Entire solutions to sublinear elliptic problems on harmonic NA groups and Euclidean spaces

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Abstract

We give necessary and sufficient conditions for the existence of entire solutions bounded or large of the equation $\mathcal{L}u - p\psi(u) = 0$, where $\mathcal{L}$ is either the Laplace operator on $\mathbb{R}^d$, $d \geq 3$ or the Laplace–Beltrami operator on the harmonic NA group and p is a function whose oscillation tends to zero at infinity at a specified rate. The results apply to noncompact rank one symmetric spaces.

Keywords Sublinear elliptic problems · Damek–Ricci spaces · Entire solutions · Large solutions

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1 Introduction

Let $\mathcal{L}$ be the Laplace operator on $\mathbb{R}^d$ ($d \geq 3$) or the Laplace–Beltrami operator on a harmonic NA group. As a Riemannian manifold such a group is $\mathbb{R}^d$ with an appropriate left-invariant metric.¹ We are interested in entire continuous solutions to the equation

$$\mathcal{L}u(x) - \varphi(x, u(x)) = 0, \quad x \in S, \quad (1.1)$$

where S is either $\mathbb{R}^d$ or harmonic NA, the equation is meant in the sense of distributions and $\varphi : S \times \mathbb{R} \rightarrow \mathbb{R}_+$ satisfies some appropriate hypotheses. A solution u to (1.1) is called *entire* if it is defined in the whole space S

¹ Harmonic NA groups are also called Damek–Ricci spaces. For their basic properties see [1, 4].

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Our aim is to give sufficient necessary and sufficient conditions for the existence of entire solutions bounded or “large”. A solution u to (1.1) is called *large* if $u(x) \rightarrow \infty$ when $d(x, 0) \rightarrow \infty$ where d is the Riemannian distance on NA or the Euclidean distance on $\mathbb{R}^d$.

Large solutions, i.e. the boundary blow-up problems are of significant interest due to its various scientific applications in different fields. Such problems arise in the study of Riemannian geometry [3], non-Newtonian fluids [2], the subsonic motion of a gas [13] and the electric potential in some bodies [12].

Throughout this paper, φ satisfies the following hypotheses:

- (H_1) For every $t_0 \in [0, \infty)$, $x \mapsto \varphi(x, t_0) \in \mathcal{K}_d^{\text{loc}}(S)$, i.e. it is locally in the Kato class in S .
- (H_2) For every $x_0 \in S$, $t \mapsto \varphi(x_0, t)$ is continuous increasing on $[0, \infty)$
- (H_3) $\varphi(x, t) = 0$ for every $x \in S$ and $t \leq 0$.

Let Ω be a domain in S . We recall that a Borel measurable function ψ on Ω is locally in the Kato class in Ω if

$$\limsup_{\alpha \rightarrow 0} \int_{D \cap \{|x-y| \leq \alpha\}} \frac{|\psi(y)|}{|x-y|^{d-2}} dy = 0$$

for every open bounded set D , $\bar{D} \subset \Omega$. (H_1) makes φ locally integrable against the Green function² for $\mathcal{L}$ which plays an important role in our approach.

There is a number of results that indicate that bounded and large solutions cannot occur at the same time. This holds, in particular, in a more general setting of elliptic operators with smooth coefficients, if for every x , $\varphi(x, \cdot)$ is concave as a function of the second variable or $\varphi(x, t) = p(x)\psi(t)$ and ψ is sublinear [5].

In our setting, we are able to say more. Both spaces $\mathbb{R}^d$ and harmonic NA have a common feature—global geodesic coordinates and a phenomenon of radially with respect to the appropriate Riemannian distance d . Suppose that, for every t , $\varphi(\cdot, t)$ is *radial* as a function of x , i.e. $\varphi(x, t) = \varphi(d(0, x), t)$, see [6]. Then an entire solution always exists, but if it is bounded or large, it depends on the growth of $\varphi(\cdot, c)$ measured by $I_{\varphi(\cdot, c)}$, where for a radial function g and $r = d(x, 0)$, I_g is defined by

$$I_g = \int_0^\infty rg(r)(r) dr, \quad (1.2)$$

on Euclidean S and

$$I_g = \int_0^\infty g(r) dr \quad (1.3)$$

on harmonic S . To obtain $I_{\varphi(\cdot, c)}$ we fix c and we put $g(r) = \varphi(r, c)$ in (1.2) or (1.3). Then we have the following characterization, see [6]:

Theorem 1 *Suppose that (H_1)–(H_3) are satisfied and φ is radial with respect to the first variable. Suppose further that either*

(H_4): For every $x_0 \in S$, $t \mapsto \varphi(x_0, t)$ is concave on $[0, \infty)$.

or

² See (4.1), (4.2) for the definition of the Green function G_Ω in a domain Ω .

