Remarks on parabolic De Giorgi classes

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Abstract

We make several remarks concerning properties of functions in parabolic De Giorgi classes of order p . There are new perspectives including a novel mechanism of propagating positivity in measure, the reservation of membership under convex composition, and a logarithmic type estimate. Based on them, we are able to give new proofs of known properties. In particular, we prove local boundedness and local Hölder continuity of these functions via Moser's ideas, thus avoiding De Giorgi's heavy machinery. We also seize this opportunity to give a transparent proof of a weak Harnack inequality for nonnegative members of some super-class of De Giorgi, without any covering argument.

Keywords Parabolic De Giorgi classes · Moser's iteration · Hölder continuity · Harnack's inequality

Mathematics Subject Classification Primary 35B65; Secondary 35K59 · 49N60

1 Introduction

De Giorgi classes consist of Sobolev functions in an open set $\Omega \subset \mathbb{R}^N$ satisfying a family of energy estimates, i.e., $u \in W_{\text{loc}}^{1,p}(\Omega)$ and for some $\gamma > 0$,

$$\int_{K_\varrho(y)} |D(u - k)_\pm|^p \, dx \leq \frac{\gamma}{(R - \varrho)^p} \int_{K_R(y)} |(u - k)_\pm|^p \, dx$$

for all $k \in \mathbb{R}$, and any pair of concentric cubes $K_\varrho(y) \subset K_R(y)$ in Ω . The significance of De Giorgi classes lies in that they are general enough to include not only weak solutions to quasi-linear elliptic equations in divergence form (cf. [3, 12]), but also local minima or quasi-minima of functionals that do not necessarily admit any Euler equations (cf. [8]). Formulated by Ladyzhenskaya and Ural'tseva (cf. [12]), it has been shown that functions in such classes (of elliptic nature) are locally Hölder continuous, using the beautiful ideas of De Giorgi in his celebrated work [1]. A probably even more striking discovery was made by DiBenedetto and Trudinger in [6] that nonnegative members of De Giorgi classes actually

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satisfy Harnack's inequality, which is a typical property of harmonic functions. In addition to De Giorgi's techniques, the main new input of [6] includes realization of pointwise lower bound of nonnegative members in De Giorgi classes with a power-like dependence on the measure distribution of their positivity. The proof uses a deep covering lemma due to Krylov and Safonov in [11].

The original consideration by De Giorgi in [1] was to obtain Hölder continuity of weak solutions to linear elliptic equations in divergence form with bounded and measurable coefficients. Later on, Moser invented a new approach in [16] to show the same kind of result. Moreover, he was able to obtain Harnack's inequality for such equations in [17]. A key idea of Moser's new proof in [16] is to show a certain logarithmic function of the solution is in fact a sub-solution and to formulate its energy estimates. The feature of Moser's approach is twofold; on the one hand, it simplifies the original proof of De Giorgi and gives a more intuitive method; on the other hand, it keeps referring to the equation. This latter point renders a question on whether we could use Moser's idea in [16] to show the Hölder regularity for functions in De Giorgi classes, where no equations are at our disposal. Recently, an affirmative answer has been given in [10] based on a result in [4]. Naturally, one wonders if Moser's idea in [17] could be used to establish Harnack's inequality for nonnegative members of De Giorgi classes. This, however, remains elusive.

A parabolic version of De Giorgi classes has been introduced in [13]. It should also be pointed out that different notions of parabolic De Giorgi classes have been introduced in the literature. See for instance [7, 9, 15]. Hölder regularity has been established in [13] employing De Giorgi's ideas. Harnack's inequality is first established in [18] using the covering lemma of Krylov and Safonov. As in [6], a weak Harnack inequality was proved in [18], which is of interest in its own right. A direct proof of Harnack's inequality is presented in [7], thus by-passing a weak Harnack inequality.

The main goals of this note are the following. In Sect. 5, we give a proof of Hölder regularity for members of certain parabolic De Giorgi classes, via Moser's ideas, thus avoiding De Giorgi's heavy machinery. This parallels the result for the elliptic De Giorgi classes in [10]. In Sect. 6, we seize this opportunity to give a transparent proof of a weak Harnack inequality for nonnegative members of a certain parabolic super-class of De Giorgi. The main tool is a measure theoretical lemma established in [5], thus by-passing the heavy covering argument of Krylov and Safonov. Last but not least, we show in Sect. 2 that local boundedness of functions in parabolic De Giorgi classes can be achieved via Moser's iteration. A similar observation has been made in [6] for the elliptic case. In Sect. 3, we show convex, non-decreasing functions of members in sub-classes of De Giorgi are still in the same classes. In Sect. 4, we present some observations on the time propagation of measure information.

1.1 Notations and definitions

Let E be an open set in $\mathbb{R}^N \times \mathbb{R}$ and $(y, s) \in E$. Let $K_\varrho(y)$ be a cube of edge 2ϱ , centered at $y \in \mathbb{R}^N$ and with faces parallel with the coordinate planes. When $y = 0$, we simply write K_ϱ . A cylinder with vertex at (y, s) , the base cube $K_\varrho(y)$ and the length τ is defined by

$$(y, s) + Q_{\varrho, \tau} = K_\varrho(y) \times (s - \tau, s].$$

When $\tau = \varrho^p$ for some $p > 1$, we write $(y, s) + Q_\varrho = K_\varrho(y) \times (s - \varrho^p, s]$. When $(y, s) = (0, 0)$, we omit it from the notation.

Suppose u is a measurable function defined in E , such that for some $p > 1$,

$$u \in L^\infty(s - T, s; L^p(K_R(y))) \cap L^p(s - T, s; W^{1,p}(K_R(y)))$$

for any $(y, s) + Q_{R,T} \subset E$. We say u belongs to the parabolic De Giorgi class $\mathfrak{A}_p^\pm(E, \gamma)$ of order p , if there exists a constant $\gamma > 0$ such that, for any $0 < \varrho < R$, $0 < \tau < T$ and $k \in \mathbb{R}$, the following integral inequalities hold:

$$\begin{aligned} \operatorname{ess\,sup}_{s-\tau < t < s} \int_{K_\varrho(y)} (u-k)_\pm^p(\cdot, t) \, dx + \iint_{(y,s)+Q_{\varrho,\tau}} |D(u-k)_\pm|^p \, dx dt \\ \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{T-\tau} \right] \iint_{(y,s)+Q_{R,T}} (u-k)_\pm^p \, dx dt. \end{aligned} \quad (1.1)$$

We define also the class $\mathfrak{A}_p(E, \gamma) := \mathfrak{A}_p^+(E, \gamma) \cap \mathfrak{A}_p^-(E, \gamma)$.

Now suppose

$$u \in C(s-T, s; L^p(K_R(y))) \cap L^p(s-T, s; W^{1,p}(K_R(y)))$$

We say u belongs to the parabolic De Giorgi class $\mathfrak{B}_p^\pm(E, \gamma)$ of order p , if $u \in \mathfrak{A}_p^\pm(E, \gamma)$ and in addition, the following integral inequalities hold for any $0 < \varrho < R$, $0 < \tau < T$ and $k \in \mathbb{R}$:

$$\begin{aligned} \operatorname{ess\,sup}_{s-T < t < s} \int_{K_\varrho(y)} (u-k)_\pm^p(\cdot, t) \, dx \leq \int_{K_R(y)} (u-k)_\pm^p(\cdot, s-T) \, dx \\ + \frac{\gamma}{(R-\varrho)^p} \iint_{(y,s)+Q_{R,T}} (u-k)_\pm^p \, dx dt. \end{aligned} \quad (1.2)$$

Analogously, we define the class $\mathfrak{B}_p(E, \gamma) := \mathfrak{B}_p^+(E, \gamma) \cap \mathfrak{B}_p^-(E, \gamma)$.

We remark that our definitions of De Giorgi classes mainly follow those in [13]. One difference is that we consider an arbitrary order $p > 1$, whereas $p = 2$ in [13]. Also, a certain non-homogeneous term is imposed in [13] for the inequalities (1.1) and (1.2). However, we decide to omit such a term for simplicity of presentation.

In the sequel, we refer to the set of parameters $\{\gamma, p, N\}$ as the *data* and use C as a generic constant that can be quantitatively determined a priori only in terms of the data. We will use $\mathcal{A}(R, T, \varrho, \tau)$ to denote a generic positive, homogeneous quantity in the sense that under the relation $\varrho = \sigma_1 R$, $\tau = \sigma_2 T$ and $T = R^p$, it becomes a quantity of σ_1 and σ_2 , possibly also depending on the data. We will say u belongs to the generalized class $\mathfrak{A}_p^\pm$, if (1.1) holds with γ replaced by $\mathcal{A}$. A similar definition holds for $\mathfrak{B}_p^\pm$.

2 Local boundedness of functions in $\mathfrak{A}_p^\pm$

In general, the membership in $\mathfrak{A}_p^\pm(E, \gamma)$ does not guarantee continuity. A Heaviside function of the time variable would be an example. Nevertheless, every function in $\mathfrak{A}_p^\pm(E, \gamma)$ is locally bounded from above or from below.

Theorem 2.1 *Suppose $u \in \mathfrak{A}_p^\pm(E, \gamma)$. Then, there is a homogeneous quantity $\mathcal{A}$, such that*

$$\operatorname{ess\,sup}_{(y,s)+Q_{\varrho,\tau}} (u-k)_\pm \leq \frac{\mathcal{A}}{(R-\varrho)^N(T-\tau)} \iint_{(y,s)+Q_{R,T}} (u-k)_\pm \, dx dt \quad (2.1)$$

for any cube $(y, s) + Q_{R,T} \subset E$ and all $k \in \mathbb{R}$. The same conclusion holds for members in the generalized classes $\mathfrak{A}_p^\pm$.

The proof is usually written using De Giorgi's iteration (cf. [13, 15]). Nevertheless, we present here a proof based on Moser's iteration.

