The second Hankel determinant for strongly convex and Ozaki close-to-convex functions

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Abstract

Let f be analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be the subclass of normalized univalent functions given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$. We give sharp bounds for the modulus of the second Hankel determinant $H_2(2)(f) = a_2 a_4 - a_3^2$ for the subclass $\mathcal{F}_O(\lambda, \beta)$ of strongly Ozaki close-to-convex functions, where $1/2 \leq \lambda \leq 1$, and $0 < \beta \leq 1$. Sharp bounds are also given for $|H_2(2)(f^{-1})|$, where f^{-1} is the inverse function of f . The results settle an invariance property of $|H_2(2)(f)|$ and $|H_2(2)(f^{-1})|$ for strongly convex functions.

Keywords Univalent function · Ozaki close-to-convex function · Convex function · Hankel determinant

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Then, for $z \in \mathbb{D}$, $f \in \mathcal{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Let $\mathcal{S}$ denote the subclass of all univalent (i.e., one-to-one) functions in $\mathcal{A}$.

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Denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of starlike functions, i.e., functions f which map $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then, it is well known that a function $f \in \mathcal{S}^*$ if, and only if, for $z \in \mathbb{D}$,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0,$$

and that $\mathcal{S}^* \subset \mathcal{S}$.

Next denote by $\mathcal{K}$ the subclass of $\mathcal{S}$ consisting of functions which are close-to-convex, i.e., functions f which map $\mathbb{D}$ onto a close-to-convex domain, if, and only if, there exist $0 \leq \delta \leq 2\pi$ and $g \in \mathcal{S}^*$ such that for $z \in \mathbb{D}$,

$$\operatorname{Re} \left\{ e^{i\delta} \frac{zf'(z)}{g(z)} \right\} > 0. \quad (2)$$

Again it is well known that $\mathcal{K} \subset \mathcal{S}$.

The class $\mathcal{C}(\alpha)$ for $0 \leq \alpha < 1$ of convex functions of order α consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

for $z \in \mathbb{D}$, is also well known and has been widely studied. When $\alpha = 0$, we obtain the class $\mathcal{C}$ of convex functions.

Little attention has been given to the case when $\alpha < 0$, but several authors have considered the class $\mathcal{C}(-1/2)$, consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\frac{1}{2}$$

for $z \in \mathbb{D}$, whose members are known to be close-to-convex, and so univalent.

The class $\mathcal{F}_O(\lambda)$, defined for $1/2 \leq \lambda \leq 1$ by

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2} - \lambda \quad (3)$$

for $z \in \mathbb{D}$, was formally introduced in [1] and is also known to be a subclass of the close-convex functions. More information concerning the coefficients of functions in $\mathcal{F}_O(\lambda)$ (called Ozaki close-to-convex functions), has also been found by Allu, Lecko and Thomas (to appear). We note that clearly $\mathcal{F}_O(1/2) = \mathcal{C}$, and $\mathcal{F}_O(1) = \mathcal{C}(-1/2)$.

Also in [1], the following notion of strongly Ozaki functions was introduced, and some basic properties were obtained.

Definition 1 Let $0 < \beta \leq 1$ and $1/2 \leq \lambda \leq 1$. Then, $f \in \mathcal{A}$ is called a strongly Ozaki close-to-convex if, and only if, for $z \in \mathbb{D}$,

$$\left| \arg \left(\frac{2\lambda - 1}{2\lambda + 1} + \frac{2}{2\lambda + 1} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right) \right| < \frac{\beta\pi}{2}. \quad (4)$$

We denote this class of functions by $\mathcal{F}_O(\lambda, \beta)$, noting that when $\beta = 1$ this reduces to (3), and when $\lambda = 1/2$ we obtain the class $\mathcal{C}^\beta$ of strongly convex functions considered in [9].

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of function $f \in \mathcal{A}$ of the form (1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Hankel matrices are used in the theory of Markov processes, of non-stationary signals, in Hamburger moment problem, and many other issues in both pure mathematics and technical applications.

To find the upper bound of $H_{q,n}(f)$ for the whole class $\mathcal{S}$ of univalent functions as well as for its subclasses is an interesting problem to study. For the class $\mathcal{S}$, a basic result was shown by Pommerenke [7], who found the growth of $H_{q,n}(f)$ dependent on q and n . In recent years there has been a great deal of attention devoted to finding bounds for the modulus of the second Hankel determinant $H_2(2)(f) = a_2a_4 - a_3^2$, when f belongs to various subclasses of $\mathcal{A}$. Most authors have used the method introduced in [4], where the sharp bound $|H_2(2)(f)| \leq 1$ was found for $f \in \mathcal{S}^*$. However, finding the sharp bound for $|H_2(2)(f)|$ when $f \in \mathcal{K}$ and even when $\delta = 0$ remains an open problem.

In [2], a sharp bound was obtained for $|H_2(2)(f)|$ when $f \in \mathcal{F}_O(\lambda)$. The main purpose of this paper is to give the sharp bounds for $|H_2(2)(f)|$, when $f \in \mathcal{F}_O(\lambda, \beta)$, together with sharp bounds for $|H_2(2)(f^{-1})|$, where f^{-1} is the inverse function of f .

2 Preliminary lemmas

Denote by $\mathcal{P}$, the class of analytic functions p in $\mathbb{D}$ with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \quad (5)$$

We will use the following properties for the coefficients of functions $\mathcal{P}$, given by (5).

Lemma 1 [5, 6] *If $p \in \mathcal{P}$ and is given by (5) with $c_1 \geq 0$, then for some complex valued ζ with $|\zeta| \leq 1$, and some complex valued η with $|\eta| \leq 1$,*

$$\begin{aligned} 2c_2 &= c_1^2 + \zeta(4 - c_1^2), \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta. \end{aligned} \quad (6)$$

Lemma 2 [3] *Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and for real numbers A, B, C , let*

$$Y(A, B, C) := \max \left\{ |A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}} \right\}. \quad (7)$$

If $AC \geq 0$, then

$$Y(A, B, C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \frac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

If $AC < 0$, then

$$Y(A, B, C) = \begin{cases} 1 - |A| + \frac{B^2}{4(1 - |C|)}, & -4AC(C^{-2} - 1) \leq B^2 \wedge |B| < 2(1 - |C|), \\ 1 + |A| + \frac{B^2}{4(1 + |C|)}, & B^2 < \min\{4(1 + |C|)^2, -4AC(C^{-2} - 1)\}, \\ R(A, B, C), & \text{otherwise,} \end{cases} \quad (8)$$

where

$$R(A, B, C) := \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}}, & \text{otherwise.} \end{cases} \quad (9)$$

