Some results of Fekete–Szegő type for Bavrin’s families of holomorphic functions in $\mathbb{C}^n$

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Abstract

In the paper there is considered a generalization of the well-known Fekete–Szegő type problem onto some Bavrin’s families of complex valued holomorphic functions of several variables. The definitions of Bavrin’s families correspond to geometric properties of univalent functions of a complex variable, like as starlikeness and convexity. First of all, there are investigated such Bavrin’s families which elements satisfy also a (j, k) -symmetry condition. As application of these results there is given the solution of a Fekete–Szegő type problem for a family of normalized biholomorphic starlike mappings in $\mathbb{C}^n$.

Keywords Holomorphic functions of scv · n -circular domains in $\mathbb{C}^n$ · Minkowski function · (j, k) -symmetry · Fekete–Szegő type estimations

Mathematics Subject Classification 32A30 · 30C45

1 Introduction

By $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N}_0, \mathbb{N}_1, \mathbb{N}_2$ let us denote the sets of complex numbers, real numbers, all integers, nonnegative integers, positive integers and the integers not smaller than 2, respectively. We say that a domain $\mathcal{G} \subset \mathbb{C}^n, n \in \mathbb{N}_1$, is complete n -circular if $z\lambda = (z_1\lambda_1, \dots, z_n\lambda_n) \in \mathcal{G}$ for each $z = (z_1, \dots, z_n) \in \mathcal{G}$ and every $\lambda = (\lambda_1, \dots, \lambda_n) \in \overline{U}^n$, where U is the unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$. From now on by $\mathcal{G}$ will be denoted a bounded complete n -circular domain in $\mathbb{C}^n, n \in \mathbb{N}_1$. Of course, only the open discs with the centre $\zeta = 0$ and the radius $r > 0$, are the bounded complete 1-circular domains $\mathcal{G} \subset \mathbb{C}$.

In our considerations the Minkowski function $\mu_{\mathcal{G}} : \mathbb{C}^n \rightarrow [0, \infty)$

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$$\mu_{\mathcal{G}}(z) = \inf \left\{ t > 0 : \frac{1}{t}z \in \mathcal{G} \right\}, z \in \mathbb{C}^n,$$

will be very useful. It is known (see e.g., [26]) that $\mu_{\mathcal{G}}$ is a norm in $\mathbb{C}^n$ if $\mathcal{G}$ is a convex bounded complete n -circular domain. This function gives the possibility to redefine the domain $\mathcal{G}$ and its boundary $\partial\mathcal{G}$ as follows:

$$\mathcal{G} = \{z \in \mathbb{C}^n : \mu_{\mathcal{G}}(z) < 1\}, \partial\mathcal{G} = \{z \in \mathbb{C}^n : \mu_{\mathcal{G}}(z) = 1\}.$$

Now, we recall some information about m -homogeneous polynomials. We say that a function $Q_m : \mathbb{C}^n \rightarrow \mathbb{C}$, $m \in \mathbb{N}_1$, is an m -homogeneous polynomial if

$$Q_m(z) = L_m(z^m) = L_m(z, \dots, z), z \in \mathbb{C}^n,$$

where $L_m : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$ is a bounded m -linear function (by Q_0 we note a complex constant). For this reason it is very natural to define (see [4]) the following generalization of the norm of m -homogeneous polynomials $Q_m : \mathbb{C}^n \rightarrow \mathbb{C}$, i.e., the $\mu_{\mathcal{G}}$ -balance of such m -homogeneous polynomials

$$\mu_{\mathcal{G}}(Q_m) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|Q_m(w)|}{(\mu_{\mathcal{G}}(w))^m} = \sup_{v \in \partial\mathcal{G}} |Q_m(v)| = \sup_{u \in \mathcal{G}} |Q_m(u)|.$$

A simple kind of 1-homogeneous polynomials are the linear functionals $J, I \in (\mathbb{C}^n)^*$ of the form

$$J(z) = \sum_{j=1}^n z_j, z = (z_1, \dots, z_n) \in \mathbb{C}^n, I(z) = (\mu_{\mathcal{G}}(J))^{-1} J(z).$$

Note that for $m \in \mathbb{N}_1$, the mapping

$$I^m : \mathbb{C}^n \rightarrow \mathbb{C}, I^m(z) = (I(z))^m, z \in \mathbb{C}^n,$$

is an m -homogeneous polynomial and $\mu_{\mathcal{G}}(I^m) = 1$.

By $\mathcal{F}(\mathcal{G}, \mathbb{C}^m)$, $m \in \mathbb{N}_1$, let us denote the space of all functions $f : \mathcal{G} \rightarrow \mathbb{C}^m$, by $\mathcal{H}_{\mathcal{G}}$ the space of all holomorphic functions $f \in \mathcal{F}(\mathcal{G}, \mathbb{C})$ and by $\mathcal{H}_{\mathcal{G}}(0)$, $\mathcal{H}_{\mathcal{G}}(1)$ the collection of all $f \in \mathcal{H}_{\mathcal{G}}$, normalized by $f(0) = 0$, $f(0) = 1$, respectively. Let us recall that every function $f \in \mathcal{H}_{\mathcal{G}}$ has a unique power series expansion

$$f(z) = \sum_{m=0}^{\infty} Q_{f,m}(z), z \in \mathcal{G}, \quad (1.1)$$

where $Q_{f,m} : \mathbb{C}^n \rightarrow \mathbb{C}$, $m \in \mathbb{N}_0$, are m -homogeneous polynomials determined uniquely by the m th Fréchet differential $D^m f(0)$ of f at zero via the formula

$$Q_{f,m}(z) = \frac{1}{m!} D^m f(0)(z^m).$$

Many authors considered some problems connected with m -homogeneous polynomials in the power series expansion (1.1) of functions from different subfamilies of $\mathcal{H}_{\mathcal{G}}$ (see for instance [2, 5, 9, 13, 24]). In particular, in the case $\mathcal{G} \subset \mathbb{C}^2$ Bavrin [2] gives the following sharp estimates

$$\sup_{z \in \mathcal{G}} |Q_{f,m}(z)| \leq 1 - |Q_{f,0}|^2, m \in \mathbb{N}_1,$$

for the homogeneous polynomials $Q_{f,m}$ of functions belonging to the family

$$\mathcal{B}_{\mathcal{G}} = \{f \in \mathcal{H}_{\mathcal{G}} : |f(z)| < 1, z \in \mathcal{G}\}.$$

Note, that the sharpness in the Bavrin's result was understood as follows: There exists a bounded complete 2-circular domain $\mathcal{G} \subset \mathbb{C}^2$ and a function $f \in \mathcal{B}_{\mathcal{G}}$ which realizes the equality in the above inequality. Let us observe that in general case $\mathcal{G} \subset \mathbb{C}^n, n \in \mathbb{N}_1$, the above inequality takes currently the following form

$$\mu_{\mathcal{G}}(Q_{f,m}) \leq 1 - |Q_{f,0}|^2, m \in \mathbb{N}_1,$$

by the definition of the $\mu_{\mathcal{G}}$ -balance $\mu_{\mathcal{G}}(Q_m)$.