2.1 Proof by Moser's iteration

Multiply both sides of (1.1)₊ by k^β with $\beta > -1$ and integrate in dk from 0 to ∞ to get that

$$\begin{aligned} & \operatorname{ess\,sup}_{-\tau < t < 0} \int_0^\infty k^\beta \, dk \int_{K_\varrho} (u(\cdot, t) - k)_+^p \, dx + \int_0^\infty k^\beta \, dk \iint_{Q_{\varrho, \tau}} |D(u - k)_+|^p \, dx \, dt \\ & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{T - \tau} \right] \int_0^\infty k^\beta \, dk \iint_{Q_{R, T}} (u - k)_+^p \, dx \, dt. \end{aligned}$$

Fixing $-\tau < t < 0$ and applying Fubini's theorem, the first term on the left-hand side is estimated by

$$\int_{K_\varrho} \int_0^u k^\beta (u - k)_+^p \, dk \, dx = \int_0^1 \lambda^\beta (1 - \lambda)^p \, d\lambda \int_{K_\varrho} u^{p+\beta+1} \, dx.$$

One could verify that there exists an absolute constant $C > 0$, such that

$$\int_0^1 \lambda^\beta (1 - \lambda)^p \, d\lambda \geq \frac{C}{(\beta + 1)^{p+1}}.$$

Similarly, the second term yields that

$$\iint_{Q_{\varrho, \tau}} \int_0^u k^\beta |Du|^p \, dx \, dt \, dk = \frac{1}{\beta+1} \iint_{Q_{\varrho, \tau}} u^{\beta+1} |Du|^p \, dx \, dt,$$

while the integral on the right-hand side is estimated from above by

$$\frac{1}{\beta+1} \iint_{Q_{R, T}} u^{p+\beta+1} \, dx \, dt.$$

Combining the above calculation gives us that for all $\beta > -1$,

$$\begin{aligned} & \frac{1}{(\beta+1)^p} \operatorname{ess\,sup}_{-\tau < t < 0} \int_{K_\varrho \times \{t\}} u^{p+\beta+1} \, dx + \iint_{Q_{\varrho, \tau}} u^{\beta+1} |Du|^p \, dx \, dt \\ & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{T - \tau} \right] \iint_{Q_{R, T}} u^{p+\beta+1} \, dx \, dt. \end{aligned}$$

Written in terms of $w \stackrel{\text{def}}{=} u^{\frac{p+\beta+1}{p}}$, the above estimate gives that

$$\begin{aligned} & \frac{1}{(\beta+1)^p} \operatorname{ess\,sup}_{-\tau < t < 0} \int_{K_\varrho \times \{t\}} w^p \, dx + \left(\frac{p}{p+\beta+1} \right)^p \iint_{Q_{\varrho, \tau}} |Dw|^p \, dx \, dt \\ & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{T - \tau} \right] \iint_{Q_{R, T}} w^p \, dx \, dt. \end{aligned}$$

This is the starting point of Moser's iteration scheme. In order to use this energy estimate, we introduce for ϱ , $\tau > 0$, $\sigma \in (0, 1)$ and $n = 0, 1, \dots$,

$$\begin{cases} \varrho_n = \sigma \varrho + \frac{(1-\sigma)\varrho}{2^n}, & \tau_n = \sigma \tau + \frac{(1-\sigma)\tau}{2^n}, \\ \tilde{\varrho}_n = \frac{\varrho_n + \varrho_{n+1}}{2}, & \tilde{\tau}_n = \frac{\tau_n + \tau_{n+1}}{2}, \\ K_n = K_{\varrho_n}, & \tilde{K}_n = K_{\tilde{\varrho}_n}, \quad Q_n = K_n \times (-\tau_n, 0], \quad \tilde{Q}_n = \tilde{K}_n \times (-\tilde{\tau}_n, 0], \\ p_n = p + \beta_n + 1, & p_{n+1} = p_n \kappa, \quad \kappa = \frac{N+p}{N}, \quad \text{i.e. } p_n = p_0 \kappa^n. \end{cases}$$

Set ζ to be a standard cutoff function that vanishes on $\partial_p \tilde{Q}_n$ and equals identity in Q_{n+1} , such that $|D\zeta| \leq 2^n/\varrho$. We apply the Sobolev imbedding (cf. [[2], Chapter I, Proposition 3.1]), together with the energy estimate and the choice $p_o = p\kappa$ such that $\beta_o > -1$, to obtain that

$$\begin{aligned} \iint_{Q_{n+1}} u^{p_{n+1}} \, dx dt &\leq \iint_{\tilde{Q}_n} (w\zeta)^{p \frac{N+p}{N}} \, dx dt \\ &\leq C \iint_{\tilde{Q}_n} |D(w\zeta)|^p \, dx dt \left(\operatorname{ess\,sup}_{-\tilde{\tau}_n < t < 0} \int_{\tilde{K}_n \times \{t\}} (w\zeta)^p \, dx \right)^{\frac{p}{N}} \\ &\leq \frac{C}{(1-\sigma)^{p\kappa}} \left(\frac{p + \beta_n + 1}{p} \right)^p (\beta_n + 1)^{\frac{p^2}{N}} \left(\frac{2^{pn}}{\varrho^p} + \frac{2^n}{\tau} \right)^\kappa \left(\iint_{Q_n} u^{p_n} \, dx dt \right)^\kappa \\ &\leq \frac{C p_n^{p\kappa} 2^{np\kappa}}{(1-\sigma)^{p\kappa}} \left(\frac{1}{\varrho^p} + \frac{1}{\tau} \right)^\kappa \left(\iint_{Q_n} u^{p_n} \, dx dt \right)^\kappa \\ &\leq \frac{C b^{2np\kappa}}{(1-\sigma)^{p\kappa}} \left(\frac{1}{\varrho^p} + \frac{1}{\tau} \right)^\kappa \left(\iint_{Q_n} u^{p_n} \, dx dt \right)^\kappa, \end{aligned}$$

for some $b, C > 1$ depending only on the data. To simplify the above iteration, we set

$$Y_n = \left(\iint_{Q_n} u^{p_n} \, dx dt \right)^{\frac{1}{p_n}},$$

take the power p_{n+1}^{-1} on both sides, and rewrite it as

$$Y_{n+1} \leq B^{\frac{n}{\kappa^n}} Y_n,$$

where

$$B = \frac{C}{(1-\sigma)^{p\kappa}} \left[\left(\frac{\tau}{\varrho^p} \right)^{\frac{p}{N+p}} + \left(\frac{\varrho^p}{\tau} \right)^{\frac{N}{N+p}} \right]^\kappa.$$

Iterating this inequality yields that

$$Y_n \leq B^{\frac{n}{\kappa^n} + \frac{n-1}{\kappa^{n-1}} + \dots + \frac{1}{\kappa}} Y_o \leq B^{\frac{\kappa}{(\kappa-1)^2}} Y_o.$$

Sending $n \rightarrow \infty$ gives that

$$\operatorname{ess\,sup}_{Q_{\sigma\varrho,\sigma\tau}} u \leq \frac{C}{(1-\sigma)^{\frac{p\kappa^2}{(\kappa-1)^2}}} \left[\left(\frac{\varrho^p}{\tau} \right)^{\frac{N}{N+p}} + \left(\frac{\tau}{\varrho^p} \right)^{\frac{p}{N+p}} \right]^{\frac{\kappa^2}{(\kappa-1)^2}} \left(\iint_{Q_o} u^{p_o} \, dx dt \right)^{\frac{1}{p_o}}.$$

To proceed, we define that

$$M = \operatorname{ess\,sup}_{Q_{\varrho,\tau}} u_+, \quad M_\sigma = \operatorname{ess\,sup}_{Q_{\sigma\varrho,\sigma\tau}} u_+.$$

Then the above estimate yields

$$M_\sigma \leq \frac{C M^{1-\frac{1}{p\kappa}}}{(1-\sigma)^{\frac{N+p}{p}}} \left[\left(\frac{\varrho^p}{\tau} \right)^{\frac{N}{N+p}} + \left(\frac{\tau}{\varrho^p} \right)^{\frac{p}{N+p}} \right]^{\frac{\kappa^2}{(\kappa-1)^2}} \left(\iint_{Q_o} u_+ \, dx dt \right)^{\frac{1}{p\kappa}}.$$

An interpolation argument (cf. [[3], Chapter I, Lemma 4.3]) would give

$$\operatorname{ess\,sup}_{Q_{\frac{\varrho}{2}, \frac{\tau}{2}}} u_+ \leq C \left[\left(\frac{\varrho^p}{\tau} \right)^{\frac{N}{N+p}} + \left(\frac{\tau}{\varrho^p} \right)^{\frac{p}{N+p}} \right]^{\frac{p\kappa^3}{(\kappa-1)^2}} \iint_{K_{\varrho} \times (-\tau, 0)} u_+ \, dx dt.$$

Fixing $\sigma_1, \sigma_2 \in (0, 1)$, it is not hard to see that there exists $(y, s) \in Q_{\sigma_1 R, \sigma_2 T}$, such that

$$\operatorname{ess\,sup}_{Q_{\sigma_1 R, \sigma_2 T}} u_+ \leq \operatorname{ess\,sup}_{Q_*} u_+,$$

where we have set

$$Q_* \stackrel{\text{def}}{=} (y, s) + Q_{\frac{(1-\sigma_1)R}{2}, \frac{(1-\sigma_2)T}{2}}.$$

Applying the above estimate to Q_* to obtain

$$\operatorname{ess\,sup}_{Q_{\sigma_1 R, \sigma_2 T}} u_+ \leq \operatorname{ess\,sup}_{Q_*} u_+ \leq \frac{CA}{(1-\sigma_1)^N R^N (1-\sigma_2)T} \iint_{K_R \times (-T, 0)} u_+ \, dx dt,$$

where

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{ \left[\frac{(1-\sigma_1)^p R^p}{(1-\sigma_2)T} \right]^{\frac{N}{(p+N)}} + \left[\frac{(1-\sigma_2)T}{(1-\sigma_1)^p R^p} \right]^{\frac{p}{N+p}} \right\}^{\frac{p\kappa^3}{(\kappa-1)^2}}.$$

Setting $\varrho = \sigma_1 R$ and $\tau = \sigma_2 T$, the desired conclusion follows.

