Vector bundles on rational homogeneous spaces

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Abstract

We consider a uniform r -bundle E on a complex rational homogeneous space X and show that if E is poly-uniform with respect to all the special families of lines and the rank r is less than or equal to some number that depends only on X , then E is either a direct sum of line bundles or unstable with respect to some numerical class of a line. So we partially answer a problem posted by Muñoz et al. (Eur J Math 6:430–452, 2020). In particular, if X is a generalized Grassmannian $\mathcal{G}$ and the rank r is less than or equal to some number that depends only on X , then E splits as a direct sum of line bundles. So we improve the main theorem of Muñoz et al. (J Reine Angew Math (Crelles J) 664:141–162, 2012, Theorem 3.1) when X is a generalized Grassmannian. Moreover, by calculating the relative tangent bundles between two rational homogeneous spaces, we give explicit bounds for the generalized Grauert–Mülich–Barth theorem on rational homogeneous spaces.

Dedicate to people who devote their lives to fight against coronaviruses.

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1 Introduction

Algebraic vector bundles on a projective variety X over the complex number field $\mathbb{C}$ are fundamental research objects in algebraic geometry. However, up to now, algebraic vector bundles are still mysterious for general projective varieties. According to Serre (GAGA), the classification of algebraic vector bundles over $\mathbb{C}$ is equivalent to the classification of holomorphic vector bundles. So there are not only algebraic ways but also analytic ways to handle the problems of vector bundles. For simplicity, we just call vector bundles of rank r or r -bundles in the context.

If X is $\mathbb{P}^1$, the structure of a vector bundle on X is quite clear because Grothendieck tells us that it splits as a direct sum of line bundles. However, if X is a projective space and the dimension of it is bigger than or equal to two, then the structures of vector bundles on X are not so easy to be determined. Since projective spaces are covered by lines, it is a natural way to consider the restriction of vector bundles to lines. From Grothendieck's result, they split after being restricted to $\mathbb{P}^1$. By semicontinuity theorem, for “almost all” lines, any r -bundle E has a constant splitting type. This means that the lines to which a vector bundle restricts having different splitting types consist a closed subset of a Grassmannian. If the closed subset is empty, such bundles are called uniform vector bundles. Uniform bundles are widely studied not only on projective spaces [3, 8–10, 26, 27, 29] but also on special Fano manifolds of Picard number one [2, 6, 11, 14, 17, 25]. Please see Introduction in [6] for the details. It is worth to mention that, in [25], the author also studied vector bundles on some Fano manifolds with higher Picard numbers. In particular, he proved that a vector bundle on rational homogeneous spaces is trivial if and only if its restriction to every Schubert line is trivial.

Instead of considering vector bundles, Occhetta-Solá Conde-Wiśniewski [22] studied flag bundles which are constructed upon the action of the defining group G on the flag manifold G/B . Recently, Muñoz-Occhetta-Solá Conde [16] studied uniform principal G -bundles with G semisimple over Fano manifolds. They presented a number of theorems that are versions of flag bundles in the theory of uniform vector bundles. More precisely, they paid special attention to homogeneous filtrations of relative tangent bundles between flag manifolds and generalized the different standard decomposability notions of vector bundles. They used an interesting concept of “tag” of a G/B -bundle, which is the set of intersection numbers of a minimal section of the G/B -bundle with the relative canonical divisors (see [22] Proposition 3.17), to describe diagonalizability of any uniform flag bundle of low rank. But they mainly focus on rational homogeneous spaces with Picard number one. In [18], the authors proposed a problem as follows.

Problem 1.1 Classify low rank uniform principal G -bundles (G semisimple algebraic group) on rational homogeneous spaces.

In this paper, we consider uniform bundles on generalized flag varieties or even rational homogeneous spaces with arbitrary Picard numbers and give partial answers to this problem.

It is well known that G/P carries a transitive G -action, it is a smooth projective variety. Borel and Remmert's classical theorem [5] states that a projective complex manifold which admits a transitive action of its automorphism group is a direct product of an abelian variety and a rational homogeneous space G/P , where G is a semi-simple algebraic group and P is a parabolic subgroup.

Let

$$X = G/P \simeq G_1/P_{I_1} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m},$$

where G_i is a simple Lie group with *Dynkin diagram* $\mathcal{D}_i$ whose set of nodes is D_i and P_{I_i} is a parabolic subgroup of G_i corresponding to $I_i \subset D_i$. We set $F(I_i) := G_i/P_{I_i}$ by marking on the Dynkin diagram $\mathcal{D}_i$ of G_i with the nodes corresponding to I_i . Let δ_i be a node in $\mathcal{D}_i$ and $N(\delta_i)$ be the set of nodes in $\mathcal{D}_i$ that are connected to δ_i .

If $\delta_i \in I_i$, we call

$$\mathcal{M}_i^{\delta_i^c} := G_i/P_{I_i}^{\delta_i^c} \times \widehat{G_i/P_{I_i}} \quad (1 \leq i \leq m),$$

the i -th *special family* of lines of class $\check{\delta}_i$, which is the coroot to the root δ_i (cf. [15] after Corollary 4.2), where $P_{I_i}^{\delta_i^c} := P_{(I_i \setminus \delta_i) \cup N(\delta_i)}$ and $\widehat{G_i/P_{I_i}}$ is $G_1/P_{I_1} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m}$ by deleting i -th term G_i/P_{I_i} . Denote

$$\mathcal{U}_i^{\delta_i^c} := G_i/P_{I_i \cup N(\delta_i)} \times \widehat{G_i/P_{I_i}}$$

the i -th *universal family* of class $\check{\delta}_i$, which has a natural $\mathbb{P}^1$ -bundle structure over $\mathcal{M}_i^{\delta_i^c}$.

We separate our discussion into two cases:

Case I: $N(\delta_i) \subseteq I_i$, then $\mathcal{U}_i^{\delta_i^c} = X$ and we have the natural projection $X \rightarrow \mathcal{M}_i^{\delta_i^c}$;

Case II: $N(\delta_i) \not\subseteq I_i$, then we have the standard diagram

$$\begin{array}{ccc} \mathcal{U}_i^{\delta_i^c} & \xrightarrow{q_2} & \mathcal{M}_i^{\delta_i^c} \\ \downarrow q_1 & & \\ X & & \end{array} \quad (1.1)$$

For $x \in X$, we call

$$\mathcal{M}_{i,x}^{\delta_i^c} = \{L \in \mathcal{M}_i^{\delta_i^c} \mid x \in L\}$$

the δ_i -th *special part of variety of minimal rational tangents at x* (sometimes we just write $\mathcal{M}_x$ if there is no confusion).

Fix $\delta_i \in I_i$. $\mathcal{M}_{i,x}^{\delta_i^c}$ is just the special family of lines of class $\check{\delta}_i$ through x on the generalized Grassmannian $\mathcal{G}^{\delta_i}$ whose Dynkin diagram $\mathcal{D}^{\delta_i}$ is the maximal sub-diagram of $(\mathcal{D}_i, I_i)$ with the only marked point δ_i . Denote $v(X, \delta_i) := \zeta(\mathcal{G}^{\delta_i})$, where $\zeta(\mathcal{G}^{\delta_i})$ are defined as in Table 1. Let

$$v(X) := \min_i \{ \min_{\delta_i \in I_i} \{ v(X, \delta_i) \} \}.$$

Definition 1.2 A vector bundle E on X is called *poly-uniform* with respect to $\mathcal{M}_i^{\delta_i^c}$ for every i ($1 \leq i \leq m$) and every $\delta_i \in I_i$ if the restriction of E to every line in $\mathcal{M}_i^{\delta_i^c}$ has the same splitting type. We also call that E *poly-uniform* with respect to all the special families of lines.

Table 1 $\varsigma(\mathcal{G})$

$\mathcal{G}$	A_n/P_1	B_n/P_1	C_n/P_1	D_n/P_1	E_6/P_1	E_7/P_1	E_8/P_1	F_4/P_1	G_2/P_1
	A_n/P_n	B_n/P_n	C_n/P_n	D_n/P_{n-1}	E_6/P_2	E_7/P_2	E_8/P_2	F_4/P_4	G_2/P_2
				D_n/P_n	E_6/P_6	E_7/P_7	E_8/P_8		
$\varsigma(\mathcal{G})$	$n-1$	$2n-3$	$2n-3$	$2n-5$	7	9	11	5	1
	$n-1$	$n-1$	$n-1$	$n-1$	5	6	7	5	1
			$n-1$	7	12	14			

Let $\mathcal{F}$ be a torsion free coherent sheaf of rank r over X . Fix integer i ($1 \leq i \leq m$) and $\delta_i \in I_i$. Since the singularity set $S(\mathcal{F})$ of $\mathcal{F}$ has codimension at least 2, there are lines $L \in \mathcal{M}_i^{\delta_i}$ which do not meet $S(\mathcal{F})$. If

$$\mathcal{F}|L \cong \mathcal{O}_L(a_1^{(\delta_i)}) \oplus \cdots \oplus \mathcal{O}_L(a_r^{(\delta_i)}),$$

we set

$$c_1^{(\delta_i)}(\mathcal{F}) = a_1^{(\delta_i)} + \cdots + a_r^{(\delta_i)}$$

and

$$\mu^{(\delta_i)}(\mathcal{F}) = \frac{c_1^{(\delta_i)}(\mathcal{F})}{\text{rk}(\mathcal{F})},$$

which are independent of the choice of L .

Definition 1.3 A torsion free coherent sheaf $\mathcal{E}$ over X is δ_i -semistable (resp. δ_i -stable) if for every coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ with $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$, we have

$$\mu^{(\delta_i)}(\mathcal{F}) \leq (\text{resp. } <) \mu^{(\delta_i)}(\mathcal{E}).$$

If E is not δ_i -semistable, then we call E is δ_i -unstable.

Now, we can state our main results.

Theorem 1.4 *On X , if an r -bundle E is poly-uniform with respect to all the special families of lines and $r \leq v(X)$, then E is δ_i -unstable for some δ_i ($1 \leq i \leq m$) or E splits as a direct sum of line bundles.*

In particular, if the Picard number of X is one, we improve an interesting theorem of Muñoz-Occhetta-Solá Conde ([17] Theorem 3.1) a little bit (see Theorem 1.5 and Table 1).

We list the values ς of special Grassmannians before Theorem 1.5 in the following table. For the values ς of other Grassmannians, see Corollary 3.4 and Remark 3.5.

Theorem 1.5 *Suppose that E is a uniform r -bundle of a generalized Grassmann $\mathcal{G}$ with respect to the special family of lines. If $r \leq \varsigma(\mathcal{G})$, then E splits as a direct sum of line bundles.*

Remark 1.6

- If δ corresponds to a long root, then the special family of lines of the generalized Grassmannian G/P_δ parameterizes *all* the lines, and thus Theorem 1.5 is valid for uniform bundles (in usual sense).
- If $\mathcal{G} = C_n/P_1 \cong \mathbb{P}^{2n-1}$ or $\mathcal{G} = G_2/P_1 \cong \mathbb{Q}^5$, then the splitting thresholds of uniform vector bundles with respect to the family of *all* lines are slightly better. If E is a uniform r -bundle ($r \leq 2n - 2$) on $C_n/P_1 \cong \mathbb{P}^{2n-1}$, then E splits. Similarly, if E is a uniform r -bundle ($r \leq 3$) on $G_2/P_1 \cong \mathbb{Q}^5$, then E splits.
- If δ corresponds to a short root, there may exist vector bundles on G/P_δ that are uniform with respect to the special family of lines, but not uniform with respect to the family of all lines. For example, the nullcorrelation bundles (cf. [23] page 40) on $C_n/P_1 \cong \mathbb{P}^{2n-1}$ which are unsplit bundles of rank $2n - 2$ (cf. paragraph after Theorem 3 in [4]) and the Cayley bundles on $G_2/P_1 \cong \mathbb{Q}^5$ (cf. [24] Theorem 3.5) which are unsplit bundles of rank 2.
- Comparing Theorem 5.6 in the paper [25], we consider uniform bundles with respect to the special family of lines in Theorem 1.5, while Pan considered uniform bundles with respect to the family of all lines. Moreover, we deal with all simple Lie algebras while Pan only considered the simple Lie algebras of types A , B and C .

By calculating the relative tangent bundles and using Descent Lemma, we have explicit bounds for the generalized Grauert–Müllich–Barth theorem on rational homogeneous spaces.

