi)^2/(4a^2\tau)} + a^{-1} e^{-a^2(x+\xi)^2/(4\tau)} \right] \\ &\quad \left. + \frac{2}{(a - a^{-1})^2(a + a^{-1})} \left[e^{-(ax+\xi/a)^2/(4\tau)} + e^{-(a\xi+x/a)^2/(4\tau)} \right] \right\} \frac{d\tau}{\sqrt{\tau}}. \end{aligned} \quad (3.9)$$

We now use analytic continuation as in Section 2 and set $a = \sqrt{i} = \frac{1+i}{\sqrt{2}}$. Then, $P(\partial) = \partial_t^2 + \partial_x^4$ and formula (3.9) yields

$$\begin{aligned} G_{(0,\xi)}(t, x) &= \frac{Y(t)}{2\sqrt{\pi}} \int_0^t \left\{ \sin \left[\frac{(x - \xi)^2}{4\tau} + \frac{\pi}{4} \right] - \sin \left[\frac{(x + \xi)^2}{4\tau} - \frac{\pi}{4} \right] \right. \\ &\quad \left. - \sqrt{2} e^{-x\xi/(2\tau)} \cos \left[\frac{x^2 - \xi^2}{4\tau} \right] \right\} \frac{d\tau}{\sqrt{\tau}}. \end{aligned}$$

in accordance with [11, p. 239].

Remark 3.5 Our faithful reader, who has followed us up to this point, might wonder how the representation of G_ξ in (3.6) would look in the case of a product of $m > 2$ factors, i.e., if $P(\partial) = \prod_{j=1}^m (Q(\partial') - a_j^2 \partial_n^2)$ for pairwise different positive numbers a_j . Since we will not use the corresponding formula, we just mention the result. For $c > 0$, let E^c denote the fundamental solution of $Q(\partial')^{m-1} (Q(\partial') - c^2 \partial_n^2)$ such that $e^{-\sigma N' x'} E$ is temperate for $\sigma \geq \sigma_0$. Then, $G_\xi(x) = \sum_{j=1}^m F_\xi^j(x)$ where

$$\begin{aligned} F_\xi^j(x) = & \left(\prod_{k \neq j} \frac{a_j^2}{a_j^2 - a_k^2} \right) E^{a_j}(x - \xi) - \left(\prod_{k \neq j} \left(\frac{a_j}{a_j - a_k} \right)^2 \right) E^{a_j}(x + \xi) \\ & + 2 \sum_{k \neq j} \left(\prod_{l \neq j, l \neq k} \frac{a_j a_k}{(a_j - a_l)(a_k - a_l)} \right) \frac{a_j^2 a_k E^{a_j}(x + a_j \xi / a_k)}{(a_j - a_k)^2 (a_j + a_k)}. \end{aligned} \quad (3.10)$$

A technically relevant application of formula (3.10) would consist in the derivation of an explicit formula for the Green function of a product $P_1(\partial)P_2(\partial)$ of two orthotropic plate operators $P_1(\partial), P_2(\partial)$ as in (1.1). This product operator describes the deflection of an orthotropic cylindrical shell, see [3, Eqs. (14) and (22), pp. 738, 739].

4 Operators with symbols that are positive outside the origin

In this final section, let us treat operators $P(\partial)$ in $\mathbb{R}^n$ such that $P(ix) > 0$ for $x \in \mathbb{R}^n \setminus \{0\}$. As has been shown in [1], the analytic distribution-valued function

$$S : \{z \in \mathbb{C}; \operatorname{Re} z > 0\} \longrightarrow \mathcal{S}'(\mathbb{R}^n) : z \longmapsto P(ix)^z$$

has a meromorphic extension $\tilde{S}$ to the whole complex plane, see also [13, Prop. 2.3.1, p. 134]. For simplicity, we shall write $P(ix)^z = \tilde{S}(z)$ if $\tilde{S}$ is analytic in $z \in \mathbb{C}$ and $P(ix)^{z_0} = \operatorname{Pf}_{z=z_0} \tilde{S}(z)$ if $z_0 \in \mathbb{C}$ is a pole of $\tilde{S}$. (Here, Pf stands for the finite part of a meromorphic distribution-valued function, compare [12].) Since $P(ix) \cdot P(ix)^z = P(ix)^{z+1}$ holds by analytic continuation for each $z \in \mathbb{C}$, $E := \mathcal{F}^{-1}(P(ix)^{-1})$ yields a temperate fundamental solution of $P(\partial)$.

