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Balanced metrics and Berezin quantization on Hartogs triangles

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Abstract

In this paper, we study balanced metrics and Berezin quantization on a class of Hartogs domains defined by $\Omega_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| < |z_2| < \cdots < |z_n| < 1\}$ which generalize the so-called classical Hartogs triangle. We introduce a Kähler metric $g(\nu)$ associated with the Kähler potential $\Phi_n(z) := -\sum_{k=1}^{n-1} \nu_k \ln(|z_{k+1}|^2 - |z_k|^2) - \nu_n \ln(1 - |z_n|^2)$ on Ω_n. As main contributions, on one hand we compute the explicit form for Bergman kernel of weighted Hilbert space, and then, we obtain the necessary and sufficient condition for the metric $g(\nu)$ on the domain Ω_n to be a balanced metric. On the other hand, by using the Calabi's diastasis function, we prove that the Hartogs triangles admit a Berezin quantization.

Keywords Hartogs triangles · Balanced metrics · Berezin quantization

Mathematics Subject Classification 32A25 · 32M15 · 32Q15

1 Introduction

Let (L, h) be a positive Hermitian line bundle over a Kähler manifold (M, g) of dimension n such that $\mathrm{Ric}(h) = \omega_g$. Here, $\mathrm{Ric}(h)$ denotes the two-form on M whose local expression is given by

$$\mathrm{Ric}(h) = -\frac{i}{2}\partial\bar{\partial} \log h(\sigma(z), \sigma(z)),$$

for a trivializing holomorphic section $\sigma : U \subset M \to L\backslash\{0\}$. In the quantum mechanics terminology, the pair (L, h) is also called a geometric quantization of the Kähler manifold (M, g).

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For all integers $\alpha > 0$, we can define a complex Hilbert space $\mathcal{H}^\alpha$ consisting of the global holomorphic sections s of the line bundle (L^α, h_α) over M with $\operatorname{Ric} h_\alpha = \alpha\omega_g$ which are bounded with respect to the following norm:

$$\langle s, s\rangle_\alpha := \int_M h_\alpha(s(z), s(z))\frac{\omega_g^n}{n!} < +\infty.$$

Let $\{s_j\}$ be an orthonormal basis of $\mathcal{H}^\alpha$ with respect to $\langle , \rangle_\alpha$. Then, one can define a smooth real-valued function on M, called Rawnsley's ε-function:

$$\varepsilon_{(\alpha,g)}(z) = \sum_{j=1}^{d_\alpha} h_\alpha(s_j(z), s_j(z)).$$

One can check that this function depends only on the Kähler metric ω_g and not on the orthonormal basis chosen. It is well known that Rawnsley's ε-function $\varepsilon_{(\alpha,g)}$ has a asymptotic expansion in terms of the parameter α (e.g., [8, 33]). There are two important branches of research on Rawnsley's ε-function. The first one is the existence of balanced metrics on complex manifolds.

Definition 1.1 The metric g on M is balanced if the Rawnsley's ε-function $\varepsilon_{(1,g)}(z)$ $(z \in M)$ is a positive constant on M.

The definition of balanced metrics was originally given by Donaldson (cf. [16]) in the case of a compact polarized Kähler manifold in 2001. Later on, it was generalized by Arezzo–Loi [1] and Engliš [20] to the noncompact case. Furthermore, balanced metrics had been widely used to study the quantization of a Kähler manifold, the expansion of the Bergman kernel function and the stability of the projective algebraic varieties. The reader is referred to Cahen–Gutt–Rawnsley [6], Engliš [20], Zhang [34] and references therein.

In fact, by Donaldson's results we know that there exist balanced metrics on compact manifold with finite automorphism group. In the noncompact case, the existence and uniqueness of balanced metrics is still an open problem. Therefore, it makes sense to study the existence and uniqueness of balanced metrics on some special noncompact manifolds.

Unfortunately, despite the extensive studies of the compact case, very little seems to be known about the existence of balanced metrics on noncompact manifolds and even on the domains in $\mathbb{C}^n$.

We want to start with the simplest situation, namely (L, h) is the trivial positive holomorphic line bundle over a domain $M \subset \mathbb{C}^n$ equipped with a Kähler metric g. In this case, the metric g can be described by a strictly plurisubharmonic real-valued function φ, called a Kähler potential for g, that is $\omega_g = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. It is not hard to see that in this case, the Hilbert space $\mathcal{H}^\alpha$ equals the weighted Hilbert space $H_{\alpha\varphi}$ of square integrable holomorphic functions on (M, g) with the weight $\exp\{-\alpha\varphi\}$ defined by

$$H_{\alpha\varphi}(M) := \left\{f \in \operatorname{Hol}(M) : \int_M |f|^2 \exp\{-\alpha\varphi\}\frac{\omega^n}{n!} < +\infty\right\},$$

where $\operatorname{Hol}(M)$ is the space of holomorphic functions on M. If $H_{\alpha\varphi}(M) \neq \{0\}$, let $K_\alpha(z, \bar{z})$ be its weighted Bergman kernel. Then, it is not difficult to see that the Rawnsley's ε-function in this case can be expressed as

$$\varepsilon_{(\alpha,g)}(z) := \exp\{-\alpha\varphi(z)\}K_\alpha(z,\bar{z}), \quad z \in M.$$

It can be easily verified that this function depends only on the metric g and not on the choice of the Kähler potential φ (which is defined up to the sum with the real part of a holomorphic function on M).