(H_5) : $\varphi(x, t) = p(x)\psi(t)$, where $p \in \mathcal{L}_{\text{loc}}^\infty(S)$ and there exists a constant $C > 0$ such that $\psi(t) \leq C(1 + t)$.

Then (1.1) has always an entire radial solution bounded or large. Moreover, there is a nontrivial bounded solution if and only if

$$\text{there exists } c > 0 \text{ such that } I_{\varphi(\cdot, c)} < \infty \quad (1.4)$$

and there is a large solution if and only if for every $c > 0$, $I_{\varphi(\cdot, c)} = \infty$. If $\varphi(x, t) = p(x)\psi(t)$ then (1.4) means $I_p < \infty$.

In both spaces, in the radial case, $I_{\varphi(\cdot, c)}$ is finite if and only if the Green potential of $\varphi(\cdot, c)$ is well defined. The difference in $I_{\varphi(\cdot, c)}$ for $\mathbb{R}^d$ (1.2) and for NA (1.3) is due to the properties of the fundamental solution G to $\mathcal{L}$ on each space. More precisely, in both spaces G is radial. On $\mathbb{R}^d$, $G(x) = a_d|x|^{-d+2}$ where a_d is a constant depending only on the dimension. For NA, we have a precise estimates for G proved in Theorems 21 and 22 in [6].

For φ radial with respect to the first variable, Theorem 1 goes considerably beyond the previous results. The aim of this paper is to prove an analogous result for $\varphi(x, t) = p(x)\psi(t)$ and (H_4) or (H_5) satisfied without assuming radiality of p . The strategy is to “radialize” p and to make use of Theorem 1. For that we have to control the oscillation of p at infinity, i.e.

$$p_{\text{osc}}(r) = p^*(r) - p_*(r),$$

where

$$p^*(r) = \sup\{p(x) : d(x, 0) = r\},$$

and

$$p_*(r) = \inf\{p(x) : d(x, 0) = r\}.$$

We define a radial function ρ [see (3.5) and (3.6)] that may grow quite fast at infinity and we assume that

$$I_{(\rho p_{\text{osc}})} < \infty \quad (1.5)$$

which means that p_{osc} is “close” at infinity to a radial function. Our result is

Theorem 2 Suppose that $\varphi(s, t) = p(s)\psi(t)$, (H_2) – (H_3) are satisfied, $p^* \in \mathcal{K}_d^{\text{loc}}(S)$, and (1.5) holds. Suppose further that (H_4) or (H_5) are satisfied. Then

$$\mathcal{L}u(x) - p(x)\psi(u(x)) = 0, \quad x \in S, \quad (1.6)$$

has always an entire continuous solution. Moreover, there is a nontrivial bounded solution if and only if $I_{p_*} < \infty$ and there is a large solution if and only if $I_{p_*} = \infty$.

For existence of nontrivial bounded solutions, weaker conditions are sufficient, see Theorem 3. For p radial the above result reduces to Theorem 1.

For the Laplace operator in $\mathbb{R}^d$ and $\varphi(x, t) = p(x)t^\gamma$ where $0 < \gamma < 1$ and p satisfying some additional hypotheses, the problem of entire solutions was considered in [7, 8, 11]. Theorems 2 and 3 not only generalize the previous results to a considerably larger class of nonlinearities but give an analogous characterization for harmonic NA groups.

In particular, the result applies to rank one symmetric spaces and

$$\mathcal{L}u(x) - p(x)u(x)^\gamma = 0, \quad \gamma \leq 1.$$

It has a bounded solution if $I_{p_*} < \infty$ and a large solution if $I_{p_*} = \infty$ provided $I_{(\rho p_{\text{osc}})} < \infty$ and, if p is additionally bounded then ρ is polynomially growing or exponentially growing depending whether $\gamma < 1$ or $\gamma = 1$.

These types of problems have been studied in a more general framework [5, 9, 10] where L is a second order elliptic operator with C^∞ coefficients defined in a Greenian domain $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) such that $L1 \leq 0$ and $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function. The present paper is based on those results. Some of them are in “Appendix”, but for most of them we refer to [5, 9, 10].

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2 Bounded solutions

Existence of entire nontrivial bounded solutions to the equation (1.1) is guaranteed by essentially weaker assumptions than those of Theorem 2. In particular φ does not need to be of the product form. We consider the radial function

$$\varphi_{\text{osc}}(r, c) = \varphi^*(r, c) - \varphi_*(r, c),$$

where

$$\varphi^*(r, c) = \sup\{\varphi(x, c) : d(x, 0) = r\},$$

and

$$\varphi_*(r, c) = \inf\{\varphi(x, c) : d(x, 0) = r\}.$$

We have

Theorem 3 (Bounded solution)

Let φ satisfies (H_1) – (H_3) . We suppose that for every $c > 0$

$$I_{\varphi_{\text{osc}}(\cdot, c)} < \infty, \tag{2.1}$$

then there exists a nontrivial nonnegative bounded solution to (1.1) if and only if

$$\text{there exists } c_1 > 0 \text{ such that } I_{\varphi_*(\cdot, c_1)} < \infty. \tag{2.2}$$

If $\varphi(x, t) = p(x)\psi(t)$ then (2.2) reduces to $I_{p_*} < \infty$.