In particular, let $Q(\partial')$ be an operator in $n-1$ variables $x' = (x_1, \dots, x_{n-1})$ such that $Q(ix') > 0$ for $x' \in \mathbb{R}^{n-1} \setminus \{0\}$. Then, $z \mapsto Q(ix')^{z/2}$ is meromorphic. Upon expanding the exponential function into a power series, this implies that, for $a > 0$, the holomorphic function

$$S^a : \{z \in \mathbb{C}; \operatorname{Re} z > 0\} \longrightarrow \mathcal{S}'(\mathbb{R}^n) : z \longmapsto S_z^a = \frac{1}{2a} Q(ix')^{z/2} e^{-\sqrt{Q(ix')} |x_n|/a} \quad (4.1)$$

can be extended as a meromorphic function to $\mathbb{C}$. From

$$Q(ix')(Q(ix') - a^2 \partial_n^2) S_z^a = Q(ix')^{(z+3)/2} \otimes \delta(x_n), \quad \operatorname{Re} z > 0,$$

we conclude, by analytic continuation, that

$$E^a := \mathcal{F}_{x'}^{-1} \left(\operatorname{Pf}_{z=-3} S_z^a \right) \quad (4.2)$$

is a fundamental solution of $Q(\partial')(Q(\partial') - a^2 \partial_n^2)$.

In the next proposition, we shall generalize Proposition 2.1 to operators of the form

$$P(\partial) = (Q(\partial') - a^2 \partial_n^2)(Q(\partial') - b^2 \partial_n^2), \quad a > 0, b > 0, a \neq b. \quad (4.3)$$

Proposition 4.1 *Let $Q(\partial') = Q(\partial_1, \dots, \partial_{n-1})$ be a linear partial differential operator with constant coefficients in $n - 1$ variables such that $Q(ix') > 0$ for $x' \in \mathbb{R}^{n-1} \setminus \{0\}$. Let $H = \{x \in \mathbb{R}^n; x_n > 0\}$, $\xi = (0, \dots, 0, \xi_n) \in \mathbb{R}^n$, $\xi_n > 0$, and $P(\partial)$ be as in (4.3). For $c > 0$, let E^c be the fundamental solution of $Q(\partial')(Q(\partial') - c^2 \partial_n^2)$ defined in (4.2).*

Then, the Green function $G_\xi \in \mathcal{D}'(H)$ of $P(\partial)$ fulfilling

$$\begin{aligned} & \text{(i) } P(\partial)G_\xi = \delta(x - \xi) \text{ in } \mathcal{D}'(H), \\ & \text{(ii) } \exists T_\xi \in \mathcal{S}'(\mathbb{R}^n) : G_\xi = T_\xi|_H, \\ & \text{(iii) } \lim_{\epsilon \searrow 0} \partial_n^k G_\xi|_{x_n=\epsilon} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}) \text{ for } k = 0, 1, \\ & \text{(iv) } \lim_{N \rightarrow \infty} N^{-2} G_\xi|_{x_n=N} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}) \end{aligned} \quad (4.4)$$

is uniquely determined and given by formula (3.6).

Proof In order to prove the uniqueness of G_ξ under the four conditions in (4.4), we proceed as in the corresponding Lemma 4.1 in [14] and we refer to it for details.