Some progress had been made in this simplest case. In 2012, Loi–Zedda [28] proved the existence of balanced metrics on bounded symmetric domains. Note that bounded symmetric domains are homogeneous domains. Inspired by this, similar results were recently generalized by Loi–Mossa [26] to all bounded homogeneous (not necessarily symmetric) domains.

Recently, Feng–Tu [21] firstly found the existence of balanced metrics on a class of nonhomogeneous domains called generalized Cartan–Hartogs domains. Later on, Bi–Feng–Tu [5] proved that balanced metric can also exist on Fock–Bargmann–Hartogs domains. For the study of the balanced metrics, see Hélène–Engliš–Youssfi [22], Loi [25], Loi–Zedda [27], and Zedda [31].

In this paper, we study the canonical metric on the Hartogs domains called n-dimensional Hartogs triangles which generalize the classical Hartogs triangles defined by

$$\Omega_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| < |z_2| < \cdots < |z_n| < 1\}, \quad (n \geq 2).$$

The Hartogs triangles have attracted many attentions and been deeply investigated by many authors from different views. In 2013, Chakrabarti–Shaw [9] focused on Sobolev regularity of the $\bar{\partial}$-equation over the Hartogs triangle. In 2016, Edholm [17] obtained the explicit form for the Bergman kernel for the generalized Hartogs triangle of exponent $\gamma > 0$, that is $H_\gamma := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\}$. By using the close form for the Bergman kernel, Edholm–McNeal [18] studied L^p boundedness of the Bergman projection on H_γ. Inspired by this, Chen [10] obtained the necessary and sufficient condition for the Bergman projection on L^p space of more general bounded Hartogs domains to be bounded. For the reference of the theories of Bergman kernel, see also Park [29].

Recently, Zapałowski [35] gave the rigidity of proper holomorphic self-mappings between generalized Hartogs triangle and obtained automorphism group of the generalized Hartogs triangle. The reader is also referred to [11–14, 23] for the studies of rigidity of the proper holomorphic mappings between Hartogs triangles. Moreover, we can see that the n-dimensional Hartogs triangles are nonhomogeneous pseudoconvex domains with nonsmooth boundary. More importantly, much less seems to be known about the geometric properties of Hartogs triangles. Thus, all the above inspire us to study the canonical metrics on n-dimensional Hartogs triangles.

Firstly, let us introduce a new Kähler metric $g(\nu)$ on Ω_n. Define the strictly plurisubharmonic function $\Phi_n(z)$ on the Hartogs triangles Ω_n as follows

$$\Phi_n(z) := -\sum_{k=1}^{n-1} \nu_k \ln(|z_{k+1}|^2 - |z_k|^2) - \nu_n \ln(1 - |z_n|^2), \tag{1.1}$$

where $\nu = (\nu_1, \dots, \nu_n)$ with $\nu_k > 0, 1 \leq k \leq n$. The Kähler form ω on Ω_n is given by

$$\omega := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi_n.$$

Hence, the Kähler metric $g(\nu)$ on Ω_n associated with ω can be expressed by

$$g(v)_{i\bar{j}} = \frac{\partial^2 \Phi_n}{\partial z_i \partial \bar{z}_j}, \quad (1 \le i, j \le n).$$

Thus, we can define the weighted Hilbert space $H_{\Phi_n}(\Omega_n)$ as follows:

$$H_{\Phi_n}(\Omega_n) := \left\{ f \in \mathrm{Hol}(\Omega_n) : \int_{\Omega_n} |f|^2 \exp\{-\Phi_n\} \frac{\omega^n}{n!} < +\infty \right\}. \tag{1.2}$$

One of the main results of our paper is the following.

Theorem 1.2 *The Kähler metric $g(v)$ on Ω_n is balanced if and only if $v_k \ge 2$ is an integer for all $k = 1, \dots, n-1$, and $v_n > 1$.*

Remark 1.3 Notice that if $v_1 = \cdots = v_n = 2$, the Kähler metric $g(v)$ is exactly the Bergman metric for Ω_n.