In the paper, we solve for $\lambda \in \mathbb{C}, k \in \mathbb{N}_2$, the problem of the sharp upper estimate

$$\mu_{\mathcal{G}}(Q_{f,2k} - \lambda(Q_{f,k})^2) \leq M(\lambda, k)$$

for the pairs $Q_{f,k}, Q_{f,2k}$ homogeneous polynomials of functions belonging to some Bavrin's subfamilies of the families $\mathcal{H}_{\mathcal{G}}(0), \mathcal{H}_{\mathcal{G}}(1)$ (the case $k = 1$ see the section Final Remarks). Moreover, here the sharpness is understand more generally. It means that for every bounded complete n -circular domain $\mathcal{G} \subset \mathbb{C}^n$ there exists a function f , which belongs to a mentioned Bavrin's subfamily and realizes the equality in the above inequality.

Note that the afore-mentioned estimate is a generalization of the well known planar Fekete–Szegő [8] result onto the s.c.v. case.

In the sequel we use a special kind of functions symmetry. Let us observe that bounded complete n -circular domains $\mathcal{G} \subset \mathbb{C}^n$ are k -symmetric sets, $k \in \mathbb{N}_2$, that is $\varepsilon \mathcal{G} = \mathcal{G}$, where $\varepsilon = \varepsilon_k = \exp \frac{2\pi i}{k}$ is a generator of the cyclic group of k th roots of unity. For $k \in \mathbb{N}_2, j \in \mathbb{Z}$ we define the collections $\mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m)$ of functions $f \in \mathcal{F}(\mathcal{G}, \mathbb{C}^m)$, (j, k) -symmetrical, i.e.,

$$f(\varepsilon z) = \varepsilon^j f(z), z \in \mathcal{G}.$$

Let us observe that $\mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m) \neq \mathcal{F}_{l,k}(\mathcal{G}, \mathbb{C}^m)$ for different $j, l \in \{0, 1, \dots, k-1\}$. Moreover, the intersections $\mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m) \cap \mathcal{F}_{l,k}(\mathcal{G}, \mathbb{C}^m)$ are the singleton $\{0\}$ for such j, l .

Now we present a functions decomposition theorem [19].

Theorem A *For every function $f \in \mathcal{F}(\mathcal{G}, \mathbb{C}^m)$ and every $k \in \mathbb{N}_2$ there exists exactly one sequence of functions $f_{j,k} \in \mathcal{F}_{j,k}(\mathcal{G}, \mathbb{C}^m), j = 0, 1, \dots, k-1$, such that*

$$f = \sum_{j=0}^{k-1} f_{j,k}. \quad (1.2)$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-jl} f(\varepsilon^l z), z \in \mathcal{G}.$$

By the uniqueness of this decomposition, the functions $f_{j,k}, j = 0, 1, \dots, k-1$, will be called (j, k) -symmetrical components of the function f .

In the next sections of the paper will be very useful the fact that $f_{0,k} \in \mathcal{H}_{\mathcal{G}}(1)$ for $f \in \mathcal{H}_{\mathcal{G}}(1)$. Note that

$$f_{0,k}(z) = 1 + \sum_{m=1}^{\infty} Q_{(f_{0,k}),m}(z) = 1 + \sum_{s=1}^{\infty} Q_{f,sk}(z), z \in \mathcal{G}.$$

Note that the above unique decomposition (1.2) of functions was used in [20] to solve some functional equations, in [21] to construction a semi power series and in [22] to obtain a uniqueness theorem of Cartan type for holomorphic mappings in $\mathbb{C}^n$.

We close this section with the following Golusin's [10] result, very useful in the proof of the first result of Fekete–Szegő type for holomorphic functions of several complex variables.

Lemma 1.1 *Let $\Phi : U \rightarrow U$ be a holomorphic function of the form*

$$\Phi(\zeta) = \sum_{v=0}^{\infty} a_v \zeta^v, \zeta \in U.$$

Then

$$|a_m| \leq 1 - |a_p|^2,$$

for every $m \in \mathbb{N}_1, p \in \mathbb{N}_0$, satisfying the condition $0 \leq 2p < m$. The estimates are optimal.

Proof Let us recall that the simplest case

$$|a_1| \leq 1 - |a_0|^2$$

follows from the well-known inequality

$$|\Phi'(\zeta)| \leq \frac{1 - |\Phi(\zeta)|^2}{1 - |\zeta|^2}, \zeta \in U.$$

For another m, p we use a Krzyż's idea [17, Chapt. 6.2] and some properties of (j, k) -symmetrical functions.

Let us take

$$\Psi(\zeta) = \zeta^{m-2p} \Phi(\zeta) = \sum_{s=0}^{\infty} a_s \zeta^{s+m-2p}, \zeta \in U.$$

Then $\Psi : U \rightarrow U$ and is holomorphic. Thus the $(0, m-p)$ -part $\Psi_{0,m-p}$ of Ψ transforms U into itself, is holomorphic and

$$\Psi_{0,m-p}(\zeta) = \sum_{s=0}^{\infty} a_{p+s(m-p)} \zeta^{(s+1)(m-p)}, \zeta \in U.$$

Hence, by the Schwarz Lemma (in version with the zero $\zeta = 0$ of multiplicity $m-p > 0$) it fulfils the inequality

$$\left| \Psi_{0,m-p}(\zeta) \right| \leq |\zeta|^{m-p}, \zeta \in U.$$

Therefore, the function

$$\Theta(\zeta) = \frac{1}{\zeta^{m-p}} \Psi_{0,m-p}(\zeta), \zeta \in U,$$

maps U into itself, is holomorphic and has the expansion

$$\Theta(\zeta) = \sum_{s=0}^{\infty} a_{p+s(m-p)} \zeta^{s(m-p)}, \zeta \in U.$$

Consequently, replacing $\zeta^{m-p} \in U$ by $\xi \in U$, we get that the function

$$\sum_{s=0}^{\infty} b_s \xi^s = \sum_{s=0}^{\infty} a_{p+s(m-p)} \xi^s, \xi \in U$$

transforms holomorphically U into itself. Hence, by the first part of the proof, we get the thesis.

Note that the equality in the inequality is attained by the functions $F(\zeta) = \zeta^m, \zeta \in U$ and $F(\zeta) = \zeta^p, \zeta \in U$. This completes the proof $\square$

2 Main results

We start this section with an n -dimensional Fekete–Szegő type theorem for bounded holomorphic functions on bounded complete n -circular domains in $\mathbb{C}^n$. Note that it is a generalization of a 1-dimensional result given by Keogh and Merkes [14].

Theorem 2.1 *Let φ be a function from the family*

$$\mathcal{B}_{\mathcal{G}}(0) = \{ \varphi \in \mathcal{H}_{\mathcal{G}}(0) : |\varphi(z)| < 1, z \in \mathcal{G} \}$$

and has the form

$$\varphi(z) = \sum_{l=1}^{\infty} Q_{\varphi,l}(z), z \in \mathcal{G}.$$

Then, for every $k \in \mathbb{N}_1$ and every $\gamma \in \mathbb{C}$, there holds the inequality

$$\mu_{\mathcal{G}} \left(Q_{\varphi,2k} - \gamma (Q_{\varphi,k})^2 \right) \leq \max \{ 1, |\gamma| \}. \quad (2.1)$$

The estimate is sharp.