As in the proof of Proposition 3.2, we consider the difference $S = G_\xi - \tilde{G}_\xi$ of two solutions of (4.4) and its partial Fourier transform

$$\tilde{S} = \mathcal{F}_{x'} S = \mathcal{F}_{x'} (T_\xi - \tilde{T}_\xi)|_H \in \mathcal{D}'(H).$$

Condition (i) in (4.4) implies that the support of $\tilde{S}$ is contained in the half-axis $\{(0, \dots, 0, x_n) \in \mathbb{R}^n; x_n > 0\}$. More precisely, we obtain the representation

$$\tilde{S} = \sum_{\alpha \in \mathbb{N}_0^{n-1}, |\alpha| \leq l} \partial^\alpha \delta(x') \otimes q_\alpha(x_n)$$

for $l \in \mathbb{N}_0$ and polynomials q_α in one variable. Hence, S is a polynomial. Condition (iii) in (4.4) implies $S = x_n^2 \cdot R$ for another polynomial and condition (iv) then yields $R = 0$, i.e., $G_\xi = \tilde{G}_\xi$.

For the verification of the representation of G_ξ in formula (3.6), we consider again S_z^a as in (4.1). Since

$$(Q(ix') - a^2 \partial_n^2) S_z^a = Q(ix')^{(z+1)/2} \otimes \delta(x_n), \quad \operatorname{Re} z > 0,$$

we obtain, by analytic continuation, that

$$V = (Q(\partial') - a^2 \partial_n^2) E^a = \mathcal{F}_{x'}^{-1} \left(\operatorname{Pf}_{x'} Q(ix')^z \right) \otimes \delta(x_n) \\ z = -1$$

vanishes in H and is independent of a . This implies that $P(\partial)E^a = P(\partial)E^b = 0$ in H and that $E = (a^2 E^a - b^2 E^b)/(a^2 - b^2) \in \mathcal{S}'(\mathbb{R}^n)$ is a fundamental solution of $P(\partial)$ due to

$$P(\partial)E = \frac{a^2}{a^2 - b^2}(\delta - b^2 \partial_n^2 V) - \frac{b^2}{a^2 - b^2}(\delta - a^2 \partial_n^2 V) = \delta.$$

Hence, conditions (i) and (ii) in (4.4) are fulfilled for G_ξ as in (3.6).

By construction, the partial Fourier transform $\mathcal{F}_{x'} G_\xi = (\mathcal{F}_{x'} T_\xi)|_H$ coincides, for fixed $x' \in \mathbb{R}^{n-1}$, with the one-dimensional Green function $g_{\xi_n}^{Q(\mathbf{i}x')}(x_n)$ given in (2.4) for $Q(\mathbf{i}x') > 0$ and in (2.5) for $Q(\mathbf{i}x') = 0$. The following lemma shows that $g_{\xi_n}^\lambda(\epsilon)$ and $N^{-2} g_{\xi_n}^\lambda(N)$ converge to 0 for $\epsilon \searrow 0$ and $N \rightarrow \infty$ uniformly with respect to $\lambda \in [0, \infty)$. The same assertion holds for $(g_{\xi_n}^\lambda)'(\epsilon)$, and this can be shown by inspection of the explicit representation of $(g_{\xi_n}^\lambda)'(\epsilon)$ by exponential functions, i.e., $(g_\eta^\lambda)'(\epsilon) = h_\eta^{a,b,\lambda}(\epsilon) + h_\eta^{b,a,\lambda}(\epsilon)$ with

$$h_\eta^{a,b,\lambda}(\epsilon) = \frac{e^{-\sqrt{\lambda}\eta/a}}{\lambda(a-b)^2(a+b)} \left[a \cosh\left(\frac{\sqrt{\lambda}\epsilon}{a}\right) - a e^{-\sqrt{\lambda}\epsilon/b} - b \sinh\left(\frac{\sqrt{\lambda}\epsilon}{a}\right) \right]$$

for $0 < \epsilon < \eta$. This implies the conditions (iii) and (iv) in (4.4) and completes the proof. $\square$

Lemma 4.2 For $a > 0, b > 0, \lambda > 0, \xi > 0, a \neq b$, let g_ξ^λ be the one-dimensional Green function of the operator in (2.1), i.e., the function satisfying the conditions in (2.2) and given by formula (2.4).