Another important application of Rawnsley's ε-function is to study whether a Berezin quantization can be established on some Kähler manifolds. In recent years, Berezin quantization has attracted a lot of attention and has been deeply studied by mathematicians and physicists, see, e.g., Cahen–Gutt–Rawnsley [6], Engliš [19], Loi–Mossa [26] and Zedda [32]. Roughly, a quantization is a construction of a quantum system from the classical mechanics of a system. In 1927, for seek of finding the purely mathematical significance of quantization, Weyl made an attempt at a quantization known as Weyl quantization. He associated a self-adjoint operators on a separable Hilbert space with functions on a symplectic manifold and some certain commutations are fulfilled. Later on, Berezin [3] raised a new quantization procedure, i.e., Berezin quantization. A Berezin quantization on a Kähler manifold (Ω, ω) is given by a family of associative algebra $\mathcal{A}_h$ where the parameter h runs through a set E of the positive reals with 0 in its closure, and moreover, there exists a subalgebra $\mathcal{A}$ of $\bigoplus\{\mathcal{A}_h; h \in E\}$ such that some properties are satisfied (refer to Berezin [3]). More precisely, we call an associative algebra with involution $\mathcal{A}$ a quantization of (Ω, ω) if the following properties are satisfied.

(i) There exist a family of associative algebras $\mathcal{A}_h$ of functions on Ω where the parameter h runs through a set E of the positive reals with 0 in its closure. Moreover, $\mathcal{A}$ is a subalgebra of $\bigoplus\{\mathcal{A}_h; h \in E\}$.

(ii) For each $f \in \mathcal{A}$ which will be written $f(h, x)$ $(h \in E, x \in \Omega)$ such that $f(h, \cdot) \in \mathcal{A}_h$, the limit

$$\lim_{h \to 0+} f(h, x) = \varphi(f)(x)$$

exists.

(iii) $\varphi(f * g) = \varphi(f) \cdot \varphi(g)$, $\varphi(h^{-1}(f * g - g * f)) = \frac{1}{i}\{\varphi(f), \varphi(g)\}$ for $f, g \in \mathcal{A}$. Here, $*$ and $\{,\}$ denote the product of $\mathcal{A}$ and the Poisson bracket.

(iv) For any two points $x_1, x_2 \in \Omega$, there exists $f \in \mathcal{A}$ such that $\varphi(f)(x_1) \ne \varphi(f)(x_2)$.

For a given Kähler manifold Ω endowed with a Kähler metric g associated with a Kähler form ω, suppose that there exists a global Kähler potential $\varphi(z) : \Omega \to \mathbb{R}$ which can extend to a sesquianalytic function $\varphi(z, \overline{w})$ on $\Omega \times \Omega$ such that $\varphi(z, \bar{z}) = \varphi(z)$. Then, the Calabi's diastasis function is defined by (see Calabi [7])

$$D_g(z,w) := \varphi(z,\bar{z}) + \varphi(w,\bar{w}) - \varphi(z,\bar{w}) - \varphi(w,\bar{z}),\ (z,w) \in \Omega \times \Omega. \tag{1.3}$$

It is not hard to see that the Calabi's diastasis function $D_g(z,w)$ is symmetric in z and w and is uniquely defined up to the real part of a holomorphic function.

Moreover, the Calabi's diastasis function has played a crucial rule in studying balanced metric, Berezin quantization and Kähler immersions (i.e., holomorphic and isometric immersions). For more details, please see [2, 7, 24].

In fact, by using the Rawnsley's ε-function and the Calabi's diastasis function, Engliš [19] gave a sufficient condition for a Kähler manifold (Ω, g) to admit a Berezin quantization.

Theorem 1.4 (see [19]) *Let Ω be a Kähler manifold endowed with a Kähler metric g associated with Kähler form ω. If*

(I) *The function $\exp\{-D_g(z,w)\}$ is globally defined on $\Omega \times \Omega$, $\exp\{-D_g(z,w)\} \leq 1$ and $\exp\{-D_g(z,w)\} = 1$ if and only if $z = w$, where $D_g(z,w)$ denotes the Calabi's diastasis function.*

(II) *There exists a subset $E \subset \mathbb{R}^+$ which has $+\infty$ in its closure such that the Rawnsley's ε-function $\varepsilon_{(\alpha,g)}(z)$ is a positive constant for $\alpha \in E$.*

Then, (Ω, g) admits a Berezin quantization.

As far as we know, the above conditions are satisfied by homogeneous Kähler manifold, a contractible homogeneous Kähler manifold (i.e., all the products $(\Omega, g) \times (\mathbb{C}^m, g_0)$, where (Ω, g) is an homogeneous bounded domain and g_0 is the standard flat metric) and some special pseudoconvex domains (cf. [4, 19, 26, 30]). So some experts are dedicated to find more noncompact Kähler manifolds which a Berezin quantization can be carried out.

In this paper, by using Theorems 1.2 and 1.4, we will show that the conditions (I) and (II) can be satisfied by the Hartogs triangles $(\Omega_n, g(\nu))$, that is

Theorem 1.5 *Let Ω_n be the Hartogs triangle endowed with the Kähler metric $g(\nu)$. If ν_k for all $k = 1, \ldots, n-1$ are positive rational numbers and $\nu_n > 0$, then $(\Omega_n, g(\nu))$ admits a Berezin quantization.*

The paper is organized as follows. In Sect. 2, we give an explicit formula for the Bergman kernel of the weighted Hilbert space of square integrable holomorphic functions on $(\Omega_n, g(\nu))$ with the weight $\exp\{-\Phi_n\}$ for some special ν_k. By using the expression of the Rawnsley's ε-function, we give the proof of Theorem 1.2. In Sect. 3, using the Calabi's diastasis function, Theorems 1.2 and 1.4, we prove Theorem 1.5.

