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Variational characterizations of ξ-submanifolds in the Eulicdean space $\mathbb{R}^{m+p}$

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Abstract

ξ-submanifold in the Euclidean space $\mathbb{R}^{m+p}$ is a natural extension of the concept of self-shrinker to the mean curvature flow in $\mathbb{R}^{m+p}$. It is also a generalization of the λ-hypersurface defined by Q.-M. Cheng et al to arbitrary codimensions. In this paper, some characterizations for ξ-submanifolds are established. First, it is shown that a submanifold in $\mathbb{R}^{m+p}$ is a ξ-submanifold if and only if its modified mean curvature is parallel when viewed as a submanifold in the Gaussian space $\left(\mathbb{R}^{m+p}, e^{-\frac{1}{m}|x|^2}\langle\cdot,\cdot\rangle\right)$; then, two generalized weighted volume functionals V_ξ and $\bar{V}_\xi$ are defined and it is proved that ξ-submanifolds can be characterized as the critical points of these two functionals; also, the corresponding second variation formulas are computed. Finally, we introduce the VP-variations and the corresponding W-stability for ξ-submanifolds which are then systematically studied. As the main result, it is proved that m-planes are the only complete, W-stable and properly immersed ξ-submanifolds with flat normal bundle.

Keywords Self-shrinker · Gaussian space · ξ-submanifold · Variation formula · Stability

Mathematics Subject Classification Primary 53C40; Secondary 53B25

Contents

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1 Introduction

Let $x : M^m \to \mathbb{R}^{m+p}$ be an m-dimensional submanifold in the $(m+p)$-dimensional Euclidean space $\mathbb{R}^{m+p}$ with the second fundamental form h. Then, x is called a *self-shrinker* to the mean curvature flow if its mean curvature vector field $H := \operatorname{tr} h$ satisfies

$$H + x^{\perp} = 0, \tag{1.1}$$

where $x^{\perp}$ is the orthogonal projection of the position vector x to the normal space $T^{\perp}M^m$ of x.

It is well known that the self-shrinker plays an important role in the study of the mean curvature flow. In fact, self-shrinkers correspond to self-shrinking solutions to the mean curvature flow and describe all possible Type I singularities of the flow. Up to now, there have been a plenty of research papers on self-shrinkers and on the asymptotic behavior of the flow. For details of this see, for example, [1–6,8,12–17,19–24,28] and references therein. In particular, the following result is well known (see Corollary 3.2 in Sect. 3):

An immersion $x : M^m \to \mathbb{R}^{m+p}$ is a self-shrinker if and only if it is minimal when viewed as a submanifold of the Gaussian space $(\mathbb{R}^{m+p}, e^{-\frac{|x|^2}{m}}\langle\cdot,\cdot\rangle)$.

In March, 2014, Cheng and Wei formally introduced ([9], finally revised in May, 2015) the definition of λ-hypersurface of weighted volume-preserving mean curvature flow in Euclidean space, giving a natural generalization of self-shrinkers in the hypersurface case. According to [9], a hypersurface $x : M^m \to \mathbb{R}^{m+1}$ is called a λ-hypersurface if its (scalar-valued) mean curvature H satisfies

$$H + \langle x, N\rangle = \lambda \tag{1.2}$$

for some constant λ, where N is the unit normal vector of x. They also found some variational characterizations for those new kind of hypersurfaces, proving that *a hypersurface x is a λ-hypersurface if and only if it is the critical point of the weighted area functional $\mathcal{A}$ preserving the weighted volume functional $\mathcal{V}$* where for any $x_0 \in \mathbb{R}^{m+1}$ and $t_0 \in \mathbb{R}$,

$$\mathcal{A}(t) = \int_M e^{-\frac{|x(t)-x_0|^2}{2t_0}}\,\mathrm{d}\mu, \quad \mathcal{V}(t) = \int_M \langle x(t) - x_0, N\rangle\, e^{-\frac{|x(t)-x_0|^2}{2t_0}}\,\mathrm{d}\mu.$$

Meanwhile, some rigidity or classification results for λ-hypersurfaces are obtained, for example, in [7,10] and [18]; for the rigidity theorems for space-like λ-hypersurfaces, see [26].

We should remark that this kind of hypersurfaces was also studied in [27] (arXiv preprint: Jul. 2013; formally published in 2015) where the authors considered the stable, two-sided, smooth, properly immersed solutions to the Gaussian Isoperimetric Problem, namely they studied hypersurfaces $\Sigma \subset \mathbb{R}^{m+1}$ that are second order stable critical points of minimizing the weighted area functional $\mathcal{A}_\mu(\Sigma) = \int_\Sigma e^{-|x|^2/4} d\mathcal{A}_\mu$ for compact (uniformly) normal variations that, in a sense, "*preserve the weighted volume* $\mathcal{V}_\mu(\Sigma) = \int_\Sigma e^{-|x|^2/4} d\mathcal{V}_\mu$". It turned out that the λ-hypersurface equation (1.2) is exactly the Euler-Lagrange equation of the variation problem in [27] of which a main result can be restated as

Hyperplanes are the only two-sided, complete and properly immersed λ-hypersurfaces in the Euclidean space that are stable under the compact normal variations "preserving the weighted volume".

In 2015, the first author and his co-author made in [25] a natural generalization of both self-shrinkers and λ-hypersurfaces by introducing the concept of ξ-submanifolds and, as the main result, a rigidity theorem for Lagrangian ξ-submanifolds in $\mathbb{C}^2$ is proved, which is motivated by a result of [23] for Lagrangian self-shrinkers in $\mathbb{C}^2$. By definition, an immersed submanifold $x : M^m \to \mathbb{R}^{m+p}$ is called a ξ-submanifold if there is a parallel normal vector field ξ such that the mean curvature vector field H satisfies

$$H + x^{\perp} = \xi. \tag{1.3}$$

We believe that if self-shrinkers and λ-hypersurfaces are taken to be parallel to minimal submanifolds and constant mean curvature hypersurfaces, respectively, then ξ-submanifolds are expected to be parallel to submanifolds of parallel mean curvature vector. So there should be many properties of ξ-submanifolds that are parallel to those of submanifolds with parallel mean curvature vectors.

In this paper, we aim at giving more characterizations of the ξ-submanifolds, including ones by variation method, the latter being more important since a differential equation usually needs a variational method to solve. For example, self-shrinker equation (1.1) has been exploited a lot by making use of variation formulas. As the main part of this paper, we shall systematically study the relevant stability problems for ξ-submanifolds, paying a particular attention on the VP-variations and the relevant W-stability.

Now, beside the various characterizations of the ξ-submanifolds and some instability results, the main theorem of this paper can be stated as

Theorem 1.1 (Theorem 7.3). *Let $x : M^m \to \mathbb{R}^{m+p}$ be a complete and properly immersed ξ-submanifold with flat normal bundle. Then, x is W-stable if and only if $x(M^m)$ is an m-plane.*

Clearly, Theorem 1.1 generalizes the main theorem for hypersurfaces in [27] which has been stated earlier.

The following uniqueness conclusion for self-shrinkers is direct from Theorem 1.1:

Corollary 1.2 *Any complete, W-stable and properly immersed self-shrinker in $\mathbb{R}^{m+p}$ with flat normal bundle must be an m-plane passing the origin.*

The organization of the present paper is as follows:

In Sect. 2, we present the necessary preliminary material, including some typical examples;

In Sect. 3, we prove a theorem (Theorem 3.1) which generalizes (to ξ-submanifolds) a well-known result that self-shrinkers are equivalent to minimal submanifolds in the Gaussian space;

In Sect. 4, we introduce, for a given manifold M^m of dimension m, two families of weighted volume functionals V_ξ and $\bar{V}_\xi$ in (4.1) parametrized by $\mathbb{R}^{m+p}$-valued functions $\xi : M^m \to \mathbb{R}^{m+p}$. Then we compute the first variation formulas (Theorem 4.1) which give that ξ-submanifolds are exactly the critical points of V_ξ and $\bar{V}_\xi$ with ξ suitably chosen (Corollary 4.2). We also compute the second variation formula of both functionals for ξ-submanifolds (Theorem 4.3), in such a situation V_ξ and $\bar{V}_\xi$ being essential the same.