Then g_ξ^λ fulfills the estimate

$$|g_\xi^\lambda(x)| \leq \frac{x\xi \min(x, \xi)}{a^2 b^2}, \quad x > 0. \quad (4.5)$$

Proof As before, let us denote by E the temperate fundamental solution of the operator $p(d/dx)$ in (2.1). Then, the Green function can be represented by

$$g_\xi^\lambda(x) = E(x - \xi) - E(x + \xi) - 4ab(a + b)\sqrt{\lambda} E'(x)E'(\xi), \quad x > 0. \quad (4.6)$$

In fact, the function on the right-hand side of formula (4.6) fulfills all the conditions in (2.2). (In parentheses, let us observe that, indeed, formula (4.6) yields a simpler and more symmetric representation of g_ξ^λ than (2.4). But this comes at the expense of the fact that the third term in (4.6), i.e., $\sqrt{\lambda} E'(x)E'(\xi)$, is multiplicative. Hence, a partial Fourier transform with respect to the variables $x_2, \dots, x_n$ results in a convolution and cannot be immediately expressed by fundamental solutions as in formula (3.6).)

Furthermore, let F^t denote the temperate fundamental solution of $(\lambda - t d^2/dx^2)^2$, $t > 0$, and write $h_\xi^{\lambda,t}$ for the Green function of $(\lambda - t d^2/dx^2)^2$, i.e.,

$$h_\xi^{\lambda,t}(x) = \lim_{b \rightarrow a = \sqrt{t}} g_\xi^\lambda(x) = F^t(x - \xi) - F^t(x + \xi) - 8\sqrt{\lambda} t^{3/2} F''(x)F''(\xi).$$

By parameter integration, see [13, Chapter 3], we have $E = (b^2 - a^2)^{-1} \int_{a^2}^{b^2} F^t dt$ and hence

$$g_\xi^\lambda(x) = \frac{1}{b^2 - a^2} \int_{a^2}^{b^2} [h_\xi^{\lambda,t}(x) + 8\sqrt{\lambda} t^{3/2} F''(x)F''(\xi)] dt \\ - 4ab(a + b)\sqrt{\lambda} E'(x)E'(\xi).$$

We now refer to [14, Eqns. (2.7), (2.10)] which imply that

$$0 \leq h_{\xi}^{\lambda,t}(x) \leq h_{\xi}^{0,t}(x) = \frac{1}{6t^2} \begin{cases} x^2(3\xi - x) : 0 < x \leq \xi \\ \xi^2(3x - \xi) : x \geq \xi \end{cases} \leq \frac{x\xi \min(x, \xi)}{2t^2}.$$

Therefore, $0 \leq A := (b^2 - a^2)^{-1} \int_a^{b^2} h_{\xi}^{\lambda,t}(x) dt \leq \frac{1}{2} a^{-2} b^{-2} x\xi \min(x, \xi)$.

On the other hand, the explicit representations

$$F^t(x) = \frac{e^{-|x|\sqrt{\lambda/t}}}{4\lambda^{3/2}\sqrt{t}} \left(1 + |x|\sqrt{\frac{\lambda}{t}} \right), \quad F^{t'}(x) = -\frac{xe^{-|x|\sqrt{\lambda/t}}}{4\sqrt{\lambda}t^{3/2}}$$

yield

$$E^t(x) = \frac{1}{b^2 - a^2} \int_a^{b^2} F^{t''}(x) dt = \frac{x}{2\sqrt{\lambda}(b^2 - a^2)} \int_{1/a}^{1/b} e^{-\sqrt{\lambda}|x|u} du$$

and hence, for $x > 0, \xi > 0$,

$$\begin{aligned} B &:= \frac{8\sqrt{\lambda}}{b^2 - a^2} \int_a^{b^2} t^{3/2} F^{t''}(x) F^{t''}(\xi) dt - 4ab(a+b)\sqrt{\lambda} E^t(x) E^t(\xi) \\ &= \frac{ab(a+b)x\xi}{(b^2 - a^2)^2 \sqrt{\lambda}} \int_{1/a}^{1/b} \int_{1/a}^{1/b} e^{-\sqrt{\lambda}xu} (e^{-\sqrt{\lambda}\xi u} - e^{-\sqrt{\lambda}\xi v}) du dv. \end{aligned} \quad (4.7)$$