2 Weighted Bergman kernel and balanced metrics on Hartogs triangles

In the following lemma, we describe the volume form of the Kähler metric $g(\nu)$. The proof is omitted since it can be obtained by a straightforward induction argument of n.

Lemma 2.1 *For $n \geq 2$, let Φ_n be defined by (1.1). Then, we have*

$$\det\left(\frac{\partial^2 \Phi_n}{\partial z^t \partial \bar{z}}\right)(z) = \frac{\prod_{j=1}^{n} \nu_j \prod_{k=1}^{n-1} |z_{k+1}|^2}{(1-|z_n|^2)^2 \prod_{k=1}^{n-1}(|z_{k+1}|^2 - |z_k|^2)^2}, \tag{2.1}$$

where $z = (z_1, \dots, z_n) \in \Omega_n$.

Since Ω_n is a Reinhardt domain, we are going to compute the squared $L^2_{\Phi_n}$-norms for some holomorphic monomials in Ω_n.

Lemma 2.2 *Let* $z = (z_1, \dots, z_n) \in \Omega_n$, *and* $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$, *we have*

$$\|z^p\|^2_{L^2_{\Phi_n}} = \prod_{k=1}^{n}(\pi \nu_k) \prod_{k=1}^{n} B\left(\sum_{j=1}^{k}(p_j + \nu_j) - \nu_k + 1, \nu_k - 1\right), \tag{2.2}$$

where $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ *is the beta function.*

Proof Combining (1.1) and (2.1), we get

$$\begin{aligned} \|z^p\|^2_{L^2_{\Phi_n}} &= \int_{\Omega_n} |z|^{2p} \exp\{-\Phi_n\} \frac{\omega^n}{n!} \\ &= \int_{\Omega_n} |z|^{2p}(1-|z_n|^2)^{\nu_n} \prod_{k=1}^{n-1}(|z_{k+1}|^2 - |z_k|^2)^{\nu_k} \\ &\quad \times \frac{\prod_{k=1}^{n} \nu_k \prod_{k=1}^{n-1} |z_{k+1}|^2}{(1-|z_n|^2)^2 \prod_{k=1}^{n-1}(|z_{k+1}|^2 - |z_k|^2)^2} dm(z), \end{aligned} \tag{2.3}$$

where $dm(z)$ is the Euclidean measure. We introduce polar coordinates in each variable by putting $z_k = t_k e^{i\theta_k}$, $1 \le k \le n$. After doing so, and integrating out the angular variables, (2.3) becomes

$$\prod_{k=1}^{n}(2\pi \nu_k) \int_{0 \le t_1 < \cdots < t_n < 1} t_1^{2p_1+1}(1-t_n^2)^{\nu_n - 2} \prod_{k=1}^{n-1} t_{k+1}^{2p_{k+1}+3}(t_{k+1}^2 - t_k^2)^{\nu_k - 2} dt_1 \cdots dt_n.$$

Next, we set $s_k = t_k^2$, $1 \le k \le n$ and then change variables again. We can obtain

$$\prod_{k=1}^{n}(\pi \nu_k) \int_{0 \le s_1 < \cdots < s_n < 1} s_1^{p_1}(1-s_n)^{\nu_n - 2} \prod_{k=1}^{n-1} s_{k+1}^{p_{k+1}+1}(s_{k+1} - s_k)^{\nu_k - 2} ds_1 \cdots ds_n. \tag{2.4}$$

Claim that

$$\|z^p\|^2_{L^2_{\Phi_n}} = \prod_{k=1}^{n}(\pi \nu_k) \prod_{k=1}^{n} B\left(\sum_{j=1}^{k}(p_j + \nu_j) - \nu_k + 1, \nu_k - 1\right).$$

We will prove this claim by induction for n. For $n = 2$, by (2.4), we learn that

$$\begin{aligned}\|z^p\|^2_{L^2_{\Phi_n}} &= \pi^2 \nu_1\nu_2 \int_{0\le s_1<s_2<1} (1-s_2)^{\nu_2-2} s_2^{p_2+1} s_1^{p_1} (s_2-s_1)^{\nu_1-2} ds_1 ds_2 \\ &= \pi^2\nu_1\nu_2 \int_0^1 (1-s_2)^{\nu_2-2} s_2^{p_2+1} ds_2 \int_0^{s_2} s_1^{p_1}(s_2-s_1)^{\nu_1-2} ds_1 \\ &= \pi^2\nu_1\nu_2 B(p_1+1,\nu_1-1) \int_0^1 (1-s_2)^{\nu_2-2} s_2^{p_2+1} s_2^{p_1+\nu_1-1} ds_2 \\ &= \pi^2\nu_1\nu_2 B(p_1+1,\nu_1-1) B(p_1+p_2+\nu_1+1,\nu_2-1).\end{aligned}$$