In Sects. 5 and 6, we study the stability problem of ξ-submanifolds. After checking that, with respect to the functional V_ξ or $\bar{V}_\xi$, many ξ-submanifolds including all the typical examples are not stable in the usual sense (Sect. 5), we define in Sect. 6 a special kind of variation for submanifolds of higher codimension, called "*VP-variation*," which is a natural generalization of "volume-preserving variation" for hypersurfaces. Accordingly, we introduce "*the W_ξ-stability*" with respect to V_ξ or $\bar{V}_\xi$ for higher codimensional submanifolds and then show that, among the typical examples given in Sect. 2, only the m-planes are W_ξ-stable (Theorem 6.1 and Theorem 6.2). In particular, we give an index estimate for the standard sphere (Theorem 6.2).

Finally, in the last section (Sect. 7), we consider the VP-variation of the standard weighted volume functional $V_w \equiv V_0$ which corresponds to a special case, i.e., $\xi = 0$, of the functional V_ξ or $\bar{V}_\xi$ defined in Sect. 4, and study the W-stability (i.e., W_0-stability, see Definition 7.1) for ξ-submanifolds. As the result, we first characterize ξ-submanifolds as critical points of V_w under VP-variations (Corollary 7.2, corresponding to the conventional extremal points with conditions) and then prove our main Theorem (Theorem 1.1).

Remark 1.1 Our discussion of variation problem for ξ-submanifolds naturally gives a new motivation of variational characterization of the submanifolds with parallel mean curvature vectors in the Euclidean space $\mathbb{R}^{m+p}$ (see Remark 4.3 at the end of Sect. 4).

Remark 1.2 Related to the present paper, it seems natural and interesting to characterize ξ-submanifolds in terms of their Gauss map, just like in the study of submanifolds in $\mathbb{R}^{m+p}$ with parallel mean curvature vectors. We shall deal this kind of problems later in the sequel.

2 ξ-submanifolds–definition and typical examples

Let $\mathbb{R}^{m+p}$ be the $(m+p)$-dimensional Euclidean space with the standard metric $\langle\cdot,\cdot\rangle$ and the standard connection D. Let $x : M^m \to \mathbb{R}^{m+p}$ be an immersion with the induced metric g, the second fundamental form h and the mean curvature vector $H := \mathrm{tr}\,_g h$. Denote by TM the tangent space of M with the Levi-Civita connection ∇, and define $T^\perp M := (x_*(TM))^\perp$ to be the normal space of x in $\mathbb{R}^{m+p}$ with the normal connection $D^\perp$.

Definition 2.1 (ξ-submanifolds, [25]). The immersed submanifold $x : M^m \to \mathbb{R}^{m+p}$ is called a ξ-submanifold if the normal vector field

$$\xi := H + x^\perp \tag{2.1}$$

is parallel in $T^\perp M$, namely $D^\perp \xi \equiv 0$.

So, self-shrinkers of the mean curvature flow are a special kind of ξ-submanifolds with $\xi = 0$.

The following are some typical examples of ξ-submanifolds:

Example 2.1 (The ξ-curves).

Let $x : (a, b) \to \mathbb{R}^{1+p}$ be a unit-speed smooth curve (that is, with an arc-length parameter s). Denote by $\{T, e_\alpha : 2 \le \alpha \le 1+p\}$ the Frenet frame with $T := \dot{x} \equiv \frac{\partial x}{\partial s}$ being the unit tangent vector, and κ_i the i-th curvature, $i = 1, \dots, p$. Then, we have the following Frenet formula:

$$\dot{T} = \kappa_1 e_2,\ \dot{e}_2 = -\kappa_1 T + \kappa_2 e_3,\ \cdots,\ \dot{e}_p = -\kappa_{p-1} e_{p-1} + \kappa_p e_{p+1},\ \dot{e}_{1+p} = -\kappa_p e_p. \tag{2.2}$$

In particular, if there exists some i such that $\kappa_i \equiv 0$, then it must hold that $\kappa_j \equiv 0$ for all $j > i$. Sometimes we call $\kappa := \kappa_1$ and $\tau := \kappa_2$ the curvature and the (first) torsion of x. Now the definition Eq. (2.1) becomes $\left(\frac{\mathrm{d}}{\mathrm{d}s}(\dot{T} + x - \langle x, T\rangle T)\right)^\perp \equiv 0$ which, by (2.2), is equivalent to

$$\dot{\kappa}_1 - \kappa_1 \langle x, T\rangle \equiv 0, \quad \kappa_1\kappa_2 \equiv 0. \tag{2.3}$$

It follows that

x is a ξ-curve if and only if it is a plane curve with the curvature κ satisfying

$$\dot{\kappa} - \kappa \langle x, \dot{x}\rangle \equiv 0. \tag{2.4}$$

In particular,

x is a self-shrinker if and only if it is a plane curve with the curvature κ satisfying

$$\kappa_r + \langle x, N\rangle \equiv 0, \tag{2.5}$$

where κ_r is the relative curvature and $N := \pm e_2$ is the unit normal of x pointing the left of T. Note that curves in the plane satisfying (2.5) are classified by U. Abresch and J. Langer in [1] which are now known as *Abresch–Langer curves* (see [23]).

Example 2.2 (The m-planes not necessarily passing through the origin).

An m-plane $x : P^m \to \mathbb{R}^{m+p}$ ($p \geq 0$) is by definition the inclusion map of a m-dimensional connected, complete and totally geodesic submanifold of $\mathbb{R}^{m+p}$. In other words, those P^ms are subplanes of dimension m in $\mathbb{R}^{m+p}$ that are not necessarily passing through the origin. Let p_0 be the orthogonal projection of the origin 0 onto P^m and ξ be the position vector of p_0 which is constant and is thus parallel along P^m. Clearly P^m is a ξ-submanifold because $H \equiv 0$ and the tangential part $x^\top$ of x is precisely $x - \xi$.

Example 2.3 (The standard spheres centered at the origin).

For a given point $x_0 \in \mathbb{R}^{m+1}$ and a positive number r. Define

$$S^m(r, x_0) = \{x \in \mathbb{R}^{m+1};\ |x - x_0| = r\},$$

the standard m-sphere in $\mathbb{R}^{m+1}$ with radius r and center x_0. In particular, we denote $S^m(r) := S^m(r, 0)$. It is easily found that $S^m(r, x_0)$ is a ξ-submanifold if and only if $x_0 = 0$.

In fact, since $x - x_0$ is a normal vector field of length r, the normal part $x^\perp$ of x is

$$x^\perp = \frac{1}{r^2}\,\langle x, x - x_0\rangle\,(x - x_0).$$

Note that $H = -\frac{m}{r^2}(x - x_0)$ is parallel. It follows that $H + x^\perp$ is parallel if and only if $x^\perp$ is. This is clearly equivalent to that $\langle x, dx\rangle \equiv 0$ which is true if and only if $x_0 = 0$.

Example 2.4 (Submanifolds in a sphere with parallel mean curvature vector).

Let $x : M^m \to S^{m+p}(a) \subset \mathbb{R}^{m+p+1}$ be a submanifold in the standard sphere $S^{m+p}(a)$ of radius a, which is of parallel mean curvature vector H. Then, as a submanifold of $\mathbb{R}^{m+p+1}$, x is a ξ-submanifold.

In fact, as the submanifold of $\mathbb{R}^{m+p+1}$, the mean curvature vector of x is $\bar H = \Delta x = H - \frac{m}{a^2}x$. Thus, $\xi := \bar H + x^\perp = H + (1 - \frac{m}{a^2})x$ which is clearly parallel. In particular, $x(M^m) \subset \mathbb{R}^{m+p+1}$ is a self-shrinker if and only if $x(M^m) \subset S^{m+p}(a)$ is a minimal submanifold.

Example 2.5 (The product of ξ-submanifolds).

Let $x_a : M^{m_a} \to \mathbb{R}^{m_a+p_a}$, $a = 1, 2$, be two immersed submanifolds. Denote $m = m_1 + m_2$, $p = p_1 + p_2$ and $M^m = M^{m_1} \times M^{m_2}$. Then, it is not hard to show that $x := x_1 \times x_2 : M^m \to \mathbb{R}^{m+p}$ is a ξ-submanifold if and only if both x_1 and x_2 are ξ-submanifolds.