We also note that from (4), we can write, for $z \in \mathbb{D}$,

$$1 + \frac{zf''(z)}{f'(z)} = \left(\frac{1}{2} + \lambda\right)p(z)^\beta + \frac{1}{2} - \lambda \quad (10)$$

for some $p \in \mathcal{P}$ and so equating coefficients we have

$$\begin{aligned} a_2 &= \frac{\beta}{4}(1 + 2\lambda)c_1, \\ a_3 &= \frac{\beta}{12}(1 + 2\lambda)\left(c_2 - \frac{1}{2}(1 - 2\beta - 2\beta\lambda)c_1^2\right), \\ a_4 &= \frac{\beta}{24}(1 + 2\lambda)\left(c_3 - \frac{1}{4}(4 - 7\beta - 6\beta\lambda)c_1c_2\right. \\ &\quad \left.+ \frac{1}{24}(8 - 21\beta + 16\beta^2 - 18\beta\lambda + 30\beta^2\lambda + 12\beta^2\lambda^2)c_1^3\right). \end{aligned} \quad (11)$$

3 $H_2(2)(f)$ for strongly Ozaki functions

We prove the following.

Theorem 1 Let $\beta \in (0, 1]$ and $\lambda \in [1/2, 1]$. If $f \in \mathcal{F}_O(\lambda, \beta)$, then

$$|H_2(2)(f)| \leq \begin{cases} \frac{1}{36}\beta^2(1 + 2\lambda)^2, & (5 + 2\lambda)\beta \leq 2, \\ \frac{\beta^2(1 + 2\lambda)^2(68 + 60\beta + 25\beta^2 + 24\beta\lambda - 12\beta^2\lambda - 60\beta^2\lambda^2)}{576(4 + 5\beta + 2\beta\lambda - 2\beta^2\lambda - 4\beta^2\lambda^2)}, & \text{otherwise.} \end{cases}$$

The inequalities are sharp.

Proof From (11) we have

$$a_2 a_4 - a_3^2 = -\frac{1}{2304} \beta^2 (1 + 2\lambda)^2 \Psi, \quad (12)$$

where

$$\begin{aligned} \Psi := & 16c_2^2 - 24c_1c_3 - 2c_1^2c_2[-4 + \beta(5 + 2\lambda)] \\ & + c_1^4[-4 + 2\beta^2\lambda(1 + 2\lambda) + \beta(5 + 2\lambda)]. \end{aligned}$$

Since both the class $\mathcal{F}_O(\lambda, \beta)$ and the functional are rotationally invariant, without loss of generality we may assume that $c = c_1 \in [0, 2]$. By Lemma 1, we obtain

$$\begin{aligned} \Psi = & 2c^4[-1 + \beta^2\lambda(1 + 2\lambda)] - c^2(4 - c^2)\beta(5 + 2\lambda)\zeta \\ & + 2(32 - 4c^2 - c^4)\zeta^2 - 12c(4 - c^2)(1 - |\zeta|^2)\eta, \end{aligned} \quad (13)$$

for some $\zeta, \eta \in \overline{\mathbb{D}}$.

Since $|\eta| \leq 1$ and $|\zeta| \leq 1$, by applying the triangle inequality to (13), we have

$$\begin{aligned} |\Psi| & \leq 2 \left| -1 + \beta^2\lambda(1 + 2\lambda) \right| c^4 + \beta(5 + 2\lambda)c^2(4 - c^2)|\zeta| \\ & \quad + 2(4 - c^2)(4 - c)(2 - c)|\zeta|^2 + 12c(4 - c^2) \\ & \leq 2 \left| -1 + \beta^2\lambda(1 + 2\lambda) \right| c^4 + \beta(5 + 2\lambda)c^2(4 - c^2) \\ & \quad + 2(4 - c^2)(4 - c)(2 - c) + 12c(4 - c^2) = G_1(c^2), \end{aligned} \quad (14)$$

where for $x \in [0, 4]$,

$$G_1(x) := 64 + 2b_1x + b_2x^2$$

with

$$b_1 := 2[\beta(5 + 2\lambda) - 2] \quad \text{and} \quad b_2 := 2 \left| \beta^2\lambda(1 + 2\lambda) - 1 \right| - \beta(5 + 2\lambda) - 2.$$

I. Consider first the case $(5 + 2\lambda)\beta \leq 2$.

Since $b_1 \leq 0$, and since $\beta \leq 2/(5 + 2\lambda)$ and $1/2 \leq \lambda \leq 1$, the inequalities

$$\beta^2\lambda(1 + 2\lambda) \leq \frac{4\lambda(1 + 2\lambda)}{(5 + 2\lambda)^2} < 1$$

hold. Thus, we get

$$b_2 = -2[\beta^2\lambda(1 + 2\lambda) - 1] - \beta(5 + 2\lambda) - 2 = -\beta[2\beta\lambda(1 + 2\lambda) + 5 + 2\lambda] < 0.$$

Hence, $G'_1(x) = 2b_1 + 2b_2x \leq 0$ holds for all $x \in [0, 4]$, and so G_1 is decreasing on $[0, 4]$. Since $c \in [0, 2]$, it follows from (14) that

$$|\Psi| \leq G_1(c^2) \leq G_1(0) = 64,$$

which establishes the first inequality in Theorem 1.

We next divide the case $(5 + 2\lambda)\beta > 2$ into two cases.