This means that the claim holds for $n=2$. Thus, assume that the claim holds for $n=\ell$; then, for $n=\ell+1$, by (2.4), we obtain

$$\begin{aligned}\|z^p\|^2_{L^2_{\Phi_n}} &= \prod_{k=1}^{\ell+1}(\pi\nu_k) \int_{0\le s_1<\cdots<s_{\ell+1}<1} s_1^{p_1}(1-s_{\ell+1})^{\nu_{\ell+1}-2} \prod_{k=1}^{\ell} s_{k+1}^{p_{k+1}+1}(s_{k+1}-s_k)^{\nu_k-2} ds_1\cdots ds_{\ell+1} \\ &= \prod_{k=1}^{\ell+1}(\pi\nu_k)\int_0^1 (1-s_{\ell+1})^{\nu_{\ell+1}-2} s_{\ell+1}^{p_{\ell+1}+1} ds_{\ell+1} \\ &\quad\times \int_{0\le s_1<\cdots<s_\ell<s_{\ell+1}} s_1^{p_1}(s_{\ell+1}-s_\ell)^{\nu_\ell-2}\prod_{k=1}^{\ell-1} s_{k+1}^{p_{k+1}+1}(s_{k+1}-s_k)^{\nu_k-2} ds_1\cdots ds_\ell \\ &= \prod_{k=1}^{\ell+1}(\pi\nu_k)\int_0^1 (1-s_{\ell+1})^{\nu_{\ell+1}-2} s_{\ell+1}^{p_{\ell+1}+1} s_{\ell+1}^{\sum_{k=1}^{\ell}(p_k+\nu_k)-1} ds_{\ell+1} \\ &\times \int_{0\le \hat s_1<\cdots<\hat s_\ell<1} \hat s_1^{p_1}(1-\hat s_\ell)^{\nu_\ell-2}\prod_{k=1}^{\ell-1}\hat s_{k+1}^{p_{k+1}+1}(\hat s_{k+1}-\hat s_k)^{\nu_k-2} d\hat s_1\cdots d\hat s_\ell.\end{aligned}$$

Therefore,

$$\begin{aligned}\|z^p\|^2_{L^2_{\Phi_n}} &= \prod_{k=1}^{\ell+1}(\pi\nu_k)\prod_{k=1}^{\ell} B\left(\sum_{j=1}^{k}(p_j+\nu_j)-\nu_k+1,\nu_k-1\right) \\ &\quad\times\int_0^1 (1-s_{\ell+1})^{\nu_{\ell+1}-2} s_{\ell+1}^{\sum_{k=1}^{\ell+1}(p_k+\nu_k)-\nu_{\ell+1}} ds_{\ell+1} \\ &= \prod_{k=1}^{\ell+1}(\pi\nu_k)\prod_{k=1}^{\ell} B\left(\sum_{j=1}^{k}(p_j+\nu_j)-\nu_k+1,\nu_k-1\right) \\ &\quad\times B\left(\sum_{k=1}^{\ell+1}(p_k+\nu_k)-\nu_{\ell+1}+1,\nu_{\ell+1}-1\right) \\ &= \prod_{k=1}^{\ell+1}(\pi\nu_k)\prod_{k=1}^{\ell+1} B\left(\sum_{j=1}^{k}(p_j+\nu_j)-\nu_k+1,\nu_k-1\right).\end{aligned}$$

The proof is completed. $\square$

Hence, by the definition of the beta function, we can easily obtain the following property.

Proposition 2.3 *$H_{\Phi_n}(\Omega_n)\neq\{0\}$ if and only if $\nu_k>1$ for all $k=1,\ldots,n$.*

Now, we give an elementary lemma for the gamma function.

Lemma 2.4 (see D'Angelo [15] Lemma 2) *Let* $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ *with* $\|x\| < 1$ *and* $s \in \mathbb{R}$ *with* $s > 0$. *Then,*

$$\sum_{q \in \mathbb{N}^m} \frac{\Gamma(|q| + s)}{\Gamma(s) \prod_{i=1}^m \Gamma(q_i + 1)} x^{2q} = \frac{1}{(1 - \|x\|^2)^s}.$$

Theorem 2.5 *Suppose that* $(\Omega_n, g(\nu))$ *is the n-dimensional Hartogs triangle endowed with the Kähler metric* $g(\nu)$. *Let* $\nu_k \geq 2$ *be integers for all* $k = 1, \dots, n-1$, *and let* $\nu_n > 1$. *Let* $H_{\Phi_n}(\Omega_n)$ *be the weighted Hilbert space of square integrable holomorphic functions on* $(\Omega_n, g(\nu))$ *with the weight* $\exp\{-\Phi_n\}$ *(see (1.2)). Then,* $H_{\Phi_n}(\Omega_n) \neq \{0\}$, *and the Bergman kernel of* $H_{\Phi_n}(\Omega_n)$ *is given by*

$$K_{\Phi_n}(z, \bar{z}) = \frac{\nu_n - 1}{\pi^n \nu_n (1 - |z_n|^2)^{\nu_n}} \prod_{k=1}^{n-1} \frac{\nu_k - 1}{\nu_k (|z_{k+1}|^2 - |z_k|^2)^{\nu_k}}. \tag{2.5}$$