2.2 Critical mass lemmas

Assume $a \in (0, 1)$ and $M > 0$ are parameters. The following lemma has been derived in [7]. It can be viewed as a direct consequence of the local boundedness estimate in Theorem 2.1.

Lemma 2.1 *Let $u \in \mathfrak{A}_p^\pm(E; \gamma)$. Suppose $(y, s) + Q_{\varrho} \subset E$ and $\mu^\pm$ satisfy*

$$\mu^+ \geq \operatorname{ess\,sup}_{(y, s) + Q_{\varrho}} u, \quad \mu^- \leq \operatorname{ess\,inf}_{(y, s) + Q_{\varrho}} u.$$

There exists $\nu > 0$ depending only on the data and a , such that if

$$|[\pm(\mu^\pm - u) < M] \cap [(y, s) + Q_{\varrho}]| \leq \nu |Q_{\varrho}|,$$

then

$$\pm(\mu^\pm - u) > aM \quad \text{a.e. in } (y, s) + Q_{\frac{\varrho}{2}}.$$

Proof Assume $(y, s) = (0, 0)$. We only treat the class $\mathfrak{A}_p^+(E; \gamma)$. An application of Theorem 2.1 in $Q_{\frac{\varrho}{2}} \subset Q_{\varrho}$, with $k = \mu^+ - M$ yields that,

$$\operatorname{ess\,sup}_{Q_{\frac{\varrho}{2}}} (u - k)_+ \leq \frac{C}{|Q_{\varrho}|} \iint_{Q_{\varrho}} (u - k)_+ \, dx dt \leq CM \frac{|[u > k] \cap Q_{\varrho}|}{|Q_{\varrho}|}.$$

Now, we choose $\nu = \frac{1}{C}(1 - a)$, such that when

$$\frac{|[u > k] \cap Q_{\varrho}|}{|Q_{\varrho}|} < \nu,$$

we have

$$CM \frac{|[u > k] \cap Q_\varrho|}{|Q_\varrho|} < (1 - a)M.$$

As a result, we arrive at the desired conclusion that

$$\operatorname{ess\,sup}_{Q_{\frac{\varrho}{2}}} u \leq k + (1 - a)M = \mu^+ - aM.$$

□

3 Additional properties of functions in $\mathfrak{A}_p^\pm$

It is known that the convex, non-decreasing function of a sub-harmonic function yields another sub-harmonic function, whereas the concave, non-increasing function of a super-harmonic function gives another super-harmonic function. Similar conclusions hold for the heat operator, and even for more general linear parabolic operators with bounded and measurable coefficients. What we are concerned with next is to show analogous properties for members of $\mathfrak{A}_p^\pm(\gamma, E)$. The elliptic version has been established in [4].

Lemma 3.1 *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be convex and non-decreasing and let $u \in \mathfrak{A}_p^+(E, \gamma)$. Then, $\varphi(u)$ belongs to the generalized class $\mathfrak{A}_p^+$.*

Proof For any such φ and $h \leq k$, observe the following elementary identity

$$(\varphi(u) - \varphi(h))_+ - \varphi'(h)(u - h)_+ = \int_{\mathbb{R}} (u - k)_+ \chi_{[k > h]} \varphi''(k) \, dk, \quad (3.1)$$

where χ is the characteristic function of the indicated set. Moreover, by the convexity and monotonicity of φ , we have that

$$(\varphi(u) - \varphi(h))_+ \geq \varphi'(h)(u - h)_+ \geq 0. \quad (3.2)$$

From (3.1), for a.e. $t \in (-\tau, 0)$, we estimate that

$$\begin{aligned} & \|(\varphi(u(\cdot, t)) - \varphi(h))_+\|_{p, K_\varrho} \\ & \leq \|\varphi'(h)(u(\cdot, t) - h)_+\|_{p, K_\varrho} + \left\| \int_{\mathbb{R}} (u(\cdot, t) - k)_+ \chi_{[k > h]} \varphi''(k) \, dk \right\|_{p, K_\varrho} \\ & = I_1 + I_2. \end{aligned}$$

For I_1 , we estimate by using (1.1) and (3.2):

$$\begin{aligned} I_1^p & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right] \iint_{Q_{R,T}} [\varphi'(h)]^p (u - h)_+^p \, dx \, dt \\ & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right] \iint_{Q_{R,T}} (\varphi(u) - \varphi(h))_+^p \, dx \, dt. \end{aligned}$$

For I_2 , we estimate by using (3.1), (3.2) and Theorem 2.1:

$$\begin{aligned}
 I_2 &\leq \int_{\mathbb{R}} \|(u-k)_+\|_{p, K_\varrho} \chi_{[k>h]} \varphi''(k) \, dk \\
 &\leq C \varrho^{\frac{N}{p}} \int_{\mathbb{R}} \|(u-k)_+\|_{\infty, K_\varrho} \chi_{[k>h]} \varphi''(k) \, dk \\
 &\leq \frac{\mathcal{A} \varrho^{\frac{N}{p}}}{(R-\varrho)^N (T-\tau)} \int_{\mathbb{R}} \iint_{Q_{R,T}} (u-k)_+ \chi_{[k>h]} \varphi''(k) \, dx dt \, dk \\
 &= \frac{\mathcal{A} \varrho^{\frac{N}{p}}}{(R-\varrho)^N (T-\tau)} \iint_{Q_{R,T}} [(\varphi(u) - \varphi(h))_+ - \varphi'(h)(u-h)_+] \, dx dt \\
 &\leq \frac{\mathcal{A} \varrho^{\frac{N}{p}}}{(R-\varrho)^N (T-\tau)} \iint_{Q_{R,T}} (\varphi(u) - \varphi(h))_+ \, dx dt \\
 &\leq \frac{\mathcal{A} \varrho^{\frac{N}{p}} (R^N T)^{1-\frac{1}{p}}}{(R-\varrho)^N (T-\tau)} \|(\varphi(u) - \varphi(h))_+\|_{p, Q_{R,T}}.
 \end{aligned}$$

Recalling that $\mathcal{A}(R, T, \varrho, \tau)$ represents a generic dimensionless quantity, we combine the above estimates to arrive at

$$\begin{aligned}
 &\operatorname{ess\,sup}_{-\tau < t < 0} \int_{K_\varrho} (\varphi(u(\cdot, t)) - \varphi(k))_+^p \, dx \\
 &\quad \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma + \mathcal{A}}{(T-\tau)} \right] \iint_{Q_{R,T}} (\varphi(u) - \varphi(k))_+^p \, dx dt.
 \end{aligned}$$

We now handle the part with the space gradient. From (3.1), taking the gradient of both sides, then taking the L^p -norm over $Q_{\varrho, \tau}$ and applying the continuous version of Minkowski's inequality, we obtain

$$\begin{aligned}
 \|D(\varphi(u) - \varphi(h))_+\|_{p, Q_{\varrho, \tau}} &\leq \|\varphi'(h) D(u-h)_+\|_{p, Q_{\varrho, \tau}} \\
 &\quad + \left\| \int_{\mathbb{R}} D(u-k)_+ \chi_{[k>h]} \varphi''(k) \, dk \right\|_{p, Q_{\varrho, \tau}} \\
 &\leq \|\varphi'(h) D(u-h)_+\|_{p, Q_{\varrho, \tau}} \\
 &\quad + \int_{\mathbb{R}} \|D(u-k)_+\|_{p, Q_{\varrho, \tau}} \chi_{[k>h]} \varphi''(k) \, dk \\
 &= I_3 + I_4.
 \end{aligned}$$

One estimates I_3 using (1.1) and (3.2):

$$\begin{aligned}
 I_3^p &\leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right] \iint_{Q_{R,T}} (u-h)_+^p [\varphi'(h)]^p \, dx dt \\
 &\leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right] \iint_{Q_{R,T}} (\varphi(u) - \varphi(h))_+^p \, dx dt
 \end{aligned}$$

One estimates I_4 by (1.1), (3.1), (3.2) and Theorem 2.1:

$$\begin{aligned}
 & \int_{\mathbb{R}} \|D(u-k)_+\|_{p, Q_{\varrho, \tau}} \chi_{[k>h]} \varphi''(k) \, dk \\
 & \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right]^{\frac{1}{p}} \int_{\mathbb{R}} \| (u-k)_+ \|_{p, Q_{\frac{R+\varrho}{2}, \frac{T+\tau}{2}}} \chi_{[k>h]} \varphi''(k) \, dk \\
 & \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right]^{\frac{1}{p}} R^{\frac{N}{p}} T^{\frac{1}{p}} \int_{\mathbb{R}} \| (u-k)_+ \|_{\infty, Q_{\frac{R+\varrho}{2}, \frac{T+\tau}{2}}} \chi_{[k>h]} \varphi''(k) \, dk \\
 & \leq \left[\frac{\gamma}{(R-\varrho)^2} + \frac{\gamma}{(T-\tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^{\frac{N}{p}} T^{\frac{1}{p}}}{(R-\varrho)^N (T-\tau)} \int_{\mathbb{R}} \iint_{Q_{R,T}} (u-k)_+ \chi_{[k>h]} \varphi''(k) \, dk \\
 & = \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^{\frac{N}{p}} T^{\frac{1}{p}}}{(R-\varrho)^N (T-\tau)} \\
 & \quad \times \iint_{Q_{R,T}} [(\varphi(u) - \varphi(h))_+ - \varphi'(s)(u-h)_+] \, dx dt \\
 & \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^{\frac{N}{p}} T^{\frac{1}{p}}}{(R-\varrho)^N (T-\tau)} \iint_{Q_{R,T}} (\varphi(u) - \varphi(h))_+ \, dx dt \\
 & \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^N T}{(R-\varrho)^N (T-\tau)} \|(\varphi(u) - \varphi(h))_+\|_{p, Q_{R,T}}.
 \end{aligned}$$

Observe that the fractional with $\mathcal{A}$ is again a dimensionless quantity. Hence, we have

$$\begin{aligned}
 & \iint_{Q_{\varrho, \tau}} |D(\varphi(u) - \varphi(h))_+|^p \, dx dt \\
 & \leq \left[\frac{\mathcal{A}}{(R-\varrho)^p} + \frac{\mathcal{A}}{(T-\tau)} \right] \iint_{Q_{R,T}} (\varphi(u) - \varphi(h))_+^p \, dx dt.
 \end{aligned}$$

Combining the above estimates gives the desired conclusion. $\square$

Lemma 3.2 *Let $\varphi : (a, \infty) \rightarrow \mathbb{R}$, for some $a < \infty$ be convex and non-increasing, such that*

$$\lim_{k \rightarrow \infty} \varphi(k) = \lim_{k \rightarrow \infty} k \varphi'(k) = 0. \quad (3.3)$$

Suppose $u \in \mathfrak{A}_p^-(E, \gamma)$, with its range in (a, ∞) . Then $\varphi(u)$ belongs to the generalized class $\mathfrak{A}_p^+$.