Proof For every arbitrarily fixed $z \in \mathcal{G}$ we define the function

$$\Phi(\zeta) = \begin{cases} \frac{\varphi(\zeta z)}{\zeta}, \zeta \in U \setminus \{0\} \\ Q_{\varphi,1}(z) = \lim_{\zeta \rightarrow 0} \frac{\varphi(\zeta z)}{\zeta}, \zeta = 0 \end{cases}.$$

Then Φ is holomorphic,

$$\Phi(\zeta) = \sum_{v=0}^{\infty} a_v \zeta^v, \zeta \in U,$$

$a_{l-1} = Q_{\varphi,l}(z)$, $l \in \mathbb{N}_1$, and $|\Phi(0)| < 1$. Moreover, applying a $\mathbb{C}^n$ -version of Schwarz Lemma, i.e., $|\varphi(z)| \leq \mu_G(z)$, $z \in \mathcal{G}$, for $\varphi \in \mathcal{B}_G(0)$, we conclude also that for $\zeta \in U \setminus \{0\}$

$$|\Phi(\zeta)| = \frac{|\varphi(\zeta z)|}{|\zeta|} \leq \frac{\mu_G(\zeta z)}{|\zeta|} = \frac{|\zeta| \mu_G(z)}{|\zeta|} < 1,$$

hence $\Phi : U \rightarrow U$. Now let us observe that from the Lemma 1.1 we get

$$|a_{m-1}| \leq 1 - |a_{p-1}|^2,$$

for arbitrarily fixed $m, p \in \mathbb{N}_1$, satisfying the condition $0 \leq 2(p-1) < m-1$. Thus

$$|Q_{\varphi,m}(z)| \leq 1 - |Q_{\varphi,p}(z)|^2, z \in \mathcal{G}.$$

Therefore, for $z \in \mathcal{G}$ and every $\gamma \in \mathbb{C}$

$$\begin{aligned} & |Q_{\varphi,m}(z) - \gamma(Q_{\varphi,p}(z))^2| \\ & \leq |Q_{\varphi,m}(z)| + |\gamma| |(Q_{\varphi,p}(z))^2| \leq 1 - |Q_{\varphi,p}(z)|^2 + |\gamma| |Q_{\varphi,p}(z)|^2 \\ & = 1 + (|\gamma| - 1) |Q_{\varphi,p}(z)|^2 \leq \max\{1, |\gamma|\}, \end{aligned}$$

because

$$\begin{cases} (|\gamma| - 1) |Q_{\varphi,p}(z)|^2 \leq 0, & \text{if } |\gamma| < 1, \\ 0 \leq (|\gamma| - 1) |Q_{\varphi,p}(z)|^2 \leq |\gamma| - 1, & \text{if } |\gamma| \geq 1. \end{cases}$$

Consequently, by the arbitrariness of $z \in \mathcal{G}$, also

$$\sup_{z \in \mathcal{G}} |Q_{\varphi,m}(z) - \gamma(Q_{\varphi,p}(z))^2| \leq \max\{1, |\gamma|\}.$$

Putting in the above $p = k$ and $m = 2k$, $k \in \mathbb{N}_1$, we obtain $m-1 > 2(p-1) \geq 1 \geq 0$ and

$$\sup_{z \in \mathcal{G}} |Q_{\varphi,2k}(z) - \gamma(Q_{\varphi,k}(z))^2| \leq \max\{1, |\gamma|\}.$$

Finally, by the fact that $Q_{\varphi,2k} - \gamma(Q_{\varphi,k})^2$ is a $2k$ -homogeneous polynomial for every $\gamma \in \mathbb{C}$ and by the definition of its μ_G -balance, we get the statements of the Theorem 2.1.

Now, we will analyse the sharpness of the above estimate.

First, we prove that in the case $|\gamma| \geq 1$, the equality in (2.1) is attained by the function $\varphi = \tilde{\varphi} \in \mathcal{B}_G(0)$, $\tilde{\varphi} = I^k$, more precisely $\tilde{\varphi} = I^k|_{\mathcal{G}}$, i.e., $\tilde{\varphi}(z) = I^k(z)$, $z \in \mathcal{G}$. Indeed, since $Q_{\tilde{\varphi},k} = I^k$, $Q_{\tilde{\varphi},2k} = 0$ and $\mu_G(I^{2k}) = 1$, we have

$$\mu_{\mathcal{G}}\left(Q_{\varphi,2k} - \gamma(Q_{\varphi,k})^2\right) = \mu_{\mathcal{G}}\left(-\gamma(Q_{\tilde{\varphi},k})^2\right) = |\gamma|\mu_{\mathcal{G}}\left((Q_{\tilde{\varphi},k})^2\right) = |\gamma| = \max\{1, |\gamma|\}.$$

Now, we show that in the case $|\gamma| < 1$, the equality in (2.1) realizes the function $\varphi = \hat{\varphi} \in \mathcal{B}_{\mathcal{G}}(0)$, $\hat{\varphi} = I^{2k}$, more precisely $\hat{\varphi} = I_{|\mathcal{G}}^{2k}$, i.e., $\hat{\varphi}(z) = I^{2k}(z)$, $z \in \mathcal{G}$. Indeed, since $Q_{\hat{\varphi},k} = 0$, $Q_{\hat{\varphi},2k} = I^{2k}$, we get

$$\mu_{\mathcal{G}}\left(Q_{\varphi,2k} - \gamma(Q_{\varphi,k})^2\right) = \mu_{\mathcal{G}}(Q_{\hat{\varphi},2k}) = 1 = \max\{1, |\gamma|\}.$$

This completes the proof. $\square$

Note that Theorem 2.1 generalizes an earlier result given by authors in [7].

In the sequel we apply Theorem 2.1 to study two Bavrin's families $\mathcal{M}_{\mathcal{G}}, \mathcal{N}_{\mathcal{G}}$ of functions $f \in \mathcal{H}_{\mathcal{G}}(1)$. These families are defined by the following family $\mathcal{C}_{\mathcal{G}}$,

$$\mathcal{C}_{\mathcal{G}} = \{f \in \mathcal{H}_{\mathcal{G}}(1) : \operatorname{Re} f(z) > 0, z \in \mathcal{G}\}$$

and by the following Temljakov [29] linear operator $\mathcal{L} : \mathcal{H}_{\mathcal{G}} \longrightarrow \mathcal{H}_{\mathcal{G}}$

$$\mathcal{L}f(z) = f(z) + Df(z)(z), z \in \mathcal{G},$$

where $Df(z)$ means the Fréchet derivative of f at the point z . Note that the operator $\mathcal{L}$ is invertible and

$$\mathcal{L}^{-1}f(z) = \int_0^1 f(zt) dt, z \in \mathcal{G}.$$

It is obvious also that for the transforms $\mathcal{L}f, \mathcal{L}\mathcal{L}f$ of the functions $f \in \mathcal{H}_{\mathcal{G}}(1)$ we have

$$\mathcal{L}f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L}f,m}(z) = 1 + \sum_{m=1}^{\infty} (m+1)Q_{f,m}(z), z \in \mathcal{G}, \quad (2.2)$$

$$\mathcal{L}\mathcal{L}f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L}\mathcal{L}f,m}(z) = 1 + \sum_{m=1}^{\infty} (m+1)^2 Q_{f,m}(z), z \in \mathcal{G}. \quad (2.3)$$