Theorem 1.7 Fix $\delta_i \in I_i$. Let E be a holomorphic r -bundle over X of general splitting type $\underline{a}_E^{(\delta_i)} = (a_1^{(\delta_i)}, \dots, a_r^{(\delta_i)})$, where $a_1^{(\delta_i)} \geq \dots \geq a_r^{(\delta_i)}$, with respect to $\mathcal{M}_i^{\delta_i}$. If for some $t < r$,

$$a_t^{(\delta_i)} - a_{t+1}^{(\delta_i)} \geq \begin{cases} 1 \text{ and } N(\delta_i) \text{ fits Case I,} \\ 2 \text{ and } N(\delta_i) \text{ fits Case II, where } \delta_i \text{ is not an exposed short root,} \\ 3 \text{ and } N(\delta_i) \text{ fits Case II, where } \delta_i \text{ is an exposed short root not in } \mathcal{D}(G_2), \\ 4 \text{ and } N(\delta_i) \text{ fits Case II, where } \delta_i \text{ is an exposed short root in } \mathcal{D}(G_2), \end{cases}$$

then there is a normal subsheaf $K \subset E$ of rank t with the following properties: over the open set $V_E = q_1(q_2^{-1}(U_E^{(\delta_i)})) \subset X$, where $U_E^{(\delta_i)}$ is an open set in $\mathcal{M}_i^{\delta_i}$, the sheaf K is a subbundle of E , which on the line $L \subset X$ given by $l \in U_E^{(\delta_i)}$ has the form

$$K|_L \cong \bigoplus_{s=1}^t \mathcal{O}_L(a_s^{(\delta_i)}).$$

Remark 1.8 A coherent sheaf $\mathcal{F}$ is said to be “normal” in the sense of Barth ([4], p. 128) if the restriction $\mathcal{F}(U) \rightarrow \mathcal{F}(U \setminus A)$ is bijective for every open subset U and every closed subset A of U of codimension at least 2.

Corollary 1.9 With the same assumption as Theorem 1.7. For a δ_i -semistable r -bundle E over X of general splitting type $\underline{a}_E^{(\delta_i)} = (a_1^{(\delta_i)}, \dots, a_r^{(\delta_i)})$, where $a_1^{(\delta_i)} \geq \dots \geq a_r^{(\delta_i)}$, with respect to $\mathcal{M}_i^{\delta_i}$ and all $s = 1, \dots, r - 1$, we have

$$a_s^{(\delta_i)} - a_{s+1}^{(\delta_i)} \leq \begin{cases} 1, \text{ if } \delta_i \text{ is not an exposed short root;} \\ 2, \text{ if } \delta_i \text{ is an exposed short root and } \delta_i \notin \mathcal{D}(G_2); \\ 3, \text{ if } \delta_i \text{ is an exposed short root and } \delta_i \in \mathcal{D}(G_2). \end{cases}$$

In particular, if $N(\delta_i)$ fits Case I, then we have $a_s^{(\delta_i)}$ ’s are constant for all $1 \leq s \leq r$.

If $X = G/B$, where G is a semi-simple Lie group and B is a Borel subgroup of G , then we have following result.

Corollary 1.10 *If an r -bundle E on X is δ_i -semistable for all i and δ_i ($1 \leq i \leq m$), then E splits as a direct sum of line bundles.*

2 Preliminaries

Throughout this paper, all algebraic varieties and morphisms will be defined over complex number field $\mathbb{C}$.

2.1 Semisimple Lie groups and algebras

In mathematics, Lie group-Lie algebra correspondence allows one to study Lie groups, which are geometric objects, in terms of Lie algebras, which are linear objects. Let G be a semi-simple Lie group. Assume V to be a nonzero finite-dimensional complex vector space, fix a maximal torus $H \subset G$, and let $\varphi : G \rightarrow GL(V)$ be a representation of G . It is well known that V decomposes into a direct sum of simultaneous eigenspaces

$$V = \bigoplus V_\lambda,$$

where the direct sum run over λ in the *character group* of H , which is the set of all holomorphic homomorphisms λ from H to $\mathbb{C}^*$, and

$$V_\lambda = \{v \in V | \varphi(h)(v) = \lambda(h)v, \text{ for all } h \in H\}.$$

Since V is a finite-dimensional vector space, we have $V_\lambda = 0$ for all but finitely many values of λ . Those values of λ for which $V_\lambda \neq 0$ are called the *weights* of V , and V_λ is called the *weight space*.

In the Lie algebra side, let $\mathfrak{g}$ be the associated semi-simple Lie algebra of G . The maximal torus corresponds to $\mathfrak{h} \subset \mathfrak{g}$ which is an abelian subalgebra of maximal dimension, i.e., Cartan subalgebra. A holomorphic representation $\varphi : G \rightarrow GL(V)$ of a complex Lie group G gives rise to a complex linear representation $\varphi_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of the Lie algebra $\mathfrak{g}$ of G . Similarly, every finite-dimensional representation of $\mathfrak{g}$ admits a decomposition

$$V = \bigoplus_{\lambda \in \{\lambda \in \mathfrak{h}^* | V_\lambda \neq 0\}} V_\lambda,$$

where

$$V_\lambda = \{v \in V | \varphi_{\mathfrak{g}}(h)(v) = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}.$$

Those λ are still called the *weights* of V , and V_λ is called the *weight space*. If we apply the above decomposition to $V = \mathfrak{g}$ and $\varphi_{\mathfrak{g}}$ the adjoint representation, we have *Cartan decomposition* of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha := \{g \in \mathfrak{g} \mid \varphi_g(h)(g) = \alpha(h)g, \text{ for all } h \in \mathfrak{h}\}.$$

The elements $\alpha \in \mathfrak{h}^\vee \setminus \{0\}$ for which $\mathfrak{g}_\alpha \neq 0$ are called roots of $\mathfrak{g}$, and the set of these elements will be denoted by Φ and be called *root system*. For every $\alpha \in \Phi$, $\mathfrak{g}_\alpha$ is called *root space* which is one-dimensional.

Fix a linear functional

$$f : \text{span}_{\mathbb{R}} \Phi \rightarrow \mathbb{R}$$

whose kernel does not intersect Φ . Let

$$\Phi^+ := \{\alpha \in \Phi \mid f(\alpha) > 0\} \text{ and } \Phi^- := \{\alpha \in \Phi \mid f(\alpha) < 0\}.$$

Then Φ^+ is called *positive system of roots* and Φ^- is called *negative system of roots*. Given such a positive system Φ^+ , we define the *fundamental system* $\Pi \subset \Phi^+$ as follows: $\alpha \in \Pi$ if and only if $\alpha \in \Phi^+$ and α cannot be expressed as the sum of two elements of Φ^+ . A nonzero representation V of $\mathfrak{g}$ is called a *highest weight representation* if it is generated by a vector $v \in V_\lambda$ such that $gv = 0$ for all $g \in \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. In this case, v is called the *highest weight vector*, and λ is the highest weight of V .

For each $\alpha \in \Phi$ there is a unique element $h_\alpha \in \mathfrak{h}$ such that

$$\alpha(h) = \langle h_\alpha, h \rangle \text{ for all } h \in \mathfrak{h}.$$

The vectors h_α for $\alpha \in \Phi$ span $\mathfrak{h}$. We denote by $\mathfrak{h}_{\mathbb{R}}$ the set of all elements of form $\sum_{i=1}^l a_i h_{\alpha_i}$ for $a_i \in \mathbb{R}$.

The *Killing form* $(\alpha, \beta) := \text{tr}(\text{ad}_\alpha \circ \text{ad}_\beta)$ defines a nondegenerated bilinear form on $\mathfrak{h}$, where $\alpha, \beta \in \mathfrak{g}$ and ad is the adjoint representation. It can be shown that $\mathfrak{h}_{\mathbb{R}}^\vee$ is a Euclidean space with respect to $(\cdot, \cdot)$. Set $n := \dim_{\mathbb{C}}(\mathfrak{h})$ and $D := \{1, 2, \dots, n\}$. We identify D with the set of fundamental roots with respect to a choice of maximal torus T and fixed Borel subgroup B . It is known that every fundamental system $\Pi \subset \Phi$ can form a basis of $\mathfrak{h}_{\mathbb{R}}^\vee$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a fundamental system. Then we define A_{ij} by

$$A_{ij} := \langle \alpha_i, \alpha_j \rangle = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \quad i, j = 1, \dots, n.$$

The matrix $A = (A_{ij})$ is called the *Cartan matrix* of $\mathfrak{g}$.

The *Dynkin diagram* of G , which we denoted by $\mathcal{D} := \mathcal{D}(G)$, is determined by the Cartan matrix. It consists of a graph whose set of nodes is D and where the nodes i and j are joined by $A_{ij}A_{ji}$ edges. When two nodes i and j are joined by a double or triple edge, we add to it an arrow pointing to i if $|A_{ij}| > |A_{ji}|$. We call α_i a short root of $\mathcal{D}$ and α_j a non-short (or long) root of $\mathcal{D}$. (Sometimes, for the sake of narrative convenience, we freely interchange the terminology “node” and “root”.) One may prove that there is a one-to-one correspondence between isomorphism classes of semisimple Lie algebras and Dynkin diagrams of reduced root systems. Moreover, every reduced root system is a disjoint union of mutually orthogonal irreducible root subsystems, each of them corresponding to one of the connected finite Dynkin diagrams A_n, B_n, C_n, D_n ($n \in \mathbb{Z}_{>0}$), E_6, E_7, E_8, F_4, G_2 . We use the Bourbaki labeling as follows.

The connected components of the Dynkin diagram $\mathcal{D}$ determine the simple algebraic groups that are factors of the semisimple Lie group G , each of them corresponding to one of the Dynkin diagrams above.

2.2 Parabolic subgroup and subalgebra

A closed subgroup P of G is called *parabolic* if the quotient space G/P is complete, hence projective. A maximal connected solvable subgroup B of G is called a *Borel* subgroup. We fix a Cartan subalgebra $\mathfrak{h}$. Let

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

be a fixed Borel subalgebra. It is easy to determine the parabolic subalgebras containing $\mathfrak{b}$. They are all of the form

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_\alpha,$$

where Φ_p^+ is a subset of Φ^+ that is closed under the addition of roots. Hence the parabolic subalgebras containing $\mathfrak{b}$ lie in bijection with the subsets of D . For some subset I of D , we write $\mathfrak{p}_I$ the parabolic subalgebra corresponding to I . We define P_I by the parabolic subgroup of G generated by the complement of a set I of fundamental positive roots such that its Lie algebra is $\mathfrak{p}_I$. Therefore the parabolic subgroup P_I corresponds to the subset I (so a maximal parabolic subgroup is defined by a single root). Then G/P_I has a minimal homogeneous embedding in projective space of the highest weight module V_λ of G corresponding

to the highest weight $\lambda = \sum_{i \in I} \omega_i$, where ω_i is the i -th *fundamental weight* dual to the root $\alpha_i \in \Pi$, by a very ample line bundle L_λ .

2.3 Rational homogeneous spaces

Every rational homogeneous space G/P can be decomposed into a product

$$G/P \simeq G_1/P_{I_1} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m}$$

of rational homogeneous spaces with simple algebraic groups $G_1, \dots, G_m$. Each rational homogeneous space G_i/P_{I_i} , called the *generalized flag manifold*, only depends on the Lie algebra $\mathfrak{g}_i$ of G_i , which is classically determined by the marked Dynkin diagram [15]. In the most common notation, we set $F_I := G/P_I$ by marking on the Dynkin diagram $\mathcal{D}$ of G the nodes corresponding to I . For instance, numbering the nodes of A_n , the usual flag manifold $F(d_1, \dots, d_s; n+1)$ corresponds to the marking of $I = \{d_1, \dots, d_s\}$ (sometimes we omit the braces and just write as $P_{d_1, \dots, d_s}$).

The two extremal cases correspond to the *generalized complete flag manifolds* (all nodes marked), and the *generalized Grassmannians* (only one node marked).

In [15], the authors explicitly describe the lines through a point of a rational homogeneous space G/P_I , where G is a simple Lie group. Let $j \in I$ and $N(j)$ be the set of nodes in $\mathcal{D}$ that are connected to j . The number the elements of the set $N(j)$ is called the degree of j in $\mathcal{D}$ and denoted by $\deg(j)$.

Definition 2.1 We call $\alpha_j (j \in I)$ an exposed short root if the connected component of j in $D \setminus (I \setminus j)$ contains root longer than α_j , i.e., if an arrow in $D \setminus (I \setminus j)$ points towards j .

Remark 2.2 Obviously, long roots of $\mathcal{D}$ in I are not exposed short roots. If I is a set of single point, i.e., $X = G/P_I$ is the *generalized Grassmannian*, then the exposed short root is just the usual short root. It's worth mentioning that if I contains all long roots of $\mathcal{D}$, then short roots of D in I are not exposed short roots.

Theorem 2.3 ([15] Theorem 4.3) *Let $I \subseteq D = \{1, \dots, n\}$. Suppose G to be a simple Lie group. Consider $X = G/P_I$ in its minimal homogeneous embedding. Denote by $F_1(X)$ the space of $\mathbb{P}^1$ s in X . Then*

1. $F_1(X) = \coprod_{j \in I} F_1^j(X)$, where $F_1^j(X)$ is the space of lines of class $\check{\alpha}_j \in H_2(X, \mathbb{Z})$.
2. If α_j is not an exposed short root, then $F_1^j(X) = G/P_{(I \setminus j) \cup N(j)}$.
3. If α_j is an exposed short root, then $F_1^j(X)$ is the union of two G -orbits, an open orbit and its boundary $G/P_{I \setminus j \cup N(j)}$.

