The obstacle problem for degenerate doubly nonlinear equations of porous medium type

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Received: 24 March 2020 / Accepted: 30 May 2020 / Published online: 17 June 2020
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Abstract

We prove the existence of nonnegative variational solutions to the obstacle problem associated with the degenerate doubly nonlinear equation

$$\partial_t b(u) - \operatorname{div}(Df(Du)) = 0,$$

where the nonlinearity $b : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is increasing, piecewise C^1 and satisfies a polynomial growth condition. The prototype is $b(u) := u^m$ with $m \in (0, 1)$. Further, $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is convex and fulfills a standard p -growth condition. The proof relies on a nonlinear version of the method of minimizing movements.

Keywords Porous medium equation · Doubly nonlinear equations · Existence · Variational solutions · Minimizing movements

Mathematics Subject Classification 35K86 · 49J40 · 49J45

1 Introduction and results

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $(0, T)$ with $0 < T < \infty$ a finite time interval. In the following, $\Omega_T := \Omega \times (0, T)$ denotes a space-time cylinder. The prototype of the equations considered in the present paper is

$$\partial_t u^m - \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } \Omega_T \quad (1.1)$$

with parameters $m \in (0, \infty)$ and $p \in (1, \infty)$. For $m = 1$ and $p \in (1, \infty)$, the preceding equation reduces to the parabolic p -Laplace equation, while it is known as the porous medium equation if $m \in (0, \infty)$ and $p = 2$. Based on the behavior of solutions, doubly nonlinear equations can be subdivided into *slow diffusion equations* with $p - 1 > m$ and *fast diffusion equations* with $p - 1 < m$. Further, we distinguish between doubly degenerate equations ($p > 2$, $0 < m < 1$), singular-degenerate equations ($1 < p < 2$, $0 < m < 1$),

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degenerate-singular equations ($p > 2$, $m > 1$) and doubly singular equations ($1 < p < 2$, $m > 1$), cf. [18]. The porous medium equation and related doubly nonlinear equations are relevant in models for fluid dynamics, filtration and soil science, cf. [4–6, 21, 28]. In the present paper, we are concerned with the obstacle problem to doubly nonlinear equations of doubly degenerate and singular-degenerate type. In order to treat (1.1), we use an approach that originates from Lichniewsky and Temam [23] and has later been developed by Bögelein, Duzaar, Marcellini and Scheven [8–11] to cover a wide range of parabolic problems. More precisely, we are concerned with *variational solutions* to the Cauchy–Dirichlet problem associated with (1.1) for given initial and boundary values $g : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$, i.e., functions $u : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ satisfying the variational inequality

$$\begin{aligned} \frac{1}{p} \iint_{\Omega_T} |Du|^p \, dx dt &\leq \iint_{\Omega_T} \partial_t v (v^m - u^m) \, dx dt + \frac{1}{p} \iint_{\Omega_T} |Dv|^p \, dx dt \\ &\quad - \mathfrak{B}[u(T), v(T)] + \mathfrak{B}[g(0), v(0)] \end{aligned}$$

associated with (1.2) for any admissible comparison map $v : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$. Here, we used the abbreviation

$$\mathfrak{B}[u, v] = \int_{\Omega} \mathfrak{b}[u, v] \, dx := \int_{\Omega} \left[\frac{1}{m+1} v^{m+1} - \frac{1}{m+1} u^{m+1} - u^m (v - u) \right] dx$$

for $u, v : \Omega \rightarrow \mathbb{R}_{\geq 0}$. Formally, the variational inequality can be derived by multiplying (1.2) by $v - u$, where $v : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ coincides with u on the lateral boundary $\partial\Omega \times (0, T)$ and then integrating the result over Ω_T . For the diffusion part, we use integration by parts and the convexity of $\frac{1}{p} |\cdot|^p$. Finally, by integration by parts the time derivative is shifted from u to v , leading in particular to the integrals over the top and bottom of the space-time cylinder on the right-hand side of the variational inequality. In the present paper, we impose an additional pointwise obstacle condition of the form $u \geq \psi$ for some obstacle function $\psi : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$. This means that u is a variational solution to the obstacle problem associated with Eq. (1.2) and initial and boundary values $g : \Omega \rightarrow \mathbb{R}_{\geq 0}$ if u coincides with g on the parabolic boundary $(\Omega \times \{0\}) \cup (\partial\Omega \times (0, T))$ and satisfies the obstacle condition $u \geq \psi$ a.e. in Ω_T and the preceding variational inequality holds true for any comparison map v with boundary values g and $v \geq \psi$ a.e. in Ω_T . For the precise definition, cf. Definition 1.1. At this stage, some words on the history of the problem are in order. First, the seminal work of Grange and Mignot [17] and Alt and Luckhaus [3] should be mentioned. In [3], the authors were among other things concerned with the obstacle problem associated with doubly nonlinear equations of the type

$$\partial_t b(u) - \operatorname{div}(\mathbf{a}(b(u), Du)) = \mathbf{f}(b(u)),$$

where b is the gradient of a convex C^1 -function with $b(0) = 0$, $\mathbf{a}(b(z), \xi)$ is continuous in z and ξ and fulfills an ellipticity and $(p-1)$ -growth condition with respect to the gradient variable and $\mathbf{f}(b(z))$ is continuous in z and satisfies a suitable growth condition. Further, a two-sided obstacle condition is imposed with obstacle functions $\psi_{\pm} \in L^p(0, T; W^{1,p}(\Omega)) \cap L^{\infty}(\Omega_T)$ with $\partial_t \psi_{\pm} \in L^1(\Omega_T)$ and $\psi_- \leq \psi_+$ a.e. in Ω_T . Under these assumptions, the existence of variational solutions has been established via time discretization for Neumann boundary values. However, the proof extends to the case of an additional Dirichlet boundary condition with zero boundary values on a part of the boundary. Later, Bernis [7] showed the existence of weak solutions to the Cauchy problem associated with higher order doubly nonlinear equations on unbounded domains. Further, Ivanov,

Mkrtychyan and Jäger [18–20] used regularization and a priori Hölder estimates to prove the existence of regular weak solutions to the Cauchy–Dirichlet problem associated with doubly nonlinear equations. The boundary values satisfy $g \in W^{1,p}(\Omega_T) \cap L^\infty(\Omega_T)$ and an additional continuity assumption with respect to space and time. A different approach has been pursued by Akagi and Stefanelli [2] in order to treat the Cauchy–Dirichlet problem with homogenous Dirichlet boundary values associated with doubly nonlinear equations of the type

$$\partial_t b(u) - \operatorname{div}(\mathbf{a}(Du)) \ni \mathbf{f},$$

where $b \subset \mathbb{R} \times \mathbb{R}$ and $\mathbf{a} \subset \mathbb{R}^n \times \mathbb{R}^n$ are maximal monotone graphs that fulfill polynomial growth conditions. The authors solve the problem by means of elliptic regularization (Weighted Energy Dissipation Functional method) after transforming it into the dual formulation $-\operatorname{div}(\mathbf{a}(Db^{-1}(v))) \ni \mathbf{f} - \partial_t v$. Recently, by a nonlinear version of the method of minimizing movements Bögelein, Duzaar, Marcellini and Scheven [11] were able to prove the existence of nonnegative variational solutions to the Cauchy–Dirichlet problem with time-independent boundary values associated with

$$\partial_t b(u) - \operatorname{div}(D_\xi f(x, u, Du)) = -D_u f(x, u, Du),$$

where $b : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is continuous, piecewise C^1 and satisfies a polynomial growth condition with $b(0) = 0$. Further, $(u, \xi) \mapsto f(x, u, \xi)$ is convex for a.e. $x \in \Omega$ and f satisfies a coercivity, but not necessarily a growth condition. This allows f to have nonstandard growth like exponential or (p, q) -growth with $1 < p < q < \infty$. Note that the required nonnegativity of the solutions is an obstacle condition with obstacle function $\psi \equiv 0$. In the case of singly nonlinear equations of p -Laplace type, Bögelein, Duzaar and Scheven [12] established the existence of variational solutions to the obstacle problem with a far more general obstacle function $\psi \in L^2(\Omega_T) \cap L^p(0, T; W^{1,p}(\Omega))$ and time-dependent boundary values via the classical method of minimizing movements. Finally, the author has been concerned with the singular equation/system

$$\partial_t(|u|^{m-1}u) - \operatorname{div}(Df(Du)) = 0,$$

where $m > 1$, f is convex and satisfies a standard p -growth and coercivity condition. By the nonlinear minimizing movements scheme developed in [11] and suitable approximation arguments, the existence of signed or vector-valued variational solutions to the Cauchy–Dirichlet problem with time-dependent boundary values and the existence of solutions to the obstacle problem with time-dependent obstacle function have been established, cf. [26, 27]. More precisely, the boundary values and the obstacle function are contained in the space $L^p(0, T; W^{1,p}(\Omega))$ with time derivative in $L^1(0, T; L^{m+1}(\Omega))$ and initial values in $L^{m+1}(\Omega)$. In the present paper, the question of uniqueness will not be discussed, since this is a delicate and widely open issue for doubly nonlinear equations. We refer to [15] for a counterexample and to [3, 15] for sufficient conditions.

1.1 The general doubly nonlinear equation

In the present paper, we are concerned with the doubly nonlinear equation

$$\partial_t b(u) - \operatorname{div}(D_\xi f(Du)) = 0 \quad \text{in } \Omega_T. \quad (1.2)$$

Here, we assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is a Borel-measurable, convex function that fulfills the growth and coercivity condition

$$\nu|\xi|^p \leq f(\xi) \leq L(1 + |\xi|^p) \quad (1.3)$$

with constants $0 < \nu \leq L$ for all $\xi \in \mathbb{R}^n$. Observe that (1.3) and the convexity of f together imply that f is locally Lipschitz continuous. More precisely,

$$|f(\xi) - f(\eta)| \leq c(n, p, L)(1 + |\xi|^{p-1} + |\eta|^{p-1})|\xi - \eta| \quad (1.4)$$

holds true for any $\xi, \eta \in \mathbb{R}^n$, cf. [24, Eq. (2.9)]. Further, the nonlinearity $b : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is continuous and piecewise C^1 in $\mathbb{R}_{>0}$. Replacing $b(u)$ by $b(u) - b(0)$, we suppose without loss of generality that $b(0) = 0$. Moreover, we assume that there exist constants $0 < \ell \leq m \leq 1$ such that

$$\ell \leq \frac{ub'(u)}{b(u)} \leq m \quad (1.5)$$

holds true whenever $u > 0$, $b(u) > 0$ and $b'(u)$ exists. In particular, this implies that $b'(u) \geq 0$ if it exists. Then, the primitive of b defined by

$$\Phi(u) := \int_0^u b(s) ds \quad \text{for any } u \geq 0$$

is a convex C^1 function with $\Phi(0) = 0$. Further, the convex conjugate (Fenchel conjugate) of Φ is defined by

$$\Phi^*(v) := \sup_{u \geq 0} \{uv - \Phi(u)\} \quad \text{for any } v \geq 0,$$

which immediately implies Fenchel's inequality

$$uv \leq \Phi(u) + \Phi^*(v) \quad \text{for all } u, v \geq 0.$$

Since Φ is convex, we easily compute that equality holds for $v = b(u)$, i.e.

$$\Phi^*(b(u)) = b(u)u - \Phi(u) \quad \text{for any } u \geq 0.$$

At this stage, we define

$$\begin{aligned} \mathfrak{b}[u, v] &:= \Phi(v) - \Phi(u) - b(u)(v - u) \\ &= \Phi(v) + \Phi^*(b(u)) - b(u)v \end{aligned}$$

for any $u, v \geq 0$. In the variational inequality associated with (1.2), we will use boundary terms

$$\mathfrak{B}[u, v] := \int_{\Omega} \mathfrak{b}[u, v] dx$$

for functions $u, v : \Omega \rightarrow \mathbb{R}_{\geq 0}$. Furthermore, in order to be able to formulate solutions to the obstacle problem, we define the Orlicz space related to Φ and some domain $A \subset \mathbb{R}^k$, $k \in \mathbb{N}$, by

$$L^\Phi(A) := \left\{ v : A \rightarrow \mathbb{R} \text{ measurable} : \int_A \Phi(a|v|) dx < \infty \text{ for some } a > 0 \right\}.$$

For details on Orlicz spaces, we refer to the monographs [1, 25]. By the assumptions on b , we obtain that both Φ and Φ^* satisfy the ∇_2 and the Δ_2 condition (see (2.3)). In particular, the Δ_2 condition on Φ implies that an equivalent definition of the Orlicz space above is given by

$$L^\Phi(A) := \left\{ v : A \rightarrow \mathbb{R} \text{ measurable} : \int_A \Phi(|v|) dx < \infty \right\}.$$

Henceforth, we often abbreviate the modular (see [25, Chapter III.3.4]) by

$$\rho_A(v) := \int_A \Phi(|v|) dx.$$

In the present paper, we assume that $L^\Phi(A)$ is equipped with the Orlicz norm

$$\|v\|_{L^\Phi(A)} := \sup \left\{ \left| \int_A vw dx \right| : \int_A \Phi^*(|w|) dx \leq 1 \right\},$$

which is equivalent to the Luxemburg norm

$$\|v\|'_{L^\Phi(A)} := \inf \left\{ \lambda > 0 : \int_A \Phi\left(\frac{|v|}{\lambda}\right) dx \leq 1 \right\}.$$

Dealing with these norms is not always straightforward. However, since Φ fulfills the Δ_2 condition, norm convergence is equivalent to modular convergence, i.e.,

$$v_i \rightarrow v \text{ strongly in } L^\Phi(A) \text{ as } i \rightarrow \infty \Leftrightarrow \lim_{i \rightarrow \infty} \rho_A(v_i - v) = 0 \quad (1.6)$$

holds true for any $v_i, v \in L^\Phi(A)$, $i \in \mathbb{N}$, and for sets $S \subset L^\Phi(A)$ we know that

$$S \text{ is norm bounded} \Leftrightarrow \sup_{v \in S} \rho_A(v) < \infty, \quad (1.7)$$

cf. [25, Chapter III.3.4]. Analogously, we define the Orlicz space $L^{\Phi^*}(A)$, the modular $\rho_A^*(\cdot)$ and the norms $\|\cdot\|_{L^{\Phi^*}(A)}$ and $\|\cdot\|'_{L^{\Phi^*}(A)}$ related to the convex conjugate Φ^* . In this setting, the *generalized Hölder's inequality*

$$\int_A |vw| dx \leq \|v\|_{L^\Phi(A)} \|w\|'_{L^{\Phi^*}(A)} \quad (1.8)$$

holds true for functions $v \in L^\Phi(A)$, $w \in L^{\Phi^*}(A)$, cf. [25, Chapter III.3.3]. Further, the Δ_2 condition on Φ implies that $L^\Phi(\Omega)$ is separable and that the dual space of $(L^\Phi(A), \|\cdot\|_{L^\Phi(A)})$ is isometrically isomorphic to $(L^{\Phi^*}(A), \|\cdot\|'_{L^{\Phi^*}(A)})$, cf. [25, Chapters III, IV].

1.2 The main result

In order to formulate a boundary condition, we consider the affine parabolic space $g + L^p(0, T; W_0^{1,p}(\Omega))$ consisting of the functions $v \in L^p(0, T; W^{1,p}(\Omega))$ such that

$v(t) \in g(t) + W_0^{1,p}(\Omega)$ for a.e. $t \in (0, T)$. In the present paper, we assume that nonnegative boundary values $g : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ are given by

$$g \in L^p(0, T; W^{1,p}(\Omega)) \text{ with } \partial_t g \in L^1(0, T; L^\Phi(\Omega)) \text{ and } g_o := g(0) \in L^\Phi(\Omega) \quad (1.9)$$

and that the nonnegative obstacle function $\psi : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ satisfies

$$\begin{cases} \psi \in g + L^p(0, T; W_0^{1,p}(\Omega)) \text{ with } \partial_t \psi \in L^1(0, T; L^\Phi(\Omega)), \psi(0) \in L^\Phi(\Omega) \\ \text{and } \psi \leq g \text{ a.e. in } \Omega_T. \end{cases} \quad (1.10)$$

Definition 1.1 (Variational solution) Assume the convex integrand f satisfies (1.3) and that (1.9) and (1.10) hold true. A measurable nonnegative map $u : \Omega_T \rightarrow \mathbb{R}_{\geq 0}$ in the class

$$u \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega))) \text{ with } u \geq \psi \text{ a.e. in } \Omega_T$$

is called a variational solution to the obstacle problem associated with (1.2) if and only if it solves the variational inequality

$$\begin{aligned} \iint_{\Omega_\tau} f(Du) \, dx dt &\leq \iint_{\Omega_\tau} \partial_t v(b(v) - b(u)) \, dx dt + \iint_{\Omega_\tau} f(Dv) \, dx dt \\ &\quad - \mathfrak{B}[u(\tau), v(\tau)] + \mathfrak{B}[g_o, v(0)] \end{aligned} \quad (1.11)$$

for a.e. $\tau \in [0, T]$ and any comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T .

At this stage, we are able to state the main result of the present paper. Note that we can conclude from (1.11) that u attains the initial datum g_o in the L^Φ -sense; see Lemma 2.18.

Theorem 1.2 Assume that the convex integrand f fulfills (1.3) and that the hypotheses (1.9) and (1.10) are satisfied. Then, there exists a variational solution

$$u \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega))) \text{ with } u \geq \psi \text{ a.e. in } \Omega_T$$

to (1.2) in the sense of Definition 1.1. Furthermore, u attains the initial datum g_o in the L^Φ -sense.