In particular, for any given positive numbers $r_1, \ldots, r_k$ ($k \geq 0$), positive integers $m_1, \ldots, m_k, n_1, \ldots, n_l$ ($l \geq 0$, $k + l > 0$) and $n \geq n_1 + \cdots + n_l$, the embedding

$$x : S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k) \times P^{n_1} \times \cdots \times P^{n_l} \to \mathbb{R}^{m_1+\cdots+m_k+k+n} \tag{2.6}$$

are all ξ-submanifolds.

Remark 2.1 Apart from these typical examples of ξ-submanifolds given above, there should certainly be other nonstandard examples. In particular, we have the so-called *λ-torus* constructed by Q.-M. Cheng and G. X. Wei in [11], which is among a general class of rotational λ-hypersurfaces. Precisely, we have

Theorem 2.1 ([11]). *For any $m \geq 2$ and $\lambda > 0$, there exists an embedded rotational λ-hypersurface $x : M^m \to \mathbb{R}^{m+1}$, which has the topology of the torus $\mathbb{S}^1 \times \mathbb{S}^{m-1}$.*

It would be interesting if one can construct similar ξ-submanifolds with certain symmetry.

3 As submanifolds of the Gaussian space

As mentioned in the introduction, m-dimensional self-shrinkers of the mean curvature flow in the Euclidean space $\mathbb{R}^{m+p} \equiv (\mathbb{R}^{m+p}, \langle\cdot,\cdot\rangle)$ is equivalent to being minimal submanifolds when viewed as submanifolds in the Gaussian metric space $(\mathbb{R}^{m+p}, \bar{g})$ where $\bar{g} := e^{-\frac{|x|^2}{m}} \langle\cdot,\cdot\rangle$. In this section, we generalize this to ξ-submanifolds to obtain our first characterization. In fact, we will prove a theorem which says that ξ-submanifolds are essentially equivalent to being submanifolds of parallel mean curvature in $(\mathbb{R}^{m+p}, \bar{g})$.

For an immersion $x : M^m \to \mathbb{R}^{m+p}$, we use $(\overline{\cdots})$ to denote geometric quantities when x is taken as an immersion into $(\mathbb{R}^{m+p}, \bar{g})$ that correspond those quantities $(\cdots)$ when x is taken as an immersion into $(\mathbb{R}^{m+p}, \langle\cdot,\cdot\rangle)$. So, for example, we have the induced metric $\bar{g}$, the second fundamental form $\bar{h}$ and the mean curvature $\bar{H}$, etc. To make things more clear, we would like to introduce a "*modified mean curvature*" for the immersion x, which is defined as $\tilde{H} = e^{-\frac{|x|^2}{2m}} \bar{H}$. Then, we have

Theorem 3.1 (The first characterization). *An immersion $x : M^m \to \mathbb{R}^{m+p}$ is a ξ-submanifold if and only if its modified mean curvature $\tilde{H}$ is parallel.*

Proof Denote by $\bar{D}$ the Levi-Civita connections of $(\mathbb{R}^{m+p}, \bar{g})$. For any given frame field $\{e_A;\ A = 1, 2\cdots, m+p\}$, the corresponding connection coefficients of the standard connection D and $\bar{D}$ are, respectively, denoted by Γ^C_{AB} and $\bar{\Gamma}^C_{AB}$ with $A, B, C, \ldots = 1, 2, \ldots m+p$. Then by the Koszul formula, we find

$$\bar{\Gamma}^C_{AB} = \Gamma^C_{AB} + \frac{1}{m}\left(g(x, e_D) g_{AB} g^{CD} - g(x, e_A)\delta^C_B - g(x, e_B)\delta^C_A\right), \tag{3.1}$$

or equivalently,

$$\bar{D}_{e_B} e_A = D_{e_B} e_A + \frac{1}{m}(g_{AB} x - g(x, e_A) e_B - g(x, e_B) e_A). \tag{3.2}$$

Now given an immersion $x : M^m \to \mathbb{R}^{m+p}$, the induced metric on M^m by x of the ambient metric $\bar{g}$ will still be denoted by $\bar{g}$. Choose a frame field $\{e_i, e_\alpha\}$ along x such that e_i, $i = 1, 2, \ldots, m$, are tangent to M^m and e_α, $\alpha = m+1, \ldots, m+p$ are normal to $x_*(TM^m)$ satisfying $\langle e_\alpha, e_\beta\rangle \equiv g(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. Then by the Gauss formula and (3.1) or (3.2), we find the relation between the second fundamental forms $\bar{h}$ and h is as follows:

$$\bar{h}_{ij} \equiv \bar{h}(e_i, e_j) = \left(\bar{D}_{e_j} e_i\right)^\perp = h_{ij} + \frac{1}{m} x^\perp g_{ij} \tag{3.3}$$

where $h_{ij} = h(e_i, e_j) = \left(D_{e_j} e_i\right)^\perp$. It follows that the mean curvature vectors satisfy

$$\bar{H} \equiv \bar{g}^{ij} \bar{h}_{ij} = e^{\frac{|x|^2}{m}} (H + x^\perp). \tag{3.4}$$

Now we compute the covariant derivative of the modified mean curvature $\tilde{H} \equiv e^{-\frac{|x|^2}{2m}} \bar{H}$ with respect to the normal connection $\bar{D}^\perp$. First we note that, since $\bar{g}$ is conformal to $\langle\cdot,\cdot\rangle$

on $\mathbb{R}^{m+p}$, $\{e_\alpha\}$ which satisfies $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$ remains a normal frame field of x considered as the immersion into $(\mathbb{R}^{m+p}, \bar{g})$, not orthonormal anymore. Thus, we can write

$$\tilde{H} = \sum \tilde{H}^\alpha e_\alpha \quad \text{with} \quad \tilde{H}^\alpha = e^{\frac{|x|^2}{2m}}(H^\alpha + \langle x, e_\alpha\rangle)$$

where $H = \sum H^\alpha e_\alpha$. Note that by (3.1),

$$\bar{\Gamma}^\alpha_{\beta i} = \Gamma^\alpha_{\beta i} - \frac{1}{m}\langle x, e_i\rangle \delta^\alpha_\beta, \quad \forall \alpha, \beta, i.$$

It follows that, for each $\alpha = m+1, \ldots, m+p$,

$$\begin{aligned}
\left(\bar{D}^\perp_{e_i}\tilde{H}\right)^\alpha &= e_i(\tilde{H}^\alpha) + \tilde{H}^\beta \bar{\Gamma}^\alpha_{\beta i} \\
&= e_i\left(e^{\frac{|x|^2}{2m}}\right)(H^\alpha + \langle x, e_\alpha\rangle) + e^{\frac{|x|^2}{2m}}(e_i(H^\alpha) + e_i\langle x, e_\alpha\rangle) \\
&\quad + e^{\frac{|x|^2}{2m}}(H^\beta + \langle x, e_\beta\rangle)\left(\Gamma^\alpha_{\beta i} - \frac{1}{m}\langle x, e_i\rangle \delta^\alpha_\beta\right) \\
&= e^{\frac{|x|^2}{2m}}(e_i(H^\alpha) + e_i\langle x, e_\alpha\rangle + H^\beta \Gamma^\alpha_{\beta i} + \langle x, e_\beta\rangle \Gamma^\alpha_{\beta i}) \\
&= e^{\frac{|x|^2}{2m}}\left(D^\perp_{e_i}(H + x^\perp)\right)^\alpha,
\end{aligned}$$

where $\bar{D}^\perp$, $D^\perp$ denote the induced normal connections accordingly. Thus, Theorem 3.1 is proved. □

The following conclusion is direct by (3.4):

Corollary 3.2 *An immersion $x : M^m \to \mathbb{R}^{m+p}$ is a self-shrinker if and only if it is minimal when viewed as a submanifold of the Gaussian space $(\mathbb{R}^{m+p}, \bar{g})$.*

4 Variational characterizations

In this section, we first define two functionals and derive the corresponding first and second variation formulas, aiming to establish variational characterizations of the ξ-submanifolds.

For a given manifold $M \equiv M^m$ of dimension m, define

$$\mathcal{M} := \{\text{all the immersions } x : M^m \to \mathbb{R}^{m+p}\}$$

and let $\xi : M^m \to \mathbb{R}^{m+p}$ be a vector-valued function on the manifold M^m. Then, we can naturally introduce, as follows, two kinds of interesting functionals V_ξ and $\bar{V}_\xi$ on $\mathcal{M}$ which are parametrized by ξ:

$$V_\xi(x) := \int_M e^{-f_x} \mathrm{d}V_x, \quad \bar{V}_\xi(x) = \int_M e^{-\bar{f}_x} \mathrm{d}V_x, \quad x \in \mathcal{M}, \tag{4.1}$$

where for any $p \in M^m$, $f_x(p) := \frac{1}{2}|x(p) - \xi(p)|^2$, $\bar{f}_x(p) = f_x(p) - \frac{1}{2}|\xi(p)|^2$ and $\mathrm{d}V_x$ is the volume element of the induced metric g_x of x.