Remark 2.4 If $I = \{j\}$ and α_j is a long root, then $F_1(X)$ is just the variety of lines on X .

Example 2.5 Let's consider the Dynkin diagram A_n , i.e., $X = SL_{n+1}/P_I$ is the *generalized flag manifold*.

1) For $I = \{k\}$, X is the usual Grassmannian and $F_1(X)$ is just the variety of lines on X .

2) For $I = \{d_1, d_2\}$, X is the usual flag manifold $F(d_1, d_2; n+1)$ and $F_1(X)$ is the disjoint union of $F_1^{d_1}(X)$ and $F_1^{d_2}(X)$.

There is a similar statement for $C_x \subseteq \mathbb{P}T_x X$, the set of tangent directions to lines on X passing through a fixed point x . It is a disjoint union of spaces of lines of class $\check{\alpha}$ through x .

Theorem 2.6 ([15] Theorem 4.8) *Let $I \subseteq D = \{1, \dots, n\}$ and $j \in I$. Suppose G to be a simple Lie group. Consider $X = G/P_I$ in its minimal homogeneous embedding. Let H be the semisimple part of P_I and $D(H)$ be the components of $(\overline{D \setminus I}) \setminus j$ containing an element of $N(j)$, where $\overline{D \setminus I}$ means $D \setminus I$ plus any nodes of I attached to a node of $D \setminus I$. Denote by C_x^j the space of lines of class $\check{\alpha}_j$ through x . Then*

1. *If α_j is not an exposed short root, then $C_x^j = H/P_{N(j)}$.*
2. *If α_j is an exposed short root, then C_x^j is a union of an open P_I -orbit and its boundary $H/P_{N(j)}$.*

Remark 2.7 If $I = \{j\}$, then the set of nodes of the Dynkin diagram H is $D(H) = D \setminus j$. $P_{N(j)}$ is a parabolic subgroup of H by marking in $D(H)$ the nodes in D that are connected to j and $C_x^j = H/P_{N(j)}$. Moreover, if α_j is a long root, then C_x^j is just the variety of lines through fixed points, i.e., so-called VMRTs. We refer to [13] for a complete account on VMRTs.

Example 2.8 Let's consider the Dynkin diagram A_n , i.e., $X = SL_{n+1}/P_I$ is the generalized flag manifold.

- 1) For $I = \{k\}$, X is the usual Grassmannian and $C_x = \mathbb{P}^{k-1} \times \mathbb{P}^{n-k}$ is just the variety of lines through x .

$$\begin{array}{cccccccccccccccc}
 X : & 1 & 2 & & k-1 & k & k+1 & & n-2 & n-1 & n \\
 & \circ & \text{---} & \circ & \cdots & \circ & \bullet & \text{---} & \circ & \cdots & \circ & \text{---} & \circ & \text{---} & \circ \\
 \\
 C_x : & 1 & 2 & & k-1 & & k+1 & & n-2 & n-1 & n \\
 & \circ & \text{---} & \circ & \cdots & \bullet & \times & \bullet & \cdots & \circ & \text{---} & \circ & \text{---} & \circ
 \end{array}$$

- 2) For $I = \{d_1, d_2\}$, X is the usual flag manifold $F(d_1, d_2; n+1)$ and C_x is the disjoint union of $C_x^{d_1}$ and $C_x^{d_2}$.

$$\begin{aligned}
 C_x^{d_1} &= \mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-d_1-1}, \\
 C_x^{d_2} &= \mathbb{P}^{d_2-d_1-1} \times \mathbb{P}^{n-d_2}.
 \end{aligned}$$

$$\begin{array}{cccccccccccccccc}
 X : & 1 & 2 & & d_1-1 & d_1 & d_1+1 & & d_2-1 & d_2 & d_2+1 & & n-2 & n-1 & n \\
 & \circ & \text{---} & \circ & \cdots & \circ & \bullet & \text{---} & \circ & \cdots & \circ & \bullet & \text{---} & \circ & \text{---} & \circ \\
 \\
 C_x^{d_1} : & 1 & 2 & & d_1-1 & & d_1+1 & & d_2-2 & d_2-1 \\
 & \circ & \text{---} & \circ & \cdots & \bullet & \times & \bullet & \cdots & \circ & \text{---} & \circ \\
 \\
 \\
 X : & 1 & 2 & & d_1-1 & d_1 & d_1+1 & & d_2-1 & d_2 & d_2+1 & & n-2 & n-1 & n \\
 & \circ & \text{---} & \circ & \cdots & \circ & \bullet & \text{---} & \circ & \cdots & \circ & \bullet & \text{---} & \circ & \text{---} & \circ \\
 \\
 C_x^{d_2} : & & & & d_1+1 & d_1+2 & d_2-1 & & d_2+1 & n-1 & n \\
 & & & & \circ & \text{---} & \circ & \cdots & \bullet & \times & \bullet & \cdots & \circ & \text{---} & \circ
 \end{array}$$

Not only $\mathbb{P}^1$ but also all linear spaces can be read from the marked Dynkin diagrams.

Theorem 2.9 ([15] Theorem 4.9, 4.14) *Let G be a simple group and $X = G/P_S$ is a rational homogeneous space. Let $F_k^\alpha(X)$ denote the variety parameterizing the α -class $\mathbb{P}^{k,s}$ on X .*

- If $\alpha \in S$ is not an exposed short root, then for all k , $F_k^\alpha(X)$ is the disjoint union of homogeneous spaces $G/P_{\Sigma_{\beta_j}}$, where $\{\beta_j\}$ is a minimal set of positive roots such that the component of $\mathcal{D} \setminus \{\beta_j\}$ containing α is isomorphic to $\mathcal{D}(A_k)$, intersects S only in α , and α is an extremal node of this component.
- If $\alpha \in S$ is an exposed short root, then for all k , $F_k^\alpha(X)$ consists of a finite number of G -orbits.

3 Uniform vector bundles

Given a smooth projective variety X and a vector bundle E on X , we denote $\mathcal{M}$ to be an unsplit family of rational curves on X . $\mathcal{M}$ is called *unsplit* if $\mathcal{M}$ is a proper $\mathbb{C}$ -scheme. We say that E is *uniform* with respect to $\mathcal{M}$ if the restriction of E to the normalization of every curve in $\mathcal{M}$ splits as a direct sum of line bundles with the same splitting type. If X is a generalized Grassmannian G/P_k , then we just call E uniform with respect to the special family of lines without mention the unsplit family $\mathcal{M}$.

3.1 Uniform vector bundles on generalized Grassmannians

Along this section we will work on uniform vector bundles on rational homogeneous spaces of Picard number one, i.e., generalized Grassmannians. Let G be a simple Lie group and $D = \{1, 2, \dots, n\}$ be the set of nodes of the Dynkin diagram $\mathcal{D}$ of G . Denote by P_k the parabolic subgroup of G corresponding to the node k . Consider the generalized Grassmannian $\mathcal{G} = G/P_k$ or, for brevity, $\mathcal{D}/P_k$. Denote by $\mathcal{M} := G/P_{N(k)}$ the generalized flag manifold defined by the marked Dynkin diagram $(\mathcal{D}, N(k))$ and by $\mathcal{U} := G/P_{k, N(k)} = G/(P_{N(k)} \cap P_k)$ the universal family, which has a natural $\mathbb{P}^1$ -bundle structure over $\mathcal{M}$, i.e., we have the natural diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{q} & \mathcal{M} \\ \downarrow p & & \\ \mathcal{G} = G/P_k & & \end{array} \quad (3.1)$$

Remarkably, $\mathcal{M}$ defined above is indeed an unsplit family of rational curves on $\mathcal{G}$. Given $x \in \mathcal{G}$, $\mathcal{M}_x = q(p^{-1}(x))$, which we call *the special family* of lines of class $\check{\alpha}_k$ through x , coincides with $H/P_{N(k)}$ by Remark 2.7, where the set of nodes of the Dynkin diagram H is $D(H) = D \setminus k$.

Assume that k is an extremal node, that is, the subdiagram $\mathcal{D}(H)$ is connected. Remarkably, in the case $\mathcal{D} = D_n$, i.e., $G = SO(2n)$, since $G/P_{n-1} \cong G/P_n$, we only need to think about the extremal node n . Similarly, since $E_6/P_1 \cong E_6/P_6$, we just consider the extremal node 1 in E_6 . According to Theorem 2.6, $\mathcal{M}_x$ has the following possibilities:

- Projective spaces $\mathbb{P}^n$ or smooth quadrics $\mathbb{Q}^n$,
- Grassmannians $G(m, n)$,
- Spinor varieties $\mathcal{S}_n$,
- $E_6/P_6, E_7/P_7, C_3/P_3$.

The possibilities are list in Table 2.

We observe that for $x \in \mathcal{G}$, the set of morphisms from $\mathcal{M}_x$ to Grassmannians plays a critical role in determining whether a uniform vector bundle with respect to special family lines can split as a direct sum of line bundles. More precisely, let ς be a positive integer. As long as we show that every morphism $\mathcal{M}_x \rightarrow G(t, \varsigma)$ is constant for any integer $t \geq 1$ and every $x \in \mathcal{G}$, then uniform r -bundles with respect to special family lines on $\mathcal{G}$ split for $r \leq \varsigma$. We suggest that interested readers refer to Theorem 3.1 in paper [17] for details.

For the proof of Theorem 1.5, we will analyze the morphisms $\mathcal{M}_x \rightarrow G(t, \varsigma)$ one by one according to the probabilities of $\mathcal{M}_x$.

Table 2 $\mathcal{M}_x$ corresponding to an extremal node

Node	$\mathcal{M}_x$						
	A_n	B_n	C_n	D_n	$E_n(n = 6, 7, 8)$	$F_n(n = 4)$	$G_n(n = 2)$
$\mathcal{D}$							
1	$\mathbb{P}^{n-1}$	$\mathbf{Q}^{2n-3}$	$\mathbb{P}^{2n-3}$	$\mathbf{Q}^{2n-4}$	S_{n-2}	C_3/P_3	$\mathbb{P}^1$
n	$\mathbb{P}^{n-1}$	$\mathbb{P}^{n-1}$	$\mathbb{P}^{n-1}$	$G(2, n)$	$E_{n-1}/P_{n-1}(n \neq 6)$	S_3	$\mathbb{P}^1$
2					$G(3, n)$		

Case I. When $\mathcal{M}_x$ is a projective space $\mathbb{P}^N$ or a smooth quadric $\mathbf{Q}^N$ ($N = 2m + 1$), then their Chow rings have the form

$$\mathbb{Z}[\mathcal{H}]/(\mathcal{H}^{N+1}),$$

where $\mathcal{H}$ is a hyperplane section. In particular, $\dim H^{2t}(\mathcal{M}_x, \mathbb{C}) = 1$ for every $1 \leq t \leq N$. By the proof of Lemma 3.4 in paper [17], every morphism $\mathcal{M}_x \rightarrow G(t, N)$ is constant for any integer t .

When $\mathcal{M}_x$ is a smooth quadric $\mathbf{Q}^N$ ($N = 2m$), since

$$A(\mathbf{Q}^{2m}) = \mathbb{Z}[\mathcal{H}, \mathcal{U}]/(\mathcal{H}^{2m+1}, 2\mathcal{H}\mathcal{U} - \mathcal{H}^{m+1}, \mathcal{H}^m\mathcal{U} - \mathcal{U}^2),$$

where $\mathcal{H}$ is a hyperplane section and $\mathcal{U}$ is a subvariety of codimension m , then we get that the morphisms $\mathcal{M}_x \rightarrow G(t, N - 1)$ can only be constant similarly.

Case II. $\mathcal{M}_x$ is Grassmannian $G(d, n)$ ($d \geq 2$). We claim that every morphism $G(d, n) \rightarrow G(t, n - d + 1)$ is constant for any integer $t \geq 1$.

Lemma 3.1 *There are nonconstant morphisms from $G(d, n)$ ($d \geq 2$) to $G(t, n - d + 1)$ for any integer $t \geq 1$.*

Proof Lemma 3.9 in paper [17] tells us that there are nonconstant morphisms from $G(2, n - d + 2)$ to $G(t, n - d + 1)$ for any integer $t \geq 1$. Since $G(d, n)$ is covered by $G(2, n - d + 2)$, there are nonconstant morphisms from $G(d, n)$ to $G(t, n - d + 1)$ for any integer $t \geq 1$. $\square$

When $\mathcal{G} = D_n/P_n$ and $\mathcal{M}_x = G(2, n)$, by the above lemma, we obtain that every morphism $G(2, n) \rightarrow G(t, n - 1)$ is constant for any integer $t \geq 1$.