Proof Suppose that there exists a nonnegative nontrivial bounded solution to (1.1). We have

$$0 \leq \varphi_*(x, c) \leq \varphi(x, c) \text{ for every } c > 0.$$

By Proposition 16 in [10], there exists a nonnegative nontrivial bounded solution to the equation

$$\mathcal{L}u = \varphi_*(\cdot, u), \text{ in } S. \tag{2.3}$$

Consequently, by Theorem 5 in [6], there exists $c_1 > 0$ such that $I_{\varphi^*(\cdot, c_1)} < \infty$.

Now let us focus on sufficiency of the condition. We are going to apply Theorem 14, see “Appendix” which is a basic tool in obtaining bounded solutions. It is enough to prove that

$$G_S(\varphi(\cdot, c_1))(0) < \infty \quad (2.4)$$

where G_S is the Green function for S .

Let $G : S \rightarrow \mathbb{R}_+$ be the fundamental solution for $\mathcal{L}$ given by $G(s) = G_S(s, 0)$, $s \in S$. G is radial and for $S = NA$ it satisfies the estimates

$$c \exp(-Qr) \leq G(r) \leq d \exp(-Qr), \text{ for } r \geq 1, \quad (2.5)$$

and

$$G(r) \asymp r^{2-d}, \text{ for } 0 < r < 1, \quad (2.6)$$

see [6]. $Q = \frac{p}{2} + q$ is the homogeneous dimension of N [4], i.e. $q = \dim \mathcal{Z}$, $p = \dim \mathcal{V}$, where $\mathcal{N} = \mathcal{Z} \oplus \mathcal{V}$ is the orthogonal decomposition of the Lie algebra of N . We define also $G(\cdot, \cdot)$ on $S \times S$ by $G(x, y) = G(x - y) = G_S(x - y, 0)$ for $S = \mathbb{R}^d$, $d \geq 3$ or $G(s, s_1) = G(s_1^{-1}s) = G_S(s_1^{-1}s, 0)$ for $S = NA$. Moreover,

$$G_S f(s) = \int_{NA} G(s, s_1) f(s_1) dm_L(s_1),$$

where dm_L is the left Haar measure on S . See [6] for more properties of G on NA .

From now we continue the proof for NA groups. For $\mathbb{R}^d$, we use estimates of the corresponding fundamental solution and follow the same argument. Suppose that there exists $c_1 > 0$ such that $I_{\varphi^*(\cdot, c_1)} < \infty$. Then by (2.1) $I_{\varphi^*(\cdot, c_1)} < \infty^3$ and so by (2.5)

$$\int_1^{+\infty} G(r) \varphi^*(r, c_1) 2^{p+q} \sinh\left(\frac{r}{2}\right)^{p+q} \cosh\left(\frac{r}{2}\right)^q dr < \infty.$$

Writing dm_L in the radial coordinates see Section 3 in [6] we have

$$\begin{aligned} & \int_1^{+\infty} G(r) \varphi^*(r, c_1) 2^{p+q} \sinh\left(\frac{r}{2}\right)^{p+q} \cosh\left(\frac{r}{2}\right)^q dr \\ &= \int_{S-B(0,1)} G(0, s) \varphi^*(s, c_1) dm_L(s) \\ &= \int_{S-B(0,1)} G_S(0, s) \varphi^*(s, c_1) dm_L(s) \end{aligned}$$

Hence

$$\int_{S-B(0,1)} G_S(0, s) \varphi(s, c_1) dm_L(s) < \infty.$$

By (H_1) , the integral on $B(0, 1)$ is finite and so we conclude that

³ φ^* does not necessarily satisfy (H_1) otherwise we could use Proposition 16 in [10] as before.

$$\int_S G_S(0, s) \varphi(s, c_1) dm_L(s) < \infty.$$

□

3 Large solutions

In this section, we prove Theorem 2. To some extent we follow the ideas of [8]. However, now the nonlinearity is more general and we require an additional regularity of ψ to define ρ and to formulate the result precisely. This is done in Proposition 8 and Corollary 10. Before, we need to develop some preparatory material.