Due to the symmetry in the left-hand side of (4.7) with respect to x and ξ , we can interchange x and ξ on the right-hand side of (4.7). From

$$|e^{-\sqrt{\lambda}xu}| < 1, \quad |e^{-\sqrt{\lambda}\xi u} - e^{-\sqrt{\lambda}\xi v}| \leq \sqrt{\lambda}\xi|u - v|, \quad u > 0, v > 0,$$

we then obtain

$$|B| \leq \frac{|a - b|x\xi}{3a^2b^2(a+b)} \min(x, \xi) \leq \frac{1}{3a^2b^2} x\xi \min(x, \xi).$$

Due to $g_{\xi}^{\lambda}(x) = A + B$ and taking into account the estimate for A above, we arrive at the inequality in (4.5). This completes the proof. $\square$

Example 4.3 In this final example, let us generalize Proposition 2.1 to n dimensions and calculate the Green function $G_{\xi}(x)$ for the operator

$$P(\partial) = (\Delta_{n-1} + a^2\partial_n^2)(\Delta_{n-1} + b^2\partial_n^2), \quad a > 0, b > 0, a \neq b,$$

with respect to Dirichlet boundary conditions at the border $x_n = 0$.

According to Proposition 4.1 above, $G_{\xi}(x)$ can be expressed by formula (3.6) if E^a denotes the fundamental solution of $\Delta_{n-1}(\Delta_{n-1} + a^2\partial_n^2)$ which is given by

$$E^a = \text{Pf}_{z=-3}(\mathcal{F}_{x'}^{-1}S_z^a), \quad S_z^a = \frac{|x'|^z}{2a} e^{-|x'|\|x_n\|/a}$$

with $x' = (x_1, \dots, x_{n-1})$. Due to $E^a = a^{-1}E^1(x', x_n/a)$, we can restrict ourselves to calculating the fundamental solution E^1 of $\Delta_{n-1}\Delta_n$.

In order to evaluate $\mathcal{F}_{x'}^{-1}S_z^1$, we use the Poisson–Bochner formula, see [15, Equ. (VII,7;22), p. 259], [13, Equ. (1.6.14), p. 97]. This yields

$$\mathcal{F}_{x'}^{-1}S_z^1 = \frac{|x'|^{(3-n)/2}}{2(2\pi)^{(n-1)/2}} \int_0^\infty \rho^{z+(n-1)/2} J_{(n-3)/2}(\rho|x'|) e^{-\rho|x_n|} d\rho$$

for $x_n \neq 0$ and $\operatorname{Re} z > 1 - n$. Note that $S_z^1 \in L^1(\mathbb{R}_{x'}^{n-1})$ for $x_n \neq 0$ and $\operatorname{Re} z > 1 - n$, and that S_z^1 continuously depends on x_n for $\operatorname{Re} z > 1 - n$. An appeal to [7, Equ. 6.621.1] furnishes

$$\begin{aligned} \mathcal{F}_{x'}^{-1}S_z^1 &= \frac{\Gamma(z+n-1)}{(4\pi)^{(n-1)/2}\Gamma(\frac{n-1}{2})} |x|^{-(z+n-1)} \\ &\times {}_2F_1\left(\frac{z+n-1}{2}, -\frac{z+1}{2}; \frac{n-1}{2}; \frac{|x'|^2}{|x|^2}\right), \quad \operatorname{Re} z > 1 - n. \end{aligned} \quad (4.8)$$

We observe that the hypergeometric function in (4.8) can also be expressed by the Legendre function, see [7, Equ. 6.621.1].