When $\mathcal{G} = E_n/P_2$ ($n = 6, 7, 8$) and $\mathcal{M}_x = G(3, n)$ ($n = 6, 7, 8$), by the above lemma, we obtain that every morphism $G(3, n) \rightarrow G(t, n - 2)$ ($n = 6, 7, 8$) is constant for any integer $t \geq 1$. Remarkably, in these cases, the value of ζ can be appropriately enlarged. Since the dimension of $G(3, n)$ ($n = 6, 7, 8$) is $3(n - 3)$, we can easily know

$$\begin{aligned} \dim(G(3, 6)) &= 9 > \dim G(t, 5), \\ \dim(G(3, 7)) &= 12 > \dim G(t, 6) \end{aligned}$$

and

$$\dim(G(3, 8)) = 15 > \dim G(t, 7) \text{ for all } t \geq 1.$$

Since the Picard number of $G(3, n)$ ($n = 6, 7, 8$) is one, the only morphisms

$$\begin{aligned} G(3, 6) &\rightarrow G(t, 5), \\ G(3, 7) &\rightarrow G(t, 6) \end{aligned}$$

and

$$G(3, 8) \rightarrow G(t, 7)$$

are all constant for any integer $t \geq 1$.

Case III. $\mathcal{M}_x$ is spinor variety $\mathcal{S}_n$ ($n = 4, 5, 6$). The Chow ring of $\mathcal{S}_n$ is presented as a quotient of $\mathbb{Z}[X_1, \dots, X_n]$ module the relations

$$X_s^2 + 2 \sum_{i=1}^{s-1} (-1)^i X_{s+i} X_{s-i} + (-1)^s X_{2s} = 0$$

for $1 \leq s \leq n$, where X_j s are the Schubert classes of codimension j , $X_0 = 1$ and $X_j = 0$ for $j < 0$ or $j > n$ (see Section 3.2 in [28]). In particular, $\dim H^{2t}(\mathcal{M}_x, \mathbb{C}) = 1$ for every $t \leq [\frac{5}{2}]$. Hence, the only morphisms $\mathcal{M}_x \rightarrow G(t, 5)$ are constant for any integer t . Remarkably, the value of ς can be appropriately enlarged. We will analyze $\mathcal{S}_n$ ($n = 4, 5, 6$) one by one and get the following conclusions.

(a) **Claim:** There is no nonconstant map from $\mathcal{S}_4$ to $G(t, 7)$ for any integer $t \geq 1$.

Proof: We can see the Chow ring of $\mathcal{S}_4$ is

$$A(\mathcal{S}_4) \cong \mathbb{Z}[X_1, X_3]/(X_3^2 - 4X_1^3X_3 + 2X_1^6, 12X_1^5X_3 - 7X_1^8).$$

(a.1) $1 \leq t \leq 2$. Since $\dim H^{2t}(\mathcal{S}_4, \mathbb{C}) = 1$, by the proof of Lemma 3.4 in paper [17], the only morphisms $\mathcal{S}_4 \rightarrow G(t, 7)$ are constant.

(a.2) $t = 3$. Let ϕ be a morphism from $\mathcal{S}_4$ to $G(3, 7)$. On $\mathcal{S}_4$, we have an exact sequence

$$0 \rightarrow \phi^*H_3 \rightarrow \mathcal{O}_{\mathcal{S}_4}^{\oplus 9} \rightarrow \phi^*Q_4 \rightarrow 0.$$

Then $c(\phi^*H_3) \cdot c(\phi^*Q_4) = 1$. According to the Chow ring of $\mathcal{S}_4$, we can expand the equation into the following form:

$$\begin{aligned} (1 + a_1X_1 + a_2X_1^2 + a_3X_1^3 + \tilde{a}_3X_3) \cdot (1 + b_1X_1 + b_2X_1^2 + \\ b_3X_1^3 + \tilde{b}_3X_3 + b_4X_1^4 + \tilde{b}_4X_1X_3) = 1. \end{aligned}$$

By analyzing the coefficients on the left-hand side of the equation above, it's easy to get $\tilde{a}_3 = 0$. Therefore, this case can boil down to case (a.1).

(b) **Claim:** There is no nonconstant map from $\mathcal{S}_5$ to $G(t, 9)$ for any integer $t \geq 1$.

Proof: We can see the Chow ring of $\mathcal{S}_5$ is

$$A(\mathcal{S}_5) \cong \mathbb{Z}[X_1, X_3, X_5]/(X_3^2 - 4X_1^3X_3 + 2X_1^6 + 2X_1X_5, 12X_1^5X_3 - 7X_1^8 - 8X_1^3X_5 - 2X_3X_5, X_5^2).$$

(b.1) $1 \leq t \leq 2$. Since $\dim H^{2t}(\mathcal{S}_5, \mathbb{C}) = 1$, by the proof of Lemma 3.4 in paper [17], the only morphisms $\mathcal{S}_5 \rightarrow G(t, 9)$ are constant.

(b.2) $t = 3$. Let ϕ be a morphism from $\mathcal{S}_5$ to $G(3, 9)$. On $\mathcal{S}_5$, we have an exact sequence

$$0 \rightarrow \phi^* H_3 \rightarrow \mathcal{O}_{\mathcal{S}_5}^{\oplus 9} \rightarrow \phi^* Q_6 \rightarrow 0.$$

Then $c(\phi^* H_3) \cdot c(\phi^* Q_6) = 1$. According to the Chow ring of $\mathcal{S}_5$, we can expand the equation into the following form:

$$(1 + a_1 X_1 + a_2 X_1^2 + a_3 X_1^3 + \tilde{a}_3 X_3) \cdot (1 + b_1 X_1 + b_2 X_1^2 + b_3 X_1^3 + \tilde{b}_3 X_3 + b_4 X_1^4 + \tilde{b}_4 X_1 X_3 + b_5 X_1^5 + \tilde{b}_5 X_1^2 X_3 + b'_5 X_5 + b_6 X_1^6 + \tilde{b}_6 X_1^3 X_3 + b'_6 X_3^2) = 1.$$

Since $A^5(\mathcal{S}_5)$ is freely generated over $\mathbb{Q}$ by the classes $X_1^5, X_1^2 X_3$ and X_5 , the above equation implies that the coefficient of X_5 is 0, i.e., $b'_5 = 0$. Since $A^6(\mathcal{S}_5)$ and $A^7(\mathcal{S}_5)$ are freely generated over $\mathbb{Q}$ by the classes $(X_1^6, X_1^3 X_3, X_3^2)$ and $(X_1^7, X_1^4 X_3, X_1 X_3^2)$ respectively, the above equation implies that the coefficients of X_3^2 and $X_1 X_3^2$ are all zero, i.e.,

$$b'_6 + \tilde{a}_3 \tilde{b}_3 = 0, \quad b'_6 a_1 + \tilde{a}_3 \tilde{b}_4 = 0.$$

It follows that

$$\tilde{a}_3 \tilde{b}_4 = \tilde{a}_3 \tilde{b}_3 a_1. \quad (3.2)$$

On the other hand, since $A^3(\mathcal{S}_5)$ and $A^4(\mathcal{S}_5)$ are freely generated over $\mathbb{Q}$ by the classes (X_1^3, X_3) and $(X_1^4, X_1 X_3)$, respectively, the above equation implies that the coefficients of X_3 and $X_1 X_3$ are all zero, i.e.,

$$\tilde{a}_3 + \tilde{b}_3 = 0, \quad \tilde{a}_3 b_1 + \tilde{b}_3 a_1 + \tilde{b}_4 = 0. \quad (3.3)$$

Combining the equations (3.2), (3.3) and $a_1 = -b_1$, we get $\tilde{a}_3 = 0$. Therefore, this case can boil down to case (b.1).

(b.3) $t = 4$. By iterating the previous process, we get $c(\phi^* H_4) \cdot c(\phi^* Q_5) = 1$, i.e.,

$$(1 + a_1 X_1 + a_2 X_1^2 + a_3 X_1^3 + \tilde{a}_3 X_3 + a_4 X_1^4 + \tilde{a}_4 X_1 X_3) \cdot (1 + b_1 X_1 + b_2 X_1^2 + b_3 X_1^3 + \tilde{b}_3 X_3 + b_4 X_1^4 + \tilde{b}_4 X_1 X_3 + b_5 X_1^5 + \tilde{b}_5 X_1^2 X_3 + b'_5 X_5) = 1.$$

In a similar way, we can prove that $b'_5 = \tilde{a}_3 = \tilde{b}_3 = 0$. Since $A^4(\mathcal{S}_5)$ and $A^8(\mathcal{S}_5)$ are freely generated over $\mathbb{Q}$ by the classes $(X_1^4, X_1 X_3)$ and $(X_1^8, X_1^5 X_3, X_1^2 X_3^2)$, respectively, the above equation implies that the coefficients of $X_1 X_3$ and $X_1^2 X_3^2$ are all zero, i.e.,

$$\tilde{a}_4 + \tilde{b}_4 = 0, \quad \tilde{a}_4 \tilde{b}_4 = 0.$$

Hence $\tilde{a}_4 = 0, \tilde{b}_4 = 0$. Therefore, this case can also boil down to case (b.1).

(c) **Claim:** There is no nonconstant map from $\mathcal{S}_6$ to $G(t, 11)$ for any integer $t \geq 1$.

Proof: The Chow ring of $\mathcal{S}_6$ is presented as a quotient of $\mathbb{Z}[X_1, X_3, X_5]$ by the relations

$$2X_3 X_5 - 5X_1^8 + 12X_1^5 X_3 - 4X_1^3 X_5 - 6X_1^2 X_3^2, \quad (3.4)$$

$$X_5^2 - 16X_1^{10} - 12X_1^5 X_5 + 32X_1^7 X_3 - 6X_1^4 X_3^2 - 4X_1 X_3^3, \quad (3.5)$$

$$X_3^4 - 86X_1^9X_3 - 4X_1^6X_3^2 + 48X_1^{12} + 40X_1^7X_5 + 20X_1^3X_3^3. \quad (3.6)$$

(c.1) $1 \leq t \leq 2$. Since $\dim H^{2t}(\mathcal{S}_6, \mathbb{C}) = 1$, by the proof of Lemma 3.4 in paper [17], the only morphisms $\mathcal{S}_6 \rightarrow G(t, 11)$ are constant.

(c.2) $t = 3$. Let ϕ be a morphism from $\mathcal{S}_6$ to $G(3, 11)$. On $\mathcal{S}_6$, we have an exact sequence

$$0 \rightarrow \phi^*H_3 \rightarrow \mathcal{O}_{\mathcal{S}_6}^{\oplus 11} \rightarrow \phi^*Q_8 \rightarrow 0.$$

Then $c(\phi^*H_4) \cdot c(\phi^*Q_5) = 1$. According to the Chow ring of $\mathcal{S}_6$, we can expand the equation into the following form:

$$\begin{aligned} (1 + a_1X_1 + a_2X_1^2 + a_3X_1^3 + \tilde{a}_3X_3) \cdot (1 + b_1X_1 + b_2X_1^2 + b_3X_1^3 + \tilde{b}_3X_3 + b_4X_1^4 + \tilde{b}_4X_1X_3 \\ + b_5X_1^5 + \tilde{b}_5X_1^2X_3 + b'_5X_5 + b_6X_1^6 + \tilde{b}_6X_1^3X_3 + b'_6X_1X_5 + b''_6X_3^2 + b_7X_1^7 \\ + \tilde{b}_7X_1^4X_3 + b'_7X_1^2X_5 + b''_7X_1X_3^2 + b_8X_1^8 + \tilde{b}_8X_1^5X_3 + b'_8X_1^3X_5 + b''_8X_1^2X_3^2) = 1. \end{aligned}$$

Since $A^3(\mathcal{S}_6)$, $A^6(\mathcal{S}_6)$ and $A^9(\mathcal{S}_6)$ are freely generated over $\mathbb{Q}$ by the classes (X_1^3, X_3) , $(X_1^6, X_1^3X_3, X_1X_5, X_3^2)$ and $(X_1^9, X_1^6X_3, X_1^4X_5, X_1^3X_3^2, X_3^3)$, respectively, the above equation implies that the coefficients of X_3 , X_3^2 and X_3^3 are all zero, i.e.,

$$\tilde{a}_3 + \tilde{b}_3 = 0, \tilde{a}_3\tilde{b}_3 + b''_6 = 0, \tilde{a}_3b''_6 = 0.$$

Hence $\tilde{a}_3 = \tilde{b}_3 = b''_6 = 0$. Therefore, this case can boil down to case (c.1).