1.3 Methods of proof

First, in Sect. 2, we collect lemmas that we need in the subsequent proofs of the existence theorems. Their proofs are already known or easy. Next, in Sect. 3, we prove a preliminary existence result for regular data, i.e., boundary values and an obstacle with time derivative in $L^2(\Omega_T) \cap L^p(0, T; W^{1,p}(\Omega))$ and initial values in $L^2(\Omega) \cap W^{1,p}(\Omega)$. Since $b'(0)$ is infinite, we assume that g and ψ are bounded away from zero. The proof relies on a nonlinear version of the method of minimizing movements. More precisely, we fix a step size $h_K := T/K$ for some $K \in \mathbb{N}$ and consider time slices of Ω_T at the time points ih_K , $i \in \{0, \dots, K\}$. Then, we set $u_0 = g(0)$ and iteratively define minimizers u_i of the elliptic variational functionals

$$F_i[v] := \int_{\Omega} f(Dv) dx + \frac{1}{h} \int_{\Omega} \mathfrak{b}[u_{i-1}, v] dx$$

in the class $v \in L^{\Phi}(\Omega) \cap (g(ih_K) + W_0^{1,p}(\Omega))$. Observe that $\mathfrak{b}[u, v] = \frac{1}{2} \|u - v\|_{L^2(\Omega)}^2$ if $b(u) = u$ and hence the scheme reduces to the classical method of minimizing movements in the linear case. Next, in Sect. 3.2, we derive suitable energy estimates for the minimizers u_i . Here, the stronger assumptions on the data are crucial. As in the classical scheme, we assemble the functions u_i to a map $u^{(K)} : \Omega \times (-h_K, T] \rightarrow \mathbb{R}_{\geq 0}$ that is piecewise constant with respect to time by setting $u^{(K)}(t) := u_i$ for $t \in ((i-1)h_K, ih_K]$, $i \in \{0, \dots, K\}$. By the energy estimates from Sect. 3.2 and the compactness result 2.21, we find a subsequence and a suitable limit map $u \in L^{\infty}(0, T; L^{\Phi}(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega)))$ such that $u^{(K)} \rightharpoonup u$ weakly in $L^p(0, T; W^{1,p}(\Omega))$ and $u^{(K)} \rightarrow u$ a.e. in Ω_T . In Sect. 3.4, we assemble the functionals F_i such that $u^{(K)}$ inherits a minimizing property and thus deduce a preliminary variational inequality for $u^{(K)}$. Finally, in Sect. 3.5, we pass to the limit $K \rightarrow \infty$ in these preliminary inequalities, which allows us to show that u is the desired variational solution. In Sect. 4, we relax the regularity assumptions on the spatial variables of the data. More precisely, the time derivatives of the boundary values and obstacle are now contained in $L^2(0, T; L^{\Phi}(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$ and the initial values in $L^{\Phi}(\Omega)$. Further, g and ψ may attain the value zero. The proof of the existence result relies on standard mollification of the boundary values and obstacle with respect to the spatial variables. Since the regularized data g_{ε} and ψ_{ε} satisfy the assumptions of Sect. 3, we find variational solutions u_{ε} , $\varepsilon > 0$, corresponding to g_{ε} and ψ_{ε} . By the energy bound from Lemma 2.20, we deduce that a subsequence converges weakly to a suitable limit map $u \in L^{\infty}(0, T; L^{\Phi}(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega)))$. Passing to the limit $\varepsilon \downarrow 0$ in the variational inequalities fulfilled by u_{ε} , we conclude that u is the desired variational inequality to g and ψ . To this end, it is important to understand that weak convergence $u_{\varepsilon} \xrightarrow{*} u$ weakly in $L^{\infty}(0, T; L^{\Phi}(\Omega))$ as $\varepsilon \downarrow 0$ in general does not imply $b(u_{\varepsilon}) \xrightarrow{*} b(u)$ weakly in $L^{\infty}(0, T; L^{\Phi^*}(\Omega))$ as $\varepsilon \downarrow 0$. Even if there is a convergent subsequence, the limit might not be $b(u)$. Therefore, we need to use a technique similar to the one in [13, Lemma 9.1] to establish the desired convergence assertion. Finally, in Sect. 5, we give the proof of Theorem 1.2. The technique is similar to the one in Sect. 4, but based on the time mollification procedure described in Sect. 2.3 instead of standard mollification.

2 Preliminaries

2.1 Technical lemmas

In this section, we collect some lemmas that we will need for the proof of the existence result. For the proofs of the lemmas 2.1, 2.3, 2.6, 2.7, 2.9 and 2.10, we refer to [11, Section 2.1].

Lemma 2.1 *For any continuous, piecewise C^1 function $b : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfying (1.5) and any $\lambda > 1, u > 0$, we have that:*

$$\lambda^{\ell} b(u) \leq b(\lambda u) \leq \lambda^m b(u), \quad (2.1)$$

$$\frac{\ell}{m} \lambda^{\ell-1} b'(u) \leq b'(\lambda u) \leq \frac{m}{\ell} \lambda^{m-1} b'(u), \quad (2.2)$$

$$\lambda^{\ell+1} \Phi(u) \leq \Phi(\lambda u) \leq \lambda^{m+1} \Phi(u), \quad (2.3)$$

$$\lambda^{\frac{m+1}{m}} \Phi^*(u) \leq \Phi^*(\lambda u) \leq \lambda^{\frac{\ell+1}{\ell}} \Phi^*(u). \quad (2.4)$$

Remark 2.2 Assuming (1.5), we infer from Lemma 2.1 that

$$b(1) \min\{u^\ell, u^m\} \leq b(u) \leq b(1) \max\{u^\ell, u^m\}$$

for any $u > 0$.

Lemma 2.3 Assume that b satisfies (1.5). Then,

$$\frac{1}{m+1} u b(u) \leq \Phi(u) \leq \frac{1}{\ell} \Phi^*(b(u)) \leq \frac{m}{\ell(m+1)} u b(u)$$

holds true for any $u \geq 0$.

Remark 2.4 Combining Lemma 2.3 with (1.7), we find that $v \in L^\infty(0, T; L^\Phi(\Omega))$ implies $b(v) \in L^\infty(0, T; L^{\Phi^*}(\Omega))$.

Lemma 2.5 For any $v \in L^\Phi(\Omega)$, we have that

$$\|v\|'_{L^\Phi(\Omega)} \leq 1 + \varrho_\Omega(v)^{\frac{1}{\ell+1}}. \quad (2.5)$$

Proof Set

$$M := \varrho_\Omega(v) < \infty.$$

If $M \leq 1$, by definition of the Luxemburg norm, we have $\|v\|'_{L^\Phi(\Omega)} \leq 1$. On the other hand, if $M > 1$, by Lemma 2.1, we find that

$$1 \geq \frac{1}{M} \varrho_\Omega(v) \geq \varrho_\Omega(M^{-\frac{1}{\ell+1}} |v|)$$

and therefore that $\|v\|'_{L^\Phi(\Omega)} \leq M^{\frac{1}{\ell+1}}$. Combining the cases implies (2.5). $\square$

Lemma 2.6 Assume that b satisfies (1.5). Then, for all $u, v \geq 0$, we have that

$$\begin{aligned} \Phi^*(b(u)) &\leq 2b[u, v] + 2^{m+2} \Phi(v), \\ \Phi(v) &\leq 2b[u, v] + 2^{2+\frac{1}{\ell}} \Phi^*(b(u)). \end{aligned}$$

Lemma 2.7 Assume that (1.5) is satisfied. Then, there exists a constant $c = c(m, \ell)$ such that the estimates

$$\begin{aligned}
\mathfrak{b}[u, v] &\leq (b(v) - b(u))(v - u) \\
&\leq |\sqrt{vb(v)} - \sqrt{ub(u)}|^2 \\
&\leq c |\sqrt{\Phi(v)} - \sqrt{\Phi(u)}|^2 \\
&\leq c^2 \mathfrak{b}[u, v]
\end{aligned}$$

hold true for all $u, v \geq 0$.

Lemma 2.8 Assume that (1.5) is in force. If $(v_i)_{i \in \mathbb{N}}$ is a sequence in $L^\Phi(\Omega)$ and $v \in L^\Phi(\Omega)$ such that $v_i \rightarrow v$ in $L^\Phi(\Omega)$ as $i \rightarrow \infty$, we also have that $v_i \rightarrow v$ in $L^{\ell+1}(\Omega)$.

Proof The Lebesgue space $L^{\ell+1}(\Omega)$ is obviously related to the function $\Psi(x) := |x|^{\ell+1}$. By (2.3) for any $\varepsilon > 0$, we obtain that

$$\left| \frac{x}{\varepsilon} \right|^{\ell+1} = \varepsilon^{-\ell-1} \Phi(1)^{-1} x^{\ell+1} \Phi(1) \leq \varepsilon^{-\ell-1} \Phi(1)^{-1} \Phi(x) \quad \forall x \geq 1.$$

Hence, Φ is completely stronger than Ψ , cf. [25, Definition 2.2.1]. By (1.6), we find that $\lim_{i \rightarrow \infty} \varrho_\Omega(v_i - v) = 0$. Therefore, from the definition of the Luxemburg norm and [25, Theorem 5.3.1], we infer

$$\lim_{i \rightarrow \infty} \|v_i - v\|_{L^{\ell+1}(\Omega)} = \lim_{i \rightarrow \infty} \|v_i - v\|'_{L^\Psi(\Omega)} = 0.$$

This concludes the proof of the lemma. $\square$

Lemma 2.9 Assume that (1.5) holds true. If $(u_i)_{i \in \mathbb{N}}$ is a sequence in $L^\Phi(\Omega)$ such that $\Phi(u_i) \rightarrow \Phi(u)$ in $L^1(\Omega)$ for some $u \in L^\Phi(\Omega)$, then $u_i \rightarrow u$ in $L^\Phi(\Omega)$ as $i \rightarrow \infty$. In particular, we have that

$$\varrho_\Omega(u_i - u_o) \leq \int_\Omega |\Phi(u_i) - \Phi(u_o)| \, dx dt.$$

Lemma 2.10 Suppose that b satisfies (1.5). If $v \in L^1(\Omega_T)$ is given with $\partial_t v \in L^\Phi(\Omega_T)$ and $v(0) \in L^\Phi(\Omega)$, we have that $v \in C^0([0, T]; L^\Phi(\Omega))$.

Lemma 2.11 Assume that the functions $v, \psi \in C^0([0, T]; L^\Phi(\Omega))$ satisfy $v \geq \psi$ a.e. in Ω_T with respect to the $(n+1)$ -dimensional Lebesgue measure $\mathcal{L}^{n+1}$. Then, $v(0) \geq \psi(0)$ a.e. in Ω with respect to the n -dimensional Lebesgue measure $\mathcal{L}^n$.

2.2 Difference quotients

First, adapting the proof of [16, Theorem 1.33], we show the following variant of Lebesgue's differentiation theorem.

Lemma 2.12 Let $(X, \|\cdot\|_X)$ be a separable Banach space and $v \in L^1(0, T; X)$. Then, for a.e. $t \in [0, T]$, we have that

$$\lim_{h \downarrow 0} \int_t^{t+h} \|v(s) - v(t)\|_X \, ds = 0.$$

Proof Since X is separable, there exists a dense subset $(v_i)_{i \in \mathbb{N}} \subset X$. Then, we know that $t \mapsto \|v(t) - v_i\|_X \in L^1(0, T)$ for any $i \in \mathbb{N}$. By Lebesgue's differentiation theorem, we conclude that

$$\lim_{h \downarrow 0} \int_t^{t+h} \|v(s) - v_i\|_X \, ds = \|v(t) - v_i\|_X$$

holds true for any $t \in [0, T] \setminus \mathcal{N}_i$, where $\mathcal{N}_i$, $i \in \mathbb{N}$, denotes a $\mathcal{L}^1$ -null set. Consequently, $\mathcal{N} := \bigcup_{i \in \mathbb{N}} \mathcal{N}_i \cup \{t \in [0, T] : v(t) \notin X\}$ is a $\mathcal{L}^1$ -null set and

$$\lim_{h \downarrow 0} \int_t^{t+h} \|v(s) - v_i\|_X \, ds = \|v(t) - v_i\|_X$$

holds true for any $t \in [0, T] \setminus \mathcal{N}$ and any $i \in \mathbb{N}$. Next, fix $t \in [0, T] \setminus \mathcal{N}$ and let $\varepsilon > 0$. Since $(v_i)_{i \in \mathbb{N}}$ is dense in X , there exists $i \in \mathbb{N}$ such that

$$\|v(t) - v_i\|_X < \frac{\varepsilon}{2}.$$

Combining the preceding considerations, we infer

$$\begin{aligned} \limsup_{h \downarrow 0} \int_t^{t+h} \|v(s) - v(t)\|_X \, ds &\leq \lim_{h \downarrow 0} \int_t^{t+h} \|v(s) - v_i\|_X \, ds + \|v_i - v(t)\|_X \\ &= 2\|v_i - v(t)\|_X < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this yields the claim. $\square$

Let $h > 0$. The difference quotient of a function v with respect to time is denoted by

$$\Delta_h v(t) := \frac{1}{h} (v(t+h) - v(t)).$$

We prove the following convergence assertion for $X = L^\Phi(\Omega)$.

Lemma 2.13 Assume that $v \in C^0([0, T]; L^\Phi(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$. Further, let $h_k := T/k$ for some $k \in \mathbb{N}$ and define the piecewise constant function $v^{(k)} : \Omega_T \rightarrow \mathbb{R}$ by

$$v^{(k)} := v(ih_k) \text{ for } t \in ((i-1)h_k, ih_k], i \in \{1, \dots, k\}.$$

Then, we have that

$$\Delta_{h_k} v^{(k)} \rightarrow \partial_t v \text{ in } L^1(0, T; L^\Phi(\Omega)) \text{ as } k \rightarrow \infty.$$

Proof Fix $t \in [0, T]$ and let $i \in \{1, \dots, k\}$ such that $t \in ((i-1)h_k, ih_k]$. Observe that

$$\Delta_{h_k} v^{(k)}(t) = \frac{1}{h_k} (v((i+1)h_k) - v(ih_k)) = \int_{ih_k}^{(i+1)h_k} \partial_t v(s) \, ds.$$

Therefore, choosing suitable $i \in \{1, \dots, k\}$ and applying Lemma 2.12 with $X = L^\Phi(\Omega)$ we compute that

$$\begin{aligned}
\|\Delta_{h_k} v^{(k)}(t) - \partial_t v(t)\|_{L^\Phi(\Omega)} &= \left\| \fint_{ih_k}^{(i+1)h_k} [\partial_t v(s) - \partial_t v(t)] \, ds \right\|_{L^\Phi(\Omega)} \\
&\leq \fint_{ih_k}^{(i+1)h_k} \|\partial_t v(s) - \partial_t v(t)\|_{L^\Phi(\Omega)} \, ds \\
&\leq 2 \fint_t^{t+2h_k} \|\partial_t v(s) - \partial_t v(t)\|_{L^\Phi(\Omega)} \, ds \rightarrow 0
\end{aligned}$$

a.e. in $[0, T]$ as $k \rightarrow \infty$. Moreover, in a similar way, we find that

$$\begin{aligned}
\|\Delta_{h_k} v^{(k)}(t) - \partial_t v(t)\|_{L^\Phi(\Omega)} &\leq 2 \fint_t^{t+2h_k} \|\partial_t v(s)\|_{L^\Phi(\Omega)} \, ds + \|\partial_t v\|_{L^\Phi(\Omega)} \\
&\rightarrow 3 \|\partial_t v\|_{L^\Phi(\Omega)}
\end{aligned}$$

in $L^1(0, T)$ in the limit $k \rightarrow \infty$. Here, we used that $\fint_t^{t+2h_k} \|\partial_t v(s)\|_{L^\Phi(\Omega)} \, ds$ is the Steklov average of the function $t \mapsto \|\partial_t v(t)\|_{L^\Phi(\Omega)} \in L^1(0, T)$. Hence, a version of the dominated convergence theorem (cf. [16, Theorem 1.20]) implies the claim. $\square$

The following statement is a slightly different version of the discrete integration by parts formula [11, Lemma 2.10].

Lemma 2.14 *Let $h \in (0, 1]$, and $u, v \in L^\Phi(\Omega \times (-h, T+h))$ be two nonnegative functions. Then, the following integration by parts formula*

$$\begin{aligned}
\iint_{\Omega_T} \Delta_{-h} b(u)(v-u) \, dxdt &\leq \iint_{\Omega_T} \Delta_h v(b(v) - b(u)) \, dxdt \\
&\quad - \frac{1}{h} \iint_{\Omega \times (T-h, T)} \mathfrak{b}[u(t), v(t+h)] \, dxdt \\
&\quad + \frac{1}{h} \iint_{\Omega \times (-h, 0)} \mathfrak{b}[u, v] \, dxdt + \delta_1(h) + \delta_2(h),
\end{aligned}$$

holds true, where the error terms $\delta_1(h)$ and $\delta_2(h)$ are given by

$$\begin{aligned}
\delta_1(h) &:= \frac{1}{h} \iint_{\Omega_T} \mathfrak{b}[v(t), v(t+h)] \, dxdt, \\
\delta_2(h) &:= \iint_{\Omega \times (-h, 0)} \Delta_h v(b(v(t+h)) - b(u(t))) \, dxdt.
\end{aligned}$$

If we assume additionally that $v \in L^\infty(-h_o, T+h_o; L^\Phi(\Omega))$ and $\partial_t v \in L^1(-h_o, T+h_o; L^\Phi(\Omega))$ for some $h_o > 0$, then we have

$$\lim_{h \downarrow 0} \delta_1(h) = 0 \quad \text{and} \quad \lim_{h \downarrow 0} \delta_2(h) = 0.$$

Proof For the proof of the integration by parts formula, we refer to [11, Lemma 2.10]. It remains to show the second assertion of the lemma. By Lemma 2.7, we conclude that

$$0 \leq \frac{1}{h} \mathfrak{b}[v(t), v(t+h)] \leq (b(v(t+h)) - b(v(t))) \Delta_h v(t).$$

First, observe that $(b(v(t+h)) - b(v(t)))\Delta_h v(t) \rightarrow 0$ a.e. in Ω_T as $h \downarrow 0$. Further, we have that

$$\sup_{h \in (0, h_o)} \sup_{t \in [0, T]} \|b(v(t+h)) - b(v(t))\|'_{L^{\Phi^*}(\Omega)} \leq 2 \sup_{t \in [0, T+h_o]} \|b(v(t))\|'_{L^{\Phi^*}(\Omega)}.$$

By Remark 2.4, this implies that the sequence $(b(v(t+h)) - b(v(t)))_{h \in (0, h_o)}$ is bounded in $L^\infty(0, T; L^{\Phi^*}(\Omega))$. Therefore, we find that

$$b(v(t+h)) - b(v(t)) \xrightarrow{*} 0 \text{ weakly } * \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } h \downarrow 0.$$

Combining this with $\Delta_h v \rightarrow \partial_t v$ in $L^1(0, T; L^\Phi(\Omega))$ as $h \downarrow 0$, we infer

$$\begin{aligned} 0 &\leq \limsup_{h \downarrow 0} \delta_1(h) = \limsup_{h \downarrow 0} \frac{1}{h} \iint_{\Omega_T} \mathbf{b}[v(t), v(t+h)] \, dx dt \\ &\leq \lim_{h \downarrow 0} \iint_{\Omega_T} (b(v(t+h)) - b(v(t))) \Delta_h v(t) \, dx dt = 0. \end{aligned}$$

Next, by the generalized Hölder's inequality (1.8) and Hölder's inequality, we compute that

$$\begin{aligned} |\delta_2(h)| &\leq \left(\sup_{t \in [0, 2h_o]} \|b(v(t))\|'_{L^{\Phi^*}(\Omega)} + \sup_{t \in [0, h_o]} \|b(u(t))\|'_{L^{\Phi^*}(\Omega)} \right) \int_0^h \|\Delta_h v\|_{L^\Phi(\Omega)} \, dt \\ &\leq \left(\sup_{t \in [0, 2h_o]} \|b(v(t))\|'_{L^{\Phi^*}(\Omega)} + \sup_{t \in [0, h_o]} \|b(u(t))\|'_{L^{\Phi^*}(\Omega)} \right) \int_0^h \|\partial_t v\|_{L^\Phi(\Omega)} \, dt \rightarrow 0 \end{aligned}$$

as $h \downarrow 0$. This concludes the proof of the lemma. $\square$

2.3 Mollification in time

In addition to standard mollification, we also consider the following mollification technique introduced by Landes [22]. We construct the regularization $[v]_h$, $h > 0$, to a given function v , such that it formally solves the ordinary differential equation

$$\partial_t [v]_h = -\frac{1}{h} ([v]_h - v) \quad (2.6)$$

with initial condition $[v]_h(0) = v_o$. The precise construction is as follows. Let X be a separable Banach space and $v_o \in X$; in the applications, we will have $X = L^r(\Omega)$ with $r \geq 1$ and $X = L^\Phi(\Omega)$. Now, we consider $v \in L^r(0, T; X)$ for some $1 \leq r \leq \infty$, and define for $h \in (0, T]$ and $t \in [0, T]$ the mollification in time by

$$[v]_h(t) := e^{-\frac{t}{h}} v_o + \frac{1}{h} \int_0^t e^{-\frac{s-t}{h}} v(s) \, ds. \quad (2.7)$$

It is easy to check that $[v]_h$ satisfies (2.6). The basic properties of the mollifications $[\cdot]_h$ are provided in the following Lemma, cf. [10, Appendix B] for the proofs of the statements.