Remark 4.1 (1) These two functionals V_ξ and $\bar{V}_\xi$ are both of weighted volumes in a sense since, for example, the weighted volume element $e^{-\frac{1}{2}|x-\xi|^2}\mathrm{d}V_x$ corresponding to the first one can be viewed as induced from an unnormalized "general Gaussian measure" on the ambient Euclidean space $\mathbb{R}^{m+p}$ with "*mean*" ξ. Note that when ξ is constant as in the case of m-planes, $\left(\frac{1}{\sqrt{2\pi}}\right)^{m+p} e^{-f_x} \mathrm{d}V_{\mathbb{R}^{m+p}}$ is nothing but the usual generalized Gaussian measure with the mean ξ (and the *variance* $\sigma^2 \equiv 1$)[1]; meanwhile, the functional $\bar{V}_\xi$ is clearly a new weighted volume obtained from V_ξ by just adding a new weight $e^{\frac{1}{2}|\xi|^2}$. Also, the weight function e^{-f_x} or $e^{-\bar{f}_x}$ naturally has a close relation with the definition of the *Hermitian Polynomials* (see, for example, [14] and [15]). These polynomials will also be used later in our stability discussion in Sect. 5.

(2) All of the typical ξ-submanifolds (that is, m-planes P^m, standard m-spheres $S^m(r)$) and their products (2.6) have finite values for both the functionals V_ξ and $\bar{V}_\xi$, where ξ is chosen to be $H + x^\perp$.

Now let $x \in \mathcal{M}$ be fixed with the induced Riemannian metric $g := x^*\langle\cdot,\cdot\rangle$ and suppose that $F : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ is a variation of x with $\eta := F_*(\frac{\partial}{\partial t})|_{t=0}$ being the corresponding variation vector field. For $p \in M$, $t \in (-\varepsilon, \varepsilon)$, denote

$$x_t(p) = F(p,t), \quad \frac{\partial F}{\partial t} = F_*\left(\frac{\partial}{\partial t}\right), \quad \frac{\partial F}{\partial u^i} = F_*(\frac{\partial}{\partial u^i}) \equiv (x_t)_*\left(\frac{\partial}{\partial u^i}\right)$$

where (u^i) is a local coordinates on M. We always assume that, for each $t \in (-\varepsilon, \varepsilon)$, $x_t : M^m \to \mathbb{R}^{m+p}$ is an immersion, that is, $x_t \in \mathcal{M}$, $t \in (-\varepsilon, \varepsilon)$.

Definition 4.1 (Compact variation). A variation $F : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ is called compactly supported, or simply *compact*, if there exists a relatively compact open domain B such that, for each $t \in (-\varepsilon, \varepsilon)$, the support set $\overline{\{p \in M^m;\ \frac{\partial F}{\partial t}(p) \neq 0\}}$ of the vector field $\frac{\partial F}{\partial t}$ is contained in B.

Denote $f_t = f_{x_t}$, $\bar{f}_t = \bar{f}_{x_t}$ and

$$\Gamma_0(T^\perp(M)) = \{\text{all smooth normal vector fields } \eta \text{ of } x \text{ with compact support}\}.$$

Theorem 4.1 (The first variation formula). *Let F be a compact variation of x. Then,*

$$V_\xi'(t) = -\int_M \left\langle (H_t + x_t^\perp - \xi) + \nabla^t \left(\langle x_t, \xi\rangle - \frac{1}{2}|\xi|^2\right), \frac{\partial F}{\partial t}\right\rangle e^{-f_t} \mathrm{d}V_t, \tag{4.2}$$

$$\bar{V}_\xi'(t) = -\int_M \left\langle (H_t + x_t^\perp - \xi) + \nabla^t \langle x_t, \xi\rangle, \frac{\partial F}{\partial t}\right\rangle e^{-\bar{f}_t} \mathrm{d}V_t, \tag{4.3}$$

where H_t is the mean curvature vector of the immersion x_t, ∇^t is the gradient operator of the induced metric g_{x_t} and $\mathrm{d}V_t = \mathrm{d}V_{x_t}$.

In particular, if F is a normal variation of x, that is, $\eta \in \Gamma_0(T^\perp(M))$, then

$$V_\xi'(0) = -\int_M \left\langle (H + x^\perp - \xi), \eta\right\rangle e^{-f_0} \mathrm{d}V, \tag{4.4}$$

$$\bar{V}_\xi'(0) = -\int_M \left\langle (H + x^\perp - \xi), \eta\right\rangle e^{-\bar{f}_0} \mathrm{d}V. \tag{4.5}$$

[1] See the explanation in Wikipedia, the free encyclopedia under the title “Gaussian measure”.

Proof From now on, we shall always write f for f_x or f_t in the computation. It is well known that

$$\frac{\partial}{\partial t}\mathrm{d}V_t = \left(\operatorname{div}\left(\frac{\partial F}{\partial t}\right)^{\top} - \left\langle H_t, \frac{\partial F}{\partial t}\right\rangle\right)\mathrm{d}V_t$$
$$= \left(\left(g_t^{ij}\left\langle\frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial t}\right\rangle\right)_{,j} - \left\langle H_t, \frac{\partial F}{\partial t}\right\rangle\right)\mathrm{d}V_t.$$

Furthermore

$$\frac{\partial}{\partial t}e^{-f} = -e^{-f}\frac{\partial f}{\partial t} = -e^{-f}\left\langle x_t - \xi, \frac{\partial F}{\partial t}\right\rangle.$$

Thus by using the divergence theorem, we find

$$\begin{aligned}
V_\xi'(t) &= \int_M \frac{\partial}{\partial t}\left(e^{-f}\mathrm{d}V_t\right) = \int_M \left(\left(\frac{\partial}{\partial t}e^{-f}\right)\mathrm{d}V_t + e^{-f}\frac{\partial}{\partial t}\mathrm{d}V_t\right)\\
&= \int_M \left(-e^{-f}\left\langle x_t - \xi, \frac{\partial F}{\partial t}\right\rangle + e^{-f}\left(\left(g_t^{ij}\left\langle\frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial t}\right\rangle\right)_{,j} - \left\langle H_t, \frac{\partial F}{\partial t}\right\rangle\right)\right)\mathrm{d}V_t\\
&= -\int_M \left(\left\langle H_t + x_t^\perp - \xi, \frac{\partial F}{\partial t}\right\rangle + g_t^{ij}\frac{\partial}{\partial u^j}\left(\langle x_t, \xi\rangle - \frac{1}{2}|\xi|^2\right)\left\langle\frac{\partial F}{\partial u^i}, \frac{\partial F}{\partial t}\right\rangle\right)e^{-f}\mathrm{d}V_t\\
&= -\int_M \left(\left\langle (H_t + x_t^\perp - \xi) + \nabla^t\left(\langle x_t, \xi\rangle - \frac{1}{2}|\xi|^2\right), \frac{\partial F}{\partial t}\right\rangle\right)e^{-f}\mathrm{d}V_t,
\end{aligned}$$

which gives (4.2). The other formula (4.3) is derived in the same way. □

Corollary 4.2 (Variational characterizations). *An immersion $x \in \mathcal{M}$ is a ξ-submanifold if and only if there exists a parallel normal vector field $\xi \in \Gamma(T^\perp M)$ such that x is the critical point of both the functionals V_ξ, $\bar{V}_\xi$ for all the compact normal variations of x.*

To find the second variational formulas, we suppose that x is a ξ-submanifold, that is, $H + x^\perp = \xi$, where ξ is a parallel normal vector of x. In particular, $|\xi|^2$ is a constant. Note that in this case, the two functionals V_ξ and $\bar{V}_\xi$ are essentially the same. So in what follows we only need to consider V_ξ.