In addition to (1.6) we consider the following equations

$$\mathcal{L}u = p_* \psi(u), \text{ in } S. \quad (3.1)$$

$$\mathcal{L}u = p^* \psi(u), \text{ in } S. \quad (3.2)$$

For a radial function g , we have

$$(\mathcal{L}g)(x) = (L_{\text{rad}}g)(r),$$

where, for S with multiplicities p, q ,

$$\begin{aligned} L_{\text{rad}} &= \partial_r^2 + \left(\frac{p+q}{2} \coth\left(\frac{r}{2}\right) + \frac{q}{2} \tanh\left(\frac{r}{2}\right) \right) \partial_r \\ &= \partial_r^2 + \left(\frac{p}{2} \coth\left(\frac{r}{2}\right) + q \coth(r) \right) \partial_r \end{aligned} \quad (3.3)$$

and $r = d(x, 0)$, see, e.g. [1].

Proposition 4 *On $S = NA$ we define a function $V(r) = V(r(x))$ by*

$$\begin{aligned} V(r) &= \int_0^r \frac{1}{\sinh\left(\frac{s}{2}\right)} \int_0^s \sinh\left(\frac{t}{2}\right) p_*(t) dt ds, \\ &= \int_0^r \sinh\left(\frac{t}{2}\right) p_*(t) \int_t^r \frac{1}{\sinh\left(\frac{s}{2}\right)} ds dt, \end{aligned}$$

$r \geq 0$. Then

$$\mathcal{L}V = L_{\text{rad}}(V) \geq p_*.$$

Remark 5 Function V is the first ingredient to control oscillation of p .

Proof We have

$$\begin{aligned}
(\partial_r V)(r) &= \frac{1}{\sinh\left(\frac{r}{2}\right)} \int_0^r \sinh\left(\frac{t}{2}\right) p_*(t) dt \\
(\partial_r^2 V)(r) &= p_*(r) + \left[\int_0^r \sinh\left(\frac{t}{2}\right) p_*(t) dt \right] \left[-\frac{1}{2} \frac{\cosh\left(\frac{r}{2}\right)}{\sinh\left(\frac{r}{2}\right)^2} \right] \\
&= p_*(r) + \left[\int_0^r \sinh\left(\frac{t}{2}\right) p_*(t) dt \right] \left[-\frac{1}{2} \frac{\coth\left(\frac{r}{2}\right)}{\sinh\left(\frac{r}{2}\right)} \right] \\
&= p_*(r) - \frac{1}{2} \coth\left(\frac{r}{2}\right) (\partial_r V)(r).
\end{aligned}$$

Therefore, by (3.3) we obtain

$$\begin{aligned}
(L_{\text{rad}} V)(r) &= p_*(r) - \frac{1}{2} \coth\left(\frac{r}{2}\right) (\partial_r V)(r) + \left(\frac{p}{2} \coth\left(\frac{r}{2}\right) + q \coth(r)\right) (\partial_r V)(r) \\
&= p_*(r) + \left(\frac{p-1}{2} \coth\left(\frac{r}{2}\right) + q \coth(r)\right) (\partial_r V)(r) \\
&\geq p_*(r).
\end{aligned}$$

□

Remark 6 For $\mathcal{L} = \Delta$ the Laplace operator in the Euclidean space $\mathbb{R}^d$, we have the following analogue of Proposition 4. Let

$$V(r) = \frac{1}{d-2} r^2 \int_0^1 (1-t^{d-2}) t p_*(rt) dt, \quad r \geq 0.$$

Then

$$\Delta(V) = p_*. \quad (3.4)$$

For the equation $\Delta u = pu^\gamma$ with $p \in \mathcal{L}_{\text{loc}}^\infty(\mathbb{R}^d)$ and $0 < \gamma < 1$, V was introduced in [8] and to prove (3.4) El Mabrouk and Hansen use the formula $\partial_r^2 + \frac{d-1}{r} \partial_r$ for the radial part of Δ .

Function ψ is the second ingredient to construct ρ . In fact, we need to dominate ψ by a more regular function ψ_1 . In [8], for $\psi(t) = t^\gamma$, such regularization was not needed.

Lemma 7 Suppose that ψ is a positive function satisfying (H_2) and $\int_0^1 \psi(y) dy > 0$. Let $0 < \theta < 1$ and let η be defined by

$$\eta : x \mapsto \begin{cases} 1, & \text{on } [-1, 0]; \\ \exp\left(\frac{1}{\theta}\right) \exp\left(\frac{1}{x^2-\theta}\right), & \text{on }]0, \sqrt{\theta}]; \\ \exp\left(\frac{1}{\theta}\right) \exp\left(\frac{1}{(x+1)^2-\theta}\right), & \text{on }]-1-\sqrt{\theta}, -1[; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi_1 = \psi * \eta \in \mathcal{C}^1(\mathbb{R})$ satisfies:

- $\psi_1' \geq \psi$ on $\mathbb{R}$.
- $\psi_1' \geq 0$ on $\mathbb{R}$.
- $\psi_1(0) > 0$.