Lemma 2.15 Suppose that X is a separable Banach space and $v_o \in X$. If $v \in L^r(0, T; X)$ for some $r \geq 1$, then the mollification $[v]_h$ defined in (2.7) fulfills $[v]_h \in L^r(0, T; X)$ and for any $t_o \in (0, T)$ there holds

$$\|[v]_h\|_{L^r(0, t_o; X)} \leq \|v\|_{L^r(0, t_o; X)} + \left[\frac{h}{r} \left(1 - e^{-\frac{t_o}{h}} \right) \right]^{\frac{1}{r}} \|v_o\|_X.$$

In the case $r = \infty$, the bracket $[\dots]^{\frac{1}{r}}$ in the preceding inequality has to be interpreted as 1. Moreover, in the case $r < \infty$ we have $[v]_h \rightarrow v$ in $L^r(0, T; X)$ as $h \downarrow 0$. Finally, if $v \in C^0([0, T]; X)$ and $v_o = v(0)$, then $[v]_h \in C^0([0, T]; X)$, $[v]_h(0) = v_o$, and moreover $[v]_h \rightarrow v$ in $C^0([0, T]; X)$ as $h \downarrow 0$. $\square$

For maps $v \in L^r(0, T; X)$ with $\partial_t v \in L^r(0, T; X)$ we have the following assertion.

Lemma 2.16 Let X be a separable Banach space and $r \geq 1$. Assume that $v \in L^r(0, T; X)$ with $\partial_t v \in L^r(0, T; X)$. Then, for the mollification in time defined by

$$[v]_h(t) := e^{-\frac{t}{h}} v(0) + \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(s) \, ds$$

, the time derivative can be computed by

$$\partial_t [v]_h(t) = \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} \partial_t v(s) \, ds,$$

and, moreover we have that

$$\|\partial_t [v]_h\|_{L^r(0, T; X)} \leq \|\partial_t v\|_{L^r(0, T; X)}$$

holds true. $\square$

Lemma 2.17 Let $r \geq 1$, $(v_i)_{i \in \mathbb{N}}$ be a sequence in $L^r(\Omega_T)$, $v \in L^r(\Omega_T)$ and $v_o \in L^r(\Omega)$. If $v_i \rightarrow v$ weakly in $L^r(\Omega_T)$ as $i \rightarrow \infty$, then $[v_i]_h \rightarrow [v]_h$ weakly in $L^r(\Omega_T)$ as $i \rightarrow \infty$ holds true for the mollifications defined by (2.7) with fixed $h > 0$ and initial values v_o .

2.4 The initial condition

Here, we show that variational solutions attain the initial datum g_o in the L^Φ -sense. For the proof of the following statement, we refer to [11, Lemma 2.9].

Lemma 2.18 Any variational solution to (1.2) in the sense of Definition 1.1 fulfills the initial condition $u(0) = g_o$ in the L^Φ -sense, i.e.

$$\lim_{h \downarrow 0} \frac{1}{h} \phi_{\Omega_h}(u - g_o) = 0.$$

Proof Since $v = g$ is admissible in the variational inequality (1.11), by the generalized Hölder's inequality (1.8), Lemmas 2.3 and 2.5, we find that

$$\begin{aligned}
& \iint_{\Omega_\tau} f(Du) \, dx \, dt + \mathfrak{B}[u(\tau), g(\tau)] \\
& \leq \sup_{t \in [0, T]} (\|b(g(t))\|'_{L^{\Phi^*}(\Omega)} + \|b(u(t))\|'_{L^{\Phi^*}(\Omega)}) \int_0^\tau \|\partial_t g\|_{L^\Phi(\Omega)} \, dt + \iint_{\Omega_\tau} f(Dg) \, dx \, dt \\
& \leq \sup_{t \in [0, T]} (2 + \varrho_\Omega^*(b(g(t)))^{\frac{1}{\varepsilon+1}} + \varrho_\Omega^*(b(u(t)))^{\frac{1}{\varepsilon+1}}) \int_0^\tau \|\partial_t g\|_{L^\Phi(\Omega)} \, dt \\
& \quad + \iint_{\Omega_\tau} f(Dg) \, dx \, dt \\
& \leq \sup_{t \in [0, T]} (2 + (m\varrho_\Omega(g(t)))^{\frac{1}{\varepsilon+1}} + (m\varrho_\Omega(u(t)))^{\frac{1}{\varepsilon+1}}) \int_0^\tau \|\partial_t g\|_{L^\Phi(\Omega)} \, dt \\
& \quad + \iint_{\Omega_\tau} f(Dg) \, dx \, dt
\end{aligned}$$

holds true for a.e. $\tau \in [0, T]$. Recalling $u, g \in L^\infty(0, T; L^\Phi(\Omega))$ and (1.7), we have that

$$C := \sup_{t \in [0, T]} (2 + (m\varrho_\Omega(g(t)))^{\frac{1}{\varepsilon+1}} + (m\varrho_\Omega(u(t)))^{\frac{1}{\varepsilon+1}}) < \infty.$$

Furthermore, we know that $\partial_t g \in L^1(0, T; L^\Phi(\Omega))$. Altogether, discarding the nonnegative energy term on the left-hand side, taking the square root, integrating over $\tau \in (0, h)$ for $h \in (0, T)$ and dividing the result by h we conclude that

$$\begin{aligned}
\frac{1}{h} \int_0^h \mathfrak{B}[u(\tau), g(\tau)]^{\frac{1}{2}} \, d\tau & \leq \frac{1}{h} \int_0^h \left(\iint_{\Omega_\tau} f(Dg) \, dx \, dt + C \int_0^\tau \|\partial_t g\|_{L^\Phi(\Omega)} \, dt \right)^{\frac{1}{2}} \, d\tau \\
& \leq \left(\iint_{\Omega_h} f(Dg) \, dx \, dt + C \int_0^h \|\partial_t g\|_{L^\Phi(\Omega)} \, dt \right)^{\frac{1}{2}} \rightarrow 0
\end{aligned} \tag{2.8}$$

in the limit $h \downarrow 0$. Next, from (2.3) and the convexity of Φ , we deduce that

$$\begin{aligned}
\Phi(|u(t) - g_o|) & \leq \Phi\left(2\left(\frac{1}{2}|u(t) - g(t)| + \frac{1}{2}|g(t) - g_o|\right)\right) \\
& \leq 2^m [\Phi(|u(t) - g(t)|) + \Phi(|g(t) - g_o|)]
\end{aligned} \tag{2.9}$$

holds true for a.e. $t \in [0, T]$. Since $g \in C^0([0, T]; L^\Phi(\Omega))$, we have that $g(t) \rightarrow g_o$ strongly in $L^\Phi(\Omega)$ as $t \downarrow 0$ and hence by (1.6) that $\Phi(|g(t) - g_o|) \rightarrow 0$ as $t \downarrow 0$. Thus, we conclude that

$$\lim_{h \downarrow 0} \frac{1}{h} \varrho_{\Omega_h}(g - g_o) = 0. \tag{2.10}$$

Hence, using the estimate from Lemma 2.9, Hölder's inequality and Lemma 2.7, we infer

$$\begin{aligned}
\frac{1}{h} \varrho_{\Omega_h}(u(t) - g(t)) &\leq \frac{1}{h} \iint_{\Omega_h} |\Phi(u(t)) - \Phi(g(t))| \, dx dt \\
&\leq \frac{1}{h} \iint_{\Omega_h} \left| \sqrt{\Phi(u(t))} - \sqrt{\Phi(g(t))} \right| \left| \sqrt{\Phi(u(t))} + \sqrt{\Phi(g(t))} \right| \, dx dt \\
&\leq \frac{1}{h} \int_0^h \left\| \sqrt{\Phi(u(t))} - \sqrt{\Phi(g(t))} \right\|_{L^2(\Omega)} \\
&\quad \left\| \sqrt{\Phi(u(t))} + \sqrt{\Phi(g(t))} \right\|_{L^2(\Omega)} \, dt \\
&\leq \frac{1}{h} \sup_{t \in (0, h)} \left(\varrho_{\Omega}(u(t))^{\frac{1}{2}} + \varrho_{\Omega}(g(t))^{\frac{1}{2}} \right) \\
&\quad \int_0^h \left(\int_{\Omega} \left| \sqrt{\Phi(u(t))} - \sqrt{\Phi(g(t))} \right|^2 \, dx \right)^{\frac{1}{2}} \, dt \\
&\leq \frac{c(m, \ell')}{h} \sup_{t \in (0, h)} \left(\varrho_{\Omega}(u(t))^{\frac{1}{2}} + \varrho_{\Omega}(g(t))^{\frac{1}{2}} \right) \int_0^h \mathfrak{B}[u(\tau), g(\tau)]^{\frac{1}{2}} \, dt.
\end{aligned}$$

By $u, g \in L^\infty(0, T; L^\Phi(\Omega))$ and (1.7), the first term on the right-hand side of the preceding inequality is bounded. Therefore, combining the preceding inequality with (2.8), we find that

$$\lim_{h \downarrow 0} \frac{1}{h} \varrho_{\Omega_h}(u - g) = 0. \quad (2.11)$$

Altogether, combining (2.9), (2.10) and (2.11), we obtain that

$$\frac{1}{h} \varrho_{\Omega_h}(u - g_o) \leq \frac{2^m}{h} \varrho_{\Omega_h}(u - g) + \frac{2^m}{h} \varrho_{\Omega_h}(g - g_o) \rightarrow 0$$

in the limit $h \downarrow 0$. This concludes the proof of the lemma. $\square$

Remark 2.19 In the case $\Phi(v) = |v|^{m+1}$, i.e., $L^\Phi(\Omega) = L^{m+1}(\Omega)$, the statement of the preceding lemma reduces to the usual convergence assertion in the L^{m+1} -sense,

$$\lim_{h \downarrow 0} \frac{1}{h} \varrho_{\Omega_h}(u - g_o) = \lim_{h \downarrow 0} \frac{1}{h} \int_0^h \|u(t) - g_o\|_{L^{m+1}(\Omega)}^{m+1} \, dt = 0.$$

2.5 An energy bound

In this section, we derive an energy bound for variational solutions.

Lemma 2.20 (Energy bound) *Assume that (1.9) and (1.10) are satisfied and that u is a variational solution to (1.2) in the sense of Definition 1.1. Then, u fulfills the energy bound*

$$\begin{aligned}
& \frac{1}{2} \sup_{t \in [0, T]} \varrho_{\Omega}(u(t)) + \iint_{\Omega_T} f(Du) \, dx \leq \frac{2\ell}{m} \\
& + \frac{1}{2} \sup_{t \in [0, T]} \varrho_{\Omega}(v(t)) + \frac{3}{\ell} \iint_{\Omega_T} f(Dv) \, dx dt \\
& + c \left(\int_0^T \|\partial_t v\|_{L^{\Phi}(\Omega)} \, dt \right)^{\frac{\ell+1}{\ell}} \\
& + \frac{3}{\ell} \mathfrak{B}[g_o, v(0)]
\end{aligned}$$

with a constant $c = c(\ell, m)$ for any comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^{\Phi}(\Omega))$, $v(0) \in L^{\Phi}(\Omega)$ and $v \geq \psi$ a.e. in Ω_T .

Proof If u is a variational solution and v an admissible comparison map in the sense of Definition 1.1, we have that

$$\begin{aligned}
& \iint_{\Omega_{\tau}} f(Du) \, dx dt + \mathfrak{B}[u(\tau), v(\tau)] \\
& \leq \iint_{\Omega_{\tau}} \partial_t v(b(v) - b(u)) \, dx dt + \iint_{\Omega_{\tau}} f(Dv) \, dx dt + \mathfrak{B}[g_o, v(0)]
\end{aligned} \tag{2.12}$$

for a.e. $\tau \in [0, T]$. By Lemmas 2.3 and 2.6, we obtain that

$$\varrho_{\Omega}(u(\tau)) \leq \frac{1}{\ell} \varrho_{\Omega}^*(b(u(\tau))) \leq \frac{2}{\ell} \mathfrak{B}[u(\tau), v(\tau)] + \frac{2^{m+2}}{\ell} \varrho_{\Omega}(v(\tau)) \tag{2.13}$$

holds true for any $\tau \in [0, T]$. For $\varepsilon > 0$, by the generalized Hölder's inequality (1.8), Hölder's inequality, Lemma 2.5 applied to $\|\cdot\|'_{L^{\Phi^*}(\Omega)}$ and Young's inequality, we infer

$$\begin{aligned}
& \iint_{\Omega_T} |\partial_t v| |b(v) - b(u)| \, dx \\
& \leq \sup_{t \in [0, T]} (\|b(v(t))\|'_{L^{\Phi^*}(\Omega)} + \|b(u(t))\|'_{L^{\Phi^*}(\Omega)}) \int_0^T \|\partial_t v\|_{L^{\Phi}(\Omega)} \, dt \\
& \leq \left(2 + \sup_{t \in [0, T]} \varrho_{\Omega}^*(b(v(t)))^{\frac{1}{\ell+1}} + \sup_{t \in [0, T]} \varrho_{\Omega}^*(b(u(t)))^{\frac{1}{\ell+1}} \right) \int_0^T \|\partial_t v\|_{L^{\Phi}(\Omega)} \, dt \\
& \leq \left(2 + \sup_{t \in [0, T]} (m\varrho_{\Omega}(v(t)))^{\frac{1}{\ell+1}} + \sup_{t \in [0, T]} (m\varrho_{\Omega}(u(t)))^{\frac{1}{\ell+1}} \right) \int_0^T \|\partial_t v\|_{L^{\Phi}(\Omega)} \, dt \\
& \leq 2^{\ell+1} \varepsilon + m\varepsilon \sup_{t \in [0, T]} \varrho_{\Omega}(v(t)) + m\varepsilon \sup_{t \in [0, T]} \varrho_{\Omega}(u(t)) \\
& + c(\varepsilon) \left(\int_0^T \|\partial_t v\|_{L^{\Phi}(\Omega)} \, dt \right)^{\frac{\ell+1}{\ell}}.
\end{aligned} \tag{2.14}$$

Inserting (2.13) and (2.14) into (2.12), we conclude that

$$\begin{aligned}
& \sup_{\tau \in [0, T]} \varrho_{\Omega}(u(\tau)) + \iint_{\Omega_T} f(Du) \, dx dt \\
& \leq \frac{2}{\ell} \sup_{\tau \in [0, T]} \mathfrak{B}[u(\tau), v(\tau)] + \iint_{\Omega_T} f(Du) \, dx dt + \frac{2^{m+2}}{\ell} \sup_{\tau \in [0, T]} \varrho_{\Omega}(v(\tau)) \\
& \leq \frac{3}{\ell} \left[\iint_{\Omega_T} |\partial_t v| |b(v) - b(u)| \, dx dt + \iint_{\Omega_T} f(Dv) \, dx dt + \mathfrak{B}[g_o, v(0)] \right] \\
& \quad + \frac{2^{m+2}}{\ell} \sup_{\tau \in [0, T]} \varrho_{\Omega}(v(\tau)) \\
& \leq \frac{3 \cdot 2^{\ell+1}}{\ell} \varepsilon + \frac{3m}{\ell} \varepsilon \sup_{t \in [0, T]} \varrho_{\Omega}(v(t)) + \frac{3m}{\ell} \varepsilon \sup_{t \in [0, T]} \varrho_{\Omega}(u(t)) \\
& \quad + \frac{3}{\ell} c(\varepsilon) \left(\int_0^T \|\partial_t v\|_{L^{\Phi}(\Omega)} \, dt \right)^{\frac{\ell+1}{\ell}} + \frac{3}{\ell} \iint_{\Omega_T} f(Dv) \, dx dt + \frac{3}{\ell} \mathfrak{B}[g_o, v(0)].
\end{aligned}$$

At this stage, we choose $\varepsilon := \frac{\ell}{6m}$. This allows us to reabsorb the term

$$\frac{1}{2} \sup_{\tau \in [0, T]} \varrho_{\Omega}(u(\tau))$$

from the right-hand side of the preceding inequality into the left-hand side, which yields the claim. $\square$

2.6 Compactness

The proof of following result can be found in [11, Proposition 3.1].

Lemma 2.21 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p > 1$, $T > 0$ and $k \in \mathbb{N}$. Suppose that for $h_k := T/k$ piecewise constant maps $u^{(k)} : \Omega \times (-h_k, T] \rightarrow \mathbb{R}_{\geq 0}$ are defined by*

$$u^{(k)}(\cdot, t) := u_i^{(k)} \text{ for } t \in ((i-1)h_k, ih_k] \text{ with } i \in \{0, \dots, k\}$$

with nonnegative functions $u_i^{(k)} \in L^{\Phi}(\Omega) \cap W^{1,p}(\Omega)$. Suppose further that there exists a constant $M > 0$ such that the energy estimate

$$\max_{i \in \{0, \dots, k\}} \left[\varrho_{\Omega}(u_i^{(k)}) + \int_{\Omega} |Du_i^{(k)}|^p \, dx \right] \leq M$$

and the continuity estimate

$$\frac{1}{h_k} \sum_{i=1}^k \int_{\Omega} \left| \sqrt{\Phi(u_i^{(k)})} - \sqrt{\Phi(u_{i-1}^{(k)})} \right|^2 \, dx \leq M$$

hold true for all $k \in \mathbb{N}$, and that $u^{(k)} \rightharpoonup u$ weakly in $L^p(0, T; W^{1,p}(\Omega))$ as $k \rightarrow \infty$. Then, there exists a subsequence $\mathfrak{K} \subset \mathbb{N}$ such that in the limit $\mathfrak{K} \ni k \rightarrow \infty$ we have the following convergences:

$$\begin{cases} \sqrt{\Phi(u^{(k)})} \rightarrow \sqrt{\Phi(u)} \text{ strongly in } L^1(\Omega_T), \\ u^{(k)} \rightarrow u \text{ a.e. in } \Omega_T. \end{cases}$$

3 Existence for regular data

First, we prove an existence result for the case of regular boundary values and obstacle. More precisely, we consider nonnegative regular boundary values given by

$$\begin{cases} g \in L^p(0, T; W^{1,p}(\Omega)), \partial_t g \in L^2(\Omega_T) \cap L^p(0, T; W^{1,p}(\Omega)), \\ g_o = g(0) \in L^2(\Omega) \cap W^{1,p}(\Omega), \kappa \leq g \end{cases} \quad (3.1)$$

with a constant $\kappa > 0$ and a nonnegative obstacle function ψ satisfying

$$\begin{cases} \psi \in g + L^p(0, T; W_0^{1,p}(\Omega)), \partial_t \psi \in L^2(\Omega_T) \cap L^p(0, T; W^{1,p}(\Omega)), \\ \psi(0) \in L^2(\Omega) \cap (g_o + W_0^{1,p}(\Omega)), \kappa \leq \psi \leq g \text{ a.e. on } \Omega_T. \end{cases} \quad (3.2)$$

Theorem 3.1 *Assume that the obstacle satisfies (3.2) and boundary values are given by (3.1). Then, there exists a variational solution to (1.2) in the sense of Definition 1.1.*

3.1 A sequence of minimizers to elliptic variational functionals

Fix a step size $h \in (0, 1]$ such that $h = T/K$ for some $K \in \mathbb{N}$. For $i \in \{0, \dots, K\}$ define $g_i := g(ih)$ and $\psi_i := \psi(ih)$. Set $u_0 = g_o$. Then, we inductively find minimizers u_i of the functionals

$$F_i[v] = \int_{\Omega} f(Dv) \, dx + \frac{1}{h} \int_{\Omega} \mathfrak{b}[u_{i-1}, v] \, dx$$

in the class $v \in L^\Phi(\Omega) \cap (g_i + W_0^{1,p}(\Omega))$ with $v \geq \psi_i$. Note that this class is not empty, since $v = g_i$ is admissible. The existence of minimizers u_i is ensured, for example, by the direct method of the calculus of variations. For convenience of the reader, we give the precise proof.