II(a). First suppose that $\beta^2\lambda(1+2\lambda) \geq 1$. Then, we see that $b_1 > 0$ and $b_2 < 0$. Indeed,

$$b_1 + 4 = 2\beta(5+2\lambda) \geq \frac{2(5+2\lambda)}{\sqrt{\lambda(1+2\lambda)}} \geq \frac{14}{\sqrt{3}} > 4$$

and

$$b_2 = 2\beta^2\lambda(1+2\lambda) - \beta(5+2\lambda) - 4 \leq 2 - \beta(5+2\lambda) < 0,$$

since $\beta^2\lambda(1+2\lambda) \leq 3$ and $(5+2\lambda)\beta > 2$. Moreover, by putting

$$\tau := -\frac{b_1}{b_2}, \quad (15)$$

we see that

$$0 < \tau < 4 \quad (16)$$

holds, since

$$b_1 + 4b_2 = 2\beta[4\beta\lambda(1+2\lambda) - (5+2\lambda)] - 20 \leq 2\beta(8\lambda^2 + 2\lambda - 5) - 20 \leq -10 < 0.$$

Therefore, $G'_1(\tau) = 0$ and $G''_1(\tau) < 0$ imply that for $x \in [0, 4]$,

$$G_1(x) \leq G_1(\tau) = \frac{4[-68 - 12(5+2\lambda)\beta + (-25 + 12\lambda + 60\lambda^2)\beta^2]}{-4 - (5+2\lambda)\beta + 2\lambda(1+2\lambda)\beta^2},$$

and so from (12) and (14), we obtain the second inequality in Theorem 1 in this case.

II(b). Now assume that $\beta^2\lambda(1+2\lambda) < 1$ and $(5+2\lambda)\beta > 2$. By (13), when $c = 2$, we have

$$|\Psi| \leq 32| -1 + \beta^2\lambda(1+2\lambda) | \leq 64,$$

and, when $c = 0$, since $\zeta \in \overline{\mathbb{D}}$, we also have

$$|\Psi| = 64|\zeta|^2 \leq 64.$$

Now let $c \in (0, 2)$. Then, since $\eta \in \overline{\mathbb{D}}$, we obtain

$$|\Psi| \leq 12c(4 - c^2)\Phi(A, B, C),$$

where

$$\Phi(A, B, C) := \left| A + B\zeta + C\zeta^2 \right| + 1 - |\zeta|^2,$$

with

$$A := \frac{c^3[-1 + \beta^2\lambda(1+2\lambda)]}{6(4 - c^2)}, \quad B := -\frac{1}{12}\beta(5+2\lambda)c, \quad C := \frac{8 + c^2}{6c}.$$

Noting that $AC < 0$ in this case, we now apply Lemma 2.

Simple calculations show that the first two alternatives when $AC < 0$ are not satisfied, and so we must consider $R(A, B, C)$.

We first show that $|C|(|B| + 4|A|) > |AB|$ holds for all $c \in (0, 2)$. A calculation shows that

$$|AB| - |C|(|B| + 4|A|) = \frac{-1}{72(4 - c^2)} H_1(c^2),$$

where for $x \in [0, 4]$,

$$H_1(x) := k_0 + 2k_1x + k_2x^2$$

with

$$k_0 := 32\beta(5 + 2\lambda), \quad k_1 := -2[-16 + 16\beta^2\lambda(1 + 2\lambda) + \beta(5 + 2\lambda)]$$

and

$$k_2 := 8 - 8\beta^2\lambda(1 + 2\lambda) - 2\beta(5 + 2\lambda) + \beta^3\lambda(5 + 12\lambda + 4\lambda^2).$$

Note that $k_2 \geq 0$ implies $k_1 > 0$. Indeed, since $\beta^2\lambda(1 + 2\lambda) < 1$, we have $3 - 2\beta^2\lambda(1 + 2\lambda) > 0$. Therefore, if $k_2 \geq 0$, then

$$k_1 = 2\beta(5 + 2\lambda)[3 - 2\beta^2\lambda(1 + 2\lambda)] + 4k_2 > 0.$$

Thus, when $k_2 \geq 0$, from $k_0 > 0$ and $k_1 > 0$, we have $H_1(x) > 0$, for $x \in [0, 4]$.

When $k_2 < 0$, since $H_1'' \equiv 2k_2 < 0$, we get for $x \in [0, 4]$,

$$H_1(x) \geq \min\{H_1(0), H_1(4)\}. \quad (17)$$

Next fix $\lambda \in [1/2, 1]$. Let $I := \left(2/(5 + 2\lambda), 1/\sqrt{\lambda(1 + 2\lambda)}\right)$ and $\varphi : I \rightarrow \mathbb{R}$ be defined by

$$\varphi(x) := 24 - (5 + 2\lambda)x - 24\lambda(1 + 2\lambda)x^2 + \lambda(1 + 2\lambda)(5 + 2\lambda)x^3.$$

Since

$$\begin{aligned} \varphi'(x) &= -(5 + 2\lambda) - 48\lambda(1 + 2\lambda)x + 3\lambda(1 + 2\lambda)(5 + 2\lambda)x^2 \\ &< -48\lambda(1 + 2\lambda)x + 2(5 + 2\lambda) \\ &< \frac{2(25 - 28\lambda - 92\lambda^2)}{5 + 2\lambda} < 0, \end{aligned}$$

so φ is decreasing on I and for $x \in I$,

$$\varphi(x) > \varphi\left(\frac{1}{\sqrt{\lambda(1 + 2\lambda)}}\right) = 0.$$

Thus, $H_1(4) = k_0 + 8k_1 + 16k_2 = 16\varphi(\beta) > 0$. Also, since $H_1(0) = k_0 > 0$, by (17), $H_1(x) > 0$ for all $x \in [0, 4]$, which implies that $|C|(|B| + 4|A|) > |AB|$.

Next a calculation gives

$$|AB| - |C|(|B| - 4|A|) = \frac{-1}{72(4 - c^2)} H_2(c^2),$$

where for $x \in [0, 4]$,

$$H_2(x) := l_0 + 2l_1x + l_2x^2$$

with

$$l_0 := 32\beta(5 + 2\lambda), \quad l_1 := 2[-16 + 16\beta^2\lambda(1 + 2\lambda) - \beta(5 + 2\lambda)]$$

and

$$l_2 := -8 + 8\beta^2\lambda(1 + 2\lambda) - 2\beta(5 + 2\lambda) + \beta^3\lambda(5 + 12\lambda + 4\lambda^2).$$

Note that $l_2 < 0$, and let

$$\xi_i := \frac{-l_1 + (-1)^{i+1}\sqrt{l_1^2 - l_0l_2}}{l_2}, \quad i = 1, 2,$$

be the zeroes of H_2 . Since $l_2 < 0$, so $\xi_1 < \xi_2$. Moreover, since $l_0 > 0$, we have $-l_1 + \sqrt{l_1^2 - l_0l_2} > 0$ and $\xi_1 < 0$. Also $\xi_2 < 4$, since this is equivalent to

$$l_0l_2 + 8l_1l_2 + 16l_2^2 > 0, \quad (18)$$

which is true. In fact, a computation shows that

$$l_0l_2 + 8l_1l_2 + 16l_2^2 = 16(8 + t)[t(2 - u) + 8(1 - u)](1 - u),$$

where $t = \beta(5 + 2\lambda)$ and $u = \beta^2\lambda(1 + 2\lambda)$. From the inequalities $t > 0$ and $0 < u < 1$, we obtain the inequality (18). Therefore, since $l_2 < 0$, $H_2(x) \geq 0$ holds for $x \in (0, \xi_2]$, and $H_2(x) \leq 0$ holds for $x \in [\xi_2, 4)$. Thus, $|AB| \leq |C|(|B| - 4|A|)$ holds when $c \in (0, \sqrt{\xi_2}]$.