Proof Since Ω_n is a Reinhardt domain, together with Lemma 2.2 and the definition of the beta function, we can obtain that $\{ \frac{z^p}{\|z^p\|_{L^2_{\Phi_n}}} \}$ forms a complete orthonormal basis of $H_{\Phi_n}(\Omega_n)$, where the multi-index $p = (p_1, \dots, p_n)$ ranges all integers that satisfy the following inequalities for all $k = 1, \dots, n$,

$$\sum_{j=1}^{k} (p_j + \nu_j) - \nu_k \geq 0.$$

Let N denote the set of all the multi-index $p = (p_1, \dots, p_n)$ satisfying such inequalities. Hence, Formula (2.2) implies that

$$\begin{aligned} K_{\Phi_n}(z, \bar{z}) &= \sum_{p \in N} \frac{|z^p|^2}{\|z^p\|^2_{L^2_{\Phi_n}}} \\ &= \frac{1}{\prod_{k=1}^n (\pi \nu_k)} \sum_{p_1=0}^{+\infty} \frac{|z_1|^{2p_1}}{B(p_1 + 1, \nu_1 - 1)} \sum_{p_2=-p_1-\nu_1}^{+\infty} \frac{|z_2|^{2p_2}}{B(p_1 + p_2 + \nu_1 + 1, \nu_2 - 1)} \\ &\cdots \sum_{p_n=-\sum_{k=1}^{n-1}(p_k+\nu_k)}^{+\infty} \frac{|z_n|^{2p_n}}{B(\sum_{k=1}^n (p_k + \nu_k) - \nu_n + 1, \nu_n - 1)}. \end{aligned}$$

Notice that by Lemma 2.4, we can learn that

$$\begin{aligned}
&\sum_{p_n=-\sum_{k=1}^{n-1}(p_k+\nu_k)}^{+\infty} \frac{|z_n|^{2p_n}}{B(\sum_{k=1}^{n}(p_k+\nu_k)-\nu_n+1,\nu_n-1)} \\
&= |z_n|^{-\sum_{k=1}^{n-1}2(p_k+\nu_k)} \sum_{m=0}^{+\infty} \frac{|z_n|^{2m}}{B(m+1,\nu_n-1)} \\
&= |z_n|^{-\sum_{k=1}^{n-1}2(p_k+\nu_k)} \frac{\Gamma(\nu_n)}{\Gamma(\nu_n-1)} \sum_{m=0}^{+\infty} \frac{\Gamma(m+\nu_n)}{\Gamma(m+1)\Gamma(\nu_n)}|z_n|^{2m} \\
&= (\nu_n-1)|z_n|^{-\sum_{k=1}^{n-1}2(p_k+\nu_k)} \frac{1}{(1-|z_n|^2)^{\nu_n}}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
K_{\Phi_n}(z,\bar{z}) =& \frac{1}{\prod_{k=1}^{n}(\pi\nu_k)} \frac{\nu_n-1}{|z_n|^{2\nu_{n-1}}(1-|z_n|^2)^{\nu_n}} \sum_{p_1=0}^{+\infty} \frac{|z_1|^{2p_1}}{B(p_1+1,\nu_1-1)} \\
&\cdots \sum_{p_{n-1}=-\sum_{k=1}^{n-2}(p_k+\nu_k)}^{+\infty} \frac{|z_{n-1}|^{2p_{n-1}}|z_n|^{-2p_{n-1}-\sum_{k=1}^{n-2}2(p_k+\nu_k)}}{B(\sum_{k=1}^{n-1}(p_k+\nu_k)-\nu_{n-1}+1,\nu_{n-1}-1)}.
\end{aligned}$$