Proposition 3.2 *Assume that nonnegative functions $g_* \in L^\Phi(\Omega) \cap W^{1,p}(\Omega)$, $\psi_* \in L^\Phi(\Omega) \cap W^{1,p}(\Omega)$ and $u_* \in L^\Phi(\Omega)$ are given. Then, there exists a minimizer u of*

$$F[v] = \int_{\Omega} f(Dv) \, dx + \frac{1}{h} \int_{\Omega} \mathfrak{b}[u_*, v] \, dx$$

in the class of functions $v \in L^\Phi(\Omega) \cap (g_ + W_0^{1,p}(\Omega))$ with $v \geq \psi_*$.*

Proof Consider a minimizing sequence $(u_j)_{j \in \mathbb{N}}$, i.e.

$$\lim_{j \rightarrow \infty} F[u_j] = \inf \{ F[v] : v \in L^\Phi(\Omega) \cap (g_* + W_0^{1,p}(\Omega)) \text{ with } v \geq \psi_* \}.$$

By means of Lemmas 2.3 and 2.6, we find that

$$\Phi(u_j) \leq 2\mathfrak{b}[u_*, u_j] + 2^{2+\frac{1}{\epsilon}} \Phi^*(b(u_*)) \leq 2\mathfrak{b}[u_*, u_j] + 2^{2+\frac{1}{\epsilon}} m\Phi(u_*).$$

Together with (1.3) and the fact that $h \in (0, 1]$, this implies

$$\begin{aligned} \int_{\Omega} v |Du_j|^p \, dx + \frac{1}{2} \varrho_{\Omega}(u_j) &\leq \int_{\Omega} [f(Du_j) + \mathfrak{b}[u_*, u_j]] \, dx + 2^{1+\frac{1}{\epsilon}} m \varrho_{\Omega}(u_*) \\ &\leq F[u_j] + 2^{1+\frac{1}{\epsilon}} m \varrho_{\Omega}(u_*). \end{aligned}$$

Hence, by (1.7) the sequence $(u_j)_{j \in \mathbb{N}}$ is bounded in $L^{\Phi}(\Omega) \cap W^{1,p}(\Omega)$. Thus, there exists a (not relabeled) subsequence and a limit map $u \in L^{\Phi}(\Omega) \cap (g_* + W_0^{1,p}(\Omega))$ such that

$$\begin{cases} u_j \rightharpoonup u \text{ weakly in } L^{\Phi}(\Omega), \\ u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \end{cases}$$

in the limit $j \rightarrow \infty$. Observe that the obstacle condition $u \geq \psi_*$ a.e. in Ω is preserved. Since F is convex and lower semicontinuous with respect to strong convergence in $L^{\Phi}(\Omega) \cap W^{1,p}(\Omega)$ by means of Fatou's Lemma, F is also lower semicontinuous with respect to weak convergence in $L^{\Phi}(\Omega) \cap W^{1,p}(\Omega)$, cf. [14, Corollary 3.9]. As a consequence, we find that

$$F[u] \leq \lim_{j \rightarrow \infty} F[u_j] = \inf \{ F[v] : v \in L^{\Phi}(\Omega) \cap (g_* + W_0^{1,p}(\Omega)) \text{ with } v \geq \psi_* \},$$

which yields the claim. $\square$

3.2 Energy estimates

Observe that $v := u_{i-1} + \psi_i - \psi_{i-1} \geq \psi_i$ is an admissible comparison function for u_i . Using the minimality of u_i with respect to F_i , we obtain that

$$\begin{aligned} \int_{\Omega} f(Du_i) \, dx + \frac{1}{h} \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] \, dx &= F_i[u_i] \\ &\leq F_i[u_{i-1} + \psi_i - \psi_{i-1}] \\ &= \int_{\Omega} f(D(u_{i-1} + \psi_i - \psi_{i-1})) \, dx + \frac{1}{h} \int_{\Omega} \mathfrak{b}[u_{i-1}, u_{i-1} + \psi_i - \psi_{i-1}] \, dx \\ &=: \text{I} + \text{II}, \end{aligned}$$

where the definition of I and II is clear in this context. First, we estimate I. To this end, using the Lipschitz estimate (1.4), Young's inequality, the assumption $h \leq 1$ and the coercivity assumption (1.3), we conclude that

$$\begin{aligned} &|f(D(u_{i-1} + \psi_i - \psi_{i-1})) - f(Du_{i-1})| \\ &\leq c(1 + |D(u_{i-1} + \psi_i - \psi_{i-1})|^{p-1} + |Du_{i-1}|^{p-1}) |D\psi_i - D\psi_{i-1}| \\ &\leq c|Du_{i-1}|^{p-1} |D\psi_i - D\psi_{i-1}| + c(1 + |D\psi_i - D\psi_{i-1}|^{p-1}) |D\psi_i - D\psi_{i-1}| \\ &\leq \nu h |Du_{i-1}|^p + c(1 + h^{1-p} |D\psi_i - D\psi_{i-1}|^{p-1}) |D\psi_i - D\psi_{i-1}| \\ &\leq hf(Du_{i-1}) + c(1 + h^{1-p} |D\psi_i - D\psi_{i-1}|^{p-1}) |D\psi_i - D\psi_{i-1}| \end{aligned}$$

holds true with a constant $c = c(p, n, L, \nu)$. Moreover, we find that

$$\int_{\Omega} |D\psi_i - D\psi_{i-1}| \, dx \leq \iint_{\Omega \times ((i-1)h, ih)} |\partial_t D\psi| \, dx dt$$

and that

$$\int_{\Omega} |D\psi_i - D\psi_{i-1}|^p \, dx \leq h^{p-1} \iint_{\Omega \times ((i-1)h, ih)} |\partial_t D\psi|^p \, dx dt.$$

Therefore, we deduce the estimate

$$I \leq (1+h) \int_{\Omega} f(Du_{i-1}) \, dx + c \iint_{\Omega \times ((i-1)h, ih)} [|\partial_t D\psi| + |\partial_t D\psi|^p] \, dx dt$$

with a constant $c = c(n, p, L, \nu)$. Next, by Lemma 2.7 and (1.5) together with Remark 2.2 and the fact that $u_{i-1} \geq \psi_{i-1} \geq \kappa$ and $0 < \ell \leq m \leq 1$, we estimate II, which leads to

$$\begin{aligned} \mathfrak{b}[u_{i-1}, u_{i-1} + \psi_i - \psi_{i-1}] &= (b(u_{i-1} + \psi_i - \psi_{i-1}) - b(u_{i-1}))(\psi_i - \psi_{i-1}) \\ &= \int_{u_{i-1} + \psi_i - \psi_{i-1}}^{u_{i-1}} b'(s) \, ds \cdot (\psi_i - \psi_{i-1}) \\ &\leq mb(1) \max\{\kappa^{\ell-1}, \kappa^{m-1}\} (\psi_i - \psi_{i-1})^2. \end{aligned}$$

Therefore, by Hölder's inequality, we conclude that

$$\begin{aligned} II &\leq \frac{mb(1)}{h} \max\{\kappa^{\ell-1}, \kappa^{m-1}\} \int_{\Omega} (\psi_i - \psi_{i-1})^2 \, dx \\ &= \frac{mb(1)}{h} \max\{\kappa^{\ell-1}, \kappa^{m-1}\} \int_{\Omega} \left| \int_{(i-1)h}^{ih} \partial_t \psi \, dt \right|^2 \, dx \\ &\leq mb(1) \max\{\kappa^{\ell-1}, \kappa^{m-1}\} \iint_{\Omega \times ((i-1)h, ih)} |\partial_t \psi|^2 \, dx dt. \end{aligned}$$

Combining the estimates for I and II, we obtain that

$$\begin{aligned} &\int_{\Omega} f(Du_i) \, dx + \frac{1}{h} \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] \, dx \\ &\leq (1+h) \int_{\Omega} f(Du_{i-1}) \, dx \\ &\quad + c \iint_{\Omega \times ((i-1)h, ih)} [|\partial_t \psi|^2 + |\partial_t D\psi| + |\partial_t D\psi|^p] \, dx dt \end{aligned} \tag{3.3}$$

holds true with a constant $c = c(\kappa, b(1), \ell, m, p, n, L, \nu)$. Summing up (3.3) from $i = 1, \dots, k$ for some $k \in \{1, \dots, K\}$ leads to

$$\begin{aligned}
& \sum_{i=1}^k \int_{\Omega} f(Du_i) \, dx + \frac{1}{h} \sum_{i=1}^k \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] \, dx \\
& \leq (1+h) \sum_{i=1}^k \int_{\Omega} f(Du_{i-1}) \, dx + c \iint_{\Omega \times (0, kh)} [|\partial_t \psi|^2 + |\partial_t D\psi| + |\partial_t D\psi|^p] \, dx dt \\
& \leq (1+h) \sum_{i=1}^{k-1} \int_{\Omega} f(Du_i) \, dx + (1+h)\Psi(kh),
\end{aligned}$$

where we used the short-hand notation

$$\Psi(\tau) = L \int_{\Omega} [1 + |Dg_o|^p] \, dx + c \iint_{\Omega \times (0, \tau)} [|\partial_t \psi|^2 + |\partial_t D\psi| + |\partial_t D\psi|^p] \, dx dt$$

for any $\tau \in (0, T]$. Reabsorbing $\sum_{i=1}^{k-1} \int_{\Omega} f(Du_i) \, dx$ into the left-hand side of the preceding inequality, we have that

$$\begin{aligned}
& \int_{\Omega} f(Du_k) \, dx + \frac{1}{h} \sum_{i=1}^k \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] \, dx \\
& \leq h \sum_{i=1}^{k-1} \int_{\Omega} f(Du_i) \, dx + (1+h)\Psi(kh).
\end{aligned} \tag{3.4}$$

In order to estimate the right-hand side of the preceding inequality, we iterate (3.3) from $j = 1, \dots, i$ for $i = 1, \dots, k-1$. This yields

$$\begin{aligned}
& \int_{\Omega} f(Du_i) \, dx \leq (1+h)^i \int_{\Omega} f(Dg_o) \, dx \\
& \quad + \sum_{j=1}^i c(1+h)^{i-j} \iint_{\Omega \times ((j-1)h, jh)} [|\partial_t \psi|^2 + |\partial_t D\psi| + |\partial_t D\psi|^p] \, dx dt \\
& \leq (1+h)^i \Psi(ih) \leq (1+h)^i \Psi(kh).
\end{aligned}$$

Inserting this into (3.4), we obtain that

$$\begin{aligned}
& \int_{\Omega} f(Du_k) \, dx + \frac{1}{h} \sum_{i=1}^k \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] \, dx \leq h \sum_{i=1}^{k-1} (1+h)^i \Psi(kh) + (1+h)\Psi(kh) \\
& = (1+h)^k \Psi(kh).
\end{aligned}$$

Since $(1+h)^k \leq (1+h)^K = (1 + \frac{T}{K})^K \leq e^T$, we conclude that

$$\int_{\Omega} f(Du_k) \, dx + \frac{1}{h} \sum_{i=1}^k \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] \, dx \leq e^T \Psi(T) \tag{3.5}$$

for any $k \in \mathbb{N}$ with $kh \leq T$. Finally, from Lemma 2.7 and (3.5), we infer the estimate

$$\begin{aligned}
\rho_{\Omega}(u_k) &\leq 2k \sum_{i=1}^k \int_{\Omega} |\sqrt{\Phi(u_i)} - \sqrt{\Phi(u_{i-1})}|^2 dx + 2\rho_{\Omega}(g_o) \\
&\leq 2kc^2(\ell, m) \sum_{i=1}^k \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] dx + 2\rho_{\Omega}(g_o) \\
&\leq 2khce^T\Psi(T) + 2\rho_{\Omega}(g_o) \\
&\leq 2Tce^T\Psi(T) + 2\rho_{\Omega}(g_o)
\end{aligned} \tag{3.6}$$

for some constant $c = c(\kappa, b(1), \ell, m, p, n, L, \nu)$.

3.3 Convergence to a limit map

In the following, we set $h_K := T/K$ for $K \in \mathbb{N}$. We define the piecewise constant function $u^{(K)} : \Omega \times (-h_K, T] \rightarrow \mathbb{R}_{\geq 0}$ by

$$u^{(K)}(\cdot, t) := u_i \text{ for } t \in ((i-1)h_K, ih_K] \text{ with } i \in \{0, \dots, K\}.$$

Analogously, we define $g^{(K)}$ and $\psi^{(K)}$. Combining estimate (3.5) with the coercivity condition (1.3)₁ and discarding the nonnegative sum on the left-hand side, we find that

$$\nu \sup_{t \in [0, T]} \int_{\Omega} |Du^{(K)}(t)|^p dx \leq e^T \Psi(T).$$

Hence, the sequence $(u^{(K)})_{K \in \mathbb{N}}$ is bounded in $L^\infty(0, T; W^{1,p}(\Omega))$. Since $u^{(K)} \in g^{(K)} + L^\infty(0, T; W_0^{1,p}(\Omega))$, there exists a subsequence $\mathfrak{K} \subset \mathbb{N}$ and a limit map $u \in g + L^p(0, T; W_0^{1,p}(\Omega))$ such that

$$u^{(K)} \rightharpoonup^* u \text{ weakly}^* \text{ in } L^\infty(0, T; W^{1,p}(\Omega)) \tag{3.7}$$

in the limit $\mathfrak{K} \ni K \rightarrow \infty$. By Lemma 2.7 and the energy estimates (3.5) and (3.6), the assumptions of Lemma 2.21 are satisfied for the sequence $(u^{(K)})_{K \in \mathfrak{K}}$. Therefore, choosing another subsequence still denoted by $\mathfrak{K}$, we obtain that

$$\begin{cases} \sqrt{\Phi(u^{(K)})} \rightarrow \sqrt{\Phi(u)} \text{ strongly in } L^1(\Omega_T), \\ u^{(K)} \rightarrow u \text{ a.e. in } \Omega_T \end{cases} \tag{3.8}$$

in the limit $\mathfrak{K} \ni K \rightarrow \infty$. At this stage, observe that $\psi^{(K)} \rightarrow \psi$ a.e. in Ω_T as $k \rightarrow \infty$. Combining this with (3.8)₂ and the fact that $u^{(K)} \geq \psi^{(K)}$ a.e. in Ω_T , we deduce that u satisfies the obstacle condition $u \geq \psi$ a.e. in Ω_T . Next, by means of Lemma 2.7 and (3.5), we conclude that

$$\begin{aligned}
\iint_{\Omega_T} |\Delta_{h_K} \sqrt{\Phi(u^{(K)})}|^2 dx dt &= \frac{1}{h_K} \sum_{i=1}^K \int_{\Omega} |\sqrt{\Phi(u_i)} - \sqrt{\Phi(u_{i-1})}|^2 dx \\
&\leq \frac{c^2(\ell, m)}{h_K} \sum_{i=1}^K \int_{\Omega} \mathfrak{b}[u_{i-1}, u_i] dx \leq c^2(\ell, m) e^T \Psi(T).
\end{aligned}$$

Thus, we find a subsequence such that $\Delta_{h_K} \sqrt{\Phi(u^{(K)})} \rightarrow w$ weakly in $L^2(\Omega_T)$. In order to characterize w , we use this fact together with (3.8)₁. More precisely, we obtain for any $\varphi \in C_0^\infty(\Omega_T)$ that

$$\begin{aligned} \iint_{\Omega_T} \sqrt{\Phi(u)} \partial_t \varphi \, dx dt &= \lim_{K \rightarrow \infty} \iint_{\Omega_T} \sqrt{\Phi(u^{(K)})} \Delta_{-h_K} \varphi \, dx dt \\ &= - \lim_{K \rightarrow \infty} \iint_{\Omega_T} \Delta_{h_K} \sqrt{\Phi(u^{(K)})} \varphi \, dx dt \\ &= - \iint_{\Omega_T} w \varphi \, dx dt. \end{aligned}$$

By a density argument, this ensures that $w = \partial_t \sqrt{\Phi(u)}$ and in particular $\partial_t \sqrt{\Phi(u)} \in L^2(\Omega_T)$. Hence, we have that $\partial_t \Phi(u) = 2\sqrt{\Phi(u)} \partial_t \sqrt{\Phi(u)} \in L^1(\Omega_T)$. This implies that $\Phi(u) \in C^0([0, T]; L^1(\Omega))$ and hence by means of Lemma 2.9 that $u \in C^0([0, T]; L^\Phi(\Omega))$.