Thus, when $c \in (0, \sqrt{\xi_2}]$, Lemma 2 gives

$$|\Psi| \leq 12c(4 - c^2)(-|A| + |B| + |C|) = G_2(c^2), \quad (19)$$

where for $x \in [0, 4]$,

$$G_2(x) := 64 + 2d_1x + d_2x^2 \quad (20)$$

with

$$d_1 := 2[-2 + (5 + 2\lambda)\beta] \quad \text{and} \quad d_2 := -4 - (5 + 2\lambda)\beta + 2\lambda(1 + 2\lambda)\beta^2.$$

Note that $d_1 > 0$ and $d_2 < 0$. We also note that $G'_2(x) = 0$ holds only for

$$x = -\frac{d_1}{d_2} =: \tau. \quad (21)$$

Clearly, $\tau > 0$. Now we claim $\tau < \xi_2$. Since

$$H_2(\tau) = \frac{1}{d_2^2}(l_0d_2^2 - 2l_1d_1d_2 + l_2d_1^2),$$

the inequality $\tau < \xi_2$ is equivalent to

$$l_0d_2^2 - 2l_1d_1d_2 + l_2d_1^2 > 0. \quad (22)$$

Moreover, by putting $t = \beta(5 + 2\lambda)$ and $u = \beta^2\lambda(1 + 2\lambda)$ again, we get

$$\begin{aligned} & l_0 d_2^2 - 2l_1 d_1 d_2 + l_2 d_1^2 \\ &= 4[t^3(4 + u) + 4t^2(7 + 2u) + 32(7 - 11u + 4u^2) + 4t(26 - 9u - 8u^2)]. \end{aligned}$$

Since $t > 0$ and $0 < u < 1$, all terms in the right side in the above are positive, and the inequality (22) follows. So, $\tau < \xi_2$ and, since $d_2 < 0$, the function G_2 has its maximum at τ in $(0, \xi_2]$. Therefore, by (19), we have

$$|\Psi| \leq G_2(\tau) = \frac{4[-68 - 12(5 + 2\lambda)\beta + (-25 + 12\lambda + 60\lambda^2)\beta^2]}{-4 - (5 + 2\lambda)\beta + 2\lambda(1 + 2\lambda)\beta^2}.$$

Thus, using (12), the second inequality in Theorem 1 holds.

We are therefore left to consider the interval $c \in [\sqrt{\xi_2}, 2)$, where Lemma 2 gives

$$|\Psi| \leq 12c(4 - c^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = 2g_1(c^2)\sqrt{g_2(c^2)}, \quad (23)$$

and where for $x \in [0, 4]$,

$$g_1(x) := 32 - 4x - \lambda(1 + 2\lambda)\beta^2 x^2,$$

and

$$g_2(x) := 1 + \frac{(5 + 2\lambda)^2 \beta^2 (4 - x)}{16[1 - \lambda\beta^2(1 + 2\lambda)](8 + x)}.$$

It is easily checked that g_1 and g_2 are both decreasing on $[\xi_2, 2)$. Hence, from (23), we have

$$|\Psi| \leq 2g_1(\xi_2)\sqrt{g_2(\xi_2)} = G_2(\xi_2), \quad (24)$$

where G_2 is defined by (20). Since $G_2(\xi_2) \leq G_2(\tau)$, it follows from (12) and (24) that the second inequality in Theorem 1 holds once more.

In order to show that the inequalities are sharp, first let f_1 be defined for $z \in \mathbb{D}$ by

$$1 + \frac{zf_1''(z)}{f_1'(z)} = \left(\frac{1}{2} + \lambda\right)p_1^\beta(z) + \frac{1}{2} - \lambda,$$

where $p_1(z) := (1 + z^2)/(1 - z^2)$ for $z \in \mathbb{D}$. Then, $f_1 \in \mathcal{F}_O(\lambda, \beta)$ with

$$f_1(z) = z + \frac{1}{6}(1 + 2\lambda)\beta z^3 + \dots$$

for $z \in \mathbb{D}$. Thus, the first bound in Theorem 1 is sharp when $(5 + 2\lambda)\beta \leq 2$.

Next let f_2 be defined for $z \in \mathbb{D}$ by

$$1 + \frac{zf_2''(z)}{f_2'(z)} = \left(\frac{1}{2} + \lambda\right)p_2^\beta(z) + \frac{1}{2} - \lambda,$$

where

$$p_2(z) := \frac{1 - z^2}{1 - bz + z^2}, \quad z \in \mathbb{D},$$

with

$$b := \sqrt{\tau} = \sqrt{\frac{2[2 - \beta(5 + 2\lambda)]}{2\beta^2\lambda(1 + 2\lambda) - \beta(5 + 2\lambda) - 4}}. \quad (25)$$

Here τ is defined by (15) or by (21), and in the first case $0 < \tau < 4$ in view of (16), or in the second case, $0 < \tau < \xi_2 < 4$ as was shown in II(b). Therefore, $b \in (0, 2)$ and $p_2 \in \mathcal{P}$. Thus, $f_2 \in \mathcal{F}_O(\lambda, \beta)$ with

$$a_2 = \frac{1}{4}(1 + 2\lambda)\beta b, \quad a_3 = \frac{1}{24}(1 + 2\lambda)\beta[-4 + (1 + 2(1 + \lambda)\beta)b^2],$$

and

$$a_4 = \frac{1}{576}(1 + 2\lambda)\beta b[2(8 + 15\lambda + 6\lambda^2)\beta^2 b^2 - 3\beta(7 + 6\lambda)(4 - \tau^2) - 8(3 - b^2)].$$