Proof Let

$$h : x \mapsto \begin{cases} \exp\left(\frac{1}{\theta}\right) \exp\left(\frac{1}{x^2 - \theta}\right), & \text{on }] -\sqrt{\theta}, \sqrt{\theta}[; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that h is $\mathcal{C}^1(\mathbb{R})$ then η is $\mathcal{C}^1(\mathbb{R})$ too. Following this $\psi_1 \in \mathcal{C}^1(\mathbb{R})$ satisfying

$$\begin{aligned} \psi_1'(x) &= \int_{\mathbb{R}} \psi(x-y) \eta'(y) dy \\ &= \int_{[-1-\sqrt{\theta}, -1]} \psi(x-y) h'(y+1) dy + \int_{[0, \sqrt{\theta}]} \psi(x-y) h'(y) dy \\ &= \int_{[0, \sqrt{\theta}]} \psi(x+1+s) h'(-s) ds + \int_{[0, \sqrt{\theta}]} \psi(x-y) h'(y) dy \end{aligned}$$

Since h is an even function, h' is an odd function and

$$\psi_1'(x) = \int_{[0, \sqrt{\theta}]} (\psi(x-y) - \psi(x+1+y)) h'(y) dy$$

in addition, $h' \leq 0$ on $[0, \sqrt{\theta}]$ and $\psi(x-y) - \psi(x+1+y) \leq 0$ if $y \in [0, \sqrt{\theta}]$, then

$$\psi_1'(x) \geq 0, \text{ on } \mathbb{R}.$$

Also

$$\begin{aligned} \psi_1(x) &= \int_{\mathbb{R}} \psi(x-y) \eta(y) dy \geq \int_{-1}^0 \psi(x-y) \eta(y) dy \\ &\geq \int_{-1}^0 \psi(x-y) dy \geq \psi(x) \end{aligned}$$

In addition,

$$\begin{aligned} \psi_1(0) &= \int_{\mathbb{R}} \psi(-y) \eta(y) dy \\ &\geq \int_{-1}^0 \psi(-y) \eta(y) dy = \int_0^1 \psi(y) dy > 0. \end{aligned}$$

□

Let us recall that a continuous function u is a supersolution (subsolution) to (3.1) if $\mathcal{L}u - p_*\psi(u) \leq 0$ ($\mathcal{L}u - p_*\psi(u) \geq 0$, respectively).

Now we are ready to prove that all radial supersolutions of (3.1) having the same value at 0 can be dominated by means of the function V .

Proposition 8 Suppose that ψ satisfies (H_2) – (H_4) . Let V and ψ_1 be as in Proposition 4 and Lemma 7. Let u be a radial supersolution of (3.1) such that $\mathcal{L}u \in L^1_{\text{loc}}(S)$ then

$$u \leq F = \zeta \circ (\zeta^{-1}(u(0)) + V), \text{ in } S,$$

where $\zeta = \mu^{-1}$ and $\mu(r) = \int_0^r \frac{1}{\psi_1(y)} dy$.

Remark 9 Notice that in view of Lemma 7, μ and ζ are well defined.⁴ For ψ satisfying (H_2) – (H_4) , we note

$$\rho = \psi(F) = \psi(\zeta \circ (\zeta^{-1}(u(0)) + V)). \quad (3.5)$$

If in addition, ψ is $\mathcal{C}^1$ integrable at 0 then we can take $\mu(r) = \int_0^r \frac{1}{\psi(y)} dy$.

In particular, for $\psi(t) = t^\gamma$ for some $0 < \gamma < 1$ we obtain

$$u \leq C (u(0))^{1-\gamma} + V)^{\frac{1}{1-\gamma}}$$

as in [8].

More generally, notice that ζ and V are strictly increasing and so is F . The growth of ψ_1 is the same as the growth of ψ and it is sublinear. So the growth of μ may be like $\log r$ which makes $\psi \circ \zeta$ to grow exponentially. If ψ grows like t^γ then $\psi \circ \zeta$ grows polynomially. Introducing F as above we cover a much larger range of nonlinearities than in [8].

Proof By the previous lemma, μ is well defined on $\mathbb{R}_+$, it belongs to $\mathcal{C}^1(\mathbb{R}_+)$ and $\mu'(x) = \frac{1}{\psi_1(x)} > 0$. We denote $\zeta = \mu^{-1}$ on $[0, \infty)$ —the inverse function to μ . Then

- $\zeta(0) = 0$, $\zeta(x) > 0$ for $x > 0$.
- $\zeta'(x) = \psi_1 \circ \zeta(x) > 0$ for $x \geq 0$.
- $\zeta''(x) = \zeta'(x)\psi_1'(\zeta(x)) \geq 0$ for $x \geq 0$.