(c.3) $t = 4$. By iterating the previous process, we get $c(\phi^*H_4) \cdot c(\phi^*Q_7) = 1$, i.e.,

$$\begin{aligned} (1 + a_1X_1 + a_2X_1^2 + a_3X_1^3 + \tilde{a}_3X_3 + a_4X_1^4 + \tilde{a}_4X_1X_3) \cdot (1 + b_1X_1 + b_2X_1^2 \\ + b_3X_1^3 + \tilde{b}_3X_3 + b_4X_1^4 + \tilde{b}_4X_1X_3 + b_5X_1^5 + \tilde{b}_5X_1^2X_3 + b'_5X_5 + b_6X_1^6 \\ + \tilde{b}_6X_1^3X_3 + b'_6X_1X_5 + b''_6X_3^2 + b_7X_1^7 + \tilde{b}_7X_1^4X_3 + b'_7X_1^2X_5 + b''_7X_1X_3^2) = 1. \end{aligned}$$

In a similar way, we can prove $\tilde{a}_3 = \tilde{b}_3 = b''_6 = 0$. Since $A^4(\mathcal{S}_6)$, $A^8(\mathcal{S}_6)$ and $A^{11}(\mathcal{S}_6)$ are freely generated over $\mathbb{Q}$ by the classes

$$(X_1^4, X_1X_3), (X_1^8, X_1^5X_3, X_1^3X_5, X_1^2X_3^2) \text{ and } (X_1^{11}, X_1^9X_3, X_1^6X_5, X_1^5X_3^2, X_1^2X_3^3)$$

, respectively, the above equation implies that the coefficients of X_1X_3 , $X_1^2X_3^2$ and $X_1^2X_3^3$ are all zero, i.e.,

$$\tilde{a}_4 + \tilde{b}_4 = 0, \tilde{a}_4\tilde{b}_4 + a_1b''_7 = 0, \tilde{a}_4b''_7 = 0.$$

It's easy to check $\tilde{a}_4 = 0$. Therefore, this case can boil down to case (c.1).

(c.4) $t = 5$. By iterating the previous process, we get $c(\phi^*H_5) \cdot c(\phi^*Q_6) = 1$, i.e.,

$$\begin{aligned} (1 + a_1X_1 + a_2X_1^2 + a_3X_1^3 + \tilde{a}_3X_3 + a_4X_1^4 + \tilde{a}_4X_1X_3 + a_5X_1^5 + \tilde{a}_5X_1^2X_3 + a'_5X_5) \\ \cdot (1 + b_1X_1 + b_2X_1^2 + b_3X_1^3 + \tilde{b}_3X_3 + b_4X_1^4 + \tilde{b}_4X_1X_3 + b_5X_1^5 + \\ \tilde{b}_5X_1^2X_3 + b'_5X_5 + b_6X_1^6 + \tilde{b}_6X_1^3X_3 + b'_6X_1X_5 + b''_6X_3^2) = 1. \end{aligned}$$

In a similar way, we can prove $\tilde{a}_3 = \tilde{b}_3 = b''_6 = \tilde{a}_4 = \tilde{b}_4 = 0$. Since $A^5(\mathcal{S}_6)$ and $A^{10}(\mathcal{S}_6)$ are freely generated over $\mathbb{Q}$ by the classes $(X_1^5, X_1^2X_3, X_5)$ and

$(X_1^{10}, X_1^7 X_3, X_1^5 X_5, X_1^4 X_3^2, X_5^2)$, respectively, the above equation implies that the coefficients of X_5 and X_5^2 are all zero, i.e.,

$$a'_5 + b'_5 = 0, \quad a'_5 b'_5 = 0.$$

Hence $a'_5 = b'_5 = 0$. It follows that the coefficients of $X_1^2 X_3$ and $X_1^4 X_3^2$ are also zero, i.e.,

$$\tilde{a}_5 + \tilde{b}_5 = 0, \quad \tilde{a}_5 \tilde{b}_5 = 0.$$

Hence $\tilde{a}_5 = \tilde{b}_5 = 0$. Therefore, this case can also boil down to case (c.1).

Case IV. $\mathcal{M}_x = E_6/P_6$. The Chow ring of E_6/P_6 has the following form (see [7] Theorem 4). Let y_1, y_4 be the Schubert classes on E_6/P_6 . Then

$$A(E_6/P_6) = \mathbb{Z}[y_1, y_4]/(r_9, r_{12}),$$

where

$$r_9 = 2y_1^9 + 3y_1 y_4^2 - 6y_1^5 y_4, \quad r_{12} = y_4^3 - 6y_1^4 y_4^2 + y_1^{12}.$$

Lemma 3.2 *There is no nonconstant map from E_6/P_6 to $G(t, 12)$ for any integer $t \geq 1$.*

Proof

- (i). $1 \leq t \leq 3$. Since $\dim H^{2t}(E_6/P_6, \mathbb{C}) = 1$, by the proof of Lemma 3.4 in paper [17], the only morphisms $E_6/P_6 \rightarrow G(t, 10)$ are constant.
- (ii). $4 \leq t \leq 6$. Let ϕ be a morphism from E_6/P_6 to $G(4, 12)$. On E_6/P_6 , we have an exact sequence

$$0 \rightarrow \phi^* H_t \rightarrow \mathcal{O}_{E_6/P_6}^{\oplus 12} \rightarrow \phi^* Q_{12-t} \rightarrow 0.$$

Then $c(\phi^* H_t) \cdot c(\phi^* Q_{12-t}) = 1$. Similar to Case III, we analyze the coefficients of the above equations and use Mathematica to conclude that these cases can boil down to case (i). □

Case V. $\mathcal{M}_x = E_7/P_7$. The Chow ring of E_7/P_7 has the following form (see [7] Theorem 6). Let y_1, y_5 and y_9 be the Schubert classes on E_7/P_7 . Then

$$A(E_7/P_7) = \mathbb{Z}[y_1, y_5, y_9]/(r_{10}, r_{14}, r_{18}),$$

where

$$\begin{aligned} r_{10} &= y_5^2 - 2y_1 y_9, \\ r_{14} &= 2y_5 y_9 - 9y_1^4 y_5^2 + 6y_1^9 y_5 - y_1^{14}, \\ r_{18} &= y_9^2 + 10y_1^3 y_5^3 - 9y_1^8 y_5^2 + 2y_1^{13} y_5. \end{aligned}$$

Lemma 3.3 *There is no nonconstant map from E_7/P_7 to $G(t, 14)$ for any integer $t \geq 1$.*

Proof

- (i). $1 \leq t \leq 4$. Since $\dim H^{2t}(E_7/P_7, \mathbb{C}) = 1$, by the proof of Lemma 3.4 in paper [17], every morphism $E_7/P_7 \rightarrow G(t, 14)$ is constant.
- (ii). $t = 5$. Let ϕ be a morphism from E_7/P_7 to $G(5, 14)$. On E_7/P_7 , we have an exact sequence

$$0 \rightarrow \phi^* H_5 \rightarrow \mathcal{O}_{E_7/P_7}^{\oplus 14} \rightarrow \phi^* Q_9 \rightarrow 0.$$

Then $c(\phi^* H_5) \cdot c(\phi^* Q_9) = 1$. According to the Chow ring of E_7/P_7 , we can expand the equation into the following form:

$$(1 + a_1 y_1 + a_2 y_1^2 + a_3 y_1^3 + a_4 y_1^4 + a_5 y_1^5 + \tilde{a}_5 y_5) \cdot (1 + b_1 y_1 + b_2 y_1^2 + b_3 y_1^3 + b_4 y_1^4 + b_5 y_1^5 + \tilde{b}_5 y_5 + b_6 y_1^6 + \tilde{b}_6 y_1 y_5 + b_7 y_1^7 + \tilde{b}_7 y_1^2 y_5 + b_8 y_1^8 + \tilde{b}_8 y_1^3 y_5 + b_9 y_1^9 + \tilde{b}_9 y_1^4 y_5 + b'_9 y_9) = 1.$$

Since $A^9(E_7/P_7)$ is freely generated over $\mathbb{Q}$ by the classes $y_1^9, y_1^4 y_5$ and y_9 , the above equation implies that the coefficient of y_9 is 0, i.e., $b'_9 = 0$. On this basis, notice that, $A^5(E_7/P_7)$ and $A^{10}(E_7/P_7)$ are freely generated over $\mathbb{Q}$ by the classes (y_1^5, y_5) and $(y_1^{10}, y_1^5 y_5, y_5^2)$, respectively, the above equation implies that the coefficients of y_5 and y_5^2 are all zero, i.e.,

$$\tilde{a}_5 + \tilde{b}_5 = 0, \quad \tilde{a}_5 \tilde{b}_5 = 0.$$

Hence $\tilde{a}_5 = \tilde{b}_5 = 0$. Therefore, this case can boil down to case (i).

- (iii). $t = 6$. By iterating the previous process, we get $c(\phi^* H_6) \cdot c(\phi^* Q_8) = 1$, i.e.,

$$(1 + a_1 y_1 + a_2 y_1^2 + a_3 y_1^3 + a_4 y_1^4 + a_5 y_1^5 + \tilde{a}_5 y_5 + a_6 y_1^6 + \tilde{a}_6 y_1 y_5) \cdot (1 + b_1 y_1 + b_2 y_1^2 + b_3 y_1^3 + b_4 y_1^4 + b_5 y_1^5 + \tilde{b}_5 y_5 + b_6 y_1^6 + \tilde{b}_6 y_1 y_5 + b_7 y_1^7 + \tilde{b}_7 y_1^2 y_5 + b_8 y_1^8 + \tilde{b}_8 y_1^3 y_5) = 1.$$

In a similar way, we can prove $\tilde{a}_5 = \tilde{b}_5 = 0$. Next, let's consider the vanishing of $\tilde{a}_6$. Since $A^6(E_7/P_7)$ and $A^{12}(E_7/P_7)$ are freely generated over $\mathbb{Q}$ by the classes $(y_1^6, y_1 y_5)$ and $(y_1^{12}, y_1^2 y_5^2, y_1^7 y_5)$, respectively, the above equation implies that the coefficients of $y_1 y_5$ and $y_1^2 y_5^2$ are all zero, i.e.,

$$\tilde{a}_6 + \tilde{b}_6 = \tilde{a}_6 \tilde{b}_6 = 0.$$

Hence, $\tilde{a}_6 = \tilde{b}_6 = 0$. Therefore, this case can boil down to case (i).

- (iv). $t = 7$. By iterating the previous process, we get $c(\phi^* H_7) \cdot c(\phi^* Q_7) = 1$, i.e.,

$$(1 + a_1 y_1 + a_2 y_1^2 + a_3 y_1^3 + a_4 y_1^4 + a_5 y_1^5 + \tilde{a}_5 y_5 + a_6 y_1^6 + \tilde{a}_6 y_1 y_5 + a_7 y_1^7 + \tilde{a}_7 y_1^2 y_5) \cdot (1 + b_1 y_1 + b_2 y_1^2 + b_3 y_1^3 + b_4 y_1^4 + b_5 y_1^5 + \tilde{b}_5 y_5 + b_6 y_1^6 + \tilde{b}_6 y_1 y_5 + b_7 y_1^7 + \tilde{b}_7 y_1^2 y_5) = 1.$$

In a similar way, we can prove $\tilde{a}_5 = \tilde{b}_5 = \tilde{a}_6 = \tilde{b}_6 = 0$. Next, let's consider the vanishing of $\tilde{a}_7$. Since $A^7(E_7/P_7)$ and $A^{14}(E_7/P_7)$ are freely generated over $\mathbb{Q}$ by the classes $(y_1^7, y_1^2 y_5)$ and $(y_1^{14}, y_1^4 y_5^2, y_1^9 y_5)$, respectively, the above equation implies that the coefficients of $y_1^2 y_5$ and $y_1^4 y_5^2$ are all zero, i.e.,

$$\tilde{a}_7 + \tilde{b}_7 = \tilde{a}_7 \tilde{b}_7 = 0.$$

Hence, $\tilde{a}_7 = \tilde{b}_7 = 0$. Therefore, this case can boil down to case (i).

□

Case VI. $\mathcal{M}_x = C_3/P_3$. The Chow ring of C_3/P_3 is

$$A(C_3/P_3) = \mathbb{Z}[y_1, y_3]/(y_1^4 - 8y_1y_3, y_3^2),$$

where y_1 and y_3 are the Schubert classes on C_3/P_3 (see Section 3.1 in [28]). Since $\dim H^{2t}(C_3/P_3, \mathbb{C}) = 1$ for $1 \leq t \leq 2$, every morphism $C_3/P_3 \rightarrow G(t, 5)$ is constant for any integer $t \geq 1$.

Now we can summarize our proof as follows.

Proof of Theorem 1.5: Theorem 1.5 follows from Lemmas 3.1, 3.2, 3.3 and the statements of Case I–VI above. $\square$

Corollary 3.4 Consider the Dynkin diagrams in Sect. 2.

- If $\mathcal{G}$ is a generalized Grassmannian of type A_n, B_n, C_n , or D_n with marked node k of degree 2, then

$$\mathcal{M}_x = Z_1 \times Z_2,$$

where Z_1 is a generalized Grassmannian corresponding to the Dynkin sub-diagram consisting of nodes $1, 2, \dots, k-1$ with marked node $k-1$ and Z_2 is a generalized Grassmannian corresponding to the Dynkin sub-diagram consisting of nodes $k+1, \dots, n$ with marked node $k+1$. Suppose Z'_1 is a generalized Grassmannian corresponding to the Dynkin sub-diagram consisting of nodes $1, 2, \dots, k$ with marked node k and Z'_2 is a generalized Grassmannian corresponding to the Dynkin sub-diagram consisting of nodes $k, k+1, \dots, n$ with marked node k . If

$$r \leq \zeta(\mathcal{G}) := \min\{\zeta(Z'_1), \zeta(Z'_2)\},$$

then uniform r -bundles with respect to the special family of lines on $\mathcal{G}$ split as a direct sum of line bundles.