Hence,

$$\begin{aligned} H_2(2)(f) = & -\frac{\beta^2(1 + 2\lambda)^2}{2304} [64 + 4(-2 + (5 + 2\lambda)\beta)b^2 \\ & + (-4 - (5 + 2\lambda)\beta + 2\lambda(1 + 2\lambda)\beta^2)b^4]. \end{aligned} \quad (26)$$

Finally, substituting b from (25) into (26), shows that equality is attained in the second inequality in Theorem 1 when $(5 + 2\lambda)\beta > 2$. $\square$

4 $H_2(2)(f^{-1})$ for strongly Ozaki functions

Since each class $\mathcal{F}_O(\lambda, \beta)$ is compact and every $f \in \mathcal{F}_O(\lambda, \beta)$ is invertible, there exists $r_0 \in (0, 1)$ such that $\mathbb{D}_{r_0} \subset f(\mathbb{D})$ for every $f \in \mathcal{F}_O(\lambda, \beta)$, where $\mathbb{D}_{r_0} := \{z \in \mathbb{C} : |z| < r_0\}$. Therefore, for $w \in \mathbb{D}_{r_0}$ the inverse function f^{-1} can be written as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \delta_n w^n. \quad (27)$$

Moreover, when $f \in \mathcal{F}_O(\lambda, \beta)$ is of the form (1), then the following relations hold (see e.g., [8])

$$\delta_2 = -a_2, \quad \delta_3 = 2a_2^2 - a_3, \quad \delta_4 = -5a_2^3 + 5a_2a_3 - a_4. \quad (28)$$

The following proposition will be used in our proof.

Proposition 1 Let $t \in (2, 11]$ and $u \in (1, 3]$. Define $H : [0, 4] \rightarrow \mathbb{R}$ by

$$H(x) := h_1(x)\sqrt{h_2(x)}, \quad (29)$$

where

$$h_1(x) := 64 - 8x + ux^2, \quad \text{and} \quad h_2(x) := 1 + \frac{t^2(4-x)}{8(2+u)(8+x)}.$$

Then, H is convex on $[0, 4]$.

Proof By differentiating H twice, we obtain

$$\begin{aligned} & (h_2(x))^{3/2} H''(x) \\ &= h_1''(x)(h_2(x))^2 + h_1'(x)h_2(x)h_2'(x) - \frac{1}{4}h_1(x)(h_2'(x))^2 + \frac{1}{2}h_1(x)h_2(x)h_2''(x) \\ &= \frac{G(x)}{32(2+u)^2(8+x)^4}, \end{aligned}$$

where

$$\begin{aligned} G(x) := & 64u(2+u)^2(8+x)^4 + t^4[1920 - 624x + u(1024 - 640x + 6x^2 + 14x^3 + x^4)] \\ & - 16t^2(8+x)[-768 + 2u(-448 + 48x + 15x^2 + x^3) \\ & + u^2(-256 + 48x + 15x^2 + x^3)]. \end{aligned}$$

We show that our assertion is true by proving that $G(x) \geq 0$ for $x \in [0, 4]$.

Let $x \in [0, 4]$, $u \in (1, 3]$ be fixed, and

$$F(s) := A_0 + A_1s + A_2s^2, \quad s \in [4, 121],$$

where

$$\begin{aligned} A_0 := & 64u(2+u)^2(8+x)^4, \\ A_1 := & -16(8+x)[-768 + 2u(-448 + 48x + 15x^2 + x^3) \\ & + u^2(-256 + 48x + 15x^2 + x^3)] \end{aligned}$$

and

$$A_2 = 1920 - 624x + u(1024 - 640x + 6x^2 + 14x^3 + x^4).$$

I. Consider first the case $A_2 \leq 0$. Then, we note that

$$\begin{aligned} F(4) = & 16[4u^3(8+x)^4 + 144(184 + 17x) \\ & + u(95232 + 32640x + 4806x^2 + 342x^3 + 9x^4) \\ & + 12u^2(6144 + 2688x + 456x^2 + 35x^3 + x^4)], \end{aligned}$$

and so clearly $F(4) > 0$.

Also when $s = 121$,

$$F(121) = k_0 + k_1u + k_2u^2 + k_3u^3,$$

where

$$\begin{aligned}k_0 &:= 17424(2296 - 439x) > 0, \\k_1 &:= 3(9972736 - 2866048x - 154782x^2 + 41370x^3 + 3675x^4), \\k_2 &:= 48(104448 + 5760x - 4728x^2 - 757x^3 - 35x^4), \quad k_3 := 64(8 + x)^4.\end{aligned}$$

Since $k_3 > 0$ and $1 < u \leq 3$, we have

$$F(121) \geq k_0 + k_1u + (k_2 + k_3)u^2,$$

and

$$k_2 + k_3 = 16(329728 + 25472x - 12648x^2 - 2143x^3 - 101x^4) > 0. \quad (30)$$

Thus, when $k_1 \geq 0$, it is clear that $F(121) > 0$.

If $k_1 < 0$, then since $3k_1 < k_1u$ we have

$$F(121) \geq k_0 + 3k_1 + (k_2 + k_3)u^2, \quad (31)$$

Now that

$$k_0 + 3k_1 = 9(14417792 - 3715952x - 154782x^2 + 41370x^3 + 3675x^4) > 0, \quad (32)$$

which, from (30), (31) and (32), implies that $F(121) > 0$. Thus, since $A_2 \leq 0$,

$$F(s) \geq \min\{F(4), F(121)\} > 0, \quad s \in [4, 121].$$

II. Next we consider the case $A_2 > 0$. In this case

$$F(s) \geq A_0 + (A_1 + 4A_2)s =: \tilde{F}(s), \quad s \in [4, 121]. \quad (33)$$

First note that

$$\begin{aligned}\tilde{F}(4) &= 16[4u^3(8 + x)^4 + 144(184 + 17x) \\&\quad + u(95232 + 32640x + 4806x^2 + 342x^3 + 9x^4) \\&\quad + 12u^2(6144 + 2688x + 456x^2 + 35x^3 + x^4)] > 0.\end{aligned}$$

Also when $s = 121$, we have

$$\tilde{F}(121) = q_0 + q_1u + q_2u^2 + q_3u^3,$$

where

$$\begin{aligned}q_0 &:= 69696(184 + 17x), \\q_1 &:= 12(1285120 + 38528x - 45774x^2 - 6174x^3 - 261x^4), \\q_2 &:= 48(104448 + 5760x - 4728x^2 - 757x^3 - 35x^4), \quad q_3 := 64(8 + x)^4.\end{aligned}$$

Since $q_3 > 0$ and $u > 1$, we obtain

$$\tilde{F}(121) > q_0 + q_1u + (q_2 + q_3)u^2,$$

and further since $q_0 > 0$, $q_1 > 0$, and $q_2 + q_3 > 0$, it follows that $\tilde{F}(121) > 0$. Hence, since the function $\tilde{F}$ is linear with respect to s , $\tilde{F}(4) > 0$ and $\tilde{F}(121) > 0$, we obtain

$$\tilde{F}(s) \geq \min\{\tilde{F}(4), \tilde{F}(121)\} > 0, \quad s \in [4, 121],$$

and so from (33) it follows that $F(s) > 0$ for $s \in [4, 121]$.