Moreover,

$$\mathcal{L}^{-1}f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L}^{-1}f,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{1}{m+1} Q_{f,m}(z), z \in \mathcal{G}. \quad (2.4)$$

We say that a function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to the Bavrin's family $\mathcal{M}_{\mathcal{G}}(\mathcal{N}_{\mathcal{G}})$ if it satisfies the factorization

$$\begin{aligned} \mathcal{L}f(z) &= f(z)h(z), \quad z \in \mathcal{G}, \\ \mathcal{L}\mathcal{L}f(z) &= \mathcal{L}f(z)h(z), \quad z \in \mathcal{G}, \end{aligned} \quad (2.5)$$

together with a function $h \in \mathcal{C}_{\mathcal{G}}$ and the transform $\mathcal{L}f, (\mathcal{L}\mathcal{L}f)$, respectively. Note that the families $\mathcal{M}_{\mathcal{G}}, \mathcal{N}_{\mathcal{G}}$ correspond with the well-known families of normalized univalent starlike (convex) functions in the disc U [2] and the family $\mathcal{M}_{\mathcal{G}}$ can be used to construction biholomorphic starlike mappings in $\mathbb{C}^n$ (see [6, 18], compare also [12, 25]). Between functions

from $\mathcal{M}_{\mathcal{G}}, \mathcal{N}_{\mathcal{G}}$ there holds a relationship, corresponding to the well-known Alexander type connexion [1], for univalent starlike and convex mappings in the unit disc. Here, this relationship is the following: if $f \in \mathcal{N}_{\mathcal{G}}$, then $\mathcal{L}f \in \mathcal{M}_{\mathcal{G}}$ and conversely, if $f \in \mathcal{M}_{\mathcal{G}}$, then $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}$ [2].

We begin the presentation of some Fekete–Szegő type results in Bavrin’s families with the following theorem.

Theorem 2.2 *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n -circular domain and let $f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$, $k \in \mathbb{N}_2$. If the expansion of the function f into a series of m -homogeneous polynomials $Q_{f,m}$ has the form (1.1), with $Q_{f,0} = 1$, then for the homogeneous polynomials $Q_{f,2k}, Q_{f,k}$ and every $\lambda \in \mathbb{C}$ there holds the following sharp estimate:*

$$\mu_{\mathcal{G}}\left(Q_{f,2k} - \lambda(Q_{f,k})^2\right) \leq \frac{1}{k(2k+1)} \max \left\{ 1, \left| \frac{4(2k+1)\lambda - (k+1)^2(k+2)}{k(k+1)^2} \right| \right\}. \quad (2.6)$$

Proof Let us recall that between the functions $p \in \mathcal{C}_{\mathcal{G}}$ and $\varphi \in \mathcal{B}_{\mathcal{G}}(0)$, there holds the following relation [2]:

$$\frac{p-1}{p+1} = \varphi \in \mathcal{B}_{\mathcal{G}}(0) \iff p \in \mathcal{C}_{\mathcal{G}}.$$

Let $k \in \mathbb{N}_2$ be arbitrarily fixed and let the function f belongs to $\mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$. Then, by the definition of the family $\mathcal{N}_{\mathcal{G}}$ and by the above relation between the families $\mathcal{C}_{\mathcal{G}}, \mathcal{B}_{\mathcal{G}}(0)$, we get

$$\frac{\mathcal{L}\mathcal{L}f(z) - \mathcal{L}f(z)}{\mathcal{L}\mathcal{L}f(z) + \mathcal{L}f(z)} = \varphi(z), z \in \mathcal{G}. \quad (2.7)$$

where $\varphi \in \mathcal{B}_{\mathcal{G}}(0) \cap \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$. On the other hand, from (1.1) we have for $z \in \mathcal{G}$:

$$\begin{aligned} \varphi(z) &= \sum_{s=1}^{\infty} Q_{\varphi,sk}(z), \\ f(z) &= 1 + \sum_{s=1}^{\infty} Q_{f,sk}(z) \end{aligned}$$

and from (2.2), (2.3) also

$$\begin{aligned} \mathcal{L}f(z) &= 1 + \sum_{s=1}^{\infty} (sk+1)Q_{f,sk}(z), \\ \mathcal{L}\mathcal{L}f(z) &= 1 + \sum_{s=1}^{\infty} (sk+1)^2Q_{f,sk}(z). \end{aligned}$$

Inserting the above expansion of functions into (2.7), we receive after computations

$$\sum_{s=1}^{\infty} sk(1+sk)Q_{f,sk}(z) = \left(\sum_{s=1}^{\infty} Q_{\varphi,sk}(z) \right) \left(2 + \sum_{s=1}^{\infty} (sk+1)(sk+2)Q_{f,sk}(z) \right).$$

Then, comparing the m -homogeneous polynomials of the same degree on both sides of the above equality, we can determine homogeneous polynomials $Q_{\varphi,k}, Q_{\varphi,2k}$, as follows

$$\begin{aligned} Q_{\varphi,k} &= \frac{1}{2}k(k+1)Q_{f,k}, \\ Q_{\varphi,2k} &= k(2k+1)Q_{f,2k} - \frac{1}{4}k(k+1)^2(k+2)(Q_{f,k})^2. \end{aligned} \quad (2.8)$$

Putting the above equalities into Theorem 2.1 and using the fact that the mapping $(Q_{f,k})^2$, is a $2k$ -homogenous polynomial, we obtain

$$\mu_G \left[k(2k+1)Q_{f,2k} - \left(\frac{1}{4}k(k+1)^2(k+2) + \gamma \frac{1}{4}k^2(k+1)^2 \right) (Q_{f,k})^2 \right] \leq \max\{1, |\gamma|\}.$$

Denoting

$$\lambda = \frac{(k+1)^2}{4(2k+1)}(k+2+\gamma k),$$

we obtain

$$\mu_G \left(Q_{f,2k} - \lambda (Q_{f,k})^2 \right) \leq \frac{1}{k(2k+1)} \max \left\{ 1, \left| \frac{4(2k+1)\lambda - (k+1)^2(k+2)}{k(k+1)^2} \right| \right\}.$$

Now, we show the sharpness of our estimate. To do it, let us consider two cases.