If $n \geq 5$, then S_z^1 is analytic near $z = -3$ and thus

$$E^1(x) = \mathcal{F}_{x'}^{-1}S_{-3}^1 = \frac{(n-5)!|x|^{4-n}}{(4\pi)^{(n-1)/2}\Gamma(\frac{n-1}{2})} {}_2F_1\left(1, \frac{n}{2} - 2; \frac{n-1}{2}; \frac{|x'|^2}{|x|^2}\right). \quad (4.9)$$

In particular, if $n \geq 5$ is odd, i.e., if $n = 1 + 2m$, $m \geq 2$, then

$${}_2F_1\left(1, m - \frac{3}{2}; m; u\right) = -\frac{2(m-1)!\sqrt{\pi}}{\Gamma(m - \frac{3}{2})u^{m-1}} \left[\sqrt{1-u} - \sum_{k=0}^{m-2} \binom{1/2}{k} (-u)^k \right]$$

according to [2, Equ. 7.3.1.123], and hence,

$$E^1(x) = \frac{(m-2)!}{8\pi^m|x'|^{2(m-1)}} \left[-|x_n| + |x| \sum_{k=0}^{m-2} \binom{1/2}{k} \left(-\frac{|x'|^2}{|x|^2}\right)^k \right].$$

If, instead, $n = 2m$ is even with $m \geq 3$, then we can use [7, Equ. 9.137.17] and [2, Equ. 7.3.1.133], which furnish

$$\begin{aligned} {}_2F_1\left(1, m-2; m-\frac{1}{2}; u\right) &= (2m-3) {}_2F_1\left(1, m-2; m-\frac{3}{2}; u\right) \\ &\quad - (2m-4) {}_2F_1\left(1, m-1; m-\frac{1}{2}; u\right) \\ &= \frac{2(\frac{1}{2})_{m-1}(u-1)}{(m-3)!u^{m-1}} \left[2\sqrt{\frac{u}{1-u}} \arctan \sqrt{\frac{u}{1-u}} - \sum_{k=1}^{m-3} \frac{(k-1)!u^k}{(\frac{1}{2})_k} \right] + \frac{2m-3}{u} \end{aligned}$$

where $(\frac{1}{2})_k = \frac{1}{2} \cdot \frac{3}{2} \cdots (\frac{1}{2} + k - 1)$ is an instance of Pochhammer's symbol. From formula (4.9), we then obtain

$$\begin{aligned} E^1(x) &= -\frac{\Gamma(m-\frac{3}{2})x_n}{4\pi^{1/2+m}|x'|^{2m-3}} \arctan\left(\frac{|x'|}{x_n}\right) \\ &\quad + \frac{\Gamma(m-\frac{3}{2})x_n^2}{8\pi^{1/2+m}} \sum_{k=1}^{m-3} \frac{(k-1)!|x'|^{2(k+1-m)}}{(\frac{1}{2})_k|x|^{2k}} + \frac{(m-3)!}{8\pi^m|x|^{2m-6}|x'|^2}. \end{aligned}$$

Let us finally treat the cases $n = 3$ and $n = 4$ where S_z^1 has a pole at $z = -3$. (The case $n = 2$ was the content of Proposition 2.1.) If $n = 4$, then formula (4.8) furnishes

$$\begin{aligned}\mathcal{F}_{x'}^{-1} S_{z-3}^1 &= \frac{\Gamma(z)}{4\pi^2} |x|^{-z} {}_2F_1\left(\frac{z}{2}, 1 - \frac{z}{2}; \frac{3}{2}; \frac{|x'|^2}{|x|^2}\right) \\ &= \frac{\Gamma(z-1)}{4\pi^2 |x'|} |x|^{1-z} \sin\left((z-1) \arcsin\left(\frac{|x'|}{|x|}\right)\right)\end{aligned}$$