Proof Under the conditions of φ , one easily verifies that

$$\varphi(u) = \int_{\mathbb{R}} (u-k)_- \varphi''(k) \, dk. \quad (3.4)$$

Since $u \in \mathfrak{A}_p^-(E; \gamma)$, it is bounded from below by Theorem 2.1. Hence, the above equation is well-defined for such u , and we may assume with no loss of generality that $u \geq 0$.

First, we take L^p -norm of both sides over K_{ϱ} to obtain that, for all $-\tau < t < 0$,

$$\|\varphi(u(\cdot, t))\|_{p, K_{\varrho}} = \left\| \int_{\mathbb{R}} (u(\cdot, t) - k)_- \varphi''(k) \, dk \right\|_{p, K_{\varrho}}.$$

The right-hand side is estimated by Minkowski's inequality and Theorem 2.1:

$$\begin{aligned}
 & \int_{\mathbb{R}} \|(u(\cdot, t) - k)_-\|_{p, K_\varrho} \varphi''(k) \, dk \\
 & \leq C \varrho^{\frac{N}{p}} \int_{\mathbb{R}} \|(u(\cdot, t) - k)_-\|_{\infty, K_\varrho} \varphi''(k) \, dk \\
 & \leq \frac{\mathcal{A} \varrho^{\frac{N}{p}}}{(R - \varrho)^N (T - \tau)} \int_{\mathbb{R}} \iint_{Q_{R,T}} (u - k)_- \varphi''(k) \, dx dt \, dk \\
 & = \frac{\mathcal{A} \varrho^{\frac{N}{p}}}{(R - \varrho)^N (T - \tau)} \iint_{Q_{R,T}} \varphi(u) \, dx dt \\
 & \leq \frac{\mathcal{A} \varrho^{\frac{N}{p}} (R^N T)^{1 - \frac{1}{p}}}{(R - \varrho)^N (T - \tau)} \|\varphi(u)\|_{p, Q_{R,T}}.
 \end{aligned}$$

As a result,

$$\operatorname{ess\,sup}_{-\tau < t < 0} \int_{K_\varrho} |\varphi(u(\cdot, t))|^p \, dx \leq \frac{\mathcal{A}}{T - \tau} \iint_{Q_{R,T}} |\varphi(u)|^p \, dx dt.$$

Next, we take the spatial gradient of both sides of (3.4), then take the power p , and integrate over $Q_{\varrho, \tau}$ to obtain that

$$\iint_{Q_{\varrho, \tau}} |D\varphi(u)|^p \, dx dt = \iint_{Q_{\varrho, \tau}} \left| \int_{\mathbb{R}} D(u - k)_- \varphi''(k) \, dk \right|^p \, dx dt.$$

The right-hand side is estimated by

$$\begin{aligned}
 \|D\varphi(u)\|_{p, Q_{\varrho, \tau}} & \leq \int_{\mathbb{R}} \|D(u - k)_-\|_{p, Q_{\varrho, \tau}} \varphi''(k) \, dk \\
 & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right]^{\frac{1}{p}} \int_{\mathbb{R}} \|(u - k)_-\|_{p, Q_{\frac{R+\varrho}{2}, \frac{T+\tau}{2}}} \varphi''(k) \, dk \\
 & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right]^{\frac{1}{p}} R^{\frac{N}{p}} T^{\frac{1}{p}} \int_{\mathbb{R}} \|(u - k)_-\|_{\infty, Q_{\frac{R+\varrho}{2}, \frac{T+\tau}{2}}} \varphi''(k) \, dk \\
 & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^{\frac{N}{p}} T^{\frac{1}{p}}}{(R - \varrho)^N (T - \tau)} \int_{\mathbb{R}} \iint_{Q_{R,T}} (u - k)_- \varphi''(k) \, dx dt \, dk \\
 & = \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^{\frac{N}{p}} T^{\frac{1}{p}}}{(R - \varrho)^N (T - \tau)} \iint_{Q_{R,T}} \varphi(u) \, dx dt \\
 & \leq \left[\frac{\gamma}{(R - \varrho)^p} + \frac{\gamma}{(T - \tau)} \right]^{\frac{1}{p}} \frac{\mathcal{A} R^N T}{(R - \varrho)^N (T - \tau)} \|\varphi(u)\|_{p, Q_{R,T}} \\
 & \leq \left[\frac{\gamma \mathcal{A}}{(R - \varrho)^p} + \frac{\gamma \mathcal{A}}{(T - \tau)} \right]^{\frac{1}{p}} \|\varphi(u)\|_{p, Q_{R,T}}.
 \end{aligned}$$

If φ is convex, non-increasing and satisfying (3.3), then $(\varphi - \ell)_+$ verifies the same properties for all ℓ in the range of φ . Hence, the desired conclusion is reached by replacing φ with $(\varphi - \ell)_+$. $\square$

Lemma 3.3 *Let $u \in \mathfrak{A}_p^-(E, \gamma)$ be nonnegative and bounded above by a positive constant M . Then,*

$$\iint_{(y,s)+Q_{\varrho,\tau}} |D \ln u|^p \, dx dt \leq \left[\frac{\gamma p}{(R-\varrho)^p} + \frac{\gamma p}{T-\tau} \right] \iint_{(y,s)+Q_{R,T}} \ln \frac{M}{u} \, dx dt$$

for any pair of cubes $(y, s) + Q_{\varrho,\tau} \subset (y, s) + Q_{R,T} \subset E$.

Proof Assume $(y, s) = (0, 0)$. According to (1.1), for all $0 < k < M$, we have that

$$\iint_{Q_{\varrho,\tau}} |D(u-k)_-|^p \, dx dt \leq \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right] \iint_{Q_{R,T}} (u-k)_-^p \, dx dt.$$

To proceed, we multiply both sides by k^{-p-1} and integrate from 0 to M . The left-hand side becomes

$$\begin{aligned} & \int_0^M \frac{dk}{k^{p+1}} \iint_{Q_{\varrho,\tau}} |D(u-k)_-|^p \, dx dt \\ &= \iint_{Q_{\varrho,\tau}} \int_0^M |D(u-k)_-|^p \frac{dk}{k^{p+1}} \, dx dt \\ &= \iint_{Q_{\varrho,\tau}} |Du|^p \int_u^M \frac{dk}{k^{p+1}} \, dx dt \\ &= \iint_{Q_{\varrho,\tau}} \left(-\frac{1}{p} \frac{|Du|^p}{M^p} + \frac{1}{p} \frac{|Du|^p}{u^p} \right) \, dx dt \\ &= \frac{1}{p} \iint_{Q_{\varrho,\tau}} |D \ln u|^p \, dx dt - \frac{1}{pM^p} \iint_{Q_{\varrho,\tau}} |Du|^p \, dx dt. \end{aligned}$$

The integral on the right-hand side is estimated by

$$\begin{aligned} & \int_0^M \frac{dk}{k^{p+1}} \iint_{Q_{R,T}} (u-k)_-^p \, dx dt \\ &= \iint_{Q_{R,T}} \int_0^M \frac{(k-u)_+^p}{k^{p+1}} \, dk \, dx dt \\ &= \iint_{Q_{R,T}} \left[-\frac{1}{p} \frac{(k-u)_+^p}{k^p} \Big|_u^M + \int_u^M \frac{(k-u)^{p-1}}{k^{p-1}} \frac{dk}{k} \right] \, dx dt \\ &\leq -\frac{1}{pM^p} \iint_{Q_{R,T}} (M-u)_+^p \, dx dt + \iint_{Q_{R,T}} \ln \frac{M}{u} \, dx dt. \end{aligned}$$

Hence, combining the above two estimates, we arrive at

$$\begin{aligned} & \iint_{Q_{\varrho,\tau}} |D \ln u|^p \, dx dt \\ &\leq \frac{1}{M^p} \left\{ \iint_{Q_{R,T}} |Du|^p \, dx dt - \left[\frac{\gamma}{(R-\varrho)^p} + \frac{\gamma}{(T-\tau)} \right] \iint_{Q_{R,T}} (u-M)_-^p \, dx dt \right\} \\ &\quad + \left[\frac{\gamma p}{(R-\varrho)^p} + \frac{\gamma p}{(T-\tau)} \right] \iint_{Q_{R,T}} \ln \frac{M}{u} \, dx dt. \end{aligned}$$

Since $u \in \mathfrak{A}_p^-(E, \gamma)$, the term in the curly bracket is non-positive and can be discarded.