Suppose that F is a compact normal variation of x. Then from (4.2), we have

$$\begin{aligned}
V_\xi''(0) &= -\int_M \left\langle D_{\frac{\partial}{\partial t}}\left((H_t + x_t^\perp - \xi) + \nabla^t\langle x_t, \xi\rangle\right), \frac{\partial F}{\partial t}\right\rangle|_{t=0}e^{-f}\mathrm{d}V\\
&\quad -\int_M \left\langle \nabla^t\langle x_t, \xi\rangle, D_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t}\right\rangle|_{t=0}e^{-f}\mathrm{d}V\\
&= -\int_M \left\langle D_{\frac{\partial}{\partial t}}\left((H_t + x_t^\perp - \xi) + \nabla^t\langle x_t, \xi\rangle\right)|_{t=0}, \eta\right\rangle e^{-f}\mathrm{d}V\\
&\quad -\int_M \left\langle \nabla\langle x, \xi\rangle, D_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t}|_{t=0}\right\rangle e^{-f}\mathrm{d}V.
\end{aligned}\tag{4.6}$$

Since

$$H_t = (g_t)^{ij}h_t\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = (g_t)^{ij}\left(D_{\frac{\partial}{\partial u^j}}\frac{\partial F}{\partial u^i} - (x_t)_*\nabla^t_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^i}\right),$$

we have

$$D_{\frac{\partial}{\partial t}} H_t = \frac{\partial}{\partial t}(g_t)^{ij} h_t \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) + (g_t)^{ij} D_{\frac{\partial}{\partial t}} \left(D_{\frac{\partial}{\partial u^j}} \frac{\partial F}{\partial u^i} - (x_t)_* \nabla^t_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}\right). \tag{4.7}$$

On the other hand,

$$\begin{aligned}
\left(\frac{\partial}{\partial t}(g_t)^{ij}\right)|_{t=0} &= -\left(\left((g_t)^{ik}(g_t)^{jl}\left\langle D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial u^k}, \frac{\partial F}{\partial u^l}\right\rangle + \left\langle \frac{\partial F}{\partial u^k}, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial u^l}\right\rangle\right)\right)|_{t=0} \\
&= -g^{ik}g^{jl}\left(\frac{\partial}{\partial u^k}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial u^l}\right\rangle - \left\langle \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial u^k}} \frac{\partial F}{\partial u^l}\right\rangle\right)|_{t=0} \\
&\quad - g^{ik}g^{jl}\left(\frac{\partial}{\partial u^l}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial u^k}\right\rangle - \left\langle \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial u^l}} \frac{\partial F}{\partial u^k}\right\rangle\right)|_{t=0} \\
&= g^{ik}g^{jl}\left\langle h(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}), \eta\right\rangle + g^{ik}g^{jl}\left\langle h(\frac{\partial}{\partial u^l}, \frac{\partial}{\partial u^k}), \eta\right\rangle,
\end{aligned}$$

and by the flatness of $\mathbb{R}^{m+p}$,

$$\begin{aligned}
D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial u^j}} \frac{\partial F}{\partial u^i}|_{t=0} &= D_{\frac{\partial}{\partial u^j}} D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial u^i} + D_{\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u^j}\right]} \frac{\partial F}{\partial u^i}|_{t=0} \\
&= D_{\frac{\partial}{\partial u^j}}\left(D^{\perp}_{\frac{\partial}{\partial u^i}}\eta - x_*\left(A_\eta \frac{\partial}{\partial u^i}\right)\right) \\
&= D^{\perp}_{\frac{\partial}{\partial u^j}} D^{\perp}_{\frac{\partial}{\partial u^i}}\eta - h\left(\frac{\partial}{\partial u^j}, A_\eta\left(\frac{\partial}{\partial u^i}\right)\right) \\
&\quad - x_*\left(A_{D^{\perp}_{\frac{\partial}{\partial u^i}}\eta} \frac{\partial}{\partial u^j}\right) - x_*\left(\nabla_{\frac{\partial}{\partial u^j}}\left(A_\eta \frac{\partial}{\partial u^i}\right)\right)
\end{aligned}$$

where A_η is the Weingarten operator of x with respect to the variation vector η. Moreover,

$$\begin{aligned}
D_{\frac{\partial}{\partial t}}\left((x_t)_* \nabla^t_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}\right)|_{t=0} &= D_{\frac{\partial}{\partial t}}\left((\Gamma_t)^k_{ij}\,(x_t)_* \frac{\partial}{\partial u^k}\right)|_{t=0} \\
&= \frac{\partial}{\partial t}((\Gamma_t)^k_{ij})|_{t=0} x_* \frac{\partial}{\partial u^k} + \Gamma^k_{ij} D_{\frac{\partial}{\partial t}}\left(\frac{\partial F}{\partial u^k}\right)|_{t=0} \\
&= \frac{\partial}{\partial t}((\Gamma_t)^k_{ij})|_{t=0} x_* \frac{\partial}{\partial u^k} + D_{\nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}}\eta.
\end{aligned}$$

It then follows that

$$\left\langle \frac{\partial}{\partial t}(g_t)^{ij} h_t\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)|_{t=0}, \eta\right\rangle = 2g^{ik}g^{jl}\left\langle h\left(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\right), \eta\right\rangle\left\langle h\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \eta\right\rangle, \tag{4.8}$$

$$g^{ij}\left\langle D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial u^j}} \frac{\partial F}{\partial u^i}|_{t=0}, \eta\right\rangle = g^{ij}\left(\left\langle D^{\perp}_{\frac{\partial}{\partial u^j}} D^{\perp}_{\frac{\partial}{\partial u^i}}\eta - h\left(\frac{\partial}{\partial u^j}, A_\eta\left(\frac{\partial}{\partial u^i}\right)\right), \eta\right\rangle\right), \tag{4.9}$$

$$g^{ij}\left(\left\langle D_{\frac{\partial}{\partial t}}\left((x_t)_* \nabla^t_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}\right)|_{t=0}, \eta\right\rangle\right) = g^{ij}\left\langle D^{\perp}_{\nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}}\eta, \eta\right\rangle. \tag{4.10}$$

Hence,

$$\begin{aligned}\left\langle D_{\frac{\partial}{\partial t}} H_t|_{t=0}, \eta\right\rangle &= \left\langle g^{ij}\left(D^{\perp}_{\frac{\partial}{\partial u^i}} D^{\perp}_{\frac{\partial}{\partial u^j}} \eta - D^{\perp}_{\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}} \eta\right), \eta\right\rangle \\ &\quad + g^{ik} g^{jl} \left\langle h\left(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\right), \eta\right\rangle \left\langle h\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \eta\right\rangle \\ &= \left\langle \triangle^{\perp}_M \eta, \eta\right\rangle + g^{ik} g^{jl} \left\langle h\left(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\right), \eta\right\rangle \left\langle h\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \eta\right\rangle \\ &= \left\langle \triangle^{\perp}_M \eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta\right\rangle h_{kl}, \eta\right\rangle,\end{aligned}$$

where $h_{ij} = h(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j})$. Furthermore,

$$\begin{aligned}\left\langle D_{\frac{\partial}{\partial t}} (x_t^{\perp} - \xi)|_{t=0}, \eta\right\rangle &= \left\langle D_{\frac{\partial}{\partial t}} x_t|_{t=0} - D_{\frac{\partial}{\partial t}} (x_t)^{\top}|_{t=0}, \eta\right\rangle \\ &= \langle \eta, \eta\rangle - \left\langle D_{\frac{\partial}{\partial t}}\left((g_t)^{ij}\left\langle x_t, \frac{\partial F}{\partial u^i}\right\rangle \frac{\partial F}{\partial u^j}\right)|_{t=0}, \eta\right\rangle \\ &= \langle \eta, \eta\rangle - \left\langle D_{x^{\top}} \eta, \eta\right\rangle = \langle \eta, \eta\rangle - \left\langle D^{\perp}_{x^{\top}} \eta, \eta\right\rangle.\end{aligned}$$

Therefore,

$$\left\langle D_{\frac{\partial}{\partial t}} (H_t + x_t^{\perp} - \xi), \frac{\partial F}{\partial t}\right\rangle|_{t=0} = \left\langle \triangle^{\perp}_M \eta - D^{\perp}_{x^{\top}} \eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta\right\rangle h_{kl} + \eta, \eta\right\rangle$$

Meanwhile,

$$\begin{aligned}\left\langle D_{\frac{\partial}{\partial t}} (\nabla^t \langle x_t, \xi\rangle)|_{t=0}, \eta\right\rangle &= \left\langle (g_t)^{ij} \frac{\partial}{\partial u^i} \langle x_t, \xi\rangle\, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial u^i}|_{t=0}, \eta\right\rangle \\ &= \left\langle g^{ij} \frac{\partial}{\partial u^i} \langle x, \xi\rangle\, D_{\frac{\partial}{\partial u^i}} \eta, \eta\right\rangle = \left\langle D^{\perp}_{\nabla\langle x, \xi\rangle} \eta, \eta\right\rangle \\ &= -\left\langle D^{\perp}_{A_{\xi}(x^{\top})} \eta, \eta\right\rangle\end{aligned}$$

since ξ is parallel along x.