3.4 Minimizing property of the approximations

Observe that for each $K \in \mathbb{N}$, the map $u^{(K)}$ is a minimizer of the functional

$$\mathbf{F}^{(K)}[v] := \iint_{\Omega_T} f(Dv) \, dx dt + \frac{1}{h_K} \iint_{\Omega_T} \mathfrak{b}[u^{(K)}(t - h_K), v(t)] \, dx dt$$

in the class of functions $v \in L^\Phi(\Omega_T) \cap (g^{(K)} + L^p(0, T; W_0^{1,p}(\Omega)))$ satisfying $v \geq \psi^{(K)}$ a.e. in Ω_T . Indeed, using the definitions of $\mathbf{F}^{(K)}$ and $u^{(K)}$ and the minimality of u_i with respect to F_i , we compute for any admissible function v as above

$$\begin{aligned} \mathbf{F}^{(K)}[u^{(K)}] &= \iint_{\Omega_T} f(Du^{(K)}) \, dx dt + \frac{1}{h_K} \iint_{\Omega_T} \mathfrak{b}[u^{(K)}(t - h_K), u^{(K)}(t)] \, dx dt \\ &= \sum_{i=1}^K \iint_{\Omega \times ((i-1)h_K, ih_K)} \left[f(Du_i) + \frac{1}{h_K} \mathfrak{b}[u_{i-1}, u_i] \right] \, dx dt \\ &= \sum_{i=1}^K \int_{(i-1)h_K}^{ih_K} F_i[u_i] \, dt \leq \sum_{i=1}^K \int_{(i-1)h_K}^{ih_K} F_i[v] \, dt \\ &= \iint_{\Omega_T} f(Dv) \, dx dt + \frac{1}{h_K} \iint_{\Omega_T} \mathfrak{b}[u^{(K)}(t - h_K), v(t)] \, dx dt \\ &= \mathbf{F}^{(K)}[v]. \end{aligned}$$

Note that for any fixed comparison map $v \in L^\Phi(\Omega_T) \cap (g^{(K)} + L^p(0, T; W_0^{1,p}(\Omega)))$ with $v \geq \psi^{(K)}$ a.e. in Ω_T and any $s \in (0, 1)$ the convex combination $w_s := u^{(K)} + s(v - u^{(K)})$ of $u^{(K)}$ and v is still admissible, since $\psi^{(K)} \leq w_s \in L^\Phi(\Omega_T) \cap (g^{(K)} + L^p(0, T; W_0^{1,p}(\Omega)))$. Then, the minimality of $u^{(K)}$ and the convexity of f imply that

$$\begin{aligned} \mathbf{F}^{(K)}[u^{(K)}] &\leq \mathbf{F}^{(K)}[w_s] \\ &\leq \iint_{\Omega_T} \left[(1-s)f(Du^{(K)}) + sf(Dv) + \frac{1}{h_K} \mathfrak{b}[u^{(K)}(t - h_K), w_s] \right] \, dx dt \end{aligned}$$

for any $s \in (0, 1)$, with equality for $s = 0$. Reabsorbing $\iint_{\Omega_T} (1-s)f(Du^{(K)}) \, dxdt$ into the left-hand side of the preceding inequality and dividing by s , we find that

$$\begin{aligned} & \iint_{\Omega_T} f(Du^{(K)}) \, dxdt \\ & \leq \iint_{\Omega_T} \left[f(Dv) + \frac{1}{sh_K} [\mathfrak{b}[u^{(K)}(t-h_K), w_s] - \mathfrak{b}[u^{(K)}(t-h_K), u^{(K)}(t)]] \right] dxdt \\ & = \iint_{\Omega_T} \left[f(Dv) + \frac{1}{h_K} \left[\frac{1}{s} (\Phi(w_s) - \Phi(u^{(K)})) - b(u^{(K)}(t-h_K)(v-u^{(K)})) \right] \right] dxdt \end{aligned}$$

holds true. Note that the map $s \mapsto \frac{1}{s} (\Phi(w_s) - \Phi(u^{(K)}))$ is monotone and converges a.e. in Ω_T to the $L^1(\Omega_T)$ -function $b(u^{(K)})(v-u^{(K)})$, since Φ is convex. Passing to the limit $s \downarrow 0$ in the preceding inequality with the aid of the dominated convergence theorem, we deduce that

$$\begin{aligned} & \iint_{\Omega_T} f(Du^{(K)}) \, dxdt \\ & \leq \iint_{\Omega_T} \left[f(Dv) + \frac{1}{h_K} [b(u^{(K)}) - b(u^{(K)}(t-h_K))](v-u^{(K)}) \right] dxdt \\ & = \iint_{\Omega_T} [f(Dv) + \Delta_{-h_K} b(u^{(K)})(v-u^{(K)})] \, dxdt \end{aligned}$$

for any map $v \in L^\Phi(\Omega_T) \cap (g^{(K)} + L^p(0, T; W_0^{1,p}(\Omega)))$ with $v \geq \psi^{(K)}$ a.e. in Ω_T . Note that in particular $u^{(K)}(t) = g_o$ for $t \in (-h_K, 0]$. Using the comparison map $\chi_{(0,T)}v + \chi_{[\tau,T]}u^{(K)}$ for some $\tau \in (0, T]$ instead of v , we infer the localized variational inequality

$$\iint_{\Omega_\tau} f(Du^{(K)}) \, dxdt \leq \iint_{\Omega_\tau} [f(Dv) + \Delta_{-h_K} b(u^{(K)})(v-u^{(K)})] \, dxdt \quad (3.9)$$

for any $\tau \in (0, T]$ and $v \in L^\Phi(\Omega_\tau) \cap (g^{(K)} + L^p(0, \tau; W_0^{1,p}(\Omega)))$ with $v \geq \psi^{(K)}$ a.e. in Ω_τ .

3.5 Variational inequality for the limit map

Here, we pass to the limit $K \rightarrow \infty$ in (3.9) in order to deduce the variational inequality for the limit map. To this end, we consider an arbitrary map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T . We extend v to negative times by setting $v(t) = v(0) \in L^\Phi(\Omega)$ for $t < 0$. Observe that the function $v_K := v + \psi^{(K)} - \psi$ is an admissible comparison map in (3.9), since $v_K \geq \psi^{(K)}$ and $v_K \in L^\Phi(\Omega_T) \cap (g^{(K)} + L^p(0, T; W_0^{1,p}(\Omega)))$ are satisfied. Hence, we obtain that

$$\begin{aligned} \iint_{\Omega_\tau} f(Du^{(K)}) \, dxdt & \leq \iint_{\Omega_\tau} [f(Dv_K) + \Delta_{-h_K} b(u^{(K)})(v-u^{(K)})] \, dxdt \\ & \quad + \iint_{\Omega_\tau} \Delta_{-h_K} b(u^{(K)})(\psi^{(K)} - \psi) \, dxdt. \end{aligned} \quad (3.10)$$

Now, we consider the terms of (3.10) separately. First, by (3.7) and since f is convex and satisfies the coercivity condition (1.3)₁, we have that

$$\iint_{\Omega_\tau} f(Du) \, dxdt \leq \liminf_{K \rightarrow \infty} \iint_{\Omega_\tau} f(Du^{(K)}) \, dxdt. \quad (3.11)$$

By (1.4) and the fact that $D\psi^{(K)} \rightarrow D\psi$ in $L^p(\Omega_T, \mathbb{R}^n)$, we conclude that

$$\begin{aligned} & \left| \iint_{\Omega_\tau} f(Dv_K) - f(Dv) \, dxdt \right| \\ & \leq c(n, p, L) \iint_{\Omega_\tau} (1 + |D\psi^{(K)} - D\psi|^{p-1} + |Dv|^{p-1}) |D\psi^{(K)} - D\psi| \, dxdt \\ & \rightarrow 0 \end{aligned} \quad (3.12)$$

in the limit $K \rightarrow \infty$. Next, shifting the difference quotient in the last term on the right-hand side from $b(u^{(K)})$ to $\psi^{(K)} - \psi$, we obtain that

$$\begin{aligned} & \iint_{\Omega_\tau} \Delta_{-h_K} b(u^{(K)}) (\psi^{(K)} - \psi) \, dxdt \\ & = \iint_{\Omega \times (0, \tau - h_K]} b(u^{(K)}) \Delta_{h_K} (\psi^{(K)} - \psi) \, dxdt \\ & \quad - \frac{1}{h_K} \iint_{\Omega \times (\tau - h_K, \tau)} b(u^{(K)}) (\psi^{(K)} - \psi) \, dxdt \\ & \quad + \frac{1}{h_K} \iint_{\Omega \times (-h_K, 0]} b(u^{(K)}) (\psi^{(K)}(t + h_K) - \psi(t + h_K)) \, dxdt \\ & =: \text{I}_K + \text{II}_K + \text{III}_K, \end{aligned} \quad (3.13)$$

where the definition of $\text{I}_K - \text{III}_K$ is clear in this context. By the generalized Hölder's inequality (1.8) and Hölder's inequality, we find that

$$|\text{I}_K| \leq \sup_{t \in [0, \tau]} \|b(u^{(K)}(t))\|'_{L^{\Phi^*}(\Omega)} \int_0^{\tau - h_K} \|\Delta_{h_K} v^{(K)} - \Delta_{h_K} v\|_{L^\Phi(\Omega)} \, dt.$$

Combining the energy estimate (3.6) with Lemma 2.3 and recalling (1.7), we conclude that $(b(u^{(K)}(t)))_{K \in \mathbb{N}}$ is bounded in $L^\infty(0, T; L^{\Phi^*}(\Omega))$. Further, by Lemma 2.13 and since $\Delta_{h_K} v \rightarrow \partial_t v$ in $L^1(0, T; L^\Phi(\Omega))$ as $K \rightarrow \infty$, we infer

$$\lim_{K \rightarrow \infty} \text{I}_K = 0. \quad (3.14)$$

Next, by the generalized Hölder's inequality (1.8), Lemma 2.12, the definition of $\psi^{(K)}$ and since $\psi \in C^0([0, T]; L^\Phi(\Omega))$, we deduce that

$$\begin{aligned} |\text{II}_K| & \leq \sup_{t \in [0, \tau]} \|b(u^{(K)}(t))\|'_{L^{\Phi^*}(\Omega)} \\ & \quad \cdot \int_{\tau - h_K}^{\tau} [\|\psi(\tau) - \psi(t)\|_{L^\Phi(\Omega)} + \|\psi(\tau) - \psi^{(K)}(t)\|_{L^\Phi(\Omega)}] \, dt \\ & \leq \sup_{t \in [0, \tau]} \|b(u^{(K)}(t))\|'_{L^{\Phi^*}(\Omega)} \\ & \quad \cdot \left(\int_{\tau - h_K}^{\tau} \|\psi(\tau) - \psi\|_{L^\Phi(\Omega)} \, dt + \sup_{t \in [\tau - h_K, \tau + h_K]} \|\psi(\tau) - \psi(t)\|_{L^\Phi(\Omega)} \right) \rightarrow 0 \end{aligned} \quad (3.15)$$

in the limit $K \rightarrow \infty$. Similarly, we obtain that

$$\limsup_{K \rightarrow \infty} |\text{III}_K| \leq \lim_{K \rightarrow \infty} \|g_o\|'_{L^{\Phi^*}(\Omega)} \int_0^{h_K} \|\psi(h_K) - \psi(t)\|_{L^{\Phi}(\Omega)} dt = 0. \quad (3.16)$$

In order to treat the remaining term in (3.10), we apply the finite integration by parts formula from Lemma 2.14. This yields

$$\begin{aligned} & \iint_{\Omega_\tau} \Delta_{-h_K} b(u^{(K)}) (v - u^{(K)}) dx dt \\ & \leq \iint_{\Omega_\tau} \Delta_{h_K} v (b(v) - b(u^{(K)})) dx dt \\ & \quad - B_\tau(h_K) + B_0(h_K) + \delta_1(h_K) + \delta_2(h_K), \end{aligned} \quad (3.17)$$

where we used the abbreviations

$$\begin{aligned} B_\tau(h_K) &:= \frac{1}{h_K} \iint_{\Omega \times (\tau - h_K, \tau)} \mathfrak{b}[u^{(K)}(t), v(t + h_K)] dx dt, \\ B_0(h_K) &:= \frac{1}{h_K} \iint_{\Omega \times (-h_K, 0)} \mathfrak{b}[u^{(K)}(t), v(t)] dx dt = \mathfrak{B}[g_o, v(0)]. \end{aligned}$$

Furthermore, the error terms $\delta_1(h_K)$ and $\delta_2(h_K)$ are given by

$$\begin{aligned} \delta_1(h_K) &:= \frac{1}{h_K} \iint_{\Omega_T} \mathfrak{b}[v(t), v(t + h_K)] dx dt, \\ \delta_2(h_K) &:= \iint_{\Omega \times (-h_K, 0)} \Delta_h v (b(v(t + h_K)) - b(u^{(K)}(t))) dx dt \\ &= \iint_{\Omega \times (-h_K, 0)} \Delta_h v (b(v(t + h_K)) - b(g_o)) dx dt. \end{aligned}$$

For the characterization of $B_0(h_K)$ and $\delta_2(h_K)$, we used that $u^{(K)}(t) = g_o$ and $v(t) = v(0)$ for $t \in (-h_K, 0]$. Since $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, Lemma 2.14 implies that

$$\lim_{\mathfrak{K} \ni K \rightarrow \infty} \delta_1(h_K) = 0 \quad \text{and} \quad \lim_{\mathfrak{K} \ni K \rightarrow \infty} \delta_2(h_K) = 0. \quad (3.18)$$

Next, we consider the first term on the right-hand side of (3.17). Since we have that $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, we find that $\Delta_{h_K} v \rightarrow \partial_t v$ strongly in $L^1(0, T; L^\Phi(\Omega))$. Further, since $(b(u^{(K)}(t)))_{K \in \mathbb{N}}$ is bounded in $L^\infty(0, T; L^{\Phi^*}(\Omega))$ and by (3.8)₂, we know that $b(u^{(K)}) \overset{*}{\rightharpoonup} b(u)$ weakly in $L^\infty(0, T; L^{\Phi^*}(\Omega))$. Therefore,

$$\lim_{\mathfrak{K} \ni K \rightarrow \infty} \iint_{\Omega_T} \Delta_{h_K} v (b(v) - b(u^{(K)})) dx dt = \iint_{\Omega_T} \partial_t v (b(v) - b(u)) dx dt. \quad (3.19)$$

Since we are not allowed to pass to the limit $\mathfrak{K} \ni K \rightarrow \infty$ in $B_\tau(h_K)$ directly, we integrate (3.10) over $\tau \in (t_o, t_o + \delta)$ for some $\delta \in (0, T)$ and $t_o \in [0, T - \delta]$ and divide the result by δ . Combining this with (3.17), we find that

$$\begin{aligned}
& \iint_{\Omega_{t_o}} f(Du^{(K)}) \, dxdt \\
& \leq \iint_{\Omega_{t_o+\delta}} f(Dv_K) \, dxdt + \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \Delta_{h_K} v(b(v) - b(u^{(K)})) \, dxdt d\tau \\
& \quad - \frac{1}{\delta} \int_{t_o}^{t_o+\delta-h_K} \int_{\Omega} \mathfrak{b}[u^{(K)}(t), v(t+h_K)] \, dxdt + \mathfrak{B}[g_o, v(0)] \\
& \quad + \delta_1(h_K) + \delta_2(h_K) + \iint_{\Omega_T} \Delta_{-h_K} b(u^{(K)}) (\psi^{(K)} - \psi) \, dxdt.
\end{aligned} \tag{3.20}$$

Note that $u^{(K)} \rightarrow u$ a.e. in Ω_T . Since $v \in C^0([0, T]; L^\Phi(\Omega))$, we have that $v(t+h_K) \rightarrow v(t)$ as $K \rightarrow \infty$ a.e. in Ω_T . Since $\mathfrak{b}$ is nonnegative, we may apply Fatou's lemma, which yields

$$\begin{aligned}
& \frac{1}{\delta} \int_{t_o}^{t_o+\delta} \int_{\Omega} \mathfrak{b}[u, v] \, dxdt \\
& \leq \liminf_{K \rightarrow \infty} \frac{1}{\delta} \int_{t_o}^{t_o+\delta-h_K} \int_{\Omega} \mathfrak{b}[u^{(K)}(t), v(t+h_K)] \, dxdt.
\end{aligned} \tag{3.21}$$

Collecting (3.11), (3.12), (3.14), (3.15), (3.16), (3.19) and (3.21), we infer

$$\begin{aligned}
& \iint_{\Omega_{t_o}} f(Du) \, dxdt \leq \iint_{\Omega_{t_o+\delta}} f(Dv) \, dxdt + \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v(b(v) - b(u)) \, dxdt d\tau \\
& \quad + \mathfrak{B}[g_o, v(0)] - \frac{1}{\delta} \int_{t_o}^{t_o+\delta} \int_{\Omega} \mathfrak{b}[u, v] \, dxdt.
\end{aligned}$$

Now, we pass to the limit $\delta \downarrow 0$. Observe that the integrals in the first line of the preceding inequality depend continuously on time by the absolute continuity of the integral. For the boundary term, by the fact that $\Phi(u(t)) \geq 0$, $b(u)v \geq 0$ and by Lemma 2.3, we find the dominating function

$$0 \leq \mathfrak{b}[u(t), v(t)] \leq \Phi(v(t)) + u(t)b(u(t)) \leq \Phi(v(t)) + (m+1)\Phi(u(t)).$$

Since $u, v \in C^0([0, T]; L^\Phi(\Omega))$ (see Lemma 2.10 for v), the right-hand side of the preceding inequality depends continuously on time. Therefore, using the dominated convergence theorem, we deduce that

$$[0, T] \ni t \mapsto \int_{\Omega} \mathfrak{b}[u(t), v(t)] \, dx \quad \text{is continuous.}$$

Altogether, passing to the limit $\delta \downarrow 0$, we conclude that

$$\begin{aligned}
& \iint_{\Omega_{t_o}} f(Du) \, dxdt \leq \iint_{\Omega_{t_o}} f(Dv) \, dxdt + \iint_{\Omega_T} \partial_t v(b(v) - b(u)) \, dxdt \\
& \quad + \mathfrak{B}[g_o, v(0)] - \mathfrak{B}[u(t_o), v(t_o)]
\end{aligned}$$

holds true for a.e. $t_o \in [0, T]$ and any comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T . Thus, u is a variational solution to (1.2) in the sense of Definition 1.1. $\square$

4 Existence for less regular data with respect to the spatial variables

Next, we prove an existence result for boundary values and obstacle, whose time derivative is less regular with respect to the spatial variables. More precisely, we assume that $0 \leq \psi \leq g$ a.e. on Ω_T ,

$$\begin{cases} g \in L^p(0, T; W^{1,p}(\Omega)), \partial_t g \in L^2(0, T; L^\Phi(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)), \\ g_o = g(0) \in L^\Phi(\Omega) \end{cases} \quad (4.1)$$

and

$$\begin{cases} \psi \in g + L^p(0, T; W_0^{1,p}(\Omega)), \partial_t \psi \in L^2(0, T; L^\Phi(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)), \\ \psi(0) \in L^\Phi(\Omega). \end{cases} \quad (4.2)$$

For the proof of the desired result, we need the following lemma, cf. [12, Lemma 4.3].

Lemma 4.1 *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and let $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ for any $\varepsilon > 0$. Then, for any $u \in L^p(0, T; W_0^{1,p}(\Omega))$ we have*

$$\iint_{(\Omega \setminus \Omega_\varepsilon) \times (0, T)} |u|^p \, dx \, dt \leq c(p, \Omega) \varepsilon^p \iint_{(\Omega \setminus \Omega_\varepsilon) \times (0, T)} |Du|^p \, dx \, dt.$$

Theorem 4.2 *Assume that hypotheses (4.1) and (4.2) are satisfied. Then, there exists a variational solution to (1.2) in the sense of Definition 1.1.*

The proof of 4.2 is divided into several steps.