Remark 3.1 The appearance of a logarithmic integral on the right-hand side is natural. Suppose $0 < u \leq M$ is a super-solution to the heat equation. If we formally multiply the equation by $-u^{-1}\zeta^2$ where ζ is a standard cutoff function in Q_ϱ vanishing on ∂Q_ϱ , then an integration over Q_ϱ followed by a standard calculation yields

$$\iint_{Q_\varrho} \zeta^2 \partial_t \ln \frac{M}{u} \, dx dt + \iint_{Q_\varrho} |D \ln u|^2 \zeta^2 \, dx dt \leq 2 \iint_{Q_\varrho} \zeta D \ln u D \zeta \, dx dt.$$

A further integration by parts in time and an application of Young's inequality would give us

$$\iint_{Q_\varrho} |D \ln u|^2 \zeta^2 \, dx dt \leq C \iint_{Q_\varrho} |D \zeta|^2 \, dx dt + C \iint_{Q_\varrho} \zeta |\partial_t \zeta| \ln \frac{M}{u} \, dx dt.$$

4 Time propagation of positivity in measure

In this section, we examine the role of (1.2). First of all, we present a standard lemma which asserts that (1.2) alone is sufficient to propagate positivity of u in measure for a short period of time (cf. [13]).

Proposition 4.1 *Suppose u is nonnegative and satisfies (1.2)₋. Assume for $M > 0$ and $\alpha \in (0, 1)$, we have $(s, s + \varrho^p] \times K_\varrho(y) \subset E$ and*

$$|[u(\cdot, s) > M] \cap K_\varrho(y)| \geq \alpha |K_\varrho|.$$

Then, there exist $\delta, \varepsilon \in (0, 1)$ depending only on the data and α , such that

$$|[u(\cdot, t) > \varepsilon M] \cap K_\varrho(y)| \geq \frac{1}{2} \alpha |K_\varrho|$$

for all times

$$s < t < s + \delta \varrho^p.$$

Proof Assume $(y, s) = (0, 0)$. We may apply (1.2)₋ with $k = M$ in the cylinders

$$K_{(1-\sigma)\varrho} \times (0, \delta \varrho^p] \subset K_\varrho \times (0, \delta \varrho^p] \stackrel{\text{def}}{=} Q_\varrho;$$

in such a case, we have for all $0 < t < \delta \varrho^p$,

$$\begin{aligned} \int_{K_{(1-\sigma)\varrho}} (u(\cdot, t) - M)_-^p \, dx &\leq \int_{K_\varrho} (u(x, 0) - M)_-^p \, dx + \frac{\gamma}{(\sigma \varrho)^p} \iint_{Q_\varrho} (u - M)_-^p \, dx dt \\ &\leq \int_{K_\varrho} (u(x, 0) - M)_-^p \, dx + \gamma \frac{M^p}{(\sigma \varrho)^p} |[u < M] \cap Q_\varrho| \\ &\leq M^p \left[1 - \alpha + \gamma \frac{\delta}{\sigma^p} \frac{|[u < M] \cap Q_\varrho|}{|Q_\varrho|} \right] |K_\varrho|. \end{aligned}$$

Set $\ell = \varepsilon M$. The left-hand side of the above estimate can be bounded from below by

$$\int_{K_{(1-\sigma)\varrho} \cap [u \leq \ell]} (u(\cdot, t) - M)_-^p \, dx \geq (1 - \varepsilon)^p M^p |A_{\ell, (1-\sigma)\varrho}(t)|$$

where we have defined, for some ε to be chosen, that

$$A_{\ell, (1-\sigma)\varrho}(t) = [u(\cdot, t) \leq \varepsilon M] \cap K_{(1-\sigma)\varrho}.$$

Notice that

$$\begin{aligned} |A_{\ell, \varrho}(t)| &= |A_{\ell, (1-\sigma)\varrho}(t) \cup (A_{\ell, \varrho}(t) - A_{\ell, (1-\sigma)\varrho}(t))| \\ &\leq |A_{\ell, (1-\sigma)\varrho}(t)| + |K_{\varrho} - K_{(1-\sigma)\varrho}| \\ &\leq |A_{\ell, (1-\sigma)\varrho}(t)| + N\sigma|K_{\varrho}|. \end{aligned}$$

Collecting all the above estimates yields that

$$|A_{\ell, \varrho}(t)| \leq \frac{1-\alpha}{(1-\varepsilon)^p} |K_{\varrho}| + C \frac{\delta}{\sigma^p} \frac{|[u < M] \cap Q_o|}{|Q_o|} |K_{\varrho}| + N\sigma|K_{\varrho}| \quad (4.1)$$

Finally, we may choose ε , σ and δ , such that

$$\frac{1-\alpha}{(1-\varepsilon)^p} \leq 1 - \frac{3}{4}\alpha, \quad N\sigma = \frac{1}{8}\alpha, \quad C \frac{\delta}{\sigma^p} \leq \frac{1}{8}\alpha.$$

□

Remark 4.1 One easily obtains the dependence of various constants on α from the above proof, namely $\varepsilon \approx \alpha$, $\sigma \approx \alpha$ and $\delta \approx \alpha^{p+1}$.

One wonders if the positivity in measure can be propagated further in time, i.e., δ can be made large by choosing a proper ε . It seems (1.2)₋ alone is insufficient. In the theory of parabolic equations, a standard tool to achieve this is a logarithmic estimate. See [[3], Chapter 2, Section 3]. We do not know if such a logarithmic estimate holds for functions in parabolic De Giorgi classes. However, we show in the following that a membership in $u \in \mathfrak{B}_p^-(E, \gamma)$ still ensures that the measure information of positivity propagates further in time.

Proposition 4.2 Suppose $u \in \mathfrak{B}_p^-(E, \gamma)$ is nonnegative. Assume for A , $M > 0$ and $\alpha \in (0, 1)$, we have $(s, s + A\varrho^p] \times K_{\varrho}(y) \subset E$ and

$$|[u(\cdot, s) > M] \cap K_{\varrho}(y)| \geq \alpha|K_{\varrho}|.$$

Then, there exist $\varepsilon > 0$ depending on the data, A and α , such that

$$|[u(\cdot, t) > \varepsilon M] \cap K_{\varrho}(y)| \geq \frac{1}{2}\alpha|K_{\varrho}|$$

for all

$$s < t < s + A\varrho^p.$$

4.1 Shrinking the measure of the set $[u \approx 0]$

We first prove the following shrinking lemma due to De Giorgi (cf. [1]).

Lemma 4.1 Let $\alpha, \delta \in (0, 1)$. Suppose there holds

$$|[u(\cdot, t) > M] \cap K_{\varrho}| \geq \alpha|K_{\varrho}| \quad \text{for all } t \in (s, s + \delta\varrho^p].$$

There exists $C > 0$ depending only on the data, such that for any positive integer j_* , we have

$$\left| \left[u \leq \frac{M}{2^{j_*}} \right] \cap Q \right| \leq \frac{C}{\alpha \delta^{\frac{1}{p}} j_*^{\frac{p-1}{p}}} |Q|, \quad \text{where } Q = K_{\varrho} \times (s, s + \delta\varrho^p].$$

Proof We assume $(y, s) = (0, 0)$ and set $k_j = 2^{-j}M$ for $j = 0, 1, \dots, j_*$. Apply (1.1)₋ for the pair of cylinders

$$K_Q \times (0, \delta Q^p] \subset K_{2Q} \times (-\delta Q^p, \delta Q^p],$$

such that

$$\iint_Q |D(u - k_j)_-|^p \, dx \, dt \leq \frac{C}{\delta Q^p} \left(\frac{M}{2^j} \right)^p |Q|. \quad (4.2)$$

Next, we apply [[3], Chapter I, Lemma 2.2] to $u(\cdot, t)$ for $t \in (0, \delta Q^p]$ over the cube K_Q , for levels $k_{j+1} < k_j$. Taking into account the measure theoretical information

$$|[u(\cdot, t) > M] \cap K_Q| \geq \alpha |K_Q| \quad \text{for all } t \in (0, \delta Q^p],$$

this gives

$$\begin{aligned} & \frac{M}{2^{j+1}} |[u(\cdot, t) < k_{j+1}] \cap K_Q| \\ & \leq \frac{C Q^{N+1}}{|[u(\cdot, t) > k_j] \cap K_Q|} \int_{[k_j < u(\cdot, t) < k_{j+1}] \cap K_Q} |Du| \, dx \\ & \leq \frac{C Q}{\alpha} \left(\int_{[k_j < u(\cdot, t) < k_{j+1}] \cap K_Q} |Du|^p \, dx \right)^{\frac{1}{p}} \\ & \quad \times |[u(\cdot, t) < k_j] - [u(\cdot, t) < k_{j+1}]] \cap K_Q|^{\frac{p-1}{p}}. \end{aligned}$$

Set

$$A_j = [u < k_j] \cap Q$$

and integrate the above estimate in dt over $(0, \delta Q^p]$; we obtain by using (4.2)

$$\begin{aligned} \frac{M}{2^j} |A_{j+1}| & \leq \frac{C Q}{\alpha} \left(\iint_Q |D(u - k_j)_-|^p \, dx \, dt \right)^{\frac{1}{p}} (|A_j| - |A_{j+1}|)^{\frac{p-1}{p}} \\ & \leq \frac{C}{\alpha \delta^{\frac{1}{p}}} \frac{M}{2^j} |Q|^{\frac{1}{p}} (|A_j| - |A_{j+1}|)^{\frac{p-1}{p}}. \end{aligned}$$

Now take the power $\frac{p}{p-1}$ on both sides of the above inequality to obtain

$$|A_{j+1}|^{\frac{p}{p-1}} \leq \frac{C}{\alpha^{\frac{p}{p-1}} \delta^{\frac{1}{p-1}}} |Q|^{\frac{1}{p-1}} (|A_j| - |A_{j+1}|).$$

Add these inequalities from 0 to $j_* - 1$ to obtain

$$j_* |A_{j_*}|^{\frac{p}{p-1}} \leq \sum_{j=0}^{j_*-1} |A_{j+1}|^{\frac{p}{p-1}} \leq \frac{C}{\alpha^{\frac{p}{p-1}} \delta^{\frac{1}{p-1}}} |Q|^{\frac{p}{p-1}}.$$

From this, we conclude

$$|A_{j_*}| \leq \frac{C}{\alpha \delta^{\frac{1}{p}} j_*^{\frac{p-1}{p}}} |Q|.$$