At the beginning, we prove that, in the case

$$\frac{|4(2k+1)\lambda - (k+1)^2(k+2)|}{k(k+1)^2} \geq 1$$

the equality in (2.6) is attained by the function $f = \mathcal{L}^{-1}\tilde{f}$, with

$$\tilde{f}(z) = (1 - I^k(z))^{\frac{-2}{k}}, z \in \mathcal{G}, \quad (2.9)$$

where the branch of the function $(1 - \xi)^{\frac{-2}{k}}$ takes value 1 at the point $\xi = 0$. Indeed. Function f belongs to $\mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$, because $\tilde{f} \in \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$. On the other hand, the function $f = \mathcal{L}^{-1}\tilde{f}$ belongs to $\mathcal{N}_{\mathcal{G}}$, because the function φ from (2.7) has the form $\varphi = \tilde{\varphi} = I^k$ and belongs to $\mathcal{B}_{\mathcal{G}}(0)$. Therefore, we can write equalities (2.8) in the following form

$$Q_{f,k} = \frac{2}{k(k+1)}I^k, Q_{f,2k} = \frac{k+2}{k^2(2k+1)}(I^k)^2. \quad (2.10)$$

From this, by the case condition for λ , we have step by step:

$$\begin{aligned}
\mu_g(Q_{f,2k} - \lambda(Q_{f,k})^2) &= \mu_g \left[\frac{(k+1)^2(k+2) - 4(2k+1)\lambda}{k^2(k+1)^2(2k+1)} (I^k)^2 \right] \\
&= \left| \frac{(k+1)^2(k+2) - 4(2k+1)\lambda}{k^2(k+1)^2(2k+1)} \right| \mu_g(I^{2k}) \\
&= \frac{1}{k(2k+1)} \max \left\{ 1, \left| \frac{4(2k+1)\lambda - (k+1)^2(k+2)}{k(k+1)^2} \right| \right\}.
\end{aligned}$$

Now, we show that, in the case

$$\frac{|4(2k+1)\lambda - (k+1)^2(k+2)|}{k(k+1)^2} < 1$$

the equality in (2.6) realizes the function $f = \mathcal{L}^{-1}\hat{f}$, with

$$\hat{f}(z) = (1 - I^{2k}(z))^{\frac{-1}{k}}, z \in \mathcal{G}, \quad (2.11)$$

where the branch of the function $(1 - \xi)^{\frac{-1}{k}}$ takes value 1 at the point $\xi = 0$. The function $f = \mathcal{L}^{-1}\hat{f}$ belongs to $\mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$, because $\hat{f} \in \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$. Also $f \in \mathcal{N}_{\mathcal{G}}$, because the function φ from (2.7) has the form $\varphi = \hat{\varphi} = I^{2k}$ and belongs to $\mathcal{B}_{\mathcal{G}}(0)$. Therefore, we can write equalities (2.8) in the following form

$$Q_{f,k} = 0, Q_{f,2k} = \frac{1}{k(2k+1)} I^{2k}. \quad (2.12)$$

From this, by the case condition for λ , we have:

$$\begin{aligned}
\mu_g(Q_{f,2k} - \lambda(Q_{f,k})^2) &= \mu_g \left(\frac{1}{k(2k+1)} I^{2k} \right) = \frac{1}{k(2k+1)} \mu_g(I^{2k}) \\
&= \frac{1}{k(2k+1)} \max \left\{ 1, \left| \frac{4(2k+1)\lambda - (k+1)^2(k+2)}{k(k+1)^2} \right| \right\}.
\end{aligned}$$

This completes the proof. $\square$

We continue the presentation of some Fekete–Szegő type results in Bavrin's families with the following theorem:

Theorem 2.3 *Let $\mathcal{G} \subset \mathbb{C}^n$ be a bounded complete n -circular domain and let $k \in \mathbb{N}_2$. If the expansion of the function $f \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$, into a series of m -homogenous polynomials $Q_{f,m}$ has the form (1.1), with $Q_{f,0} = 1$, then for the homogeneous polynomials $Q_{f,2k}, Q_{f,k}$ and $\lambda \in \mathbb{C}$ there holds the following sharp estimate:*

$$\mu_{\mathcal{G}}(Q_{f,2k} - \lambda(Q_{f,k})^2) \leq \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}. \quad (2.13)$$

Proof Let $k \in \mathbb{N}_2$ be arbitrarily fixed. Then, it is obvious that $\mathcal{L}^{-1}f \in \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$, because $f \in \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$. Also the assumption that $f \in \mathcal{M}_{\mathcal{G}}$, by the relationship of the Alexander type, gives that $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}}$. Hence, we have that $\mathcal{L}^{-1}f \in \mathcal{N}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$. On the other hand, its expansion into a series of m -homogenous polynomials $Q_{f,m}$, by (2.4), has the form

$$\mathcal{L}^{-1}f(z) = 1 + \sum_{s=1}^{\infty} \frac{1}{sk+1} Q_{f,sk}(z), z \in \mathcal{G}.$$

Thus, in view of (2.6) from Theorem 2.2, we get that for every $\delta \in \mathbb{C}$

$$\begin{aligned} & \mu_{\mathcal{G}} \left(\frac{1}{2k+1} Q_{f,2k} - \delta \left(\frac{1}{k+1} Q_{f,k} \right)^2 \right) \\ & \leq \frac{1}{k(2k+1)} \max \left\{ 1, \left| \frac{4(2k+1)\delta - (k+1)^2(k+2)}{k(k+1)^2} \right| \right\}. \end{aligned}$$

Hence,

$$\mu_{\mathcal{G}} \left(Q_{f,2k} - \delta \frac{2k+1}{(k+1)^2} (Q_{f,k})^2 \right) \leq \frac{1}{k} \max \left\{ 1, \left| \frac{4(2k+1)\delta - (k+1)^2(k+2)}{k(k+1)^2} \right| \right\}$$

and denoting $\delta \frac{2k+1}{(k+1)^2} = \lambda$, finally

$$\mu_{\mathcal{G}} \left(Q_{f,2k} - \lambda (Q_{f,k})^2 \right) \leq \frac{1}{k} \max \left\{ 1, \left| \frac{4\lambda - (k+2)}{k} \right| \right\},$$

which is the same as (2.13).

It remains to show the sharpness of the estimate (2.13).

First, we prove that, in the case

$$\frac{|2+k-4\lambda|}{k} \geq 1$$

the equality in (2.13) is attained by the function $f = \tilde{f}$ defined in (2.9). Of course $\tilde{f} \in \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$ and, by the Alexander type relationship and the fact that $\mathcal{L}^{-1}\tilde{f} \in \mathcal{N}_{\mathcal{G}}$ (see the proof of the sharpness in Theorem 2.2), the function $\tilde{f}$ belongs to $\mathcal{M}_{\mathcal{G}}$. Therefore, in view of (2.2) and (2.10), we achieve

$$Q_{f,k} = Q_{\tilde{f},k} = \frac{2}{k} I^k, \quad Q_{f,2k} = Q_{\tilde{f},2k} = \frac{k+2}{k^2} (I^k)^2.$$

From this (see the proof of the sharpness in Theorem 2.2) and by the case condition for λ , we have step by step:

$$\begin{aligned} \mu_{\mathcal{G}}(Q_{f,2k} - \lambda (Q_{f,k})^2) &= \mu_{\mathcal{G}} \left[\frac{(k+2) - 4\lambda}{k^2} (I^k)^2 \right] \\ &= \left| \frac{(k+2) - 4\lambda}{k^2} \right| \mu_{\mathcal{G}}(I^{2k}) = \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}. \end{aligned}$$

Now, we show that, in the case

$$\frac{|2+k-4\lambda|}{k} < 1$$

the equality in (2.13) realizes the function $f = \hat{f}$ defined in (2.11). Of course $\hat{f} \in \mathcal{F}_{0,k}(\mathcal{G}, \mathbb{C})$ and, by the Alexander type relationship and the fact that $\mathcal{L}^{-1}\hat{f} \in \mathcal{N}_{\mathcal{G}}$ (see the proof of the

sharpness in Theorem 2.2), the function $\hat{f}$ belongs to $\mathcal{M}_G$. Therefore, in view of (2.2) and (2.12) we achieve

$$Q_{f,k} = Q_{\hat{f},k} = 0, \quad Q_{f,2k} = Q_{\hat{f},2k} = \frac{1}{k} I^{2k}.$$