4.1 Approximation

Let $\varepsilon > 0$, extend g to $\mathbb{R}^n \times (0, T)$ by zero and define $g_\varepsilon := g * \eta_\varepsilon + \varepsilon$, where η_ε denotes the standard mollifier in $\mathbb{R}^n$. Then, we have that $g_\varepsilon \in L^p(0, T; W^{1,p}(\Omega))$, $g_{o,\varepsilon} := g_\varepsilon(0) \in C^\infty(\mathbb{R}^n) \subset (L^2(\Omega) \cap W^{1,p}(\Omega))$ and

$$\partial_t g_\varepsilon = \partial_t g * \eta_\varepsilon \in L^2(0, T; C^\infty(\mathbb{R}^n)) \cap L^p(0, T; W^{1,p}(\Omega)).$$

In order to define mollifications of the obstacle function and comparison maps, we consider the cutoff function $\zeta_\varepsilon \in W^{1,\infty}(\Omega, \mathbb{R}_{\geq 0})$ with $\zeta_\varepsilon \equiv 0$ in $\Omega \setminus \Omega_\varepsilon$, $\zeta_\varepsilon \equiv 1$ in $\Omega_{2\varepsilon}$ and

$$\zeta_\varepsilon(x) := \frac{\text{dist}(x, \partial\Omega) - \varepsilon}{\varepsilon} \quad \text{for } x \in \Omega_\varepsilon \setminus \Omega_{2\varepsilon}.$$

Then, we define

$$\psi_\varepsilon := (\zeta_\varepsilon \psi + (1 - \zeta_\varepsilon)g) * \eta_\varepsilon + \varepsilon.$$

Note that

$$D\psi_\varepsilon = (\zeta_\varepsilon D\psi + (1 - \zeta_\varepsilon)Dg + D\zeta_\varepsilon(\psi - g)) * \eta_\varepsilon.$$

Since $\text{spt}(\zeta_\varepsilon) = \overline{\Omega_\varepsilon}$, we conclude that $\psi_\varepsilon \in g_\varepsilon + L^p(0, T; W_0^{1,p}(\Omega))$ and $\psi_\varepsilon(0) \in L^2(\Omega) \cap (g_{0,\varepsilon} + W_0^{1,p}(\Omega))$. Further, since ζ_ε and η_ε are independent of time, we have that

$$\partial_t \psi_\varepsilon = (\zeta_\varepsilon \partial_t \psi + (1 - \zeta_\varepsilon) \partial_t g) * \eta_\varepsilon \in L^2(0, T; C^\infty(\Omega)) \cap L^p(\Omega_T)$$

and

$$\partial_t D\psi_\varepsilon = (\zeta_\varepsilon \partial_t D\psi + (1 - \zeta_\varepsilon) \partial_t Dg + D\zeta_\varepsilon (\partial_t \psi - \partial_t g)) * \eta_\varepsilon \in L^p(\Omega_T, \mathbb{R}^n).$$

Finally, by a standard property of the applied mollification procedure, we find that $0 < \varepsilon \leq \psi_\varepsilon \leq g_\varepsilon$. More generally, for any comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T , we define the mollification

$$v_\varepsilon := (\zeta_\varepsilon v + (1 - \zeta_\varepsilon)g) * \eta_\varepsilon + \varepsilon.$$

Then, we obtain that $v_\varepsilon \in g_\varepsilon + L^p(0, T; W_0^{1,p}(\Omega))$, $\partial_t v_\varepsilon \in L^1(0, T; L^\Phi(\Omega))$, $v_\varepsilon(0) \in L^\Phi(\Omega)$ and $v_\varepsilon \geq \psi_\varepsilon$ a.e. in Ω_T . Next, we prove that

$$v_\varepsilon \rightarrow v \text{ in } L^\Phi(\Omega_T) \text{ as } \varepsilon \downarrow 0. \quad (4.3)$$

Indeed, we conclude that

$$\begin{aligned} \|v_\varepsilon - v\|_{L^\Phi(\Omega_T)} &\leq \|v_\varepsilon - v * \eta_\varepsilon\|_{L^\Phi(\Omega_T)} + \|v * \eta_\varepsilon - v\|_{L^\Phi(\Omega_T)} \\ &= \|((1 - \zeta_\varepsilon)(g - v)) * \eta_\varepsilon + \varepsilon\|_{L^\Phi(\Omega_T)} + \|v * \eta_\varepsilon - v\|_{L^\Phi(\Omega_T)} \\ &\leq \|(1 - \zeta_\varepsilon)(g - v)\|_{L^\Phi(\Omega_T)} + \|\varepsilon\|_{L^\Phi(\Omega_T)} + \|v * \eta_\varepsilon - v\|_{L^\Phi(\Omega_T)} \rightarrow 0 \end{aligned}$$

in the limit $\varepsilon \downarrow 0$. Further, (4.3) in particular implies that there exists a (not relabeled) subsequence such that $v_\varepsilon \rightarrow v$ a.e. in Ω_T . A similar computation shows that

$$v_\varepsilon(0) \rightarrow v(0) \text{ in } L^\Phi(\Omega) \text{ and a.e. in } \Omega \text{ as } \varepsilon \downarrow 0. \quad (4.4)$$

Moreover, observe that

$$\begin{aligned} \sup_{t \in [0, T]} \|v_\varepsilon(t)\|_{L^\Phi(\Omega)} &\leq \sup_{t \in [0, T]} \|\zeta_\varepsilon v(t) + (1 - \zeta_\varepsilon)g(t)\|_{L^\Phi(\Omega)} + \|\varepsilon\|_{L^\Phi(\Omega)} \\ &\leq \sup_{t \in [0, T]} \|v(t)\|_{L^\Phi(\Omega)} + \sup_{t \in [0, T]} \|g(t)\|_{L^\Phi(\Omega)} + \|1\|_{L^\Phi(\Omega)}. \end{aligned} \quad (4.5)$$

Hence, by Remark 2.4 the sequence $(b(v_\varepsilon))_{\varepsilon > 0}$ is bounded in $L^\infty(0, T; L^{\Phi^*}(\Omega))$. Together, the preceding considerations prove that there exists another subsequence such that

$$b(v_\varepsilon) \overset{*}{\rightharpoonup} b(v) \text{ weakly }^* \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } \varepsilon \downarrow 0. \quad (4.6)$$

Next, we compute that

$$\begin{aligned} \|\partial_t v_\varepsilon - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} &\leq \|\partial_t v_\varepsilon - \partial_t v * \eta_\varepsilon\|_{L^1(0, T; L^\Phi(\Omega))} + \|\partial_t v * \eta_\varepsilon - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} \\ &\leq \|(1 - \zeta_\varepsilon)(\partial_t g - \partial_t v)\|_{L^1(0, T; L^\Phi(\Omega))} + \|\partial_t v * \eta_\varepsilon - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} \rightarrow 0 \end{aligned}$$

in the limit $\varepsilon \downarrow 0$, which yields

$$\partial_t v_\varepsilon \rightarrow \partial_t v \text{ in } L^1(0, T; L^\Phi(\Omega)) \text{ as } \varepsilon \downarrow 0. \quad (4.7)$$

Finally, we show that

$$Dv_\varepsilon \rightarrow Dv \text{ in } L^p(\Omega_T, \mathbb{R}^n) \text{ as } \varepsilon \downarrow 0. \quad (4.8)$$

To this end, we first compute

$$\begin{aligned} \|Dv_\varepsilon - Dv\|_{L^p(\Omega_T, \mathbb{R}^n)} &\leq \|Dv_\varepsilon - Dv * \eta_\varepsilon\|_{L^p(\Omega_T, \mathbb{R}^n)} + \|Dv * \eta_\varepsilon - Dv\|_{L^p(\Omega_T, \mathbb{R}^n)} \\ &\leq \|(1 - \zeta_\varepsilon)(Dg - Dv) + D\zeta_\varepsilon(v - g)\|_{L^p(\Omega_T, \mathbb{R}^n)} \\ &\quad + \|Dv * \eta_\varepsilon - Dv\|_{L^p(\Omega_T, \mathbb{R}^n)}. \end{aligned}$$

The second term on the right-hand side of the preceding inequality clearly vanishes as $\varepsilon \downarrow 0$. For the first term, by definition of ζ_ε and Lemma 4.1 we conclude that

$$\begin{aligned} &\iint_{\Omega_T} |(1 - \zeta_\varepsilon)(Dg - Dv) + D\zeta_\varepsilon(v - g)|^p \, dxdt \\ &\leq 2^{p-1} \iint_{\Omega_T} [(1 - \zeta_\varepsilon)^p |Dg - Dv|^p + |D\zeta_\varepsilon|^p |v - g|^p] \, dxdt \\ &\leq 2^{p-1} \iint_{(\Omega \setminus \Omega_{2\varepsilon}) \times (0, T)} [|Dg - Dv|^p + \varepsilon^{-p} |v - g|^p] \, dxdt \\ &\leq c(n, p, \Omega) \iint_{(\Omega \setminus \Omega_{2\varepsilon}) \times (0, T)} |Dg - Dv|^p \, dxdt \rightarrow 0 \end{aligned}$$

in the limit $\varepsilon \downarrow 0$. This proves (4.8).

4.2 Solutions of the regularized problem

For any $\varepsilon > 0$, the mollifications g_ε and ψ_ε satisfy the assumptions of Theorem 3.1. Hence, there exist corresponding variational solutions $u_\varepsilon \in C^0([0, T]; L^\Phi(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega)))$. Applying Lemma 2.20 with the admissible comparison map g_ε , we obtain the energy bound

$$\frac{1}{2} \sup_{t \in [0, T]} \varrho_\Omega(u_\varepsilon(t)) + \iint_{\Omega_T} f(Du_\varepsilon) \, dx \leq C, \quad (4.9)$$

where the constant C is defined by

$$\begin{aligned} C &:= \frac{2^\ell}{m} + \frac{1}{2} \sup_{\varepsilon > 0} \sup_{t \in [0, T]} \varrho_\Omega(g_\varepsilon(t)) + c(\ell, m) \sup_{\varepsilon > 0} \left(\int_0^T \|\partial_t g_\varepsilon\|_{L^\Phi(\Omega)} \, dt \right)^{\frac{\ell+1}{\ell}} \\ &\quad + \frac{3}{\ell} \sup_{\varepsilon > 0} \iint_{\Omega_T} f(Dg_\varepsilon) \, dxdt. \end{aligned}$$

By (4.5) and (1.7), (4.7) and (4.8) together with the growth condition (1.3) C is finite. Therefore, there exists a (not relabeled) subsequence and a limit map $u \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega)))$ such that

$$\begin{cases} u_\varepsilon \overset{*}{\rightharpoonup} u \text{ weakly}^* \text{ in } L^\infty(0, T; L^\Phi(\Omega)), \\ u_\varepsilon \rightharpoonup u \text{ weakly in } L^p(0, T; W^{1,p}(\Omega)) \end{cases} \quad (4.10)$$

as $\varepsilon \downarrow 0$. By (4.3) applied to $v = \psi$ and (4.10)₂, we obtain that

$$\iint_{\Omega_T} (u - \psi) \varphi \, dx = \lim_{\varepsilon \downarrow 0} \iint_{\Omega_T} (u_\varepsilon - \psi_\varepsilon) \varphi \, dx \geq 0 \quad \forall \varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0}).$$

Hence, the obstacle condition $u \geq \psi$ is satisfied.

4.3 Improved convergence of the solutions

Next, we need to establish

$$b(u_\varepsilon) \rightharpoonup^* b(u) \text{ weakly }^* \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } \varepsilon \downarrow 0. \quad (4.11)$$

Since $(b(u_\varepsilon))_\varepsilon$ is bounded in $L^\infty(0, T; L^{\Phi^*}(\Omega))$ by (4.10) and Remark 2.4, there exists a subsequence such that

$$b(u_\varepsilon) \rightharpoonup^* w \text{ weakly }^* \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } \varepsilon \downarrow 0 \quad (4.12)$$

for some limit map $w \in L^\infty(0, T; L^{\Phi^*}(\Omega))$. Therefore, it remains to prove that w has the structure $b(u)$. To this end, for $h > 0$ we consider mollifications $[u_\varepsilon - \psi_\varepsilon]_h$ and $[u - \psi]_h$ according to (2.7) with zero initial values and define

$$w_{\varepsilon, h} := [u_\varepsilon - \psi_\varepsilon]_h + \psi_\varepsilon \text{ and } w_h := [u - \psi]_h + \psi.$$

Since $L^\Phi(\Omega)$ is separable, by Lemma 2.15 we obtain that $w_{\varepsilon, h}, w_h \in L^\infty(0, T; L^\Phi(\Omega))$. Further, we find that $w_{\varepsilon, h} \in g_\varepsilon + L^p(0, T; W_0^{1,p}(\Omega))$ and $w_h \in g + L^p(0, T; W_0^{1,p}(\Omega))$. Since $u_\varepsilon \geq \psi_\varepsilon$, we have that $w_{\varepsilon, h} \geq \psi_\varepsilon$. Next, note that (2.6) implies

$$\partial_t [u_\varepsilon - \psi_\varepsilon]_h = \frac{1}{h} ((u_\varepsilon - \psi_\varepsilon) - [u_\varepsilon - \psi_\varepsilon]_h) \in L^\Phi(\Omega_T) \cap L^p(\Omega_T).$$

Thus, by (4.10)₂ and since $\psi_\varepsilon \rightarrow \psi$ in $L^p(0, T; W^{1,p}(\Omega))$, we deduce that for any fixed $h > 0$ the sequence $(\partial_t [u_\varepsilon - \psi_\varepsilon]_h)_\varepsilon$ is bounded in $L^p(\Omega_T)$. Further, by Lemma 2.17

$$w_{\varepsilon, h} \rightharpoonup w_h \text{ weakly in } L^p(0, T; W^{1,p}(\Omega)) \text{ as } \varepsilon \downarrow 0 \quad (4.13)$$

holds true for fixed $h > 0$. Therefore, from Rellich's theorem, we infer

$$w_{\varepsilon, h} \rightarrow w_h \text{ in } L^p(\Omega_T) \text{ as } \varepsilon \downarrow 0. \quad (4.14)$$

We did not have to pass to a subsequence, since the limit is determined by (4.13). Next, we use $w_{\varepsilon, h}$ as comparison map in the variational inequality satisfied by u_ε . Discarding the nonnegative terms $\iint_{\Omega_T} f(Du_\varepsilon) \, dx \, dt$ and $\mathfrak{B}[u_\varepsilon(T), w_{\varepsilon, h}(T)]$, we deduce that

$$\begin{aligned} & - \iint_{\Omega_T} \partial_t w_{\varepsilon, h} (b(w_{\varepsilon, h}) - b(u_\varepsilon)) \, dx \, dt \\ & = \frac{1}{h} \iint_{\Omega_T} (w_{\varepsilon, h} - u_\varepsilon) (b(w_{\varepsilon, h}) - b(u_\varepsilon)) \, dx \, dt - \iint_{\Omega_T} \partial_t \psi_\varepsilon (b(w_{\varepsilon, h}) - b(u_\varepsilon)) \, dx \, dt \\ & \leq \iint_{\Omega_T} f(Dw_{\varepsilon, h}) \, dx \, dt + \mathfrak{B}[g_{0, \varepsilon}, \psi_\varepsilon(0)], \end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \iint_{\Omega_T} (w_{\varepsilon,h} - u_\varepsilon)(b(w_{\varepsilon,h}) - b(u_\varepsilon)) \, dx dt \\
& \leq h \iint_{\Omega_T} \partial_t \psi_\varepsilon (b(w_{\varepsilon,h}) - b(u_\varepsilon)) \, dx dt + h \iint_{\Omega_T} f(Dw_{\varepsilon,h}) \, dx dt \\
& \quad + h \mathfrak{B}[g_{o,\varepsilon}, \psi_\varepsilon(0)] \\
& =: h\mathrm{I}_h + h\mathrm{II}_h + h\mathrm{III},
\end{aligned} \tag{4.15}$$

where the definition of I_h , II_h and III is clear in this context. By the generalized Hölder's inequality (1.8), Hölder's inequality, Lemmas 2.3 and 2.5, we infer

$$\begin{aligned}
|\mathrm{I}_h| & \leq \|\partial_t \psi_\varepsilon\|_{L^1(0,T;L^\Phi(\Omega))} \left[\sup_{t \in [0,T]} \|b(w_{\varepsilon,h}(t))\|'_{L^{\Phi^*}(\Omega)} + \sup_{t \in [0,T]} \|b(u_\varepsilon(t))\|'_{L^{\Phi^*}(\Omega)} \right] \\
& \leq \|\partial_t \psi_\varepsilon\|_{L^1(0,T;L^\Phi(\Omega))} \left[2 + \sup_{t \in [0,T]} \varrho_\Omega^*(b(w_{\varepsilon,h}(t)))^{\frac{1}{\varepsilon+1}} + \sup_{t \in [0,T]} \varrho_\Omega^*(b(u_\varepsilon(t)))^{\frac{1}{\varepsilon+1}} \right] \\
& \leq \|\partial_t \psi_\varepsilon\|_{L^1(0,T;L^\Phi(\Omega))} \left[2 + \sup_{t \in [0,T]} (m\varrho_\Omega(w_{\varepsilon,h}(t)))^{\frac{1}{\varepsilon+1}} + \sup_{t \in [0,T]} (m\varrho_\Omega(u_\varepsilon(t)))^{\frac{1}{\varepsilon+1}} \right].
\end{aligned}$$

Note that $\sup_{t \in [0,T]} \varrho_\Omega(u_\varepsilon(t)) \leq 2C$ and $(\|\partial_t \psi_\varepsilon\|_{L^1(0,T;L^\Phi(\Omega))})_\varepsilon$ is bounded by (4.7). Further, by Lemma 2.15, (4.5) and (4.10), we find that

$$\begin{aligned}
\sup_{\varepsilon > 0} \|w_{\varepsilon,h}\|_{L^\infty(0,T;L^\Phi(\Omega))} & \leq \sup_{\varepsilon > 0} \|u_\varepsilon - \psi_\varepsilon\|_{L^\infty(0,T;L^\Phi(\Omega))} + \sup_{\varepsilon > 0} \|\psi_\varepsilon\|_{L^\infty(0,T;L^\Phi(\Omega))} \\
& \leq \sup_{\varepsilon > 0} \|u_\varepsilon - \psi_\varepsilon\|_{L^\infty(0,T;L^\Phi(\Omega))} + \sup_{\varepsilon > 0} \|\psi_\varepsilon\|_{L^\infty(0,T;L^\Phi(\Omega))} \\
& < \infty.
\end{aligned}$$

Thus, taking into account (1.7), we conclude that

$$|\mathrm{I}_h| \leq c \tag{4.16}$$

for a constant c independent of ε and h . Next, by the growth condition (1.3), Lemma 2.15, (4.8) and (4.9), we obtain that

$$\begin{aligned}
|\mathrm{II}_h| & \leq L \iint_{\Omega_T} [1 + |Dw_{\varepsilon,h}|^p] \, dx dt \\
& \leq L|\Omega|T + 2^{p-1}L \left[\|Du_\varepsilon - D\psi_\varepsilon\|_{L^p(\Omega_T, \mathbb{R}^n)}^p + \|D\psi_\varepsilon\|_{L^p(\Omega_T, \mathbb{R}^n)}^p \right] \\
& \leq L|\Omega|T + c(p)L \left[\|Du_\varepsilon\|_{L^p(\Omega_T, \mathbb{R}^n)}^p + \|D\psi_\varepsilon\|_{L^p(\Omega_T, \mathbb{R}^n)}^p \right] \\
& \leq L|\Omega|T + c(p)L \left[C + \sup_{\varepsilon > 0} \|D\psi_\varepsilon\|_{L^p(\Omega_T, \mathbb{R}^n)}^p \right] < \infty.
\end{aligned} \tag{4.17}$$