Let $\epsilon > 0$, $a = \zeta^{-1}(u(0) + \epsilon)$ and $g(r) = \zeta(a + V(r))$. Then g is radially symmetric satisfying $g(0) = u(0) + \epsilon > u(0)$ and $g > 0$ on S . We will do the calculation on NA . For Δ on $\mathbb{R}^d$ it is analogous. In addition,

$$\begin{aligned} (\partial_r g)(r) &= (\partial_r V)(r)\zeta'(a + V(r)) \\ (\partial_r^2 g)(r) &= (\partial_r^2 V)(r)\zeta'(a + V(r)) + (\partial_r V)^2(r)\zeta''(a + V(r)) \end{aligned}$$

then

$$\begin{aligned} (L_{\text{rad}} g)(r) &= \zeta'(a + V(r))(L_{\text{rad}} V)(r) + (\partial_r V)^2(r)\zeta''(a + V(r)) \\ &\geq \zeta'(a + V(r))(L_{\text{rad}} V)(r) \\ &\geq \zeta'(a + V(r))p_*(r) = \psi_1(g(r))p_*(r) \\ &\geq \psi(g(r))p_*(r) \end{aligned}$$

Consequently, g is a subsolution of (3.1) and $L_{\text{rad}}(g)$

⁴ In view of (H_2) – (H_4) , if $\int_0^1 \psi(y) dy = 0$ then $\psi \equiv 0$.

is continuous. By Lemma 14 of [6] we get

$$u \leq \zeta(\zeta^{-1}(u(0) + \epsilon) + V).$$

Letting ϵ tend to zero we get

$$u \leq \zeta(\zeta^{-1}(u(0)) + V).$$

□

In fact, the condition (H_4) may be replaced by sublinearity of ψ . Given ψ that satisfies $(H_2) - (H_3)$ and there exists a constant $C > 0$ such that $\psi(t) \leq C(1 + t)$ we can find $\tilde{\psi}$ satisfying $(H_2) - (H_4)$ and such that $\psi \leq \tilde{\psi}$, see [5].

Corollary 10 *Suppose that ψ satisfies $(H_2) - (H_3)$ and $\psi(t) \leq C(1 + t)$. Let $\tilde{\psi}$ be a function satisfying $(H_2) - (H_4)$ and such that $\psi \leq \tilde{\psi}$. Let $\tilde{F}$ the function constructed in Proposition 8 but for $\tilde{\psi}$. Let u be a radial supersolution of (3.1) such that $\mathcal{L}u \in L^1_{\text{loc}}(S)$ then*

$$u \leq \tilde{F} = \tilde{\zeta} \circ (\tilde{\zeta}^{-1}(u(0)) + V), \text{ in } S.$$

Proof Let $\tilde{\zeta}$ and $\tilde{g}$ be as in the proof of Proposition 8 but defined for $\tilde{\psi}$. Then

$$\mathcal{L}u - p_*\tilde{\psi}(u) \leq \mathcal{L}u - p_*\psi(u) \leq 0 \leq \mathcal{L}g - p_*\tilde{\psi}(g).$$

As before, we may apply Lemma 14 of [6] and we get

$$u \leq \tilde{\zeta}(\tilde{\zeta}^{-1}(u(0) + \epsilon) + V).$$

Letting ϵ tend to zero we get

$$u \leq \tilde{\zeta}(\tilde{\zeta}^{-1}(u(0)) + V) = \tilde{F}.$$

□

Remark 11 Hence for ψ satisfying $(H_2) - (H_3)$ and $\psi(t) \leq C(t + 1)$ we note

$$\rho = \psi(\tilde{F}) = \psi(\tilde{\zeta}(\tilde{\zeta}^{-1}(u(0)) + V)) \quad (3.6)$$

Now we are ready to formulate our main result. Later on

$$\rho = \begin{cases} \psi(F), & \text{if } (H_4) \text{ is satisfied, see (3.5),} \\ \psi(\tilde{F}), & \text{if } (H_5) \text{ is satisfied, see (3.6).} \end{cases}$$

Theorem 12 (Large solution) *We suppose that $\varphi(s, t) = p(s)\psi(t)$, $(H_1) - (H_3)$ are satisfied and $p^* \in \mathcal{K}_d^{\text{loc}}(S)$. We assume in addition that (H_4) or (H_5) is satisfied and*

$$I_{(\rho p_{\text{osc}})} < \infty, \quad (3.7)$$

Then (1.1) has a large solution in S if and only if

$$I_{p^*} = \infty.$$

Remark 13 Notice that V defined as in Remark 6 is dominated by $(d-2)^{-1} \int_0^r tp_*(t) dt$ so we may take

$$\rho = \psi \left(\left(1 + \int_0^r tp_*(t) dt \right)^{1/(1-\gamma)} \right)$$

as it was done in [8].