From this (see the proof of the sharpness in Theorem 2.2) and by the case condition for λ , we conclude that:

$$\mu_G(Q_{f,2k} - \lambda(Q_{f,k})^2) = \mu_G\left(\frac{1}{k} I^{2k}\right) = \frac{1}{k} = \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}.$$

This completes the proof. $\square$

Now, we transfer the statement of Theorem 2.3, onto a family $\mathcal{M}_G^k \cap \mathcal{F}_{0,k}(G, \mathbb{C}), k \in \mathbb{N}_2$. Here, $\mathcal{M}_G^k$ is defined by the factorization similar as for the elements from $\mathcal{M}_G$. More precisely, the function f of the right hand side in (2.5) is replaced by a function from $\mathcal{F}_{0,k}(G, \mathbb{C})$, generated by f . Formally, we say that a function $f \in \mathcal{H}_G(1)$ belongs to $\mathcal{M}_G^k, k \in \mathbb{N}_2$, (see [3, 5]) if there exists a function $h \in \mathcal{C}_G$ such that

$$\mathcal{L}f(z) = f_{0,k}(z)h(z), z \in G,$$

where $f_{0,k} \in \mathcal{F}_{0,k}(G, \mathbb{C})$ is the $(0, k)$ -symmetrical component of the function f in the decomposition (1.2) from Theorem A. This family for $k = 2$ corresponds to a well-known Sakaguchi family [27] of a complex variable functions; strictly speaking of functions univalent starlike with respect to two symmetric points. In the paper [5] it was shown that for $k \in \mathbb{N}_2$ the inclusions $\mathcal{M}_G \subset \mathcal{M}_G^k, \mathcal{M}_G^k \subset \mathcal{M}_G$ do not hold, but

$$\mathcal{M}_G \cap \mathcal{F}_{0,k}(G, \mathbb{C}) = \mathcal{M}_G^k \cap \mathcal{F}_{0,k}(G, \mathbb{C}).$$

This identity and Theorem 2.3 implies directly the next result of Fekete–Szegő type in Bavrín's families:

Theorem 2.4 *Let $G \subset \mathbb{C}^n$ be a bounded complete n -circular domain and let $f \in \mathcal{M}_G^k \cap \mathcal{F}_{0,k}(G, \mathbb{C}), k \in \mathbb{N}_2$. If the expansion of the function f into a series of m -homogeneous polynomials $Q_{f,m}$ has the form (1.1), with $Q_{f,0} = 1$, then for the homogeneous polynomials $Q_{f,2k}, Q_{f,k}$ and $\lambda \in \mathbb{C}$ there holds the following sharp estimate:*

$$\mu_G(Q_{f,2k} - \lambda(Q_{f,k})^2) \leq \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}.$$

The equality in the above inequality realize the same functions $f = \tilde{f}, f = \hat{f}$ as in the previous Theorem 2.2.

3 Applications

In this section we apply Theorems 2.3 and 2.4, to obtain a Fekete–Szegő type results for two families of biholomorphic mappings in $\mathbb{C}^n$. By $S^*(\mathbb{B}^n)$ let us denote the family of biholomorphic mappings $F \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n), F(0) = 0, DF(0) = I$ onto starlike domains

$F(\mathbb{B}^n)$. For a wide collection of references in this area see the monographs [11, 16]. In a Kikuchi–Matsuno–Suffridge characterization [15, 23, 28] of the family $S^*(\mathbb{B}^n)$, the collection $P(\mathbb{B}^n)$ of all holomorphic mappings $H \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n)$, $H(0) = 0$, $DH(0) = I$, such that $\operatorname{Re}\langle H(z), z \rangle > 0$, $z \in \mathbb{B}^n \setminus \{0\}$, plays the main role (here $\langle \cdot, \cdot \rangle$ means the Euclidean inner product). This characterization is included in the following theorem:

Theorem B *A locally biholomorphic mapping $F \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n)$ normalized by the conditions $F(0) = 0$, $DF(0) = I$, belongs to $S^*(\mathbb{B}^n)$, iff there exist a mapping $H \in P(\mathbb{B}^n)$ such that*

$$F(z) = DF(z)H(z), z \in \mathbb{B}^n. \quad (3.1)$$

Let $\tilde{S}^*(\mathbb{B}^n)$ be the family of mappings $F \in S^*(\mathbb{B}^n)$ with the factorization

$$F(z) = zf(z), z \in \mathbb{B}^n, \quad (3.2)$$

where $f \in \mathcal{H}_{\mathbb{B}^n}(1)$.

In the paper [6] the authors considered a family of biholomorphic mappings $S^k(\mathbb{B}^n)$, $k \in \mathbb{N}_2$, defined by an equation similar to (3.1). More precisely, the mapping F of the left hand side in (3.1) is replaced by a function from $\mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, generated by F .

Formally, we say that a locally biholomorphic mapping $F \in \mathcal{F}(\mathbb{B}^n, \mathbb{C}^n)$, normalized by $F(0) = 0$, $DF(0) = I$, belongs to the family $S^k(\mathbb{B}^n)$, $k \in \mathbb{N}_2$, if it satisfies the equation

$$F_{1,k}(z) = DF(z)H(z), z \in \mathbb{B}^n,$$

where $H \in P(\mathbb{B}^n)$ and $F_{1,k} \in \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$ is the $(1, k)$ -symmetrical part of F in the decomposition (1.2). Also in the paper [6] the authors proved, that for every $k \in \mathbb{N}_2$ any inclusions $S^k(\mathbb{B}^n) \subset S^*(\mathbb{B}^n)$, $S^*(\mathbb{B}^n) \subset S^k(\mathbb{B}^n)$ do not hold. However, for the same k there holds the following identity:

$$S^k(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n) = S^*(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n). \quad (3.3)$$

Let $\tilde{S}^k(\mathbb{B}^n)$, $k \in \mathbb{N}_2$, be the family of mappings $F \in S^k(\mathbb{B}^n)$ with the factorization (3.2). Now, we present the main theorem, in this section of the paper. It is a Fekete–Szegő type result for locally biholomorphic mappings in $\mathbb{C}^n$, compare [30].

Theorem 3.1 *For mappings $F \in \tilde{S}^k(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, $k \in \mathbb{N}_2$, the parameter $\lambda \in \mathbb{C}$ and points $z \in \mathbb{B}^n \setminus \{0\}$ there holds the following sharp estimate*

$$\left| \frac{T_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}} - \lambda \left(\frac{T_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}} \right)^2 \right| \leq \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}, \quad (3.4)$$

where $T_z \in (\mathbb{C}^n)^*$, $z \in \mathbb{B}^n \setminus \{0\}$, is arbitrary functional satisfying the conditions $\|T_z\| = 1$, $T_z(z) = \|z\|$.