□

4.2 Proof of Proposition 4.2

We come back at (4.1) and choose

$$\sigma = \delta^{\frac{1}{p+1}} \left(\frac{|[u < k] \cap Q_o|}{|Q_o|} \right)^{\frac{1}{p+1}},$$

such that (4.1) becomes

$$|A_{\ell, Q}(t)| \leq \left[\frac{1-\alpha}{(1-\varepsilon)^p} + C\delta^{\frac{1}{p+1}} \left(\frac{|[u < k] \cap Q_o|}{|Q_o|} \right)^{\frac{1}{p+1}} \right] |K_Q|.$$

We choose δ and ε such that

$$C\delta^{\frac{1}{p+1}} = \frac{1}{8}\alpha, \quad \frac{1-\alpha}{(1-\varepsilon)^p} < \frac{1-\frac{1}{2}\alpha}{(1-\varepsilon)^p} \leq 1 - \frac{1}{4}\alpha. \quad (4.3)$$

As a result, we obtain

$$|A_{\ell, Q}(t)| \leq \left(1 - \frac{1}{8}\alpha\right) |K_Q| \quad \text{for all } s \leq t \leq s_1 \stackrel{\text{def}}{=} s + \delta Q^p.$$

Having ε and δ determined in (4.3), we use (1.2)_– again and repeat the above argument with

$$M_1 = \varepsilon M, \quad \ell_1 = \frac{M_1}{2^{n_1+j_1}}, \quad k_1 = \frac{M_1}{2^{j_1}},$$

where j_1 and n_1 are positive numbers to be determined. We may use the above measure theoretical information for $t \in [s, s_1]$, and apply Lemma 4.1 to obtain a refined estimate:

$$|A_{\ell_1, Q}(t)| \leq \left[\frac{1-\alpha}{(1-2^{-n_1})^2} + C\delta^{\frac{1}{p+1}} \left(\frac{1}{\alpha\delta^{\frac{1}{p}} j_1^{\frac{p-1}{p}}} \right)^{\frac{1}{p+1}} \right] |K_Q| \quad \text{for all } t \in [0, s_1].$$

We choose j_1 and n_1 , such that

$$C\delta^{\frac{1}{p+1}} \left(\frac{1}{\alpha\delta^{\frac{1}{p}} j_1^{\frac{p-1}{p}}} \right)^{\frac{1}{p+1}} \leq \frac{\delta\alpha}{4A}, \quad \frac{1-\alpha}{(1-2^{-n_1})^2} \leq 1 - \alpha + \frac{\delta\alpha}{4A}.$$

As a result, we obtain that

$$|A_{\ell_1, Q}(t)| \leq \left(1 - \alpha + \frac{\delta\alpha}{2A}\right) |K_Q| \quad \text{for all } t \in [s, s_1].$$

Now, we may proceed by induction. Suppose the construction has been made up to the $(i-1)$ -th step: the sequences $\{M_i\}$, $\{n_i\}$ and $\{j_i\}$ have been chosen up to the $(i-1)$ -th step, and we have the measure theoretical information

$$|A_{\ell_{i-1}, Q}(t)| \leq \left(1 - \alpha + (i-1)\frac{\delta\alpha}{2A}\right) |K_Q| \quad \text{for all } t \in [s_{i-1}, s_i],$$

where

$$\ell_{i-1} = \frac{M_{i-1}}{2^{n_{i-1}+j_{i-1}}} \stackrel{\text{def}}{=} \widehat{M}_{i-1}.$$

Setting

$$\ell_{i-1}^\varepsilon = \varepsilon \widehat{M}_{i-1}, \quad s_{i+1} = s_i + \delta \varrho^p, \quad Q_i = K_\varrho \times (s_i, s_{i+1}],$$

and using the above measure theoretical information at $t = s_i$, we can repeat the above argument to obtain that, for all $t \in [s_i, s_{i+1}]$,

$$|A_{\ell_{i-1}^\varepsilon, \varrho}(t)| \leq \left[\frac{1 - \alpha + (i-1) \frac{1}{2A} \delta \alpha}{(1 - \varepsilon)^2} + C \delta^{\frac{1}{p+1}} \left(\frac{|[u < \ell_{i-1}] \cap Q_i|}{|Q_i|} \right)^{\frac{1}{p+1}} \right] |K_\varrho|.$$

Assuming $(i-1)\delta < A$, we may choose ε and δ as in (4.3); this ensures

$$|A_{\ell_{i-1}^\varepsilon, \varrho}(t)| \leq \left(1 - \frac{1}{8}\alpha\right) |K_\varrho| \quad \text{for all } t \in [s_i, s_{i+1}].$$

Now, we set

$$M_i = \varepsilon \widehat{M}_{i-1}, \quad \ell_i = \frac{M_i}{2^{n_i+j_i}}, \quad k_i = \frac{M_i}{2^{j_i}},$$

where j_i and n_i are to be determined. Then, we use the above measure theoretical information in Lemma 4.1 to obtain a refined estimate: for all $t \in [s_i, s_{i+1}]$,

$$|A_{\ell_i, \varrho}(t)| \leq \left[\frac{1 - \alpha + (i-1) \frac{1}{2A} \delta \alpha}{(1 - 2^{-n_i})^2} + C \delta^{\frac{1}{p+1}} \left(\frac{1}{\alpha \delta^{\frac{1}{p}} j_i^{\frac{p-1}{p}}} \right)^{\frac{1}{p+1}} \right] |K_\varrho|.$$

We choose j_i and n_i , such that

$$\begin{aligned} C \delta^{\frac{1}{p+1}} \left(\frac{1}{\alpha \delta^{\frac{1}{p}} j_i^{\frac{p-1}{p}}} \right)^{\frac{1}{p+1}} &\leq \frac{\delta \alpha}{4A}, \\ \frac{1 - \alpha + (i-1) \frac{1}{2A} \delta \alpha}{(1 - 2^{-n_i})^2} &\leq 1 - \alpha + (i-1) \frac{\delta \alpha}{2A} + \frac{\delta \alpha}{4A}. \end{aligned}$$

As a result, we obtain that for all times $t \in [s_i, s_{i+1}]$,

$$|A_{\ell_i, \varrho}(t)| \leq \left(1 - \alpha + i \frac{\delta \alpha}{2A}\right) |K_\varrho|.$$

The above argument terminates if $i\delta \geq A$, and we reach the desired conclusion with the choice

$$\varepsilon M = \frac{M_i}{2^{n_i+j_i}}.$$

5 Hölder continuity for functions in $\mathfrak{B}_p$

Theorem 5.1 *If $u \in \mathfrak{B}_p(E; \gamma)$, there are constants $C > 0$ and $0 < \beta < 1$ depending only on the data, such that for every pair of cylinders $(y, s) + Q_\varrho \subset (y, s) + Q_R \subset E$, we have*

$$\operatorname{ess\,osc}_{(y,s)+Q_\varrho} u \leq C \operatorname{ess\,osc}_{(y,s)+Q_R} u \cdot \left(\frac{\varrho}{R}\right)^\beta$$

For a function $u \in \mathfrak{B}_p(E; \gamma)$ and $(y, s) + Q_{2\varrho} \subset E$, we set

$$\mu^+ = \operatorname{ess\,sup}_{(y,s)+Q_{2\varrho}} u, \quad \mu^- = \operatorname{ess\,inf}_{(y,s)+Q_{2\varrho}} u, \quad \omega(2\varrho) = \operatorname{ess\,osc}_{(y,s)+Q_{2\varrho}} u = \mu^+ - \mu^-.$$

5.1 Proof by Moser's approach

The purpose of this section is to prove Theorem 5.1 using an intuitive idea of Moser. Thus, the heavy machinery of De Giorgi, such as Lemmas 2.1 and 4.1, is avoided. A similar adaption has been made to parabolic equations in [14], which however cannot be directly generalized to parabolic De Giorgi classes.

Without loss of generality, we may take $(y, s) = (0, 0)$. For ease of notation, we write $\omega = \omega(2\varrho)$. Let ε be the number determined in Proposition 4.1 with $\alpha = 1/2$. We introduce two functions:

$$w_1 \stackrel{\text{def}}{=} \varphi_1(u) = \ln \frac{\varepsilon\omega}{2(\mu^+ - u)}, \quad w_2 \stackrel{\text{def}}{=} \varphi_2(u) = \ln \frac{\varepsilon\omega}{2(u - \mu^-)}.$$

We first apply Lemma 3.3 to w_1 and w_2 . Indeed, since $u \in \mathfrak{A}_p(E; \gamma)$ we have both $\mu^+ - u$ and $u - \mu^-$ members of $\mathfrak{A}_p^-(E; \gamma)$. Therefore, we may apply Lemma 3.3 to $\mu^+ - u$ in Q_R with $\varrho < R < 2\varrho$, to obtain

$$\iint_{Q_\varrho} \left| D \ln \frac{\varepsilon\omega}{2(\mu^+ - u)} \right|^p dx dt \leq \frac{C}{(R - \varrho)^p} \iint_{Q_R} \ln \frac{\omega}{\mu^+ - u} dx dt,$$

that is, in terms of w_1 ,

$$\iint_{Q_\varrho} |Dw_1|^p dx dt \leq \frac{C}{(R - \varrho)^p} \iint_{Q_R} |w_1| dx dt + \frac{CR^{N+p}}{(R - \varrho)^p}. \quad (5.1)$$

Similar inequality holds for w_2 .