Finally, note that **I** and **II** implied that $F(s) > 0$ for $s \in (4, 121]$, $x \in [0, 4]$, and $u \in (1, 3]$, which therefore shows that $G(x) \geq 0$, and the proof of Proposition 1 is complete. $\square$

Theorem 2 Let $\beta \in (0, 1]$ and $\lambda \in [1/2, 1]$. If $f \in \mathcal{F}_O(\lambda, \beta)$, then

$$|H_2(2)(f^{-1})| \leq \begin{cases} \frac{\beta^2(1+2\lambda)^2}{36}, & (1+10\lambda)\beta \leq 2, \\ \frac{\beta^2(1+2\lambda)^2}{576} \left(16 + \frac{((1+10\lambda)\beta - 2)^2}{4 + (1+10\lambda)\beta + (8\lambda^2 - 3 - 2\lambda)\beta^2} \right), & (1+10\lambda)\beta > 2. \end{cases} \quad (34)$$

Both inequalities are sharp.

Proof Note first that since $H_2(2)$ and the class $\mathcal{F}_O(\lambda, \beta)$ are rotationally invariant, by (28) together with (11) and (6) we can write

$$H_2(2)(f^{-1}) = \delta_2\delta_4 - \delta_3^2 = \frac{1}{2304}\beta^2(1+2\lambda)^2\Phi, \quad (35)$$

where

$$\Phi := [2 - (3 + 2\lambda - 8\lambda^2)\beta^2]c^4 - (1 + 10\lambda)\beta c^2(4 - c^2)\zeta - 2(c^2 + 8)(4 - c^2)\zeta^2 + 12c(4 - c^2)(1 - |\zeta|^2)\eta, \quad (36)$$

for some $c \in [0, 2]$ and $\zeta, \eta \in \overline{\mathbb{D}}$.

I. Assume first that $(1 + 10\lambda)\beta \leq 2$. Then, applying the triangle inequality to (36), and since $|\zeta| \leq 1$, we obtain

$$\begin{aligned} |\Phi| &\leq [2 - (3 + 2\lambda - 8\lambda^2)\beta^2]c^4 + 12c(4 - c^2) \\ &\quad + (1 + 10\lambda)\beta c^2(4 - c^2)|\zeta| + 2(4 - c^2)(2 - c)(4 - c)|\zeta|^2 \\ &\leq 64 + 4[-2 + (1 + 10\lambda)\beta]c^2 + \beta[-1 - 3\beta + 8\lambda^2\beta - 2\lambda(5 + \beta)]c^4. \end{aligned} \quad (37)$$

Since $-1 - 3\beta + 8\lambda^2\beta - 2\lambda(5 + \beta) < 0$, and $-2 + (1 + 10\lambda)\beta \leq 0$, by (37), it follows that

$$|\Phi| \leq 64,$$

and so from (35), the first inequality in (34) is proved.

II. Next assume that $(1 + 10\lambda)\beta > 2$. We consider two cases.

II(A) When $\beta = 1/2$ and $\lambda = 1$, from (36), we obtain

$$\Phi = -2(4 - c^2)\zeta[8\zeta + c^2(3 + \zeta)].$$

Therefore, since $|\zeta| \leq 1$ and $c \in [0, 2]$, we have

$$|\Phi| \leq 2(4 - c^2)|\zeta|[8|\zeta| + c^2(3 + |\zeta|)] \leq 8(4 - c^2)(2 + c^2) \leq 72,$$

and so from (35), we obtain inequality (34).

II(B) Now assume that $\beta \neq 1/2$ or $\lambda \neq 1$, so that $-2 < (-3 - 2\lambda + 8\lambda^2)\beta^2 \leq 3$.

When $c = 0$, (36), gives

$$|\Phi| = 64|\zeta|^2 \leq 64, \quad (38)$$

and when $c = 2$, we have

$$|\Phi| = 16[2 - (3 + 2\lambda - 8\lambda^2)\beta^2]. \quad (39)$$

We suppose therefore that $c \in (0, 2)$ and use Lemma 2. Applying the triangle inequality to (36), we obtain

$$|\Phi| \leq 12c(4 - c^2)\Psi(A, B, C),$$

where for $\zeta \in \overline{\mathbb{D}}$,

$$\Psi(A, B, C) := |A + B\zeta + C\zeta^2| + 1 - |\zeta|^2,$$

with

$$A := \frac{-[2 - (3 + 2\lambda - 8\lambda^2)\beta^2]c^3}{12(4 - c^2)}, \quad B := \frac{1}{12}(1 + 10\lambda)\beta c, \quad C := \frac{c^2 + 8}{6c}.$$

Clearly, $A < 0$, $B > 0$ and $C > 0$, and so $AC < 0$. Furthermore, the following conditions hold for all λ and β such that $(1 + 10\lambda)\beta > 2$:

- (i) $B^2 > -4AC(C^{-2} - 1)$,
- (ii) $2(1 - |C|) < |B|$,
- (iii) $|C|(|B| + 4|A|) > |AB|$.

Indeed (i) follows from

$$C^{-2} - 1 = \frac{-(16 - c^2)(4 - c^2)}{(c^2 + 8)^2} < 0 \quad \text{and} \quad AC < 0,$$

and (ii) written as $B - 2(1 - C) > 0$ is true, since

$$12c[B - 2(1 - C)] = [(1 + 10\lambda)\beta + 4]c^2 - 24c + 32 > 6c^2 - 24c + 32 > 0.$$

Finally, note that since $B - 4C < 0$,

$$12c(B - 4C) = [(1 + 10\lambda)\beta - 8]c^2 - 64 \leq 3c^2 - 64 < 0$$

and $BC > 0$, so condition (iii) written as $BC + A(B - 4C) > 0$ is true.