- If $\mathcal{G}$ is a generalized Grassmannian E_7/P_5 or E_8/P_6 , then all uniform 2-bundles on $\mathcal{G}$ split as a direct sum of line bundles.
- If $\mathcal{G}$ is a generalized Grassmannian E_8/P_5 , then all uniform 2-bundles and 3-bundles on $\mathcal{G}$ split as a direct sum of line bundles.

Proof If $\mathcal{G}$ is a generalized Grassmannian of type A_n, B_n, C_n , or D_n , in order to prove E splits, we just need to prove that every morphism $\mathcal{M}_x \rightarrow G(t, r)$ is constant. Let ϕ be a morphism from $\mathcal{M}_x$ to $G(t, r)$. Because every Z_i ($1 \leq i \leq 2$) can be regarded as the subvariety of $\mathcal{M}_x$ corresponding to the special family of lines of class $\check{\alpha}_k$ through x on Z'_i , we can consider the restriction of ϕ to Z_i ($1 \leq i \leq 2$). By assumption $r \leq \min\{\zeta(Z'_1), \zeta(Z'_2)\}$, all the restriction maps are constant. Hence, ϕ is also constant. For the cases of E_7 and E_8 , the proofs are same. $\square$

Remark 3.5

- For the case $\mathcal{M}_x = \mathbb{P}^1$ or $\mathcal{M}_x = \mathbb{P}^1 \times \dots$, we can say nothing about the splitting result of other cases according to our theorem.

- In some cases, the value of $\zeta(\mathcal{G})$ cannot be big anymore, which means that there exist uniform but nonsplitting $(\zeta(\mathcal{G}) + 1)$ -bundles. For instance, Grassmannian $A_{n-1}/P_d = G(d, n)$ ($d \leq n - d$) has uniform but nonsplitting d -bundle H_d (the universal bundle of $G(d, n)$); Spinor variety $D_{n+1}/P_{n+1} = \mathcal{S}_n$ has uniform but nonsplitting $(n + 1)$ -bundle $\mathcal{Q}_{n+1}$ (the universal quotient bundle of $\mathcal{S}_n$).
- Compare to the main theorem (Theorem 3.1) in [17], we improve their results. In particular, we enlarge the splitting threshold for uniform bundles on Hermitian symmetric spaces E_6/P_6 from ≥ 5 to ≥ 7 and E_7/P_7 from ≥ 7 to ≥ 12 (see Table 1 in [17]).

It is known from Theorem 2.9 that any generalized Grassmannian $\mathcal{G}$ is covered by projective planes except for $\mathbb{P}^1$, C_n/P_n and G_2/P_2 . In order to test the splitting property of a vector bundle E on $\mathcal{G}$ covered by projective planes, one only needs to test the splitting property of E restricting to every projective plane $\mathbb{P}^2 \subseteq \mathcal{G}$. Note that a vector bundle on $\mathbb{P}^n$ splits precisely when its restriction to some projective plane $\mathbb{P}^2 \subseteq \mathbb{P}^n$ splits (see Theorem 2.3.2 in [23]). However, for a general generalized Grassmannian, we need the condition “restricting to every projective plane $\mathbb{P}^2 \subseteq \mathcal{G}$ ” because the “positions” of $\mathbb{P}^2$ ’s in a general generalized Grassmannian are different usually.

Corollary 3.6 *Suppose $\mathcal{G}$ to be a generalized Grassmannian of the form A_n/P_k ($n \neq 1$), B_n/P_k ($k \neq n - 1$), C_n/P_k ($k \neq n$), D_n/P_k , E_n/P_k ($n = 6, 7, 8$) or F_4/P_k ($k \neq 2$). Let E be an r -bundle on $\mathcal{G}$. If E splits as a direct sum of line bundles when it restricts to every $\mathbb{P}^2 \subseteq \mathcal{G}$, then E splits as a direct sum of line bundles on $\mathcal{G}$.*

Proof The condition implies that E is uniform with respect to the special family of lines. The reason for this is the fact every special line L is contained in two different $\mathbb{P}^2$ by Theorem 2.9.

We prove the corollary by induction on r . If we have the exact sequence of vector bundles

$$0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0 \quad (3.7)$$

on $\mathcal{G}$, where the rank of M and N are both smaller than r , such that

$$M|_Z = \bigoplus_{i=1}^{r-t} \mathcal{O}_Z(a_{t+i}), \quad N|_Z = \mathcal{O}_Z^{\oplus t}, \quad (a_{t+i} > 0)$$

for every $Z \simeq \mathbb{P}^2$, then by the induction hypothesis, M and N split. Since $H^1(\mathcal{G}, N^\vee \otimes M) = 0$, the above exact sequence splits and hence also E .

Similar to the proof of Theorem 3.3 and Lemma 3.4 in [6], by using the notations in the diagram (3.1), on $\mathcal{U} = G/(P_k \cap P_{N(k)})$, we can obtain an exact sequence

$$0 \rightarrow \tilde{M} \rightarrow p^*E \rightarrow \tilde{N} \rightarrow 0$$

and the morphism

$$\varphi : \mathcal{M}_x \rightarrow \mathbb{G}(t-1, \mathbb{P}(E_x^\vee)),$$

where $\mathbb{G}(t-1, \mathbb{P}(E_x^\vee))$ is the set of $(t-1)$ -dimensional projective subspaces of $\mathbb{P}(E_x^\vee)$. If we prove that the morphism φ is constant for every $x \in \mathcal{G}$, then $\tilde{M}$ and $\tilde{N}$ are trivial on all p -fibers. So there exist two bundles M, N over $\mathcal{G}$ with $\tilde{M} = p^*M, \tilde{N} = p^*N$. By projecting the bundle sequence

$$0 \rightarrow p^*M \rightarrow p^*E \rightarrow p^*N \rightarrow 0$$

onto $\mathcal{G}$, we can get the desired exact sequence (3.7). Thus, to prove the existence of the above exact sequence, it suffices to show that every morphism

$$\varphi : \mathcal{M}_x \rightarrow \mathbb{G}(t-1, \mathbb{P}(E_x^\vee))$$

is constant for any $x \in \mathcal{G}$.

Given a projective subspace Z of dimension 2 and a line $L \subseteq Z$, we take any point $x \in L$ and denote Z' to be the subspace of $\mathcal{M}_x$ corresponding to the tangent directions to Z at x . By the hypothesis, ELZ is a direct sum of line bundles, so every morphism

$$\varphi|_{Z'} : Z' \rightarrow \mathbb{G}(t-1, \mathbb{P}(E_x^\vee))$$

is constant. Since $\mathcal{G}$ is covered by linear projective subspaces of dimension 2 and $\mathcal{M}_x$ is chain-connected by $\mathbb{P}^1$ (see [20] Section 2.4), every morphism φ is constant for any $x \in \mathcal{G}$. $\square$

3.2 Uniform vector bundles on rational homogeneous spaces

First, let's recall the notations in Sect. 1.

Assume

$$X = G/P \simeq G_1/P_{I_1} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m},$$

where G_i is a simple Lie group with Dynkin diagram $\mathcal{D}_i$ whose set of nodes is D_i and P_{I_i} is a parabolic subgroup of G_i corresponding to $I_i \subset D_i$. We set $F(I_i) := G_i/P_{I_i}$ by marking on the Dynkin diagram $\mathcal{D}_i$ of G_i the nodes corresponding to I_i . Let δ_i be a node in $\mathcal{D}_i$ and $N(\delta_i)$ be the set of nodes in $\mathcal{D}_i$ that are connected to δ_i .

If $\delta_i \in I_i$,

$$\mathcal{M}_i^{\delta_i^c} := G_i/P_i^{\delta_i^c} \times \widehat{G_i/P_{I_i}} \quad (1 \leq i \leq m),$$

is the i -th special family of lines of class $\check{\delta}_i$ by Theorem 2.3, where $P_i^{\delta_i^c} := P_{(I_i \setminus \delta_i) \cup N(\delta_i)}$ and $\widehat{G_i/P_{I_i}}$ is $G_i/P_{I_i} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m}$ by deleting i -th term G_i/P_{I_i} . For $i = 1$ and $\delta \in I$, $\mathcal{M}^\delta$ denotes the special family of lines of class $\check{\delta}$.

For $x \in X$,

$$\mathcal{M}_{i,x}^{\delta_i^c} = \{L \in \mathcal{M}_i^{\delta_i^c} \mid x \in L\}$$

is the δ_i -th special part of variety of minimal rational tangents at x (sometimes we just write $\mathcal{M}_x$ if there is no confusion).

Fix $\delta_i \in I_i$. $\mathcal{M}_{i,x}^{\delta_i^c}$ is just the special family of lines of class $\check{\delta}_i$ through x on the generalized Grassmannian $\mathcal{G}^{\delta_i}$ whose Dynkin diagram $\mathcal{D}^{\delta_i}$ is the maximal sub-diagram of $(\mathcal{D}_i, I_i)$ with the only marked point δ_i . Denote $v(X, \delta_i) := \zeta(\mathcal{G}^{\delta_i})$ and

$$v(X) := \min_i \{ \min_{\delta_i \in I_i} \{ v(X, \delta_i) \} \}.$$

Proposition 3.7 Fix integer i ($1 \leq i \leq m$) and $\delta_i \in I_i$. Let E be a uniform r -bundle on X of type $(a_1^{(\delta_i)}, \dots, a_r^{(\delta_i)})$, $a_1^{(\delta_i)} \leq \dots \leq a_r^{(\delta_i)}$ with respect to $\mathcal{M}_i^{\delta_i}$. If $r \leq v(X, \delta_i)$ and these $a_j^{(\delta_i)}$'s are not all same, then E can be expressed as an extension of uniform bundles with respect to $\mathcal{M}_i^{\delta_i}$. In particular, E is δ_i -unstable.

Proof After twisting with an appropriate line bundle, we can assume that E has the splitting type

$$\underline{a}_E^{(\delta_i)} = (0, \dots, 0, a_{t+1}^{(\delta_i)}, \dots, a_r^{(\delta_i)}), \quad a_{t+i}^{(\delta_i)} > 0, \text{ for } i = 1, \dots, r-t.$$

with respect to $\mathcal{M}_i^{\delta_i}$.

Let's consider the standard diagram

$$\begin{array}{ccc} \mathcal{U}_i^{\delta_i^c} = G_i/P_{I_i \cup N(\delta_i)} \times \widehat{G_i/P_{I_i}} & \xrightarrow{q_2} & \mathcal{M}_i^{\delta_i^c} \\ \downarrow q_1 & & \\ X & & \end{array} \quad (3.8)$$

For $L \in \mathcal{M}_i^{\delta_i}$, the q_2 -fiber

$$\tilde{L} = q_2^{-1}(L) = \{(x, L) \in X \times \mathcal{M}_i^{\delta_i} \mid x \in L\},$$

is mapped under q_1 to the line L identically in X and we have

$$q_1^*E|_{\tilde{L}} \cong E|_L.$$

For $x \in X$, the q_1 -fiber $q_1^{-1}(x)$ is mapped isomorphically under q_2 to the subvariety

$$\mathcal{M}_x = \{L \in \mathcal{M}_i^{\delta_i} \mid x \in L\}.$$

Because

$$\begin{aligned} E|_L &\cong \mathcal{O}_L^{\oplus t} \oplus \bigoplus_{i=1}^{r-t} \mathcal{O}_L(a_{t+i}^{(\delta_i)}), \quad a_{t+i}^{(\delta_i)} > 0, \\ h^0\left(q_2^{-1}(L), q_1^*(E^\vee)|_{q_2^{-1}(L)}\right) &= t \end{aligned}$$

for all $L \in \mathcal{M}_i^{\delta_i}$. Thus the direct image $q_{2*}q_1^*(E^\vee)$ is a vector bundle of rank t over $\mathcal{M}_i^{\delta_i}$. The canonical homomorphism of sheaves

$$q_2^*q_{2*}q_1^*(E^\vee) \rightarrow q_1^*(E^\vee)$$

makes $\tilde{N}^\vee := q_2^*q_{2*}q_1^*(E^\vee)$ to be a subbundle of $q_1^*(E^\vee)$. Because over each q_2 -fiber $\tilde{L}$, the evaluation map

$$\tilde{N}^\vee|_{\tilde{L}} = H^0(\tilde{L}, q_1^*(E^\vee)|_{\tilde{L}}) \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{L}} \rightarrow q_1^*(E^\vee)|_{\tilde{L}}$$

identifies $\tilde{N}^\vee|_{\tilde{L}}$ with $\mathcal{O}_L^{\oplus t} \subseteq \mathcal{O}_L^{\oplus t} \oplus \bigoplus_{i=1}^{r-t} \mathcal{O}_L(-a_{t+i}^{(\delta_i)}) = E^\vee|_L$. Over $\mathcal{U}_i^{\delta_i^c}$, we thus obtain an exact sequence

$$0 \rightarrow \tilde{M} \rightarrow q_1^*E \rightarrow \tilde{N} \rightarrow 0 \quad (3.9)$$

of vector bundles, whose restriction to q_2 -fibers $\tilde{L}$ looks as follows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \widetilde{M}|_{\widetilde{L}} & \longrightarrow & q_1^*E|_{\widetilde{L}} & \longrightarrow & \widetilde{N}|_{\widetilde{L}} \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \bigoplus_{i=1}^{r-t} \mathcal{O}_L(a_{t+i}^{(\delta_i)}) & \longrightarrow & \mathcal{O}_L^{\oplus t} \oplus \bigoplus_{i=1}^{r-t} \mathcal{O}_L(a_{t+i}^{(\delta_i)}) & \longrightarrow & \mathcal{O}_L^{\oplus t} \longrightarrow 0.
 \end{array}$$