due to [2, Equ. 7.3.1.91]. Here $x' = (x_1, x_2, x_3)$. Using $\text{Res}_{z=0} \Gamma(z-1) = -1$ and $\text{Pf}_{z=0} \Gamma(z-1) = -\psi(2)$ and $E^a = a^{-1} (\text{Pf}_{z=0} \mathcal{F}_{x'}^{-1} S_{z-3}^1)(x', x_4/a)$, we obtain

$$E^a(x) = \frac{1}{4\pi^2} \left[\frac{\psi(2)}{a} - \frac{1}{2a} \log\left(|x'|^2 + \frac{x_4^2}{a^2}\right) - \frac{x_4}{a^2 |x'|} \arctan\left(\frac{a|x'|}{x_4}\right) \right]. \quad (4.10)$$

Note that the terms corresponding to the constant $\psi(2)/a$ in formula (4.10) cancel in the linear combination of fundamental solutions making up the Green function $G_\varepsilon(x)$ according to formula (3.6).

For $n = 3$, the easiest derivation consists in descending from the five-dimensional case by integration with respect to the (x_3, x_4) -plane and by renaming afterwards x_5 as x_3 . When regularizing the corresponding integral, this furnishes

$$E^1(x) = \frac{1}{4\pi} [|x_3| \log(|x| + |x_3|) - |x|].$$

Funding Open access funding provided by University of Innsbruck and Medical University of Innsbruck.

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References

- Bernstein, I.N.: Modules over a ring of differential operators. Study of fundamental solutions of equations with constant coefficients. *Funct. Anal. Appl.* **5**, 89–101 (1971)
- Brychkov, Y.A., Marichev, O.I., Prudnikov, A.P.: Integrals and series. More special functions. Gordon & Breach, New York (1990)
- Cheng, S., He, F.B.: Theory of orthotropic and composite cylindrical shells, accurate and simple fourth-order governing equations. *ASME J. Appl. Mech.* **51**, 736–744 (1984)
- Delache, S., Leray, J.: Calcul de la solution élémentaire de l'opérateur d'Euler-Poisson-Darboux et de l'opérateur de Tricomi-Clairaut, hyperbolique, d'ordre 2. *Bull. Soc. Math. France* **99**, 313–336 (1971)
- Friedlander, G., Joshi, M.: Introduction to the theory of distributions, 2nd edn. Cambridge University Press, Cambridge (1998)
- Girkmann, K.: Flächenträgerwerke, 6th edn. Springer, Wien (1963)
- Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series and products. Academic Press, New York (1980)

8. Hörmander, L.: The analysis of linear partial differential operators. Vol. I (Distribution theory and Fourier analysis), Grundlehren Math. Wiss. 256, 2nd edn. Springer, Berlin (1990)
9. Lavoine, J.: Transformation de Fourier des pseudo-fonctions. Editions du CNRS, Paris (1963)
10. Michell, J.H.: On the flexure of a circular plate. Proc. London Math. Soc. **34**, 223–228 (1901)
11. Ortner, N., Wagner, P.: The Green's functions of clamped semi-infinite vibrating beams and plates. Int. J. Solids Struct. **26**, 237–249 (1990)
12. Ortner, N., Wagner, P.: Distribution-valued analytic functions. Tredition, Hamburg (2013)
13. Ortner, N., Wagner, P.: Fundamental solutions of linear partial differential operators. Springer, New York (2015)
14. Ortner, N., Wagner, P.: Green functions and Poisson kernels for iterated operators. Dirichlet and Cauchy–Dirichlet problems in half- and quarter spaces. Pure and Applied Functional Analysis, Special issue in honour of S. Agmon (to appear)
15. Schwartz, L.: Théorie des distributions, 2nd edn. Hermann, Paris (1966)
16. Stein, P.: Die Anwendung der Singularitätenmethode zur Berechnung orthogonaler anisotroper Rechteckplatten, einschließlich Trägerrosten. Stahlbau-Verlag, Köln (1959)
17. Wagner, P.: Parameterintegration zur Berechnung von Fundamentallösungen. Diss. Math. **230**, 1–50 (1984)

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