Next note that the condition $|AB| \leq |C|(|B| - 4|A|)$ is equivalent to $BC + 4AC + AB \geq 0$, and a computation gives

$$144(4 - c^2)(BC + 4AC + AB) = \varphi(c^2),$$

where for $x \in [0, 4]$,

$$\varphi(x) := b_0 + 2b_1x + b_2x^2 \quad (40)$$

with

$$b_0 := 64\beta(1 + 10\lambda), \quad b_1 := -4[16 + (1 + 10\lambda)\beta + 8(-3 - 2\lambda + 8\lambda^2)\beta^2],$$

$$b_2 := -16 - 4(1 + 10\lambda)\beta - 8(-3 - 2\lambda + 8\lambda^2)\beta^2 + (3 + 32\lambda + 12\lambda^2 - 80\lambda^3)\beta^3.$$

We note that $b_0 > 0$, $b_1 < 0$ and $b_2 < 0$. Furthermore, $\varphi'(x) = 2b_1 + 2b_2x < 0$ for $x \in (0, 4)$. Clearly, $\varphi(0) = b_0 > 0$, and $\varphi(4) = 16b_2 + 8b_1 + b_0 < 0$. Indeed, by putting $t = (1 + 10\lambda)\beta$ and $u = (-3 - 2\lambda + 8\lambda^2)\beta^2$, since $t > 2$ and $u > -2$, we have

$$16b_2 + 8b_1 + b_0 = -16(u + 2)(24 + t) < 0.$$

Therefore, the function φ has a unique zero $x_0 \in (0, 4)$, where

$$x_0 := \frac{-b_1 - \sqrt{b_1^2 - b_0b_2}}{b_2}.$$

Thus,

$$\begin{cases} \varphi(x) \geq 0, & \text{when } x \in (0, x_0], \\ \varphi(x) \leq 0, & \text{when } x \in [x_0, 4), \end{cases} \quad (41)$$

and it follows that

$$\begin{cases} |AB| \leq |C|(|B| - 4|A|), & \text{when } c \in (0, \sqrt{x_0}], \\ |AB| \geq |C|(|B| - 4|A|), & \text{when } c \in [\sqrt{x_0}, 2). \end{cases}$$

II(B)-a. Let $c \in (0, \sqrt{x_0}]$. Then, by Lemma 2,

$$|\Phi| \leq 12c(4 - c^2)(-|A| + |B| + |C|) = g(c^2), \quad (42)$$

where for $x \in [0, 4]$,

$$g(x) := 64 + 2d_1x + d_2x^2, \quad (43)$$

with

$$d_1 := 2[-2 + (1 + 10\lambda)\beta], \quad d_2 := -[4 + (1 + 10\lambda)\beta + (-3 - 2\lambda + 8\lambda^2)\beta^2].$$

We note that $g'(x) = 0$ only when $x = \tilde{\tau} := -d_1/d_2$, and clearly, $\tilde{\tau} > 0$.

We now show that $\tilde{\tau} < x_0$. Since

$$\varphi(\tilde{\tau}) = \frac{1}{d_2^2}(b_0d_2^2 - 2b_1d_1d_2 + b_2d_1^2),$$

by (41), the inequality $\tilde{\tau} < x_0$ is equivalent to

$$b_0d_2^2 - 2b_1d_1d_2 + b_2d_1^2 > 0. \quad (44)$$

Moreover, by putting $t = (1 + 10\lambda)\beta$, and $u = (-3 - 2\lambda + 8\lambda^2)\beta^2$, we obtain

$$\begin{aligned} b_0d_2^2 - 2b_1d_1d_2 + b_2d_1^2 \\ = 128(14 + 11u + 2u^2) + 16(52 + 9u - 4u^2)t + 32(7 - u)t^2 \\ + 4(8 - u)t^3 =: L(u, t). \end{aligned} \quad (45)$$

Since $t > 2$ and $-2 < u \leq 3$, the quantities $16(52 + 9u - 4u^2)$, $32(7 - u)$ and $4(8 - u)$ are all positive. Thus, we obtain

$$L(u, t) > L(u, 2) = 128(6 + u)^2 > 0,$$

which, by (45), implies that (44) holds. So $\tilde{\tau} < x_0$, and since $d_2 < 0$, the function g has its maximum at $\tilde{\tau}$ in $(0, x_0]$. Thus, by (42), we have

$$|\Phi| \leq g(\tilde{\tau}) = 64 + \frac{4((1 + 10\lambda)\beta - 2)^2}{4 + (1 + 10\lambda)\beta + (8\lambda^2 - 3 - 2\lambda)\beta^2}. \quad (46)$$

II(B)-b. Let $c \in [\sqrt{x_0}, 2)$. Then, by Lemma 2, we obtain

$$|\Phi| \leq 12c(4 - c^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = h_1(c^2)\sqrt{h_2(c^2)}, \quad (47)$$

where for $x \in [0, 4]$,

$$h_1(x) := 64 - 8x + (8\lambda^2 - 3 - 2\lambda)\beta^2 x^2,$$

and

$$h_2(x) := 1 + \frac{\beta^2(1 + 10\lambda)^2(4 - x)}{8[2 + (-3 - 2\lambda + 8\lambda^2)\beta^2](8 + x)}.$$

It is easy to see that h_2 is decreasing in $[x_0, 2)$.

(i) Moreover, when $(8\lambda^2 - 3 - 2\lambda)\beta^2 \leq 1$, h_1 is also decreasing in $[x_0, 2)$. Therefore, from (47), we obtain

$$|\Phi| \leq h_1(x_0)\sqrt{h_2(x_0)} = g(x_0) \leq g(\tilde{\tau}), \quad (48)$$

where g is the function defined by (43).

(ii) When $(8\lambda^2 - 3 - 2\lambda)\beta^2 > 1$, by putting $t = (1 + 10\lambda)\beta$ and $u = (8\lambda^2 - 3 - 2\lambda)\beta^2$ so that $2 < t \leq 11$ and $1 < u \leq 3$, we obtain

$$h_1(c^2)\sqrt{h_2(c^2)} = H(c^2), \quad (49)$$

where H is the function defined by (29). By Proposition 1, we deduce that

$$\begin{aligned} H(c^2) &\leq \max\{H(x_0), H(4)\} \\ &= \max\{g(x_0), 32 + 16(8\lambda^2 - 3 - 2\lambda)\beta^2\} \leq g(\tilde{\tau}), \end{aligned} \quad (50)$$

and so by (47), (49) and (50), we obtain the inequality (48) once more.