Now, we go with two alternatives: either

$$|[\mu^+ - u(\cdot, -\delta\varrho^p) \geq \tfrac{1}{2}\omega] \cap K_\varrho| \geq \tfrac{1}{2}|K_\varrho|,$$

or

$$|[u(\cdot, -\delta\varrho^p) - \mu^- \geq \tfrac{1}{2}\omega] \cap K_\varrho| \geq \tfrac{1}{2}|K_\varrho|,$$

where δ is the constant appearing in Proposition 4.1 with $\alpha = 1/2$. Let us suppose for instance the first case holds. According to Proposition 4.1, we have

$$|[\mu^+ - u(\cdot, t) \geq \tfrac{1}{2}\varepsilon\omega] \cap K_\varrho| \geq \tfrac{1}{4}|K_\varrho| \quad \text{for all } -\delta\varrho^p < t < 0.$$

In terms of w_1 , this may be rephrased as

$$|[w_1(\cdot, t) \leq 0] \cap K_\varrho| \geq \tfrac{1}{4}|K_\varrho| \quad \text{for all } -\delta\varrho^p < t < 0.$$

We may employ the Poincaré type inequality (cf. [[3], Chapter 10, Proposition 5.2]) for each time slice to $w_1(\cdot, t)$, and then a time integration over $(-\delta\varrho^p, 0)$ on both sides, and the fact that $w_1 \geq -\ln(2/\varepsilon)$ to obtain that

$$\begin{aligned} \int_{-\delta\varrho^p}^0 \int_{K_\varrho} |w_1| dx dt &= \int_{-\delta\varrho^p}^0 \int_{K_\varrho} w_{1+} dx dt + \int_{-\delta\varrho^p}^0 \int_{K_\varrho} w_{1-} dx dt \\ &\leq C\varrho \int_{-\delta\varrho^p}^0 \int_{K_\varrho} |Dw_{1+}| dx dt + C\varrho^{N+p}. \end{aligned}$$

The integral term on the right-hand side is estimated by Hölder's inequality, Young's inequality and (5.1) as

$$\begin{aligned}
 & C\varrho \int_{-\delta\varrho^p}^0 \int_{K_\varrho} |Dw_{1+}| \, dx dt \\
 & \leq C\varrho^{1+N+p-\frac{N+p}{p}} \left(\iint_{Q_\varrho} |Dw_{1+}|^p \, dx dt \right)^{\frac{1}{p}} \\
 & \leq C\varrho^{1+N+p-\frac{N+p}{p}} \left(\frac{C}{(R-\varrho)^p} \iint_{Q_R} |w_1| \, dx dt + \frac{CR^{N+p}}{(R-\varrho)^p} \right)^{\frac{1}{p}} \\
 & \leq C \frac{\varrho^{1+N+p-\frac{N+p}{p}}}{R-\varrho} \left(\iint_{Q_R} |w_1| \, dx dt \right)^{\frac{1}{p}} + C \frac{\varrho^{1+N+p}}{R-\varrho} \left(\frac{R}{\varrho} \right)^{\frac{N+p}{p}}.
 \end{aligned}$$

Thus combining above estimates, we obtain

$$\begin{aligned}
 \iint_{Q_{\delta\varrho}} |w_1| \, dx dt & \leq C \frac{\varrho^{1+N+p-\frac{N+p}{p}}}{R-\varrho} \left(\iint_{Q_R} |w_1| \, dx dt \right)^{\frac{1}{p}} \\
 & \quad + C \frac{\varrho^{1+N+p}}{R-\varrho} \left(\frac{R}{\varrho} \right)^{\frac{N+p}{p}} + C\varrho^{N+p}.
 \end{aligned}$$

An interpolation argument (cf. [[6], Theorem 1]) yields that

$$\frac{1}{(\delta\varrho)^{N+p}} \iint_{Q_{\delta\varrho}} |w_1| \, dx dt \leq C(\text{data}). \quad (5.2)$$

An application of Lemma 3.1 gives that w_{1+} belongs to the generalized $\mathfrak{A}_p^+$. As a result, Theorem 2.1 holds for w_{1+} . The supreme estimate together with (5.2) yields that

$$\operatorname{ess\,sup}_{Q_{\frac{\delta\varrho}{2}}} w_{1+} \leq \frac{C}{(\delta\varrho)^{N+p}} \iint_{Q_{\delta\varrho}} |w_1| \, dx dt \leq C(\text{data}),$$

which implies

$$\operatorname{ess\,sup}_{Q_{\frac{\delta\varrho}{2}}} u \leq \mu^+ - \frac{\varepsilon}{2e^C} \omega.$$

Therefore,

$$\operatorname{ess\,osc}_{Q_{\frac{\delta\varrho}{2}}} u \leq \left(1 - \frac{\varepsilon}{2e^C} \right) \omega.$$

A standard iteration finishes the proof.

5.2 A revisit to De Giorgi's approach

The purpose of this section is to point out that the Hölder regularity could be established with less assumptions. Namely, it suffices to assume that u is a member of $\mathfrak{A}_p^+(E; \gamma) \cap \mathfrak{B}_p^-(E; \gamma)$ or $\mathfrak{A}_p^-(E; \gamma) \cap \mathfrak{B}_p^+(E; \gamma)$. The argument is modeled on the one in [[2], Chapter III].

5.2.1 Expansion of positivity

Suppose $K_{4\varrho}(y) \times (s, s + \varrho^p] \subset E$. We show in the following that the measure information on the positivity of a non-negative member u of $\mathfrak{B}_p^-(E; \gamma)$ at $t = s$ translates into pointwise information forward in time and over a larger space cube.

Proposition 5.1 *Let $u \in \mathfrak{B}_p^-(E; \gamma)$ be non-negative. Suppose for some $M > 0$ and $\alpha \in (0, 1)$,*

$$|[u(\cdot, s) > M] \cap K_{\varrho}(y)| \geq \alpha |K_{\varrho}|.$$

Then, there exist $\eta, \delta \in (0, 1)$ depending on the data and α , such that

$$u(\cdot, t) \geq \eta M \quad \text{a.e. in } K_{2\varrho}(y),$$

for all

$$s + \frac{1}{2}\delta(4\varrho)^p < t < s + \delta(4\varrho)^p.$$

Proof Assume $(y, s) = (0, 0)$. We rephrase the starting information in a larger cube:

$$|[u(\cdot, 0) > M] \cap K_{4\varrho}(y)| \geq \alpha 4^{-N} |K_{4\varrho}|.$$

By Proposition 4.1, there exist $\delta, \varepsilon > 0$ depending only on the data and α , such that

$$|[u(\cdot, t) > \varepsilon M] \cap K_{4\varrho}| \geq \frac{1}{2} 4^{-N} \alpha |K_{4\varrho}|$$

for all

$$0 < t < \delta(4\varrho)^p.$$

Next, by Lemma 4.1, there exists $C > 0$ depending only on the data, such that for any positive integer j_* , we have

$$\left| \left[u \leq \frac{\varepsilon M}{2^{j_*}} \right] \cap Q \right| \leq \frac{C}{\alpha \delta^{\frac{1}{p}} j_*^{\frac{p-1}{p}}} |Q|, \quad \text{where } Q = K_{4\varrho} \times (0, \delta(4\varrho)^p].$$

Finally, let ν be the number claimed in Lemma 2.1. Choose j_* so large that

$$\frac{C}{\alpha \delta^{\frac{1}{p}} j_*^{\frac{p-1}{p}}} \leq \nu.$$

Thus by Lemma 2.1 with $\mu^- = 0$, we conclude that

$$u(\cdot, t) \geq \frac{\varepsilon M}{2^{j_*+1}} \quad \text{a.e. in } K_{2\varrho}$$

for all times

$$\frac{1}{2}\delta(4\varrho)^p < t < \delta(4\varrho)^p.$$

The proof is finished by choosing $\eta = \varepsilon 2^{-j_*-1}$. $\square$

Remark 5.1 By repeated applications of Proposition 5.1, for any $A > 0$ there exist $\bar{\eta} \in (0, 1)$ depending on the data, α and A , such that

$$u(\cdot, t) \geq \bar{\eta} M \quad \text{a.e. in } K_{2\varrho}(y),$$

for all times

$$s + \varrho^p < t < s + A\varrho^p.$$

5.2.2 Another Proof of Theorem 5.1

Let $\nu > 0$ be the number fixed in Lemma 2.1 with $a = 1/2$, and suppose

$$|[\mu^+ - u \geq \tfrac{1}{2}\omega] \cap Q_\varrho| \leq \nu|Q_\varrho|.$$

Then since $u \in \mathfrak{A}_p^+(E; \gamma)$, Lemma 2.1 would give us that

$$\mu^+ - u \geq \tfrac{1}{4}\omega \quad \text{a.e. in } Q_{\frac{\varrho}{2}},$$

which in turn gives the reduction in oscillation:

$$\operatorname{ess\,osc}_{Q_{\frac{\varrho}{2}}} u \leq \tfrac{3}{4}\omega.$$

Now suppose to the contrary that

$$|[\mu^+ - u \geq \tfrac{1}{2}\omega] \cap Q_\varrho| > \nu|Q_\varrho|.$$

Then, there exists some

$$-\varrho^p \leq s \leq -\tfrac{1}{2}\nu\varrho^p,$$

such that

$$|[\mu^+ - u(\cdot, s) \geq \tfrac{1}{2}\omega] \cap K_\varrho| > \tfrac{1}{2}\nu|K_\varrho|.$$

Indeed, if the above inequality were not to hold for any s in the given interval, then we would have arrived at a contradiction:

$$\begin{aligned} |[\mu^+ - u \geq \tfrac{1}{2}\omega] \cap Q_\varrho| &= \int_{-\varrho^p}^{-\frac{1}{2}\nu\varrho^p} |[\mu^+ - u(\cdot, s) \geq \tfrac{1}{2}\omega] \cap K_\varrho| \, ds \\ &\quad + \int_{-\frac{1}{2}\nu\varrho^p}^0 |[\mu^+ - u(\cdot, s) \geq \tfrac{1}{2}\omega] \cap K_\varrho| \, ds \\ &\leq \nu|Q_\varrho|. \end{aligned}$$

Since $\mu^+ - \tfrac{1}{2}\omega > \mu^- + \tfrac{1}{2}\omega$ always holds, this implies

$$|[u(\cdot, s) - \mu^- > \tfrac{1}{2}\omega] \cap K_\varrho| \geq \tfrac{1}{2}\nu|K_\varrho|.$$

Then, since $u \in \mathfrak{B}_p^-(E; \gamma)$, Proposition 5.1 (see also Remark 4.1) gives $\eta \in (0, 1)$ depending only on the data, such that

$$u(\cdot, s) - \mu^- > \eta\omega \quad \text{a.e. in } Q_{\frac{\varrho}{2}},$$

which in turn gives

$$\operatorname{ess\,osc}_{Q_{\frac{\varrho}{2}}} u \leq (1 - \eta)\omega.$$

Hence in either case, the oscillation is reduced in a smaller, quantified cylinder and the proof now may be concluded in a standard way.