Proof We start with a few facts, very useful in the proof:

1. A mapping F , satisfying (3.2), belongs to $S^k(\mathbb{B}^n)$, iff $f \in \mathcal{M}_{\mathbb{B}^n}^k$ (see [6, 18]).
2. A mapping F of the form (3.2) belongs to $\mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, iff $f \in \mathcal{F}_{0,k}(\mathbb{B}^n, \mathbb{C})$.
3. For $z \in \mathbb{B}^n$ there hold (see for instance [30]) the equalities:

$$\frac{D^{2k+1}F(0)(z^{2k+1})}{(2k+1)!} = z \frac{D^{2k}f(0)(z^{2k})}{(2k)!},$$

$$\frac{D^{k+1}F(0)(z^{k+1})}{(k+1)!} = z \frac{D^kf(0)(z^k)}{(k)!}.$$

4. For $\in \mathbb{C}$ the mapping $Q_{f,2k} - \lambda Q_{f,k}^2 \in \mathcal{F}(\mathbb{B}^n, \mathbb{C})$, is a $2k$ th homogeneous polynomial.

5. There hold the identity $\mu_{\mathbb{B}^n}(\cdot) = \|\cdot\|$ in $\mathbb{C}^n$.

Applying the above facts and Theorem 2.5, we get step by step

$$\begin{aligned} & \left| \frac{T_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)! \|z\|^{2k+1}} - \lambda \left(\frac{T_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)! \|z\|^{k+1}} \right)^2 \right| \\ &= \left| \frac{T_z(z) D^{2k}f(0)(z^{2k})}{(2k)! \|z\|^{2k+1}} - \lambda \left(\frac{T_z(z) D^kf(0)(z^k)}{(k)! \|z\|^{k+1}} \right)^2 \right| \\ &= \left| \frac{D^{2k}f(0)(z^{2k})}{(2k)! \|z\|^{2k}} - \lambda \left(\frac{D^kf(0)(z^k)}{(k)! \|z\|^k} \right)^2 \right| \\ &= \left| \frac{Q_{f,2k}(z)}{\|z\|^{2k}} - \lambda \left(\frac{Q_{f,k}(z)}{\|z\|^k} \right)^2 \right| \\ &\leq \sup_{z \in \mathbb{B}^n} \frac{|Q_{f,2k}(z) - \lambda(Q_{f,k}(z))^2|}{\|z\|^{2k}} = \mu_{\mathbb{B}^n}(Q_{f,2k} - \lambda(Q_{f,k})^2) \\ &\leq \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}. \end{aligned}$$

It remains to show the sharpness of the estimate (3.4).

To do it, observe first that there exist points $z^0 \in \mathbb{B}^n \setminus \{0\}$ such that $|I(\frac{z^0}{\|z^0\|})| = 1$. It follows from the maximum principle for the modulus of holomorphic functions of several complex variables, because $\|I\| = \mu_{\mathbb{B}^n}(I) = 1$. We will show that the equality in (3.4) in such points are attained by the mappings $\tilde{F}, \hat{F}$ of the form (3.2), where $f = \tilde{f}, f = \hat{f}$ are defined in $\mathcal{G} = \mathbb{B}^n$ by (2.9), (2.11) in the cases $|2+k-4\lambda| \geq k$ and $|2+k-4\lambda| < k$, respectively. The mappings $\tilde{F}, \hat{F}$ belong to $\mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, by the enumerated above fact 2, because $\tilde{f}, \hat{f} \in \mathcal{F}_{0,k}(\mathbb{B}^n, \mathbb{C})$. Also $\tilde{F}, \hat{F} \in S^k(\mathbb{B}^n)$, by the enumerated above fact 1 and the relations $\tilde{f}, \hat{f} \in \mathcal{M}_{\mathbb{B}^n}^k$.

First, we assume that $|2+k-4\lambda| \geq k$, i.e., $f = \tilde{f}$. It is easy to check that in this case

$$Q_{f,k} = Q_{\tilde{f},k} = \frac{2}{k} I^k, \quad Q_{f,2k} = Q_{\tilde{f},2k} = \frac{k+2}{k^2} (I^k)^2.$$

For $z = z^0$ and $F = \tilde{F}$, i.e., $(f = \tilde{f})$, we obtain (for the first below equality see the previous part of the proof)

$$\begin{aligned}
& \left| \frac{T_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}} - \lambda \left(\frac{T_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}} \right)^2 \right| \\
&= \frac{1}{\|z\|^{2k}} \left| Q_{f,2k}(z) - \lambda (Q_{f,2k}(z))^2 \right| \\
&= \frac{1}{\|z\|^{2k}} \left| \frac{k+2}{k^2} (I^k(z))^2 - \lambda \frac{4}{k^2} (I^k(z))^2 \right| = \left| \left(I^k\left(\frac{z}{\|z\|}\right) \right)^2 \right| \left| \frac{k+2}{k^2} - \lambda \frac{4}{k^2} \right| \\
&= \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}.
\end{aligned}$$

Now, we assume that $|2+k-4\lambda| < k$, i.e., $f = \hat{f}$. In this case it is easy to check that

$$Q_{f,k} = Q_{\hat{f},k} = 0, \quad Q_{f,2k} = Q_{\hat{f},2k} = \frac{1}{k} I^{2k}.$$

For $z = z^0$ and $F = \hat{F}$, i.e., $(f = \hat{f})$, we obtain similarly

$$\begin{aligned}
& \left| \frac{T_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}} - \lambda \left(\frac{T_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}} \right)^2 \right| = \frac{1}{\|z\|^{2k}} \left| \frac{1}{k} I^{2k}(z) \right| \\
&= \frac{1}{k} \left| I^{2k}\left(\frac{z}{\|z\|}\right) \right| = \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\}.
\end{aligned}$$

□

The identity (3.3) and Theorem 3.1 implies the following result of Fekete–Szegő type for starlike biholomorphic mappings in $\mathbb{C}^n$.

Theorem 3.2 *For mappings $F \in \tilde{\mathcal{S}}^*(\mathbb{B}^n) \cap \mathcal{F}_{1,k}(\mathbb{B}^n, \mathbb{C}^n)$, $k \in \mathbb{N}_2$, points $z \in \mathbb{B}^n \setminus \{0\}$ and parameter $\lambda \in \mathbb{C}$, there holds the following sharp estimate*

$$\left| \frac{T_z(D^{2k+1}F(0)(z^{2k+1}))}{(2k+1)!\|z\|^{2k+1}} - \lambda \left(\frac{T_z(D^{k+1}F(0)(z^{k+1}))}{(k+1)!\|z\|^{k+1}} \right)^2 \right| \leq \frac{1}{k} \max \left\{ 1, \frac{|2+k-4\lambda|}{k} \right\},$$

where $T_z \in (\mathbb{C}^n)^*$, $z \in \mathbb{B}^n \setminus \{0\}$, is arbitrary functional satisfying the conditions: $\|T_z\| = 1$, $T_z(z) = \|z\|$.

In the other way a similar theorem was proved by Xu [30].