Because $\widetilde{N}^\vee$ is a subbundle of $q_1^*(E^\vee)$ of rank t , for every point $x \in X$, it provides a morphism

$$\varphi : \mathcal{M}_x \rightarrow \mathbb{G}(t-1, \mathbb{P}(E_x^\vee)),$$

where $\mathbb{G}(t-1, \mathbb{P}(E_x^\vee))$ is the set of $(t-1)$ -dimensional projective subspaces of $\mathbb{P}(E_x^\vee)$. Since $r \leq v(X, \delta_i)$, every morphism φ is constant by the proof of Corollary 3.4 and the definition of $v(X, \delta_i)$. It follows that $\widetilde{M}$ and $\widetilde{N}$ are trivial on all q_1 -fibers. So the canonical morphisms $q_1^*q_{1*}\widetilde{M} \rightarrow \widetilde{M}$ and $q_1^*q_{1*}\widetilde{N} \rightarrow \widetilde{N}$ are isomorphisms. Hence there are uniform bundles $M = q_{1*}\widetilde{M}$, $N = q_{1*}\widetilde{N}$ with respect to $\mathcal{M}_i^{\delta_i}$ over X with

$$\widetilde{M} = q_1^*M, \widetilde{N} = q_1^*N.$$

Then we obtain the exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow N \rightarrow 0 \quad (3.10)$$

by projecting the exact sequence (3.9) onto X . So $\mu^{(\delta_i)}(N) > \mu^{(\delta_i)}(E)$ and thus E is δ_i -unstable $\square$

Proposition 3.8 *On X , if an r -bundle E is poly-uniform with respect to all the special families of lines such that the splitting type with respect to $\mathcal{M}_i^{\delta_i}$ is $(a^{(\delta_i)}, \dots, a^{(\delta_i)})$ for each i and δ_i , then E splits as a direct sum of line bundles.*

Proof After twisting with an appropriate line bundle, we can assume that E is trivial on all the special families of lines on X . We are going to show that E is trivial.

Let's first consider the case that X is a generalized flag manifold, which corresponds to a connected marked Dynkin diagram $\mathcal{D}(X)$ with l black nodes. We prove the lemma by induction on l . For $l = 1$, X is just a generalized Grassmannian. Then the result holds by Proposition 1.2 in [1]. Suppose the assertion is true for all generalized flag manifolds with connected marked Dynkin diagram and l' black nodes ($1 \leq l' < l$). Let's consider the natural projection

$$\pi : X \rightarrow X',$$

where X' is corresponding to the marked Dynkin diagram $\mathcal{D}(X)$ by changing the first black node δ to white. It's not hard to see that every π -fiber $\pi^{-1}(x)$ is isomorphic to the generalized Grassmannian $\mathcal{G}^\delta$ with the only marked point δ . Since the restriction of E to every line in $\mathcal{M}^\delta$ is trivial, so is to every line in $\pi^{-1}(x)$. Thus E is trivial on all π -fibers by Proposition 1.2 in [1]. It follows that $E' = \pi_*E$ is an algebraic vector bundle of rank r over X' and $E \cong \pi^*E'$.

Claim. E' is trivial on all the special families of lines on X' .

In fact, let γ be a black node in $\mathcal{D}(X)$ that is different from δ and L be a line in $\mathcal{M}^\gamma$. Then $\pi(L)$ is also a line in X' . When γ runs through all black nodes except δ and L runs through all lines in $\mathcal{M}^\gamma$ in X , $\pi(L)$ also runs through all lines in all the special families of lines of X' . The projection π induces an isomorphism

$$E'|_{\pi(L)} \cong \pi^* E'|_L \cong E|_L.$$

We identify L with $\pi(L)$. Since $E|_L$ is trivial for every line L in $\mathcal{M}'$ by assumption, $E'|_L$ is trivial for every line L in all the special families of lines of X' . By the induction hypothesis, E' is trivial. Thus $E \cong q^* E'$ is trivial.

Now let's think about the general case that the marked Dynkin diagram of X is not connected. Assume X can be decomposed into a product

$$X = G/P \simeq G_1/P_{I_1} \times G_2/P_{I_2} \times \cdots \times G_m/P_{I_m},$$

where $m \geq 2$ and each G_i ($1 \leq i \leq m$) is a simple Lie group with a connected Dynkin diagram. We prove that E is trivial by induction on m . For $m = 1$, the result holds from the previous analysis. Considering the natural projection

$$f : X \rightarrow X' := G_2/P_{I_2} \times \cdots \times G_m/P_{I_m},$$

one is easy to see that every f -fiber $f^{-1}(x)$ is isomorphic to G_1/P_{I_1} . By assumption, E is trivial on all the special families of lines on $f^{-1}(x) \cong G_1/P_{I_1}$. Thus E is trivial on all f -fibers by the previous analysis. It follows that $E' = f_* E$ is an algebraic vector bundle of rank r over X' and $E \cong f^* E'$. Similarly, we can prove that E' is trivial and so is E . $\square$

Theorem 3.9 *On X , if r -bundle E is poly-uniform with respect to all the special families of lines and $r \leq v(X)$, then E is δ_i -unstable for some δ_i ($1 \leq i \leq m$) or E splits as a direct sum of line bundles.*

Proof It is obviously from Proposition 3.7 and 3.8. $\square$

4 Semistable vector bundles on rational homogeneous spaces

The construction of subsheaves in holomorphic vector bundles plays an important role in the proof of the generalized Grauert–Mülich–Barth theorem. A generalization of the Grauert–Mülich–Barth theorem to normal projective varieties in characteristic zero is proved (see [12] Theorem 3.1.2). Since the theorem in the book is for any normal projective variety, the bound is pretty coarse. In this section, we will find the explicit bounds for rational homogeneous spaces.

The following Descent Lemma provides a way for us to prove the existence of subsheaves.

Lemma 4.1 (Descent Lemma [23]) *Let X, Y be complex manifolds, $f : X \rightarrow Y$ be a surjective submersion with connected fibers and E be an algebraic r -bundle over Y . Let $\tilde{K} \subset f^* E$ be a subbundle of rank t in $f^* E$ and $\tilde{Q} = f^* E / \tilde{K}$ be its quotient. If*

$$\mathrm{Hom}(T_{X/Y}, \mathrm{Hom}(\tilde{K}, \tilde{Q})) = 0,$$

then $\tilde{K}$ is the form $\tilde{K} = f^ K$ for some algebraic subbundle $K \subset E$ of rank t .*

Let G be a simple Lie group and $D = \{1, 2, \dots, n\}$ be the set of nodes of the Dynkin diagram $\mathcal{D}$ of G . Denote by P_k the parabolic subgroup of G corresponding to the node k . Consider $X = G/P_k$ in its minimal homogeneous embedding. Let $\mathcal{M} := G/P_{N(k)}$ be

the generalized flag manifold defined by the marked Dynkin diagram $(\mathcal{D}, N(k))$ and $\mathcal{U} := G/P_{k, N(k)} = G/(P_{N(k)} \cap P_k)$ be the universal family, which has a natural $\mathbb{P}^1$ -bundle structure over $\mathcal{M}$, i.e., we have the natural diagram:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{q} & \mathcal{M} \\ \downarrow p & & \\ X = G/P_k. & & \end{array} \quad (4.1)$$

Given $x \in X$, $\mathcal{M}_x = q(p^{-1}(x))$ coincides with $H/P_{N(k)}$ by Remark 2.7 where the set of nodes of the Dynkin diagram H is $D(H) = D \setminus k$.

To apply the Descent Lemma above, we need to calculate the splitting types of $T_{\mathcal{U}/X}|_{q^{-1}(l)}$ for later use. More precisely, we will show that the splitting types of $T_{\mathcal{U}/X}|_{q^{-1}(l)}$ take the forms $(-1, \dots, -1)$, $(-1, \dots, -1, -2, \dots, -2)$ or (-3) . Building upon this assert, we will generalize the Grauert–Mülich–Barth theorem to any rational homogeneous spaces and give explicit numerical bounds.

It's well known that homogeneous vector bundles especially the tangent bundle on a rational homogeneous space G/P are determined by representations of the Lie algebra $\mathfrak{p}$. Restricting the relative tangent bundle $T_{\mathcal{U}/X}$ to a line $\tilde{L} = q^{-1}(l)$, we will get a flag bundle $H/P_{N(k)}$ and $D(H) = D \setminus k$ with tag, where the tag is the set of intersection numbers of a minimal section of the associated $H/P_{N(k)}$ -flag bundle with the relative canonical divisors (see [22] Proposition 3.17). We refer to [19, 22] for a complete account on the tag of a flag bundle. We also need to calculate the weights of the representation. Assume $H/P_{N(k)} := G_1/P_{I_1} \times G_2/P_{I_2} \times \dots \times G_m/P_{I_m}$ ($m \leq 3$) where G_i is simple Lie group. The weights are those giving the tangent bundle of G_i/P_{I_i} ($1 \leq i \leq m$), which are all the positive roots of the Lie algebra of G_i that contain the roots adjacent to α_k , as a summand. Putting together the tag and the weights, we will get the splitting type of $T_{\mathcal{U}/X}|_{\tilde{L}}$. We refer to ([19], page 5–7) for a complete account.

Professor L. E. Solá Conde told us the following example for calculating the relative tangent bundle restricting to an isotropic line by personal correspondence. Using his method, we can figure out all the splitting types of $T_{\mathcal{U}/X}|_{q^{-1}(l)}$.

Example 4.2 Let $X = E_7/P_7$ and $\mathcal{U} = E_7/P_{6,7}$. Then $H/P_{N(7)} = E_6/P_6$. Let $E_7/P_I := E_7/P_{\{1,2,3,4,5,6,7\}}$. The isotropic lines in E_7/P_i are the image of the fibers of the $\mathbb{P}^1$ -bundle

$$p^i := E_7/P_I \rightarrow E_7/P_{I \setminus i}, \quad (i = 1, 2, \dots, 7)$$

into E_7/P_7 via the natural map

$$\pi^i : E_7/P_I \rightarrow E_7/P_i.$$

Let $-K_i$ ($i = 1, 2, \dots, 7$) be the relative anticanonical bundles of p^i and C_i be a fiber of p^i . When mapped to the varieties of Picard number one E_7/P_i 's, C_i 's are the isotropic lines, where $1 \leq i \leq 7$. The C_7 can be regarded as a minimal section of the fibration morphism $E_7/P_{6,7} \rightarrow E_7/P_7$ when it restricts to $\pi^7(C_7)$. By [21] Proposition 2.13, the matrix $(-K_i C_j)$ is the Cartan matrix of E_7 . The tag is exactly

$$(K_1 C_7, \dots, K_6 C_7) = (0, 0, 0, 0, 0, 1),$$

which is the 7-th column of the Cartan matrix without the last term and with the signs changed. The weights are coefficients of all the positive roots of the Lie algebra of E_6 in term of the linear combinations of the fundamental roots that contain the 6-th root as a summand; that is the roots of the form $m_1\alpha_1 + \dots + m_6\alpha_6$ with $m_i \geq 0$, $m_6 > 0$, where α_i 's are the fundamental roots ($i = 1, \dots, 6$). So

$$T_{\mathcal{U}/X}|_{\tilde{L}} = \mathcal{O}_{\tilde{L}}(-1)^{\oplus 16}.$$

However, we can use basic arguments in Lie algebra to simplify the calculations for $T_{\mathcal{U}/X}|_{q^{-1}(l)}$.

Proposition 4.3 *The splitting type of $T_{\mathcal{U}/X}|_{q^{-1}(l)}$ is one of the following forms:*

- $(-1, \dots, -1)$ if α_k is long;
- $(-1, \dots, -1, -2, \dots, -2)$ if α_k is short and $\mathcal{D} \neq G_2$;
- (-3) if α_k is short and $\mathcal{D} = G_2$.