Equation (3.7) means that p_{osc} must decay quite fast at ∞ and the rate of decay depends both on ψ and p_* . Suppose that p_* is bounded and not integrable (at infinity). Then by (2.5) and the formula for m_L , $I_{p_*} = \infty$. Then $V(r)$ basically grows linearly. If ψ is linear at ∞ , then $\psi \circ \zeta$ or $\psi \circ \tilde{\zeta}$ grows exponentially; if $\psi(t) = t^\gamma$, then $\psi \circ \zeta$ grows polynomially. The same applies to ρ which determines the behavior of p_{osc} at ∞ . If p_* is unbounded $V(r)$ may grow faster than linearly and ρ faster than exponentially.

Proof Since $F \geq 1$ (resp. $\tilde{F} \geq 1$), necessity of condition follows from Theorem 3. Indeed, suppose that there is a large solution and $I_{p_*} < \infty$. Then by Theorem 3 there is a bounded solution to (1.1). But this is in contradiction with Theorem 2 (resp. Theorem 1) in [5]. So let us focus on sufficiency. Suppose that $I_{p_*} = \infty$. By Theorem 1, there exists a large radial solution v to (3.1) satisfying $v(0) = 1^5$. Moreover,

$$\mathcal{L}v - p^*\psi(v) \leq \mathcal{L}v - p_*\psi(v) = 0, \text{ in } S,$$

so v is a radial supersolution to Eq. (3.2). Let

$$D_n = \{x \in S, d(x, 0) = n\}.$$

By Theorem 14, we define a sequence of functions (u_n) by

$$u_n = U_{D_n}^{p^*} v, \quad n \in \mathbb{N}.$$

, i.e. $U_{D_n}^{p^*} v$ is a radial solution of Eq. (3.2) in D_n having the boundary values v .

Consequently, by Lemma 15, (u_n) is a positive decreasing sequence bounded above by v . Let $u = \lim_{n \rightarrow +\infty} u_n$. By Lemma 8 in [10], u is a radial solution of (3.2) satisfying $u \leq v$. Furthermore

$$\mathcal{L}u - p\psi(u) \geq \mathcal{L}u - p^*\psi(u) = 0, \text{ in } S,$$

and

$$\mathcal{L}v - p\psi(v) \leq \mathcal{L}v - p_*\psi(v) = 0, \text{ in } S,$$

, i.e. u is a subsolution to (1.1) and v is a supersolution to (1.1). We argue that there is a solution w to (1.1) such that

$$u \leq w \leq v, \text{ in } S.$$

Indeed, Let $w_n = U_{D_n}^p v$. Due to Lemma 15, (w_n) is a positive decreasing sequence satisfying $u \leq w_n \leq v$. Hence again by Lemma 8 in [10], $w = \lim_{n \rightarrow +\infty} w_n$ is a solution to (1.1).

⁵ If $I_{p_*} = \infty$, for any $\alpha \in \mathbb{R}_+^*$, we can get a large solution v to $\mathcal{L}u - p_*\psi(u) = 0$ such that $v(0) = \alpha$, see Theorem 8 in [6].

Now, we need to prove that w is a large solution. Actually, it is enough to prove that

$$\lim_{d(x,0) \rightarrow +\infty} u(x) = +\infty. \quad (3.8)$$

However, $I_{p_*} = \infty \Rightarrow I_{p^*} = \infty$, then by Theorem 1, u is either trivial or unbounded. Moreover, u is $\mathcal{L}$ -subharmonic radial in S , hence, by the maximum principle for elliptic operators, it follows that

$$u(x) \leq u(x_0), \quad \text{if } d(x, 0) \leq d(x_0, 0). \quad (3.9)$$

hence u is either trivial or (3.8) is satisfied. It remains to prove that u is nontrivial. By Theorem 14, we have in D_n

$$\begin{aligned} u_n(0) + G_{D_n}(p^* \psi(u_n))(0) &= H_{D_n} v(0) \\ &= v(0) + G_{D_n}(p_* \psi(v))(0). \end{aligned}$$

Though $v(0) = 1$ then in D_n

$$u_n(0) + G_{D_n}(p_{\text{osc}} \psi(u_n))(0) = 1 + G_{D_n}(p_*(\psi(v) - \psi(u_n)))(0).$$

Additionally, $u_n \leq v$ in D_n so

$$u_n(0) + G_{D_n}(p_{\text{osc}} \psi(u_n))(0) \geq 1, \text{ in } D_n. \quad (3.10)$$

In view of Proposition 8 (resp. Proposition 10) applied to v , we have⁶

$$\begin{aligned} G_{D_n}(0, s) p_{\text{osc}}(s) \psi(u_n)(s) &\leq G_S(0, s) p_{\text{osc}}(s) \psi(v)(s) \\ &\leq G(0, s) p_{\text{osc}}(s) \rho(r(s)) \end{aligned}$$