By summing up, we have proved the following second variation formulas for ξ-submanifolds:

Theorem 4.3 *Let $x : M^m \to \mathbb{R}^{m+p}$ be a ξ-submanifold. Then for any compact normal variation $F : M^m \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$, we have*

$$\begin{aligned}V''_{\xi}(0) = -\int_M &\left(\left\langle \triangle^{\perp}_M(\eta) - D^{\perp}_{x^{\top} + A_{\xi}(x^{\top})} \eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta\right\rangle h_{kl} + \eta, \eta\right\rangle\right. \\ &\left. + \left\langle \nabla\langle x, \xi\rangle, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}|_{t=0}\right\rangle\right) e^{-f} \mathrm{d}V, \qquad (4.11)\end{aligned}$$

$$\begin{aligned}\bar{V}''_{\xi}(0) = -\int_M &\left(\left\langle \triangle^{\perp}_M(\eta) - D^{\perp}_{x^{\top} + A_{\xi}(x^{\top})} \eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta\right\rangle h_{kl} + \eta, \eta\right\rangle\right. \\ &\left. + \left\langle \nabla\langle x, \xi\rangle, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}|_{t=0}\right\rangle\right) e^{-\bar{f}} \mathrm{d}V. \qquad (4.12)\end{aligned}$$

In order to simplify the second variation formulas, we introduce the following definition:

Definition 4.2 (SN-variation). A variation $F : M^m \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ of an immersion $x : M^m \to \mathbb{R}^{m+p}$ is called specially normal (or simply SN) if it is normal and $\frac{\partial^2 F}{\partial t^2}|_{t=0} = 0$.

Remark 4.2 The introduction of the SN*-variation* is based on the observation that the Hessian $\mathrm{Hess}(f)$ at a given point p of a smooth function f on a Riemannian manifold $\tilde{N}$, $p \in \tilde{N}$, is determined only by those local values of f along the *simplest* curves $\tilde{\gamma}$ passing through the point p. For example, if we choose $\tilde{\gamma}$ to be geodesic ones, then the second derivatives can be computed as

$$\left.\frac{d^2}{dt^2}\right|_{t=0} (f(\tilde{\gamma})) = \mathrm{Hess}(f)(\tilde{\gamma}'(0), \tilde{\gamma}'(0)),$$

implying that f is (semi-)convex at p if and only if $\left.\frac{d^2}{dt^2}\right|_{t=0} (f(\tilde{\gamma})) \geq 0$ for all of these geodesics $\tilde{\gamma}$.

Clearly, for any $\eta \in \Gamma(T^{\perp}M)$, SN-variations with variation vector field η always exist in our present case. For example, we can choose

$$F(p, t) = x(p) + \psi(t)\eta(p), \quad \forall (p, t) \in M^m \times (-\varepsilon, \varepsilon)$$

where ψ is any smooth function satisfying $\psi(0) = \psi''(0) = 0$, $\psi'(0) = 1$.

Corollary 4.4 (The simplified second variation formulas). *Let $x : M^m \to \mathbb{R}^{m+p}$ be a ξ-submanifold. Then for any compact SN-variation $F : M^m \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ it holds that*

$$V_\xi''(0) = -\int_M \left(\left\langle (\Delta_M^{\perp} - D^{\perp}_{x^\top + A_\xi(x^\top)} + 1)\eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta \right\rangle h_{kl}, \eta \right\rangle\right) e^{-f} \mathrm{d}V, \tag{4.13}$$

$$\bar{V}_\xi''(0) = -\int_M \left(\left\langle (\Delta_M^{\perp} - D^{\perp}_{x^\top + A_\xi(x^\top)} + 1)\eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta \right\rangle h_{kl}, \eta \right\rangle\right) e^{-\bar{f}} \mathrm{d}V. \tag{4.14}$$

Remark 4.3 From the above discussion, one may naturally think of the variational characterization of the usual submanifolds with parallel mean curvature vector in the Euclidean space. In fact, our computations and argument in this section essentially apply to this situation. For example, a suitable functional $\tilde{V}_\xi$ may be defined by

$$\tilde{V}_\xi = \int_M e^{\langle x, \xi \rangle} \mathrm{d}V_x, \quad \forall x \in \mathcal{M}$$

and the first variation formula of $\tilde{V}_\xi$ is given in the following

Proposition 4.5 *Let $x \in \mathcal{M}$ be fixed and $\xi : M^m \to \mathbb{R}^{m+p}$ be a smooth map. Suppose that F is a compact variation of x. Then*

$$\tilde{V}_\xi'(t) = -\int_M \left\langle (H_t - \xi) + \nabla^t \langle x_t, \xi \rangle, \frac{\partial F}{\partial t} \right\rangle e^{\langle x, \xi \rangle} \mathrm{d}V_t. \tag{4.15}$$

In particular, if F is a normal variation of x, then

$$\tilde{V}_\xi'(0) = -\int_M \langle H - \xi, \eta \rangle \, e^{\langle x, \xi \rangle} \mathrm{d}V. \tag{4.16}$$

Corollary 4.6 *An immersion $x \in \mathcal{M}$ has a parallel mean curvature vector if and only if there exists a parallel normal vector field $\xi \in \Gamma(T^{\perp}M)$ such that x is the critical point of the functional $\tilde{V}_\xi$ for all the compact normal variations of x.*

Accordingly, the second variation formula for a submanifold $x : M^m \to \mathbb{R}^{m+p}$ with parallel mean curvature vector $H \equiv \xi$ may be described as

Theorem 4.7 *Let $x : M^m \to \mathbb{R}^{m+p}$ be an immersed submanifold with parallel mean curvature H. Then for any compact normal variation $F : M^m \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ we have*

$$\tilde{V}''_H(0) = -\int_M \left(\left\langle \triangle^\perp_M(\eta) + D^\perp_{\nabla\langle x, H\rangle}\eta, \eta \right\rangle + |A_\eta|^2 + \left\langle \nabla\langle x, H\rangle, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}|_{t=0} \right\rangle \right) e^{\langle x, H\rangle} \mathrm{d}V. \tag{4.17}$$

5 The instabilities of the typical examples

The most natural stability definition to the functional V_ξ is as follows:

Definition 5.1 A ξ-submanifold $x : M^m \to \mathbb{R}^{m+p}$ is called stable if $V_\xi(x) < +\infty$ and for every compact SN-variation $F : M^m \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ of x it holds that $V''_\xi(0) \geq 0$ or, equivalently, $\bar{V}''_\xi(0) \geq 0$.

In this section, we shall show that, as ξ-submanifolds, all the typical examples given in Sect. 2 are not stable in the sense of Definition 5.1.