Similarly, we can see that

$$\begin{aligned}
&\sum_{p_{n-1}=-\sum_{k=1}^{n-2}(p_k+\nu_k)}^{+\infty} \frac{|z_{n-1}|^{2p_{n-1}}|z_n|^{-2p_{n-1}-\sum_{k=1}^{n-2}2(p_k+\nu_k)}}{B(\sum_{k=1}^{n-1}(p_k+\nu_k)-\nu_{n-1}+1,\nu_{n-1}-1)} \\
&= |z_{n-1}|^{-\sum_{k=1}^{n-2}2(p_k+\nu_k)} \sum_{m=0}^{+\infty} \frac{|z_{n-1}/z_n|^{2m}}{B(m+1,\nu_{n-1}-1)} \\
&= (\nu_{n-1}-1)|z_{n-1}|^{-\sum_{k=1}^{n-2}2(p_k+\nu_k)} \frac{1}{(1-|\frac{z_{n-1}}{z_n}|^2)^{\nu_{n-1}}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
K_{\Phi_n}(z,\bar{z}) =& \frac{1}{\prod_{k=1}^{n}(\pi\nu_k)} \frac{(\nu_n-1)(\nu_{n-1}-1)}{|z_{n-1}|^{2\nu_{n-2}}(1-|z_n|^2)^{\nu_n}(|z_n|^2-|z_{n-1}|^2)^{\nu_{n-1}}} \\
&\times \sum_{p_1=0}^{+\infty} \frac{|z_1|^{2p_1}}{B(p_1+1,\nu_1-1)} \cdots \sum_{p_{n-2}}^{+\infty} \frac{|z_{n-2}|^{2p_{n-2}}|z_{n-1}|^{-2p_{n-2}-\sum_{k=1}^{n-3}2(p_k+\nu_k)}}{B(\sum_{k=1}^{n-2}(p_k+\nu_k)-\nu_{n-2}+1,\nu_{n-2}-1)}.
\end{aligned}$$

Therefore, by induction, we conclude that

$$K_{\Phi_n}(z,\bar{z}) = \frac{\nu_n-1}{\pi^n\nu_n(1-|z_n|^2)^{\nu_n}} \prod_{k=1}^{n-1} \frac{\nu_k-1}{\nu_k(|z_{k+1}|^2-|z_k|^2)^{\nu_k}}.$$

The proof is completed. □

Now, we are able to prove Theorem 1.2.

Proof of Theorem 1.2 By the definition of balanced metric and Theorem 2.5, we see that

$$\begin{aligned}\varepsilon_{(1,g(\nu))}(z) &= \exp\{-\Phi_n(z)\}K_{\Phi_n}(z,\bar{z})\\ &= \frac{1}{\pi^n}\prod_{k=1}^{n}\frac{\nu_k-1}{\nu_k}.\end{aligned}$$

Thus, the metric $g(\nu)$ is balanced. On the other hand, now assume that $g(\nu)$ is balanced. This means that there exists a constant $C > 0$ such that

$$\begin{aligned}K_{\Phi_n}(z,\bar{z}) &= C\exp\{\Phi_n(z)\}\\ &= C(1-|z_n|^2)^{-\nu_n}\prod_{k=1}^{n-1}(|z_{k+1}|^2-|z_k|^2)^{-\nu_k}.\end{aligned}$$

Notice that by Lemma 2.4, we get

$$(|z_{k+1}|^2-|z_k|^2)^{-\nu_k} = \sum_{p_k=0}^{+\infty}\frac{\Gamma(p_k+\nu_k)}{\Gamma(\nu_k)\Gamma(p_k+1)}|z_k|^{2p_k}|z_{k+1}|^{-2(\nu_k+p_k)}.$$

Thus, for any $p_1 \in \mathbb{N}$, consider the coefficient of $|z_1|^{2p_1}$ in the series expansion of $K_{\Phi_n}(z,\bar{z})$, and then, one can see that

$$\text{the coefficient of } |z_1|^{2p_1} = \widetilde{C}\sum_{k=2}^{n}\sum_{p_k=0}^{+\infty}\frac{\Gamma(p_k+\nu_k)}{\Gamma(p_k+1)}|z_k|^{2(p_k-p_{k-1}-\nu_{k-1})}. \tag{2.6}$$

where $\widetilde{C}$ is a constant which is independent of z. Since $z_1^{p_1}$ belongs to the basis of $H_{\Phi_n}(\Omega_n)$, we can conclude that the right hand of (2.6) must contain a positive constant term. This means that we can find some term in (2.6) which is independent of z_k, for all $k = 2, \dots, n$. Thus, for any $1 \le k \le n-1$, there exist p_k and p_{k+1} such that

$$\nu_k = p_{k+1} - p_k.$$

Notice that for any $1 \le k \le n$, p_k is an integer, and thus, $\nu_1, \dots, \nu_{n-1}$ are forced to be integers. Thus, the proof follows by Proposition 2.3. □

In 2016, Edholm [17] introduced a new domain named the generalized Hartogs triangle of exponent $\gamma > 0$ and obtained the closed form of Bergman kernel for this domain with some special γ. And then Park [29] extended Edholm's result to three-dimensional case. The method can even be applied to n-dimensional case as well. Inspired by their work, we state the following open problem:

Problem 2.6 *Consider the generalized Hartogs triangle*

$$\mathbb{H}_p := \{z \in \mathbb{C}^n : |z_1|^{p_1} < |z_2|^{p_2} < \cdots < |z_n|^{p_n} < 1\}$$

where $p_1, \dots, p_n$ are any positive integers. Can we give some conditions of p to find balanced metrics on $\mathbb{H}_p$, even for the case $n = 2$?

3 Berezin quantization of Hartogs triangles

Now, we consider the Berezin quantization on $(\Omega_n, g(\nu))$. At first, we give some useful lemma.