Proof Set $P = P_{\alpha_k}$. Let $\mathfrak{g}$ (resp. $\mathfrak{p}$) be the associated simple Lie algebra (resp. parabolic subalgebra) of G (resp. P), i.e.

- $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$,
- $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi_p^+} \mathfrak{g}_{\alpha}$.

As is well known, the tangent bundle of G/P corresponds to the module $\mathfrak{g}/\mathfrak{p}$. Similarly, the tangent bundle of $\mathcal{U}$ corresponds to the module $\mathfrak{g}/\mathfrak{q}$, where

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi^+, \langle \alpha, \alpha_k \rangle = 0} \mathfrak{g}_{\alpha}.$$

Thus the relative tangent bundle corresponds to the module

$$\mathfrak{p}/\mathfrak{q} \cong \bigoplus_{\alpha \in \Phi_p^+, \langle \alpha, \alpha_k \rangle < 0} \mathfrak{g}_{\alpha},$$

and the splitting type of $T_{\mathcal{U}/X}|_{q^{-1}(l)}$ is determined by considering the restriction of this module to the Borel subalgebra of the algebra generated by $\mathfrak{g}_{\alpha_k}$.

Then, from basic facts on the root systems, the following holds for $\alpha \in \Phi_p^+$ with $\langle \alpha, \alpha_k \rangle < 0$:

$$\begin{cases} \langle \alpha, \alpha_k \rangle = -2, & \text{if } ||\alpha||^2 / ||\alpha_k||^2 = 2, \\ \langle \alpha, \alpha_k \rangle = -3, & \text{if } ||\alpha||^2 / ||\alpha_k||^2 = 3, \\ \langle \alpha, \alpha_k \rangle = -1, & \text{otherwise.} \end{cases}$$

This proves the lemma. $\square$

Theorem 4.4 *Let X be a generalized Grassmannian by marking on $\mathcal{D}$ the node k and α_k be a long root of $\mathcal{D}$. Let E be a holomorphic r -bundle over X of general splitting type $a_E = (a_1, \dots, a_r)$, $a_1 \geq \dots \geq a_r$. If for some $t < r$,*

$$a_t - a_{t+1} \geq 2 \text{ for some } t < r,$$

then there is a normal subsheaf $K \subset E$ of rank t with the following properties: over the open set $V_E = p(q^{-1}(U_E)) \subset X$, where U_E is an open set in $\mathcal{M}$, the sheaf K is a subbundle of E , which on the line $L \subset X$ given by $l \in U_E$ has the form

$$K|_L \cong \bigoplus_{j=1}^t \mathcal{O}_L(a_j).$$

Proof Lemma 4.1 and Proposition 4.3 play important roles in our proof and our proof applies almost verbatim the proof of Chapter 2, Theorem 2.1.4 given in [23]. $\square$

This theorem has far reaching consequences. We give first a series of immediate deductions.

Corollary 4.5 *Let X be a generalized Grassmannian with long root α_k . For a semistable r -bundle E over X of general splitting type $\underline{a}_E = (a_1, \dots, a_r)$, $a_1 \geq \dots \geq a_r$, we have*

$$a_i - a_{i+1} \leq 1 \text{ for } i = 1, \dots, r-1.$$

Proof If $a_t - a_{t+1} \geq 2$ for some $t < r$, then we could find a normal subsheaf $K \subset E$ which is of the form

$$K|_L \cong \bigoplus_{j=1}^t \mathcal{O}_L(a_j)$$

over a general line $L \subset X$. Then we would have $\mu(E) < \mu(K)$ contrary to the hypothesis. $\square$

In particular, we get the generalization theorem of Grauert–Mülich–Barth as follows.

Corollary 4.6 *Let X be a generalized Grassmannian with long root α_k . The general splitting type of a semistable normalized 2-bundle E over generalized Grassmannian X is*

$$a_E = \begin{cases} (0, 0) & \text{if } c_1(E) = 0 \\ (0, -1) & \text{if } c_1(E) = -1. \end{cases}$$

Corollary 4.7 *Let X be a generalized Grassmannian with long root α_k . For a uniform r -bundle E ($r = \zeta(X) + 1$) over X of general splitting type (see Sect. 3 for the notation $\zeta(X)$)*

$$\underline{a}_E = (a_1, \dots, a_r), \quad a_1 \geq \dots \geq a_r,$$

which does not split, we have

$$a_i - a_{i+1} \leq 1 \text{ for } i = 1, \dots, r-1.$$

Proof If $a_t - a_{t+1} \geq 2$ for some $t < r$, then we could find a uniform subbundle $K \subset E$ of general splitting type $\underline{a}_K = (a_1, \dots, a_t)$ (because $V_E = X$). Then the quotient bundle $Q = E/K$ would be uniform of general splitting type $(a_{s+1}, \dots, a_r)$. According to Theorem 1.5, the bundle K and Q must be direct sums of line bundles. The exact sequence

$$0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$$

would therefore split and hence E would be a direct sum of line bundles contrary to the hypothesis. $\square$

When α_k is a short root of $\mathcal{D}$, $\mathcal{M}$ is not the variety of lines on X , but only a closed G -orbit. Therefore we're just going to think about the general splitting types and semistability of vector bundles with respect to $\mathcal{M}$.

Theorem 4.8 *Let X be a generalized Grassmannian by marking on $\mathcal{D}$ the node k and α_k be a short root of $\mathcal{D}$. Let E be a holomorphic r -bundle over X of general splitting type $\underline{a}_E = (a_1, \dots, a_r)$, $a_1 \geq \dots \geq a_r$ with respect to $\mathcal{M}$. If for some $t < r$,*

$$a_t - a_{t+1} \geq \begin{cases} 3, & \text{if } \mathcal{D} \neq G_2 \\ 4, & \text{if } \mathcal{D} = G_2, \end{cases}$$

then there is a normal subsheaf $K \subset E$ of rank t with the following properties: over the open set $V_E = p(q^{-1}(U_E)) \subset X$, where U_E is an open set in $\mathcal{M}$, the sheaf K is a subbundle of E , which on the line $L \subset X$ given by $l \in U_E$ has the form

$$K|_L \cong \bigoplus_{j=1}^t \mathcal{O}_L(a_j).$$

Proof Lemma 4.1 and Proposition 4.3 play important roles in our proof and our proof applies almost verbatim the proof of Chapter 2, Theorem 2.1.4 given in [23]. $\square$

Similarly, we have the following corollary.

Corollary 4.9 *Let X be a generalized Grassmannian with short root α_k . For a semistable r -bundle E over X of general splitting type $\underline{a}_E = (a_1, \dots, a_r)$, $a_1 \geq \dots \geq a_r$ with respect to $\mathcal{M}$, we have*

$$a_i - a_{i+1} \leq 3 \text{ for } i = 1, \dots, r-1.$$

In particular, if $\mathcal{D} \neq G_2$, then $a_i - a_{i+1} \leq 2$ for $i = 1, \dots, r-1$.

From now, let

$$X = G/P \simeq G_1/P_{I_1} \times G_2/P_{I_2} \times \dots \times G_m/P_{I_m},$$

where G_i is a simple Lie group with Dynkin diagram $\mathcal{D}_i$ whose set of nodes is D_i and P_{I_i} is a parabolic subgroup of G_i corresponding to $I_i \subset D_i$. We set $F(I_i) := G_i/P_{I_i}$ by marking on the Dynkin diagram $\mathcal{D}_i$ of G_i the nodes corresponding to I_i . Let δ_i be a node in $\mathcal{D}_i$ and $N(\delta_i)$ be the set of nodes in $\mathcal{D}_i$ that are connected to δ_i .

If $\delta_i \in I_i$, recall that

$$\mathcal{M}_i^{\delta_i^c} := G_i/P_{I_i}^{\delta_i^c} \times \widehat{G_i/P_{I_i}} \quad (1 \leq i \leq m),$$

the i -th special family of lines of class $\check{\delta}_i$, where $P_{I_i}^{\delta_i^c} := P_{(I_i \setminus \delta_i) \cup N(\delta_i)}$ and $\widehat{G_i/P_{I_i}}$ is $G_1/P_{I_1} \times G_2/P_{I_2} \times \dots \times G_m/P_{I_m}$ by deleting i -th term G_i/P_{I_i} . Denote

$$\mathcal{U}_i^{\delta_i^c} := G_i/P_{I_i \cup N(\delta_i)} \times \widehat{G_i/P_{I_i}}$$

the i -th universal family of class $\check{\delta}_i$, which has a natural $\mathbb{P}^1$ -bundle structure over $\mathcal{M}_i^{\delta_i^c}$.

We separate our discussion into two cases:

Case I: $N(\delta_i) \subseteq I_i$, then $\mathcal{U}_i^{\delta_i^c} = X$ and we have the natural projection $X \rightarrow \mathcal{M}_i^{\delta_i^c}$;

Case II: $N(\delta_i) \not\subseteq I_i$, then we have the standard diagram:

$$\begin{array}{ccc} \mathcal{U}_i^{\delta_i^c} & \xrightarrow{q_2} & \mathcal{M}_i^{\delta_i^c} \\ \downarrow q_1 & & \\ X & & \end{array} \quad (4.2)$$

Notice that for $x \in X$, $\mathcal{M}_{i,x}^{\delta_i^c} = q_2(q_1^{-1}(x))$ coincides with $H/P_{N(\delta_i)}$ where $D(H)$ is the components of $(D_i \setminus I_i) \setminus \delta_i$ containing an element of $N(\delta_i)$ by Theorem 2.6.

Let $\mathcal{G}^{\delta_i}$ be a generalized Grassmannian whose Dynkin diagram $\mathcal{D}^{\delta_i}$ is the maximal sub-diagram of $(\mathcal{D}_i, I_i)$ with the only marked point δ_i . Let's consider the standard diagram associated to $\mathcal{G}^{\delta_i}$

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{q} & \mathcal{M} \\ \downarrow p & & \\ \mathcal{G}^{\delta_i} & & \end{array} \quad (4.3)$$

It's not hard to see that $\mathcal{M}_{i,x}^{\delta_i^c}$ is isomorphic to $\mathcal{M}_y = q(p^{-1}(y))$ for every $y \in \mathcal{G}^{\delta_i}$.

We have the following lemma.

Lemma 4.10 *Let $\tilde{L} = q_2^{-1}(l) \subset \mathcal{U}_i^{\delta_i^c}$ for $l \in \mathcal{M}_i^{\delta_i^c}$. For the relative tangent bundle $T_{\mathcal{U}_i^{\delta_i^c}/X}$, we have*

$$T_{\mathcal{U}_i^{\delta_i^c}/X}|_{\tilde{L}} = T_{\mathcal{U}/\mathcal{G}^{\delta_i}}|_{\tilde{L}}.$$

If δ_i is an exposed short root of $(\mathcal{D}_i, I_i)$, then δ_i is a short root of $\mathcal{D}^{\delta_i}$ and vice versa. Combining this fact and the above lemma, we get the following proposition:

Proposition 4.11 *Let $\tilde{L} = q_2^{-1}(l) \subset \mathcal{U}_i^{\delta_i^c}$ for $l \in \mathcal{M}_i^{\delta_i^c}$. Then the splitting type of $T_{\mathcal{U}_i^{\delta_i^c}/X}$ on $\tilde{L}$ is one of the following forms:*

$$\begin{cases} (-1, \dots, -1), & \text{if } \delta_i \text{ is not an exposed short root;} \\ (-1, \dots, -1, -2, \dots, -2), & \text{if } \delta_i \text{ is an exposed short root and } \delta_i \notin \mathcal{D}(G_2); \\ (-3), & \text{if } \delta_i \text{ is an exposed short root and } \delta_i \in \mathcal{D}(G_2). \end{cases}$$

Proof It's obviously from Proposition 4.3. □

According to Theorem 2.3, 2.6, if δ_i is not an exposed short root, then $\mathcal{M}_i^{\delta_i^c}$ is the space of lines of class $\check{\delta}_i$ and $\mathcal{M}_{i,x}^{\delta_i^c}$ is the space of lines of class $\check{\delta}_i$ through x .

Using the above proposition, we have the similar result for rational homogeneous spaces. When δ_j is an exposed short root of $(\mathcal{D}, I)$, $\mathcal{M}_i^{\delta_i^c}$ is not the space of lines of class $\check{\delta}_i$, but only a closed G -orbit.

Proposition 4.11 and Lemma 4.1 play important roles in the proof of Theorem 1.7 and our proof applies almost verbatim the proof of Theorem 5.7 given in [6]. Moreover, Corollary 1.9 follows easily.

Proof of Corollary 1.10:

The assumption tells us that E is poly-uniform with respect to $\mathcal{M}_i^{\delta_i^c}$ for each i and δ_i and the splitting type is $(a^{(\delta_1)}, \dots, a^{(\delta_l)})$ by Corollary 1.9. The statement follows from Proposition 3.8.

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