Write the second fundamental form h of x locally as $h = h_{ij}\omega^i\omega^j = h^\alpha_{ij}e_\alpha$ with respect to an orthonormal tangent frame field $\{e_i;\ 1 \leq i \leq m\}$ with dual $\{\omega^i\}$ and an orthonormal normal frame field $\{e_\alpha;\ m+1 \leq \alpha \leq m+p\}$, and denote

$$\mathcal{L} = \triangle^\perp_{M^m} - D^\perp_{x^\top + A_\xi(x^\top)}, \quad L = \mathcal{L} + \langle h_{ij}, \cdot\rangle h_{ij} + 1, \quad \tilde{\mathcal{L}} = \triangle_{M^m} - \nabla_{x^\top + A_\xi(x^\top)}, \tag{5.1}$$

where $\triangle^\perp_{M^m}$, $\triangle_{M^m}$ are Laplacians on $T^\perp M^m$, TM^m, respectively, and sometimes we shall omit the subscript "$_{M^m}$" if no confusion is made. It follows that

$$Q(\eta, \eta) :\equiv -\int_M \langle L(\eta), \eta\rangle\, e^{-f} \mathrm{d}V, \tag{5.2}$$

and that, for any parallel normal vector field N,

$$L(N) = N + \langle h_{ij}, N\rangle h_{ij}. \tag{5.3}$$

Lemma 5.1

$$L(\phi\eta) = (\tilde{\mathcal{L}}\phi)\eta + \phi L(\eta) + 2D^\perp_{\nabla\phi}\eta, \quad \phi \in C^\infty(M^m),\ \eta \in \Gamma(T^\perp M^m). \tag{5.4}$$

Proof We compute directly

$$\begin{aligned} L(\phi\eta) &= \triangle^\perp(\phi\eta) - D^\perp_{x^\top + A_\xi(x^\top)}(\phi\eta) + \langle h_{ij}, \phi\eta\rangle h_{ij} + \phi\eta \\ &= (\triangle\phi)\eta + 2D^\perp_{\nabla\phi}\eta + \phi\triangle^\perp\eta - (\nabla_{x^\top + A_\xi(x^\top)}\phi)\eta \\ &\quad - \phi(D^\perp_{x^\top + A_\xi(x^\top)}\eta) + \phi\langle h_{ij}, \eta\rangle h_{ij} + \phi\eta \\ &= (\triangle - \nabla_{x^\top + A_\xi(x^\top)})\phi\eta + \phi(\triangle^\perp - D^\perp_{x^\top + A_\xi(x^\top)} + \langle h_{ij}, \cdot\rangle h_{ij} + 1)\eta + 2D^\perp_{\nabla\phi}\eta \\ &= (\tilde{\mathcal{L}}\phi)\eta + \phi(L\eta) + 2D^\perp_{\nabla\phi}\eta. \end{aligned}$$

□

Lemma 5.2 *Let $x : M^m \to \mathbb{R}^{m+p}$ be a ξ-submanifold. Then for any $\eta_1, \eta_2 \in \Gamma(T^\perp M^m)$ one of which is compactly supported, it holds that*

$$\int_M \langle \eta_1, \mathcal{L}\eta_2 \rangle e^{-f} \mathrm{d}V = -\int_M \left\langle D^\perp \eta_1, D^\perp \eta_2 \right\rangle e^{-f} \mathrm{d}V. \tag{5.5}$$

Similarly, for any $\phi_1, \phi_2 \in C^\infty(M^m)$ one of which is compactly supported, it holds that

$$\int_M \phi_1 \tilde{\mathcal{L}} \phi_2 e^{-f} \mathrm{d}V = -\int_M \langle \nabla\phi_1, \nabla\phi_2 \rangle e^{-f} \mathrm{d}V. \tag{5.6}$$

Proof To prove the two formulas, it suffices to use the Divergence Theorem and the following equalities:

$$\langle \eta_1, \mathcal{L}\eta_2 \rangle e^{-f} = \mathrm{div}\left(\left\langle \eta_1, D^\perp_{e_i} \eta_2 \right\rangle e^{-f} e_i\right) - \left\langle D^\perp \eta_1, D^\perp \eta_2 \right\rangle e^{-f}, \tag{5.7}$$

$$\phi_1 \tilde{\mathcal{L}} \phi_2 e^{-f} = \mathrm{div}\left(\phi_1 \nabla_{e_i} \phi_2 e^{-f} e_i\right) - \langle \nabla\phi_1, \nabla\phi_2 \rangle e^{-f}. \tag{5.8}$$

□

Lemma 5.3 *For any $\phi \in C^\infty_0(M^m)$ and $\eta \in \Gamma(T^\perp M^m)$, it holds that*

$$\int_M \langle \phi\eta, L(\phi\eta) \rangle e^{-f} \mathrm{d}V = \int_M \phi^2 \langle \eta, L(\eta) \rangle e^{-f} \mathrm{d}V - \int_M |\nabla\phi|^2 |\eta|^2 e^{-f} \mathrm{d}V. \tag{5.9}$$

Proof By (5.4) and (5.6), we find

$$\begin{aligned}
\int_M \langle \phi\eta, L(\phi\eta) \rangle e^{-f} \mathrm{d}V &= \int_M \left\langle \phi\eta, (\tilde{\mathcal{L}}\phi)\eta + \phi L\eta + 2D^\perp_{\nabla\phi}\eta \right\rangle e^{-f} \mathrm{d}V \\
&= \int_M (\phi|\eta|^2)\tilde{\mathcal{L}}\phi e^{-f} \mathrm{d}V + \int_M \phi^2 \langle \eta, L\eta \rangle e^{-f} \mathrm{d}V + \int_M \left\langle \eta, D^\perp_{\nabla\phi^2}\eta \right\rangle e^{-f} \mathrm{d}V \\
&= -\int_M ((|\nabla\phi|^2|\eta|^2) + \frac{1}{2}\langle \nabla\phi^2, \nabla|\eta|^2 \rangle) e^{-f} \mathrm{d}V + \int_M \phi^2 \langle \eta, L\eta \rangle e^{-f} \mathrm{d}V \\
&\quad + \frac{1}{2}\int_M \nabla_{\nabla\phi^2}|\eta|^2 e^{-f} \mathrm{d}V \\
&= \int_M \phi^2 \langle \eta, L\eta \rangle e^{-f} \mathrm{d}V - \int_M |\nabla\phi|^2 |\eta|^2 e^{-f} \mathrm{d}V.
\end{aligned}$$

□

Proposition 5.4 *As ξ-submanifolds, all m-planes in $\mathbb{R}^{m+p}$ are not stable.*

Proof For an m-plane $x : P^m \subset \mathbb{R}^{m+p}$, let o be the orthogonal projection on P^m of the origin O. Then $\xi = \vec{Oo}$. Denote by $B_R(o) \subset P$ the closed ball of radius $R > 0$ centered at the fixed point o:

$$B_R(o) = \{x \in P;\ |x^\top| \equiv |x - \xi| \le R\}.$$

Let N be a unit constant vector in $\mathbb{R}^m$ orthogonal to P^m and ϕ_R be a cut-off function on P^m satisfying

$$(\phi_R)|_{B_R(o)} \equiv 1, \quad (\phi_R)|_{P^m \setminus B_{R+2}(o)} \equiv 0, \quad |\nabla\phi| \le 1, \quad R > 0.$$

Define $\eta_R = \phi_R N$. Then η_R is compactly supported and can be chosen to be a variation vector field for some SN-variation. By (5.9) and (5.3),

$$\begin{aligned} Q(\eta_R, \eta_R) &= -\int_M \langle \phi_R N, L(\phi_R N)\rangle\, e^{-f} \mathrm{d}V \\ &= -\int_{P^m} \phi_R^2 \langle N, L(N)\rangle\, e^{-f} \mathrm{d}V + \int_{P^m} |\nabla \phi_R|^2 e^{-f} \mathrm{d}V \\ &= -\int_{P^m} \phi_R^2 \left\langle N, N + \left\langle h_{ij}, N\right\rangle h_{ij}\right\rangle e^{-f} \mathrm{d}V + \int_{P^m} |\nabla \phi_R|^2 e^{-f} \mathrm{d}V \\ &\leq -\int_{P^m} \phi_R^2 e^{-f} \mathrm{d}V + \int_{B_{R+2}(o)\setminus B_R(o)} e^{-f} \mathrm{d}V \;\to -\int_{P^m} e^{-f} \mathrm{d}V < 0 \end{aligned}$$

when $R \to +\infty$ since $\int_{P^m} e^{-f} \mathrm{d}V < +\infty$. Thus for large R, we have $Q(\eta_R, \eta_R) < 0$. □

Proposition 5.5 *As ξ-submanifolds, the standard m-spheres $S^m(r)$ are all non-stable.*

Proof For the standard sphere $S^m(r) \subset \mathbb{R}^{m+1} \subset \mathbb{R}^{m+p}$, we have $h = -\frac{1}{r^2} g\, x$, $x^\perp = x$ and $\xi = \left(-\frac{m}{r^2} + 1\right) x$. Choose the variation vector field $\eta = x$ so that $\mathcal{L}\eta = 0$. It follows that

$$\begin{aligned} Q(\eta, \eta) &\leq -\int_{S^m(r)} \langle \eta, L(\eta)\rangle\, e^{-f} \mathrm{d}V_{S^m(r)} = -\int_{S^m(r)} \left(\sum \left\langle h_{ij}, \eta\right\rangle^2 + |x|^2\right) e^{-f} \mathrm{d}V_{S^m(r)} \\ &= -(m + r^2) \int_{S^m(r)} e^{-f} \mathrm{d}V_{S^m(r)} < 0. \end{aligned}$$