Finally, by (4.4) applied to $\psi(0)$ and g_o , we have that $\mathfrak{b}[g_{o,\varepsilon}, \psi_\varepsilon(0)] \rightarrow \mathfrak{b}[g_o, \psi(0)]$ a.e. in Ω_T as $\varepsilon \downarrow 0$. Further, by Lemma 2.3 and since $b(g_{o,\varepsilon}), \psi_\varepsilon(0) \geq 0$ by Lemma 2.11, we find the dominating function

$$\begin{aligned}
0 & \leq \mathfrak{b}[g_{o,\varepsilon}, \psi_\varepsilon(0)] \leq \Phi(\psi_\varepsilon(0)) + \Phi^*(b(g_{o,\varepsilon})) \leq \Phi(\psi_\varepsilon(0)) + m\Phi(g_{o,\varepsilon}) \\
& \rightarrow \Phi(\psi(0)) + m\Phi(g_o)
\end{aligned}$$

in $L^1(\Omega)$ as $\varepsilon \downarrow 0$. Thus, we conclude that $\mathfrak{B}[g_{o,\varepsilon}, \psi_\varepsilon(0)] \rightarrow \mathfrak{B}[g_o, \psi(0)]$ in the limit $\varepsilon \downarrow 0$. In particular, this implies

$$\sup_{\varepsilon > 0} \text{III} < \infty. \quad (4.18)$$

Inserting (4.16), (4.17) and (4.18) into (4.15), we infer

$$\iint_{\Omega_T} (w_{\varepsilon,h} - u_\varepsilon)(b(w_{\varepsilon,h}) - b(u_\varepsilon)) \, dxdt \leq ch,$$

where the constant c is independent of ε and h . By Lemma 2.9, Hölder's inequality and Lemma 2.7, we bound the left-hand side of the preceding inequality from below,

$$\begin{aligned} \varrho_{\Omega_T}(w_{\varepsilon,h} - u_\varepsilon) &\leq \iint_{\Omega_T} |\Phi(w_{\varepsilon,h}) - \Phi(u_\varepsilon)| \, dxdt \\ &\leq \left(\iint_{\Omega_T} |\sqrt{\Phi(w_{\varepsilon,h})} + \sqrt{\Phi(u_\varepsilon)}|^2 \, dxdt \right)^{\frac{1}{2}} \left(\iint_{\Omega_T} |\sqrt{\Phi(w_{\varepsilon,h})} - \sqrt{\Phi(u_\varepsilon)}|^2 \, dxdt \right)^{\frac{1}{2}} \\ &\leq c [\varrho_{\Omega_T}(w_{\varepsilon,h})^{\frac{1}{2}} + \varrho_{\Omega_T}(u_\varepsilon)^{\frac{1}{2}}] \left(\iint_{\Omega_T} (w_{\varepsilon,h} - u_\varepsilon)(b(w_{\varepsilon,h}) - b(u_\varepsilon)) \, dxdt \right)^{\frac{1}{2}} \end{aligned}$$

with a constant $c = c(\ell, m)$. Since we have already shown that $\iint_{\Omega_T} \Phi(w_{\varepsilon,h}) \, dxdt$ and $\iint_{\Omega_T} \Phi(u_\varepsilon) \, dxdt$ stay bounded in the limits $\varepsilon \downarrow 0$ and $h \downarrow 0$, we deduce that

$$\varrho_{\Omega_T}(w_{\varepsilon,h} - u_\varepsilon) \leq ch^{\frac{1}{2}},$$

where the constant c does not depend on ε or h . Further, by (1.6) and Lemma 2.8 for any $\delta > 0$, there exists $h_\delta > 0$ such that

$$\|w_{\varepsilon,h} - u_\varepsilon\|_{L^{\ell+1}(\Omega_T)} < \frac{\delta}{2} \quad (4.19)$$

holds true for any $0 < h < h_\delta$ and any $\varepsilon > 0$. Choosing h_δ smaller if necessary, by Lemma 2.15 we may assume that

$$\|w_h - u\|_{L^{\ell+1}(\Omega_T)} < \frac{\delta}{2}. \quad (4.20)$$

Let $q := \min\{\ell + 1, p\}$. Then, by (4.19) and (4.20) with a suitable choice of $h > 0$ and (4.14), we infer

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \|u - u_\varepsilon\|_{L^q(\Omega_T)} &\leq \lim_{\varepsilon \downarrow 0} [\|u - w_h\|_{L^q(\Omega_T)} + \|w_h - w_{\varepsilon,h}\|_{L^q(\Omega_T)} + \|w_{\varepsilon,h} - u_\varepsilon\|_{L^q(\Omega_T)}] \\ &< \delta + \lim_{\varepsilon \downarrow 0} \|w_h - w_{\varepsilon,h}\|_{L^q(\Omega_T)} = \delta. \end{aligned}$$

Since δ was arbitrary, this leads to

$$u_\varepsilon \rightarrow u \text{ in } L^{\min\{\ell+1, p\}}(\Omega_T) \text{ as } \varepsilon \downarrow 0.$$

Passing to a (not relabeled) subsequence, we have that $u_\varepsilon \rightarrow u$ a.e. in Ω_T as $\varepsilon \downarrow 0$. Together with (4.12), this implies (4.11).

4.4 Passage to the limit

Consider a comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T . Define mollifications v_ε as in Sect. 4.1. Because of the boundary term, we are not allowed to pass to the limit $\varepsilon \downarrow 0$ in the variational inequality satisfied by u_ε and v_ε directly. Instead, we integrate over $\tau \in (t_o, t_o + \delta)$ for some $\delta \in (0, T)$ and $t_o \in (0, T - \delta]$. This leads to

$$\begin{aligned} & \iint_{\Omega_{t_o}} f(Du_\varepsilon) \, dxdt + \int_{t_o}^{t_o+\delta} \mathfrak{B}[u_\varepsilon(\tau), v_\varepsilon(\tau)] \, d\tau \\ & \leq \iint_{\Omega_{t_o+\delta}} f(Dv_\varepsilon) \, dxdt + \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v_\varepsilon (b(v_\varepsilon) - b(u_\varepsilon)) \, dxdt d\tau \\ & \quad + \mathfrak{B}[g_{o,\varepsilon}, v_\varepsilon(0)]. \end{aligned} \quad (4.21)$$

Since f is convex and satisfies the coercivity condition (1.3)₁, by (4.10)₂ we have that

$$\iint_{\Omega_{t_o}} f(Du) \, dxdt \leq \liminf_{\varepsilon \downarrow 0} \iint_{\Omega_{t_o}} f(Du_\varepsilon) \, dxdt. \quad (4.22)$$

Further, note that

$$\mathfrak{B}[u_\varepsilon(\tau), v_\varepsilon(\tau)] = \int_{\Omega} [\Phi(v_\varepsilon(\tau)) + \Phi^*(b(u_\varepsilon(\tau))) - b(u_\varepsilon(\tau))v_\varepsilon(\tau)] \, dx.$$

By (1.6), the functionals $\int_{t_o}^{t_o+\delta} \int_{\Omega} \Phi(\cdot) \, dx d\tau$ and $\int_{t_o}^{t_o+\delta} \int_{\Omega} \Phi^*(\cdot) \, dx d\tau$ are continuous with respect to strong convergence in $L^\Phi(\Omega_T)$, respectively $L^{\Phi^*}(\Omega_T)$. Together with the convexity of Φ and Φ^* , this implies that they are lower semicontinuous with respect to weak convergence in $L^\Phi(\Omega_T)$, respectively $L^{\Phi^*}(\Omega_T)$, cf. [14, Corollary 3.9]. Therefore, by (4.3) and (4.11), we conclude that

$$\int_{t_o}^{t_o+\delta} \mathfrak{B}[u(\tau), v(\tau)] \, d\tau \leq \liminf_{\varepsilon \downarrow 0} \int_{t_o}^{t_o+\delta} \mathfrak{B}[u_\varepsilon(\tau), v_\varepsilon(\tau)] \, d\tau. \quad (4.23)$$

Next, by the local Lipschitz condition (1.4) and (4.8), we find that

$$\lim_{\varepsilon \downarrow 0} \iint_{\Omega_{t_o+\delta}} f(Dv_\varepsilon) \, dxdt = \iint_{\Omega_{t_o+\delta}} f(Dv) \, dxdt. \quad (4.24)$$

From (4.6), (4.7), (4.11) and the dominated convergence theorem, we infer

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v_\varepsilon (b(v_\varepsilon) - b(u_\varepsilon)) \, dxdt d\tau \\ & = \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v (b(v) - b(u)) \, dxdt d\tau. \end{aligned} \quad (4.25)$$

Finally, by (4.4) applied to v and g and the dominated convergence theorem, we have that

$$\lim_{\varepsilon \downarrow 0} \mathfrak{B}[g_{o,\varepsilon}, v_\varepsilon(0)] = \mathfrak{B}[g_o, v(0)]. \quad (4.26)$$

Inserting (4.22), (4.23), (4.24), (4.25) and (4.26) into (4.21), we deduce that

$$\begin{aligned} & \iint_{\Omega_{t_0}} f(Du) \, dx \, dt + \int_{t_0}^{t_0+\delta} \mathfrak{B}[u(\tau), v(\tau)] \, d\tau \\ & \leq \iint_{\Omega_{t_0+\delta}} f(Dv) \, dx \, dt + \int_{t_0}^{t_0+\delta} \iint_{\Omega_\tau} \partial_t v (b(v) - b(u)) \, dx \, dt \, d\tau + \mathfrak{B}[g_o, v(0)] \end{aligned}$$

holds true for any $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T . Passing to the limit $\delta \downarrow 0$, we conclude that u is a variational solution to (1.2) in the sense of Definition 1.1. $\square$

5 Proof of Theorem 1.2

5.1 Regularization

We consider a sequence $0 < h_i \downarrow 0$ as $i \rightarrow \infty$ and set $\varepsilon_i := h_i^{\frac{1}{2(n+1)p}}$. Then, we extend the initial datum g_o by zero to a function $g_o \in L^\Phi(\mathbb{R}^n)$ and define mollifications of the extension by

$$g_{o,i} := g_o * \eta_{\varepsilon_i},$$

where η_ε denotes a standard mollifier in $\mathbb{R}^n$. Thus, we have that $g_{o,i} \in L^\Phi(\Omega) \cap W^{1,p}(\Omega)$. Further, we set

$$g_i := [g]_{h_i},$$

where $[g]_{h_i}$ denotes the mollification in time according to (2.7) with initial values $g_{o,i}$. Since the space $L^\Phi(\Omega)$ is separable, Lemma 2.15 implies $g_i \in C^0([0, T]; L^\Phi(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$. Moreover, by (2.6) we obtain

$$\partial_t g_i = \frac{1}{h_i} (g - g_i) \in L^\infty(0, T; L^\Phi(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)).$$

Next, we define mollifications of comparison maps $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T . To this end, we set

$$v_{o,i} := (\chi_{\Omega \setminus \Omega_{\varepsilon_i}} g_o + \chi_{\Omega_{\varepsilon_i}} v(0)) * \eta_{\varepsilon_i},$$

where χ is the indicator function, i.e., for $A \subset \mathbb{R}^n$ we have that $\chi_A \equiv 1$ in A and $\chi_A \equiv 0$ in $\mathbb{R}^n \setminus A$. By definition of η_{ε_i} , we obtain that $v_{o,i} \in L^\Phi(\Omega) \cap (g_{o,i} + W_0^{1,p}(\Omega))$. Then, by

$$v_i := [v]_{h_i}$$

we denote the mollification given by (2.7) with initial datum $v_{o,i}$. Lemma 2.15 implies that $v_i \in C^0([0, T]; L^\Phi(\Omega)) \cap (g_i + L^p(0, T; W_0^{1,p}(\Omega)))$. Moreover, by (2.6), we conclude that

$$\partial_t v_i = \frac{1}{h_i} (v - v_i) \in L^\infty(0, T; L^\Phi(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)).$$

In particular, we apply the preceding mollification procedure to $v = \psi$. Since $\psi_{o,i} \leq g_{o,i}$ by Lemma 2.11 and the definition of η_{ε_i} , we have that $0 \leq \psi_i \leq g_i$. In the following, we will also use the mollification

$$\tilde{v}_i := [v]_{h_i}$$

according to (2.7) with $v_o = v(0)$. In particular, we know that $\tilde{v}_i = v_i + e^{-\frac{t}{h_i}}(v(0) - v_{o,i})$. First, we find that

$$v_{o,i} \rightarrow v(0) \text{ a.e. and in } L^\Phi(\Omega) \text{ as } i \rightarrow \infty. \quad (5.1)$$

Then, we claim that

$$v_i \rightarrow v \text{ in } L^\infty(0, T; L^\Phi(\Omega)) \text{ as } i \rightarrow \infty. \quad (5.2)$$

Indeed, from Lemma 2.15, we conclude that $\tilde{v}_i \rightarrow v$ in $L^\infty(0, T; L^\Phi(\Omega))$ as $i \rightarrow \infty$. Together with (5.1), we obtain that

$$\begin{aligned} \lim_{i \rightarrow \infty} \|v_i - v\|_{L^\infty(0, T; L^\Phi(\Omega))} &\leq \lim_{i \rightarrow \infty} \|v - \tilde{v}_i\|_{L^\infty(0, T; L^\Phi(\Omega))} + \lim_{i \rightarrow \infty} \|e^{-\frac{t}{h_i}}(v(0) - v_{o,i})\|_{L^\infty(0, T; L^\Phi(\Omega))} \\ &\leq \lim_{i \rightarrow \infty} \|\tilde{v}_i - v\|_{L^\infty(0, T; L^\Phi(\Omega))} + \lim_{i \rightarrow \infty} \|v_{o,i} - v(0)\|_{L^\Phi(\Omega)} = 0. \end{aligned}$$

Moreover, from Lemma 2.3, (1.7) and (5.2), we conclude that

$$\sup_{t \in [0, T]} \varrho_\Omega^*(b(v_i(t))) \leq \sup_{t \in [0, T]} m\varrho_\Omega(v_i(t)) < \infty.$$

Together with (5.2), this implies that

$$b(v_i) \overset{*}{\rightharpoonup} b(v) \text{ weakly } ^* \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } i \rightarrow \infty. \quad (5.3)$$

Next, we show that

$$\partial_t v_i \rightarrow \partial_t v \text{ in } L^1(0, T; L^\Phi(\Omega)) \text{ as } i \rightarrow \infty. \quad (5.4)$$

To this end, from Lemma 2.16, we infer

$$\partial_t \tilde{v}_i(t) = \frac{1}{h_i} \int_0^t e^{-\frac{s-t}{h_i}} \partial_t v(s) \, ds,$$

i.e., the mollification of $\partial_t v$ according to (2.7) with zero initial datum. Therefore, by Lemma 2.15, we obtain that $\partial_t \tilde{v}_i \rightarrow \partial_t v$ in $L^1(0, T; L^\Phi(\Omega))$ as $i \rightarrow \infty$, which allows us to compute

$$\begin{aligned} \lim_{i \rightarrow \infty} \|\partial_t v_i - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} &\leq \lim_{i \rightarrow \infty} \|\partial_t \tilde{v}_i - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} + \lim_{i \rightarrow \infty} \|\partial_t e^{-\frac{t}{h_i}}(v(0) - v_{o,i})\|_{L^1(0, T; L^\Phi(\Omega))} \\ &= \lim_{i \rightarrow \infty} \|\partial_t \tilde{v}_i - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} + \lim_{i \rightarrow \infty} \int_0^T \frac{1}{h_i} e^{-\frac{t}{h_i}} \, dt \cdot \|v(0) - v_{o,i}\|_{L^\Phi(\Omega)} \\ &\leq \lim_{i \rightarrow \infty} \|\partial_t \tilde{v}_i - \partial_t v\|_{L^1(0, T; L^\Phi(\Omega))} + \lim_{i \rightarrow \infty} \|v(0) - v_{o,i}\|_{L^\Phi(\Omega)} = 0. \end{aligned}$$

Finally, we establish the assertion

$$Dv_i \rightarrow Dv \text{ in } L^p(\Omega_T, \mathbb{R}^n) \text{ as } i \rightarrow \infty. \quad (5.5)$$

To this end, we consider the mollification according to (2.7) with zero initial values, i.e., $v_i^o := v_i - e^{-\frac{\cdot}{h_i}} v_{o,i}$. Lemma 2.15 implies

$$Dv_i^o \rightarrow Dv \text{ in } L^p(\Omega_T, \mathbb{R}^n) \text{ as } i \rightarrow \infty. \quad (5.6)$$

Further, observe that

$$\iint_{\Omega_T} \left| D(e^{\frac{\cdot}{h_i}} v_{o,i}) \right|^p dx dt = \int_0^T e^{-\frac{pt}{h_i}} dt \int_{\Omega} |Dv_{o,i}|^p dx \leq \frac{h_i}{p} \int_{\Omega} |Dv_{o,i}|^p dx.$$

Using Young's inequality for convolutions, we estimate the last integral. This yields

$$\begin{aligned} \int_{\Omega} |Dv_{o,i}|^p dx &= \int_{\Omega} |(\chi_{\Omega \setminus \Omega_{\varepsilon_i}} g_o + \chi_{\Omega_{\varepsilon_i}} v(0)) * D\eta_{\varepsilon_i}|^p dx \\ &\leq \|\chi_{\Omega \setminus \Omega_{\varepsilon_i}} g_o + \chi_{\Omega_{\varepsilon_i}} v(0)\|_{L^1(\Omega)}^p \|D\eta_{\varepsilon_i}\|_{L^p(\mathbb{R}^n)}^p \\ &\leq (\|1\|'_{L^{\Phi^*}(\Omega)})^p [\|g_o\|_{L^{\Phi}(\Omega)} + \|v(0)\|_{L^{\Phi}(\Omega)}]^p \varepsilon_i^{-(n+1)p+n} \int_{\mathbb{R}^n} |D\eta|^p dx \\ &\leq c [\|g_o\|_{L^{\Phi}(\Omega)} + \|v(0)\|_{L^{\Phi}(\Omega)}]^p \varepsilon_i^{-(n+1)p} \end{aligned}$$

with $c = c(n, p, m, |\Omega|)$. Joining the two preceding estimates and recalling the definition of ε_i , we obtain that

$$\begin{aligned} \iint_{\Omega_T} \left| D(e^{\frac{\cdot}{h_i}} v_{o,i}) \right|^p dx dt &\leq c [\|g_o\|_{L^{\Phi}(\Omega)} + \|v(0)\|_{L^{\Phi}(\Omega)}]^p \varepsilon_i^{-(n+1)p} h_i \\ &= c [\|g_o\|_{L^{\Phi}(\Omega)} + \|v(0)\|_{L^{\Phi}(\Omega)}]^p \sqrt{h_i} \rightarrow 0 \end{aligned}$$

in the limit $i \rightarrow \infty$. Finally, combining the preceding assertion with (5.6), we obtain (5.5).