Moreover, writing the left Haar measure in radial coordinates, we have

$$\begin{aligned} &\int_{NA} G(0, s) p_{\text{osc}}(s) \rho(r(s)) dm_L(s) \\ &= \int_0^{+\infty} G(r) p_{\text{osc}}(r) \rho(r) 2^{p+q} \sinh^{p+q} \left(\frac{r}{2} \right) \cosh^q \left(\frac{r}{2} \right) dr. \end{aligned}$$

Since $G(r) \sinh^{p+q} \left(\frac{r}{2} \right) \cosh^q \left(\frac{r}{2} \right)$, is bounded on $\mathbb{R}_+$ then by (3.7)

$$\int_0^{+\infty} G(r) p_{\text{osc}}(r) \rho(r) 2^{p+q} \sinh^{p+q} \left(\frac{r}{2} \right) \cosh^q \left(\frac{r}{2} \right) dr < \infty.$$

We conclude that

$$\int_{NA} G_S(0, s) p_{\text{osc}}(s) \psi(v)(s) dm_L(s) < \infty,$$

Letting $n \rightarrow \infty$ in (3.10), by the dominated convergence theorem we have

$$u(0) + G(p_{\text{osc}} \psi(u))(0) \geq 1.$$

And so u is not trivial. □

⁶ From now we continue the proof in NA groups. For $\mathbb{R}^d$, we use estimates of the corresponding fundamental solution and follow the same argument.

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Appendix

For the readers convenience we recall two basic facts about elliptic operators in a Greenian domains that have been used in the paper. For the rest of the material we refer to [5, 6, 9, 10].

Let L is a second order elliptic operator with C^∞ coefficients defined in an open domain $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) such that $L1 \leq 0$. We say that Ω is *Greenian* if there is a function $G_\Omega(x, y)$ smooth on $\Omega \times \Omega \setminus \{(x, x) : x \in \Omega\}$ such that for every $y \in \Omega$

$$LG_\Omega(\cdot, y) = -\delta_y, \quad \text{in the sense of distributions,} \quad (4.1)$$

and

$$G_\Omega(\cdot, y), \quad \text{is a potential,} \quad (4.2)$$

, i.e. every positive L -harmonic function h such that $h(x) \leq G_\Omega(x, y)$ is equal 0. We write

$$G_\Omega(\varphi(\cdot, c))(x) = \int_\Omega G_\Omega(x, y)\varphi(y, c) \, dy.$$

We have the following characterization of bounded solutions to

$$Lu(x) - \varphi(x, u(x)) = 0 \quad (4.3)$$

Theorem 14 Suppose that $L1 \leq 0$ and φ satisfies (H_1) , (H_2) and (H_3) . Assume further that $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) and there is $c > 0$ such that $G_\Omega(\varphi(\cdot, c))$ is finite at least on one point⁷. Then positive continuous solutions to (4.3) bounded by c are in one-to-one correspondence with positive L -harmonic functions bounded by c given by

$$h = u + G_\Omega(\varphi(\cdot, u)), \quad \text{in } \Omega. \quad (4.4)$$

For a bounded regular domain D satisfying $\bar{D} \subset \Omega$ and $f \in C^+(\partial D)$ (4.4) becomes

$$H_D f = u + G_D(\varphi(\cdot, u)), \quad \text{in } D. \quad (4.5)$$

where $H_D f$ is the L -harmonic function in D having f as its boundary values.⁸

In what follows the solution u to (4.3) in a bounded regular domain D taking the values of f at the boundary is denoted $u = U_D^\varphi f$.

⁷ Then $G_\Omega(\varphi(\cdot, c))$ is finite at any point because φ is locally in the Kato class.

⁸ In a bounded regular domain D (H_1) implies that $G_D(\varphi(\cdot, c))$ vanishes on ∂D .

We need also a Lemma that gives comparison between sub-solutions and super-solutions to (4.3) in D . It holds in a considerable generality: we require only that the function $\varphi : D \times \mathbb{R} \mapsto \mathbb{R}$ is increasing with respect to the second variable, see [10]:

Lemma 15 (Comparison with values on the boundary) *Let $u, v \in \mathcal{C}(D)$, $Lu, Lv \in \mathcal{L}_{\text{loc}}^1(D)$ and let $\varphi : D \times \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function with respect to the second variable. If*

$$\begin{cases} Lu - \varphi(\cdot, u) \leq Lv - \varphi(\cdot, v), \\ \liminf_{\substack{x \rightarrow y \\ y \in \partial D}} (u - v)(x) \geq 0. \end{cases}$$

Then:

$$u - v \geq 0 \text{ in } D.$$

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