4 Final remarks

It is possible to allow also $k = 1$ in definition of (j, k) -symmetrical, $j \in \mathbb{Z}$, functions, from $\mathcal{F}(\mathcal{G}, \mathbb{C}^m)$. Then $\varepsilon = 1$ and we should take the convention $\mathcal{F}_{j,1}(\mathcal{G}, \mathbb{C}^m) = \mathcal{F}(\mathcal{G}, \mathbb{C}^m)$ for $j \in \mathbb{Z}$. Consequently, in the case $m = 1$

$$\begin{aligned}\mathcal{N}_G \cap \mathcal{F}_{0,1}(\mathcal{G}, \mathbb{C}) &= \mathcal{N}_G, \\ \mathcal{M}_G \cap \mathcal{F}_{0,1}(\mathcal{G}, \mathbb{C}) &= \mathcal{M}_G = \mathcal{M}_G^1 = \mathcal{M}_G^1 \cap \mathcal{F}_{0,1}(\mathcal{G}, \mathbb{C}),\end{aligned}$$

while in the case $m = n$

$$\tilde{S}^*(\mathbb{B}^n) \cap \mathcal{F}_{1,1}(\mathbb{B}^n, \mathbb{C}^n) = \tilde{S}^*(\mathbb{B}^n) = \tilde{S}^1(\mathbb{B}^n) = \tilde{S}^1(\mathbb{B}^n) \cap \mathcal{F}_{1,1}(\mathbb{B}^n, \mathbb{C}^n).$$

Therefore, for $\lambda \in \mathbb{C}$, we obtain the following sharp estimates

$$\begin{aligned}\mu_G\left(Q_{f,2} - \lambda(Q_{f,1})^2\right) &\leq \frac{1}{3} \max\{1, 3|\lambda - 1|\}, \\ \mu_G\left(Q_{f,2} - \lambda(Q_{f,1})^2\right) &\leq \max\{1, |3 - 4\lambda|\}, \\ \left|\frac{T_z(D^3F(0)(z^3))}{3!\|z\|^3} - \lambda\left(\frac{T_z(D^2F(0)(z^2))}{2!\|z\|^2}\right)^2\right| &\leq \max\{1, |3 - 4\lambda|\}, z \in \mathbb{B}^n \setminus \{0\},\end{aligned}$$

with $T_z \in (\mathbb{C}^n)^*$, $\|T_z\| = 1$, $T_z(z) = \|z\|$, for the families $\mathcal{N}_G, \mathcal{M}_G = \mathcal{M}_G^1, \tilde{S}^*(\mathbb{B}^n) = \tilde{S}^1(\mathbb{B}^n)$, respectively.

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References

1. Alexander, J.W.: Functions which map the interior of the unit circle upon simple regions. *Ann. Math.* **17**, 12–22 (1915)
2. Bavin, I.I.: A Classes of regular functions in the case of several complex variables and extremal problems in these classes. *Moskov Obl. Ped. Inst., Moscov*, pp. 1–99 (1976) (**In Russian**)
3. Długosz, R.: Embedding theorems for holomorphic functions of several complex variables. *J. Appl. Anal.* **19**, 153–165 (2013)
4. Długosz, R., Leś, E.: Embedding theorems and extreme problems for holomorphic functions on circular domains of $\mathbb{C}^n$. *Complex Variables Elliptic Equ.* **59**, 883–899 (2014)
5. Długosz, R., Liczberski, P.: An application of hypergeometric functions to a construction in several complex variables. *J. d'Analyse Math.* **137**, 707–721 (2019)
6. Długosz, R., Liczberski, P.: Relations among starlikeness, convexity and k-fold symmetry of locally biholomorphic mappings in $\mathbb{C}^n$. *J. Math. Anal. Appl.* **450**, 169–179 (2017)
7. Długosz, R., Liczberski, P.: Some results of Fekete–Szegő type. Some holomorphic Functions of several complex variables. *Symmetry* **12**(1707), 1–10 (2020)
8. Fekete, M., Szegő, G.: Eine Bemerkung über ungerade schlichte Funktionen. *J. Lond. Math. Soc.* **8**, 85–89 (1933)
9. Fukui, S.: On the estimates of coefficients of analytic functions. *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A* **10**, 216–218 (1969)
10. Golusin, G.M.: Estimates for analytic functions with bounded mean of the modulus. *Trav. Inst. Math. Stekloff* **18**, 3–88 (1946)
11. Graham, I., Kohr, G.: *Geometric Function Theory in One and Higher Dimensions*. Marcel Dekker Inc, New York (2003)

12. Hamada, H., Honda, T., Kohr, G.: Parabolic starlike mappings in several complex variables. *Manuscr. Math.* **123**, 301–324 (2007)
13. Higuchi, T.: On coefficients of holomorphic functions of several complex variables. *Sci. Rep. Tokyo Kyoiku Daigaku* **8**, 251–258 (1965)
14. Keogh, F.R., Merkes, E.P.: A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **20**, 8–12 (1969)
15. Kikuchi, K.: Starlike and convex mappings in several complex variables. *Pac. J. Math.* **44**, 569–580 (1973)
16. Kohr, G., Liczberski, P.: *Univalent Mappings of Several complex Variables*. Cluj University Press, Cluj-Napoca (1998)
17. Krzyż, J.G.: *Problems in Complex Variable Theory*. Elsevier, New York (1971)
18. Liczberski, P.: On the subordination of holomorphic mappings in $\mathbb{C}^n$. *Demonstr. Math.* **2**, 293–301 (1986)
19. Liczberski, P., Połubiński, J.: On (j, k) -symmetrical functions. *Math. Bohem.* **120**, 13–28 (1995)
20. Liczberski, P., Połubiński, J.: Functions (j, k) -symmetrical and functional equations with iterates of the unknown function. *Publ. Math. Debr.* **60**, 291–305 (2002)
21. Liczberski, P., Połubiński, J.: Symmetrical series expansion of Complex valued functions. *N. Z. J. Math.* **27**, 245–253 (1998)
22. Liczberski, P., Połubiński, J.: A uniqueness theorem of Cartan–Gutzmer type for holomorphic mappings in $\mathbb{C}^n$. *Ann. Pol. Math.* **79**, 121–127 (2002)
23. Matsuno, T.: Star-like and convex-like theorems in the complex space. *Sci. Rep. Tokyo Kyoiku Daj-gaku Sect. A* **5**, 88–95 (1955)
24. Michiwaki, Y.: Note on some coefficients in a starlike functions of two complex variables. *Res. Rep. Nagaoka Tech. College* **1**, 151–153 (1963)
25. Pfaltzgraff, J.A., Suffridge, T.J.: An extension theorem and linear invariant families generated by star-like maps. *Ann. UMCS Sect. Math.* **53**, 193–207 (1999)
26. Rudin, W.: *Functional Analysis*. McGraw-Hill Inc., New York (1991)
27. Sakaguchi, K.: On certain univalent mapping. *J. Math. Soc. Jpn.* **11**, 72–75 (1959)
28. Suffridge, T.J.: The principle of subordination applied to functions of several variables. *Pac. J. Math.* **33**, 241–248 (1970)
29. Temljakov, A.: Integral representation of functions of two complex variables. *Izv. Acad. Sci. SSSR Ser. Math.* **21**, 89–92 (1957)
30. Xu, Q.H.: A refinement of the coefficient inequalities for a subclass of starlike mappings in several complex variables. *Results Math.* **74**, 156 (2019)

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