Lemma 3.1 (see Lemma 3.2 in [30]) *Assume that Ω is a domain in $\mathbb{C}^n$. Let g be a Kähler metric on Ω associated to the Kähler form $\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. Then, the following formula is established*

$$\varepsilon_{(\alpha\beta,g)}(z) = \beta^n \varepsilon_{(\alpha,\beta g)}(z).$$

Now, we can give the proof of Theorem 1.5.

Proof of Theorem 1.5 Firstly, we prove that $(\Omega_n, g(\nu))$ satisfies condition (I) in Theorem 1.4. In fact, it is easy to see that

$$\begin{aligned}
&\exp\{-D_{g(\nu)}(z,w)\}\\
&= \frac{\prod_{k=1}^{n-1} |z_{k+1}\overline{w_{k+1}} - z_k\overline{w_k}|^{-2\nu_k}}{\prod_{k=1}^{n-1}\Big((|z_{k+1}|^2 - |z_k|^2)(|w_{k+1}|^2 - |w_k|^2)\Big)^{-\nu_k}} \times \frac{|1 - z_n\overline{w_n}|^{-2\nu_n}}{\Big((1-|z_n|^2)(1-|w_n|^2)\Big)^{-\nu_n}}\\
&= \frac{\prod_{k=1}^{n-1} \left|1 - \frac{z_k}{z_{k+1}}\overline{\frac{w_k}{w_{k+1}}}\right|^{-2\nu_k}}{\prod_{k=1}^{n-1}\Big((1-|\frac{z_k}{z_{k+1}}|^2)(1-|\frac{w_k}{w_{k+1}}|^2)\Big)^{-\nu_k}} \times \frac{|1 - z_n\overline{w_n}|^{-2\nu_n}}{\Big((1-|z_n|^2)(1-|w_n|^2)\Big)^{-\nu_n}}.
\end{aligned}$$

By Taylor expansion, we know that

$$\left(1 - \frac{z_k}{z_{k+1}}\overline{\frac{w_k}{w_{k+1}}}\right)^{-\nu_k} = \sum_{\alpha=0} c_\alpha(\nu_k)\left(\frac{z_k}{z_{k+1}}\right)^\alpha \left(\overline{\frac{w_k}{w_{k+1}}}\right)^\alpha,$$

where $c_\alpha(\nu_k)$ are the constants depending on α and ν_k. By Cauchy–Schwarz inequality, we get

$$\left|1 - \frac{z_k}{z_{k+1}}\overline{\frac{w_k}{w_{k+1}}}\right|^{-2\nu_k} \leq \left(\left(1 - \left|\frac{z_k}{z_{k+1}}\right|^2\right)\left(1 - \left|\frac{w_k}{w_{k+1}}\right|^2\right)\right)^{-\nu_k} \tag{3.1}$$

for $1 \leq k \leq n-1$. Similarly, we also have

$$|1 - z_n\overline{w_n}|^{-2\nu_n} \leq \Big((1-|z_n|^2)(1-|w_n|^2)\Big)^{-\nu_n}. \tag{3.2}$$

Hence, we must have

$$\exp\{-D_{g(\nu)}(z,w)\} \leq 1,\ (z,w) \in \Omega_n \times \Omega_n.$$

Furthermore, by (3.1) and (3.2), we get $\exp\{-D_{g(\nu)}(z,w)\} = 1$ if and only if for $1 \leq k \leq n-1$, we have

$$\frac{z_k}{z_{k+1}} = \frac{w_k}{w_{k+1}} \quad \text{and} \quad z_n = w_n.$$

It follows that $z = w$.

Now, we are in position to check the condition (II) in Theorem 1.4. Let $E \subset \mathbb{R}^+$ be a set defined by

$$E := \{\alpha \in \mathbb{N}^+; \alpha\nu_k \text{ are integers for } 1 \le k \le n-1, \quad \text{and} \quad \alpha\nu_n > 1\}.$$

Since ν_k $(1 \le k \le n-1)$ are positive rational numbers and $\nu_n > 0$, we can learn that $+\infty$ is in the closure of the subset E. Then, we want to prove that this subset E satisfies condition (II) in Theorem 1.4. Actually, since $\alpha \in E$, this means that $\alpha\nu_k$ are integers for all $1 \le k \le n-1$, and $\alpha\nu_n > 1$; thus, by Theorem 1.2, we can conclude that $\alpha g(\nu)$ is the balanced metric on Ω_n, i.e., $\varepsilon_{(1,\alpha g(\nu))}(z)$ is a positive constant for all $\alpha \in E$. Then, by Lemma 3.1, we can obtain that $\varepsilon_{(\alpha, g(\nu))}(z)$ is a positive constant for all $\alpha \in E$. This follows that E satisfies condition (II) in Theorem 1.4. Therefore, we conclude that $(\Omega_n, g(\nu))$ admit Berezin quantization by Theorem 1.4. The proof is complete. $\square$

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