5.2 Solutions corresponding to the approximations

As we have shown in the previous section, the approximations g_i and ψ_i satisfy the assumptions of Theorem 4.2. Consequently, for each $i \in \mathbb{N}$ there exists a variational solution $u_i \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g_i + L^p(0, T; W_0^{1,p}(\Omega)))$ corresponding to g_i and ψ_i . By Lemma 2.20 with $v = g_i$ we deduce the energy bound

$$\frac{1}{2} \sup_{t \in [0, T]} \varrho_\Omega(u_i(t)) + \iint_{\Omega_T} f(Du_i) dx \leq C, \quad (5.7)$$

where the constant C is given by

$$\begin{aligned} C &:= \frac{2^\ell}{m} + \frac{1}{2} \sup_{i \in \mathbb{N}} \sup_{t \in [0, T]} \varrho_\Omega(g_i(t)) + c(\ell, m) \sup_{i \in \mathbb{N}} \left(\int_0^T \|\partial_t g_i\|_{L^{\Phi}(\Omega)} dt \right)^{\frac{\ell+1}{\ell}} \\ &\quad + \frac{3}{\ell} \sup_{i \in \mathbb{N}} \iint_{\Omega_T} f(Dg_i) dx dt. \end{aligned}$$

By (5.2) together with (1.6), (5.4) and (5.5) together with the growth condition (1.3), C is finite. Hence, there exists a limit map $u \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g + L^p(0, T; W_0^{1,p}(\Omega)))$ and a (not relabeled) subsequence such that

$$\begin{cases} u_i \overset{*}{\rightharpoonup} u \text{ weakly } ^* \text{ in } L^\infty(0, T; L^\Phi(\Omega)), \\ u_i \rightharpoonup u \text{ weakly in } L^p(0, T; W^{1,p}(\Omega)) \end{cases} \quad (5.8)$$

in the limit $i \rightarrow \infty$. The obstacle condition is preserved as $i \rightarrow \infty$. Indeed, by (5.8) and (5.2) applied to $v_i = \psi_i$, we find that

$$\iint_{\Omega_T} (u - \psi) \varphi \, dx dt = \lim_{i \rightarrow \infty} \iint_{\Omega_T} (u_i - \psi_i) \varphi \, dx dt \geq 0 \quad \forall \varphi \in C_0^\infty(\Omega_T, \mathbb{R}_{\geq 0}).$$

5.3 Convergence of solutions

In this step, we wish to establish

$$b(u_i) \overset{*}{\rightharpoonup} b(u) \text{ weakly } ^* \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } i \rightarrow \infty. \quad (5.9)$$

Since $(b(u_i))_{i \in \mathbb{N}}$ is bounded in $L^\infty(0, T; L^{\Phi^*}(\Omega))$ by Lemma 2.3, (1.7) and (5.7), we know that there exists $w \in L^\infty(0, T; L^{\Phi^*}(\Omega))$ such that (for a subsequence)

$$b(u_i) \overset{*}{\rightharpoonup} w \text{ weakly } ^* \text{ in } L^\infty(0, T; L^{\Phi^*}(\Omega)) \text{ as } i \rightarrow \infty. \quad (5.10)$$

However, it remains to prove that w has the structure $b(u)$. To this end, let $\lambda > 0$ and consider mollifications $[u_i - \psi_i]_\lambda$ and $[u - \psi]_\lambda$ according to (2.7) with zero initial datum. Then, we define

$$w_{i,\lambda} := [u_i - \psi_i]_\lambda + \psi_i \text{ and } w_\lambda := [u - \psi]_\lambda + \psi.$$

By Lemma 2.15 we obtain that $w_{i,\lambda} \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g_i + L^p(0, T; W^{1,p}(\Omega)))$ and $w_\lambda \in L^\infty(0, T; L^\Phi(\Omega)) \cap (g + L^p(0, T; W^{1,p}(\Omega)))$. Furthermore, (2.6) implies that

$$\partial_t [u_i - \psi_i]_\lambda = -\frac{1}{h_i} ([u_i - \psi_i]_\lambda - (u_i - \psi_i)) \in L^\infty(0, T; L^\Phi(\Omega)) \cap L^p(\Omega_T).$$

Since $\psi_i \rightarrow \psi$ in $L^p(\Omega_T)$, by (5.8)₂ and Lemma 2.17, the sequence $([u_i - \psi_i]_\lambda)_{i \in \mathbb{N}}$ is bounded in $L^p(\Omega_T)$ for any fixed $\lambda > 0$. Further, by (5.5), (5.8)₂ and Lemma 2.17

$$w_{i,\lambda} \rightharpoonup w_\lambda \text{ weakly in } L^p(0, T; W^{1,p}(\Omega)) \text{ as } i \rightarrow \infty \quad (5.11)$$

holds true for fixed $\lambda > 0$. Therefore, we conclude from Rellich's theorem that

$$w_{i,\lambda} \rightarrow w_\lambda \text{ in } L^p(\Omega_T) \text{ as } i \rightarrow \infty \quad (5.12)$$

for any fixed $\lambda > 0$. Here, we did not have to pass to a subsequence, since the limit is determined by (5.11). Since $u_i \geq \psi_i$ for any $i \in \mathbb{N}$, we know that $w_{i,\lambda} \geq \psi_i$. Using $w_{i,\lambda}$ as comparison map in the variational inequality for u_i leads us to

$$\begin{aligned}
 & - \iint_{\Omega_T} \partial_t w_{i,\lambda} (b(w_{i,\lambda}) - b(u_i)) \, dx dt \\
 & = \frac{1}{\lambda} \iint_{\Omega_T} (w_{i,\lambda} - u_i) (b(w_{i,\lambda}) - b(u_i)) \, dx dt - \iint_{\Omega_T} \partial_t \psi_i (b(w_{i,\lambda}) - b(u_i)) \, dx dt \\
 & \leq \iint_{\Omega_T} f(Dw_{i,\lambda}) \, dx dt + \mathfrak{B}[g_{\partial,i}, \psi_i(0)].
 \end{aligned}$$

The preceding inequality is equivalent to

$$\begin{aligned}
 & \iint_{\Omega_T} (w_{i,\lambda} - u_i) (b(w_{i,\lambda}) - b(u_i)) \, dx dt \\
 & \leq \lambda \iint_{\Omega_T} \partial_t \psi_i (b(w_{i,\lambda}) - b(u_i)) \, dx dt + \lambda \iint_{\Omega_T} f(Dw_{i,\lambda}) \, dx dt \quad (5.13) \\
 & \quad + \lambda \mathfrak{B}[g_{\partial,i}, \psi_i(0)] \\
 & =: \lambda \text{I}_\lambda + \lambda \text{II}_\lambda + \lambda \text{III},
 \end{aligned}$$

where the definition of I_λ , II_λ and III is clear in this context. By the generalized Hölder's inequality (1.8), (2.5) and Lemma 2.3, we estimate

$$\begin{aligned}
 |\text{I}_\lambda| & \leq \|\partial_t \psi_i\|_{L^1(0,T;L^\Phi(\Omega))} \left[\sup_{t \in [0,T]} \|b(w_{i,\lambda})\|'_{L^{\Phi^*}(\Omega)} + \sup_{t \in [0,T]} \|b(u_i)\|'_{L^{\Phi^*}(\Omega)} \right] \\
 & \leq \|\partial_t \psi_i\|_{L^1(0,T;L^\Phi(\Omega))} \left[2 + \sup_{t \in [0,T]} \varrho_\Omega^*(b(w_{i,\lambda}))^{\frac{1}{\varepsilon+1}} + \sup_{t \in [0,T]} \varrho_\Omega^*(b(u_i))^{\frac{1}{\varepsilon+1}} \right] \\
 & \leq \|\partial_t \psi_i\|_{L^1(0,T;L^\Phi(\Omega))} \left[2 + \sup_{t \in [0,T]} (m\varrho_\Omega(w_{i,\lambda}))^{\frac{1}{\varepsilon+1}} + \sup_{t \in [0,T]} (m\varrho_\Omega(u_i))^{\frac{1}{\varepsilon+1}} \right].
 \end{aligned}$$

By (5.4) applied to $v_i = \psi_i$, the first factor on the right-hand side of the preceding inequality stays bounded in the limit $i \rightarrow \infty$. Further, by the energy bound (5.7) we know that $\sup_{t \in [0,T]} \int_\Omega \Phi(u_i) \, dx \leq 2C$ for any $i \in \mathbb{N}$. By Lemma 2.15, (5.2) and (5.8)₁ we find that

$$\begin{aligned}
 \sup_{i \in \mathbb{N}} \|w_{i,\lambda}\|_{L^\infty(0,T;L^\Phi(\Omega))} & \leq \sup_{i \in \mathbb{N}} \|u_i - \psi_i\|_{L^\infty(0,T;L^\Phi(\Omega))} + \sup_{i \in \mathbb{N}} \|u_i\|_{L^\infty(0,T;L^\Phi(\Omega))} \\
 & \leq \sup_{i \in \mathbb{N}} \|u_i - \psi_i\|_{L^\infty(0,T;L^\Phi(\Omega))} + \sup_{i \in \mathbb{N}} \|u_i\|_{L^\infty(0,T;L^\Phi(\Omega))} < \infty.
 \end{aligned}$$

Consequently, by (1.7), we conclude that $\sup_{t \in [0,T]} \int_\Omega \Phi(w_{i,\lambda}) \, dx$ is bounded by a constant independent of $i \in \mathbb{N}$ and $\lambda > 0$. Altogether, we obtain that

$$|\text{I}_\lambda| \leq c \quad (5.14)$$

with a constant c independent of i and λ . Next, by the growth condition (1.3), Lemma 2.15 and (5.5), we deduce that

$$\begin{aligned}
 |\text{II}_\lambda| & \leq L \int_\Omega [1 + |Dw_{i,\lambda}|^p] \, dx dt \\
 & \leq L|\Omega|T + 2^{p-1}L \left[\|Du_i - D\psi_i\|_{L^p(\Omega_T, \mathbb{R}^n)}^p + \|D\psi_i\|_{L^p(\Omega_T, \mathbb{R}^n)}^p \right] \quad (5.15) \\
 & \leq L|\Omega|T + c(p)L \left[\|Du_i\|_{L^p(\Omega_T, \mathbb{R}^n)}^p + \|D\psi_i\|_{L^p(\Omega_T, \mathbb{R}^n)}^p \right] \\
 & \leq L|\Omega|T + c(p)L \left[C + \sup_{i \in \mathbb{N}} \|D\psi_i\|_{L^p(\Omega_T, \mathbb{R}^n)}^p \right] < \infty.
 \end{aligned}$$

Finally, by (5.1), we have that $\mathfrak{h}[g_{o,i}, \psi_i(0)] \rightarrow \mathfrak{h}[g_o, \psi(0)]$ a.e. in Ω_T . Further, by Lemma 2.3, the nonnegativity of $b(g_{o,i})$ and $\psi_i(0)$ and (5.1), we find the dominating function

$$\begin{aligned} 0 &\leq \mathfrak{h}[g_{o,i}, \psi_i(0)] \leq \Phi(\psi_i(0)) + \Phi^*(b(g_{o,i})) \leq \Phi(\psi_i(0)) + m\Phi(g_{o,i}) \\ &\rightarrow \Phi(\psi(0)) + m\Phi(g_o). \end{aligned}$$

Consequently, by a version of the dominated convergence theorem (cf. [16, Theorem 1.20]), we know that $\mathfrak{B}[g_{o,i}, \psi_i(0)] \rightarrow \mathfrak{B}[g_o, \psi(0)]$ as $i \rightarrow \infty$ and hence

$$\sup_{i \in \mathbb{N}} \text{III} < \infty. \quad (5.16)$$

Inserting (5.14), (5.15) and (5.16) into (5.13), we obtain

$$\iint_{\Omega_T} (w_{i,\lambda} - u_i)(b(w_{i,\lambda}) - b(u_i)) \, dxdt \leq c\lambda$$

with a constant c independent of $i \in \mathbb{N}$ and $\lambda > 0$. In order to bound the left-hand side of the preceding inequality from below, we apply Lemma 2.9, Hölder's inequality and Lemma 2.7. This yields

$$\begin{aligned} \varrho_{\Omega_T}(w_{i,\lambda} - u_i) &\leq \iint_{\Omega_T} |\Phi(w_{i,\lambda}) - \Phi(u_i)| \, dxdt \\ &\leq \left(\iint_{\Omega_T} |\sqrt{\Phi(w_{i,\lambda})} + \sqrt{\Phi(u_i)}|^2 \, dxdt \right)^{\frac{1}{2}} \left(\iint_{\Omega_T} |\sqrt{\Phi(w_{i,\lambda})} - \sqrt{\Phi(u_i)}|^2 \, dxdt \right)^{\frac{1}{2}} \\ &\leq c[\varrho_{\Omega_T}(w_{i,\lambda})^{\frac{1}{2}} + \varrho_{\Omega_T}(u_i)^{\frac{1}{2}}] \left(\iint_{\Omega_T} (w_{i,\lambda} - u_i)(b(w_{i,\lambda}) - b(u_i)) \, dxdt \right)^{\frac{1}{2}} \end{aligned}$$

We have already established that the first factor on the right-hand side of the preceding inequality is bounded independent of i and λ . Therefore, we obtain that

$$\varrho_{\Omega_T}(w_{i,\lambda} - u_i) \leq c\lambda^{\frac{1}{2}},$$

where the constant is independent of i and λ . Thus, by (1.6) and Lemma 2.8 for any $\delta > 0$, there exists $\lambda_o > 0$ such that

$$\|w_{i,\lambda} - u_i\|_{L^{\ell+1}(\Omega_T)} < \frac{\delta}{2} \quad (5.17)$$

holds true for any $0 < \lambda < \lambda_o$. Decreasing λ_o if necessary, by Lemma 2.15, we may assume that

$$\|w_\lambda - u\|_{L^{\ell+1}(\Omega_T)} < \frac{\delta}{2} \quad (5.18)$$

holds true for any $0 < \lambda < \lambda_o$ as well. Finally, we abbreviate $q := \min\{p, \ell + 1\}$. Combining (5.12), (5.17) and (5.18) with the choice $0 < \lambda < \lambda_o$, we infer

$$\begin{aligned}
& \limsup_{i \rightarrow \infty} \|u_i - u\|_{L^q(\Omega_T)} \\
& \leq \limsup_{i \rightarrow \infty} [\|u_i - w_{i,\lambda}\|_{L^q(\Omega_T)} + \|w_{i,\lambda} - w_\lambda\|_{L^q(\Omega_T)} + \|w_\lambda - u\|_{L^q(\Omega_T)}] \\
& < \delta + \lim_{i \rightarrow \infty} \|w_{i,\lambda} - w_\lambda\|_{L^q(\Omega_T)} = \delta.
\end{aligned}$$

Since $\delta > 0$ was arbitrary, the preceding consideration yields

$$u_i \rightarrow u \text{ in } L^{\min\{p, \ell+1\}}(\Omega_T) \text{ as } i \rightarrow \infty.$$

Passing to a subsequence, we further have that $u_i \rightarrow u$ a.e. in Ω_T as $i \rightarrow \infty$. Together with (5.10), this proves (5.9).

5.4 Conclusion of the proof

Consider a comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_T . We define mollifications v_i to v as in Sect. 5.1. Since we are not able to pass to the limit $i \rightarrow \infty$ in the variational inequality satisfied by u_i and v_i directly, we integrate over $\tau \in (t_o, t_o + \delta)$ for some $\delta \in (0, T)$ and $t_o \in (0, T - \delta]$. This leads to

$$\begin{aligned}
& \iint_{\Omega_{t_o}} f(Du_i) \, dxdt + \int_{t_o}^{t_o+\delta} \mathfrak{B}[u_i(\tau), v_i(\tau)] \, d\tau \\
& \leq \iint_{\Omega_{t_o+\delta}} f(Dv_i) \, dxdt + \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v_i (b(v_i) - b(u_i)) \, dxdt d\tau \\
& \quad + \mathfrak{B}[g_{o,i}, v_i(0)].
\end{aligned} \tag{5.19}$$

In the following, we consider the terms of (5.19) separately. Since f is convex and satisfies the growth condition (1.3), we conclude from (5.8)₂ that

$$\iint_{\Omega_{t_o}} f(Du) \, dxdt \leq \liminf_{i \rightarrow \infty} \iint_{\Omega_{t_o}} f(Du_i) \, dxdt. \tag{5.20}$$

In order to treat the second term on the left-hand side of (5.19), note that

$$\mathfrak{B}[u_i(\tau), v_i(\tau)] = \Phi(v_i(\tau)) + \Phi^*(b(u_i(\tau))) - b(u_i(\tau))v_i(\tau).$$

Since Φ is convex and the functional $\int_{t_o}^{t_o+\delta} \int_{\Omega} \Phi(\cdot) \, dx d\tau$ is continuous with respect to strong convergence in $L^\Phi(\Omega_T)$ by (1.6), we conclude that it is lower semicontinuous with respect to weak convergence in $L^\Phi(\Omega_T)$, cf. [14, Corollary 3.9]. Similarly, we find that $\int_{t_o}^{t_o+\delta} \int_{\Omega} \Phi^*(\cdot) \, dx d\tau$ is lower semicontinuous with respect to weak convergence in $L^{\Phi^*}(\Omega_T)$. Thus, as a consequence of (5.2) and (5.9), we obtain that

$$\int_{t_o}^{t_o+\delta} \mathfrak{B}[u(\tau), v(\tau)] \, d\tau \leq \liminf_{i \rightarrow \infty} \int_{t_o}^{t_o+\delta} \mathfrak{B}[u_i(\tau), v_i(\tau)] \, d\tau. \tag{5.21}$$

By the Lipschitz condition (1.4) and (5.5), we have that

$$\lim_{i \rightarrow \infty} \iint_{\Omega_{t_o+\delta}} f(Dv_i) \, dx dt = \iint_{\Omega_{t_o+\delta}} f(Dv) \, dx dt. \quad (5.22)$$

Next, by (5.3), (5.4) and (5.8)₁, we conclude that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v_i (b(v_i) - b(u_i)) \, dx d\tau \\ = \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v (b(v) - b(u)) \, dx d\tau. \end{aligned} \quad (5.23)$$

Finally, by (5.1) applied to $g_{o,i}$ and $v_i(0)$ and the dominated convergence theorem, we infer

$$\lim_{i \rightarrow \infty} \mathfrak{B}[g_{o,i}, v_i(0)] = \mathfrak{B}[g_o, v(0)]. \quad (5.24)$$

Collecting (5.20), (5.21), (5.22), (5.23) and (5.24), we deduce from (5.19) that

$$\begin{aligned} \iint_{\Omega_{t_o}} f(Du) \, dx dt + \int_{t_o}^{t_o+\delta} \mathfrak{B}[u(\tau), v(\tau)] \, d\tau \\ \leq \iint_{\Omega_{t_o+\delta}} f(Dv) \, dx dt + \int_{t_o}^{t_o+\delta} \iint_{\Omega_\tau} \partial_t v (b(v) - b(u)) \, dx d\tau + \mathfrak{B}[g_o, v(0)] \end{aligned}$$

holds true for any admissible comparison map $v \in g + L^p(0, T; W_0^{1,p}(\Omega))$ with $\partial_t v \in L^1(0, T; L^\Phi(\Omega))$, $v(0) \in L^\Phi(\Omega)$ and $v \geq \psi$ a.e. in Ω_τ . Passing to the limit $\delta \downarrow 0$, we conclude that u is a variational solution to (1.2) in the sense of Definition 1.1. $\square$

Acknowledgements Open Access funding provided by Projekt DEAL.

Funding L. Schätzler has been supported by Studienstiftung des deutschen Volkes.

Compliance with ethical standards

Conflict of interest No conflicts of interest have occurred.

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