Thus, from (38), (39), (46) and (48), inequality (46) holds for all $c \in [0, 2]$ and $\zeta, \eta \in \overline{\mathbb{D}}$, and so from (35), the second inequality in (34) is proved.

In order to show that the inequalities are sharp, first let f_1 be defined for $z \in \mathbb{D}$ by

$$1 + \frac{zf_1''(z)}{f_1'(z)} = \left(\frac{1}{2} + \lambda\right)p_1^\beta(z) + \frac{1}{2} - \lambda,$$

where $p_1(z) := (1 + z^2)/(1 - z^2)$ for $z \in \mathbb{D}$. Then, $f_1 \in \mathcal{F}_O(\lambda, \beta)$ with

$$f_1(z) = z + \frac{1}{6}(1 + 2\lambda)\beta z^3 + \dots$$

for $z \in \mathbb{D}$. Hence, f_1^{-1} is given by

$$f_1^{-1}(w) = w - \frac{1}{6}(1 + 2\lambda)\beta w^3 + \dots$$

for $w \in \mathbb{D}_{r_0}$. Thus, $H_{2,2}(f_1^{-1}) = -\beta^2(1 + 2\lambda)^2/36$, which shows that the first bound in (34) is sharp when $(1 + 10\lambda)\beta \leq 2$.

Next let f_2 be defined, for $z \in \mathbb{D}$ by

$$1 + \frac{zf_2''(z)}{f_2'(z)} = \left(\frac{1}{2} + \lambda\right)p_2^\beta(z) + \frac{1}{2} - \lambda,$$

where

$$p_2(z) := \frac{1 + bz + z^2}{1 - z^2}, \quad z \in \mathbb{D},$$

with

$$b := \sqrt{\frac{2[(1 + 10\lambda)\beta - 2]}{4 + (1 + 10\lambda)\beta + (8\lambda^2 - 3 - 2\lambda)\beta^2}}. \quad (51)$$

Since $b \in (0, 2)$, it follows that $p_2 \in \mathcal{P}$. Therefore, $f_2 \in \mathcal{F}_O(\lambda, \beta)$ with

$$a_2 = \frac{1}{4}(1 + 2\lambda)\beta b, \quad a_3 = \frac{1}{24}(1 + 2\lambda)\beta [4 + (-1 + 2(1 + \lambda)\beta)b^2]$$

and

$$\begin{aligned} a_4 &= \frac{1}{576}(1 + 2\lambda)\beta b \\ &\times [2(8 + 15\lambda + 6\lambda^2)\beta^2 b^2 - 3(7 + 6\lambda)\beta(-4 + b^2) + 8(-3 + b^2)]. \end{aligned}$$

A simple calculation gives

$$\begin{aligned} H_2(2)(f_2^{-1}) &= a_2^4 - a_2^2 a_3 + a_2 a_4 - a_3^2 \\ &= -\frac{\beta^2(1 + 2\lambda)^2}{2304} [64 + 4(-2 + (1 + 10\lambda)\beta)b^2 \\ &\quad - (4 + (1 + 10\lambda)\beta + (-3 - 2\lambda + 8\lambda^2)\beta^2)b^4], \end{aligned} \quad (52)$$

and substituting for b in (51) into (52) we obtain

$$H_2(2)(f_2^{-1}) = -\frac{\beta^2(1 + 2\lambda)^2}{2304} \left(64 + \frac{4((1 + 10\lambda)\beta - 2)^2}{4 + (1 + 10\lambda)\beta + (8\lambda^2 - 3 - 2\lambda)\beta^2} \right).$$

This shows that the bound in (34) is sharp for the case $(1 + 10\lambda)\beta > 2$. $\square$

5 Strongly convex functions ($\lambda = 1/2$)

When $\lambda = 1/2$, i.e. when $f \in \mathcal{C}^\beta$, the class of convex functions of order β , it was shown in [9] that the bounds for $|H_2(2)(f)|$ and $|H_2(2)(f^{-1})|$ were the same, reflecting other invariant properties concerning the coefficients of f and f^{-1} . It was shown that when $f \in \mathcal{C}^\beta$,

$$|H_2(2)(f)| (|H_2(2)(f^{-1})|) \leq \begin{cases} \frac{\beta^2}{9}, & 0 < \beta \leq \frac{1}{3}, \\ \frac{\beta(1+\beta)(1+17\beta)}{72(3+\beta)}, & \frac{1}{3} < \beta \leq 1. \end{cases}$$

Although these bounds are correct, they are not (as claimed), best possible, since $\lambda = 1/2$ in Theorems 1 and 2 give the following sharp bounds.

Theorem 3 *If $f \in \mathcal{C}^\beta$, then for $0 < \beta \leq 1$,*

$$|H_2(2)(f)| (|H_2(2)(f^{-1})|) \leq \begin{cases} \frac{\beta^2}{9}, & 0 < \beta \leq \frac{1}{3}, \\ \frac{\beta^2(1+\beta)(17+\beta)}{72(2+3\beta-\beta^2)}, & \frac{1}{3} < \beta \leq 1. \end{cases}$$

When $\beta = 1$, we deduce the following [4].

Corollary 1 *If $f \in \mathcal{C}$, then*

$$|H_2(2)(f)| (|H_2(2)(f^{-1})|) \leq \frac{1}{8}.$$

We note that the proof of the weaker result above used Lemma 1, which alone was not strong enough to give the sharp estimate, whereas the additional use of Lemma 2 produces the correct estimates given in Theorem 3.

Remark 1 We finally note that when $\lambda = 1$, Theorems 1 and 2 show that there is no invariance between $|H_2(2)(f)|$ and $|H_2(2)(f^{-1})|$; however, a simple calculation gives the following strange invariance property when $\beta = 1$.

Corollary 2 *If $f \in \mathcal{F}_O(\lambda)$ with $\lambda = \frac{1}{18}(8 + \sqrt{73})$, then*

$$|H_2(2)(f)| (|H_2(2)(f^{-1})|) \leq \frac{2395 + 233\sqrt{73}}{15552}.$$

Both inequalities are sharp.

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