6 Harnack's inequalities for functions in $\mathfrak{B}_p$

Assume

$$K_{4\varrho}(y) \times [s - (4\varrho)^p, s + (4\varrho)^p] \subset E.$$

The following Harnack's inequality is shown in [7]. See also [18].

Theorem 6.1 *Let $u \in \mathfrak{B}_p(E; \gamma)$ be non-negative. There exist $\theta \in (0, 1)$ and $C > 1$ depending only on the data, such that*

$$C^{-1} \sup_{K_\varrho(y)} u(\cdot, s - \theta\varrho^p) \leq u(y, s) \leq C \inf_{K_\varrho(y)} u(\cdot, s + \theta\varrho^p).$$

The approach used in [7] is a direct one, thus by-passing a weak Harnack inequality. On the other hand, a weak Harnack inequality is established in [18] for $p = 2$ using the Krylov–Safonov covering argument (cf. [11]). Here, we give a transparent proof of a weak Harnack inequality for the class $\mathfrak{B}_p^-(E, \gamma)$, via a measure theoretical lemma in [5], thus avoiding the heavy covering argument.

6.1 Weak Harnack inequality for functions in $\mathfrak{B}_p^-(E, \gamma)$

Assume that the cylinder

$$K_{4\varrho} \times (s, s + (4\varrho)^p] \subset E.$$

Theorem 6.2 *Let $u \in \mathfrak{B}_p^-(E; \gamma)$ be nonnegative. Then, there exist $\delta_o, q \in (0, 1)$ and $C > 1$ depending only on the data, such that*

$$C \operatorname{ess\,inf}_{K_{2\varrho}(y)} u(\cdot, t) \geq \left(\int_{K_\varrho(y)} u^q(\cdot, s) \, dx \right)^{\frac{1}{q}},$$

for all times

$$s + \frac{1}{2}\delta_o\varrho^p < t < s + \delta_o\varrho^p.$$

A combination of Theorems 6.2 and 2.1 would give another proof of Theorem 6.1 (cf. [6, 18]). The key to proving Theorem 6.2 is to show an expansion of positivity with a power-like dependence on the measure distribution of the positivity. The main tool is a certain clustering lemma from [5].

Proposition 6.1 *Let $u \in \mathfrak{B}_p^-(E; \gamma)$ be nonnegative. Suppose for some $M > 0$ and $\alpha \in (0, 1)$, there holds*

$$|[u(\cdot, s) > M] \cap K_\varrho(y)| \geq \alpha |K_\varrho|.$$

Then, there exist $\delta_o, \eta_o \in (0, 1)$ and $d > 1$ depending only on the data, such that

$$u(\cdot, t) \geq \eta_o \alpha^d M \quad \text{a.e. in } K_{2\varrho}(y),$$

for all times

$$s + \frac{1}{2}\delta_o\varrho^p < t < s + \delta_o\varrho^p.$$

Proof Assume $(y, s) = (0, 0)$. By Proposition 4.1, there exist $\delta = C^{-1}\alpha^{p+1}$ and $\varepsilon = C^{-1}\alpha$, where $C > 1$ depends only on the data, such that

$$|[u(\cdot, t) > \varepsilon M] \cap K_Q| \geq \frac{1}{2}\alpha|K_Q| \quad \text{for all } 0 < t < \delta Q^p. \quad (6.1)$$

Now, we set $Q' = K_{2Q} \times (0, \delta Q^p]$ and $Q = K_Q \times (\frac{1}{2}\delta Q^p, \delta Q^p]$. Apply (1.1) to $(u - M)_-$ with the pair of cylinders $Q \subset Q'$ to obtain

$$\iint_Q |D(u - M)_-|^p \, dx \, dt \leq C \frac{M^p}{\delta Q^p} |Q|.$$

Under the change of variables

$$x \rightarrow \frac{x}{Q}, \quad t \rightarrow \frac{t}{\delta Q^p}, \quad w = \frac{(u - M)_-}{M},$$

the above estimate reads

$$\iint_{K_1 \times (\frac{1}{2}, 1)} |Dw|^p \, dx \, dt \leq \frac{C}{\alpha^{p+1}}. \quad (6.2)$$

In order to use the lemma in [5], we introduce $v = (1 - w)/\varepsilon$. Then, in terms of v , the measure information (6.1) reads

$$|[v(\cdot, t) > 1] \cap K_1| \geq \frac{1}{2}\alpha|K_1| \quad \text{for all } \frac{1}{2} < t < 1. \quad (6.3)$$

Combining (6.2) and (6.3), there exists $\tau_1 \in (\frac{1}{2}, 1]$ satisfying

$$\int_{K_1} |Dv(\cdot, \tau_1)| \, dx \leq \frac{C}{\alpha^{p+1}}, \quad |[v(\cdot, \tau_1) > 1] \cap K_1| \geq \frac{1}{2}\alpha|K_1|.$$

Now, an application of the lemma in [5] yields that there exist $y_o \in K_1$ and $\sigma = C^{-1}\alpha^{4+\frac{1}{p}}$ for some absolute constant $C > 1$, such that

$$|[v(\cdot, \tau_1) > \frac{1}{2}] \cap K_\sigma(y_o)| \geq \frac{1}{2}|K_\sigma|.$$

Returning to the original coordinates gives

$$|[u(\cdot, t_1) > \frac{1}{2}\varepsilon M] \cap K_{\sigma Q}(x_o)| \geq \frac{1}{2}|K_{\sigma Q}|$$

for some $x_o \in K_Q$ and $\frac{1}{2}\delta Q^p < t_1 < \delta Q^p$. Using this measure information, we may apply Proposition 5.1 repeatedly (choosing $\alpha = 1/2$ in Proposition 5.1) to obtain $\bar{\eta}, \bar{\delta} \in (0, 1)$ depending only on the data, such that for $n = 1, 2, \dots$,

$$u(\cdot, t) \geq \frac{1}{2}\varepsilon \bar{\eta}^n M \quad \text{a.e. in } K_{2^n \sigma Q}(x_o)$$

for all

$$t_{n-1} + \frac{1}{2}\bar{\delta}(2^n \sigma Q)^p < t < t_{n-1} + \bar{\delta}(2^n \sigma Q)^p \stackrel{\text{def}}{=} t_n.$$

Finally, we choose n so large that $2^n \sigma = 3$, such that $K_{2Q} \subset K_{2^n \sigma Q}(x_o)$. At the same time, taking into consideration of the power-like dependence on α of ε and σ , there exist $\eta_o \in (0, 1)$ and $d > 1$ depending only on the data, such that

$$\varepsilon \bar{\eta}^n = \varepsilon 2^{n \ln \bar{\eta}} = \varepsilon \left(\frac{\sigma}{3}\right)^{-\ln \bar{\eta}} \stackrel{\text{def}}{=} \eta_o \alpha^d.$$

The time interval for such positivity is

$$t_n - \frac{1}{2}\bar{\delta}(3Q)^p = t_n - \frac{1}{2}\bar{\delta}(2^n \sigma Q)^p < t < t_n.$$

We calculate t_n :

$$t_n = t_1 + \sum_{i=1}^{n-1} \bar{\delta} (2^i \sigma \varrho)^p = t_1 + \frac{\bar{\delta} (2^n \sigma \varrho)^p - \bar{\delta} (2 \sigma \varrho)^p}{2^p - 1} = t_1 + \bar{\delta} \varrho^p \frac{3^p - (2\sigma)^p}{2^p - 1}.$$

With no loss of generality, we assume $\sigma < 1/4$. In this way, it is not hard to see that there exist $\delta_o, \eta_o \in (0, 1)$ depending only on the data, such that

$$u(\cdot, t) \geq \eta_o \alpha^d M \quad \text{a.e. in } K_{2\varrho}$$

for all

$$t_1 + \frac{1}{2} \delta_o \varrho^p < t < t_1 + \delta_o \varrho^p.$$

The qualitative location of $t_1 \in (0, \varrho^p)$ may be removed by repeated applications of this conclusion. The proof is then finished by properly redefining δ_o . $\square$

6.1.1 Proof of Theorem 6.2

Assume $(y, s) = (0, 0)$ and define

$$I \stackrel{\text{def}}{=} \operatorname{ess\,inf}_{K_{2\varrho} \times (\frac{1}{2} \delta_o \varrho^p, \delta_o \varrho^p]} u.$$

We first estimate the L^q -norm of $u(\cdot, 0)$ by its measure distribution:

$$\begin{aligned} \int_{K_\varrho} u^q(\cdot, 0) \, dx &= q \int_0^\infty |[u(\cdot, 0) > M] \cap K_\varrho| M^{q-1} \, dM \\ &\leq q \int_I |[u(\cdot, 0) > M] \cap K_\varrho| M^{q-1} \, dM + I^q |K_\varrho|. \end{aligned} \quad (6.4)$$

By Theorem 6.2, there exist $d > 1$ and $\eta_o \in (0, 1)$ depending only on the data, such that

$$I \geq \eta_o M \left(\frac{|[u(\cdot, 0) > M] \cap K_\varrho|}{|K_\varrho|} \right)^d.$$

Thus, we may estimate the first term on the right-hand side of (6.4) by

$$q \int_I |[u(\cdot, 0) > M] \cap K_\varrho| M^{q-1} \, dM \leq \frac{q I^{\frac{1}{d}}}{\eta_o^{\frac{1}{d}} |K_\varrho|} \int_I M^{q-\frac{1}{d}-1} \, dM.$$

Now, we stipulate to take $q < 1/d$, such that the improper integral on the right-hand side converges. In such a way, the right-hand side of the above inequality is bounded above by $C I^q |K_\varrho|$. Hence, putting everything back in (6.4), we obtain the desired conclusion.

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