□

From Proposition 5.4 and Proposition 5.5, we easily find

Corollary 5.6 *The product ξ-submanifolds $S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k) \times P^{n_1} \times \cdots \times P^{n_l}$ are not stable.*

A more general conclusion than Proposition 5.5 is the following

Proposition 5.7 *Let $x : M^m \to \mathbb{R}^{m+p}$ be a compact ξ-submanifold. If x has a non-trivial parallel normal vector field, then x is not stable. In particular, all compact λ-hypersurfaces and compact ξ-submanifold with $\xi \neq 0$ are not stable.*

Proof Let $\eta \neq 0$ be a parallel normal vector field. Then η can be chosen to be the variation vector field of some SN-variation F of x. Since $\Delta^\perp \eta = D^\perp_{x^\top + A_\xi(x^\top)} \eta = 0$, it then follows from (4.13) that

$$Q(\eta, \eta) = -\int_M \left(\sum \left\langle h_{ij}, \eta\right\rangle^2 + |\eta|^2\right) e^{-f} \mathrm{d}V < 0.$$

□

Corollary 5.8 *Any compact and simply connected ξ-submanifold with flat normal bundle is not stable.*

To end this section, we would like to remark that, by using suitably chosen cut-off functions, say, the cut-off functions ϕ_R introduced in Sect. 7 for large enough numbers $R > 0$, we can extend the above instability conclusions to more general complete case. For example, the following conclusion is also true:

Theorem 5.9 *Any complete and properly immersed ξ-submanifold with a non-trivial parallel normal vector field η is not stable.*

Proof Let $x : M^m \to \mathbb{R}^{m+p}$ be a complete and properly immersed ξ-submanifold and N be a non-trivial parallel normal vector field of x. Without loss of generality, we assume that $\int_M e^{-f}\mathrm{d}V < \infty$ and $|N|^2 = 1$. For a Large $R > 0$, define $\eta_R := \phi_R N$. Choose an SN-variation of x with η_R being its variation vector field. Then, by (5.2), (5.3) and Lemma 5.3, we have

$$\begin{aligned}
Q(\eta_R, \eta_R) &= -\int_M \langle \phi_R N, L(\phi_R N)\rangle\, e^{-f}\mathrm{d}V \\
&= -\int_M \phi_R^2 \langle N, L(N)\rangle\, e^{-f}\mathrm{d}V + \int_M |\nabla\phi_R|^2 e^{-f}\mathrm{d}V \\
&= -\int_M \phi_R^2 \left\langle N, N + \langle h_{ij}, N\rangle h_{ij}\right\rangle e^{-f}\mathrm{d}V + \int_{B_{2R}(o)\setminus B_R(o)} |\nabla\phi_R|^2 e^{-f}\mathrm{d}V \\
&\leq -\int_M \phi_R^2 e^{-f}\mathrm{d}V + \int_{M\setminus B_R(o)} e^{-f}\mathrm{d}V \ \to -\int_M e^{-f}\mathrm{d}V < 0 \quad (R \to +\infty),
\end{aligned}$$

since $\lim_{R\to+\infty}\int_{M\setminus B_R(o)} e^{-f}\mathrm{d}V = 0$. So that there is an R large enough such that we have $Q(\eta_R, \eta_R) < 0$. □

Corollary 5.10 *Let $x : M^m \to \mathbb{R}^{m+p}$ be a complete and properly immersed ξ-submanifold. Then, x is not stable if any of the following three holds:*

(1) *the codimension $p = 1$;*
(2) *$p \geq 2$ and $\xi \neq 0$;*
(3) *M^m is simply connected and the normal bundle of x is flat.*

Remark 5.1 Up to now, it is still unclear for the existence of stable ξ-submanifolds in the sense of Definition 5.1 . Other stability problems have been previously discussed for both self-shrinker hypersurfaces and λ-hypersurfaces. For example, Colding and Minicozzi introduced a notion of $\mathcal{F}$-functional and proved that self-shrinkers are exactly critical points of the $\mathcal{F}$-functional ([13]). They also proved that the standard sphere and hyperplane are the only two complete $\mathcal{F}$-stable hypersurface self-shrinkers of polynomial volume growth. Furthermore, in [9], Cheng and Wei extended the above $\mathcal{F}$-functional to λ-hypersurfaces and studied the corresponding $\mathcal{F}$-stability. In particular, they proved that the standard sphere $\mathbb{S}^m(r)$ of radius r is $\mathcal{F}$-unstable as a λ-hypersurface if and only if $\sqrt{m} < r \leq \sqrt{m+1}$.

6 The W_ξ-stability of ξ-submanifolds

By the discussion of last section, it turns out that the concept of stability given in Definition 5.1 is over-strong in a sense. So it is natural and interesting to find a suitably weaker stability definition for ξ-submanifolds. Motivated by the "weighted-volume-preserving" variations of hypersurfaces (see [27]), we can introduce the W_ξ-stability in the following way.

Note that, by [27], a compact variation F of a hypersurface $x : M^m \to \mathbb{R}^{m+1}$ is called "weighted-volume-preserving" if $\int_M \left\langle \frac{\partial}{\partial F}\Big|_{t=0}, n\right\rangle e^{-\frac{1}{2}|x|^2} = 0$ where n is the unit normal vector field. Since a normal vector field $N = \lambda n$ is parallel if and only if $\lambda = \mathrm{const}$, it follows that F is "weighted-volume-preserving" if and only if $\int_M \left\langle \frac{\partial}{\partial F}\Big|_{t=0}, N\right\rangle e^{-\frac{1}{2}|x|^2}\mathrm{d}V = 0$ for all parallel normal vector field N. This recommends us to make the following generalization:

Definition 6.1 Let $x : M^m \to \mathbb{R}^{m+p}$ be an immersion. A compact SN-variation $F : M^m \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+p}$ of x is called VP ("*weighted-volume-preserving*") if the corresponding variation vector $\eta \equiv \frac{\partial F}{\partial t}\big|_{t=0}$ satisfies

$$\int_M \langle \eta, N\rangle e^{-f} \mathrm{d}V = 0, \quad \forall N \in \Gamma(T^\perp M) \text{ satisfying } D^\perp N \equiv 0. \tag{6.1}$$

Remark 6.1 It is clear that, in the special case of codimension 1, VP-variations defined here are nothing but the "weighted-volume-preserving" ones that were considered in [27].

Definition 6.2 A ξ-submanifold $x : M^m \to \mathbb{R}^{m+p}$ is called W_ξ-stable if $V_\xi(x) < +\infty$ and for every VP-variation it holds that $V''_\xi(0) \geq 0$.

Then, we have

Theorem 6.1 *Any of the m-planes is W_ξ-stable.*

Proof For an m-plane $x : P^m \subset \mathbb{R}^{m+p}$, let η be an arbitrary normal vector field on P^m with compact support. Then, we have $A_\eta \equiv 0$, $x - \xi = x^\top$ and

$$L = \triangle^\perp_{P^m} - D^\perp_{x^\top} + 1.$$

Clearly, there are constant normal basis e_α, $\alpha = m+1, \ldots, m+p$. So η can be expressed by $\eta = \sum \eta^\alpha e_\alpha$ with $\eta^\alpha \in C^\infty_0(P^m)$. Consequently,

$$L(\eta) = \sum \tilde{L}(\eta^\alpha) e_\alpha, \quad \langle L\eta, \eta\rangle = \sum \eta^\alpha \tilde{L}\eta^\alpha,$$

where $\tilde{L} = \triangle_{P^m} - \nabla_{x^\top} + 1$. Now we make the following

Claim: *The eigenvalues of the operator $-\tilde{L}$ are $\lambda_n = n - 1$ with $n = 0, 1, \ldots$.*

To prove this claim, we need to make use of the *multivariable Hermitian polynomials* $\mathcal{H}_{n_1 \cdots n_m}$ on $\mathbb{R}^m$, labelled with $0 \leq n_1, \ldots, n_m < +\infty$, which are defined by the expansion (see [14] and [15] for the detail)

$$e^{-\frac{|u-t|^2}{2}} = e^{-\frac{|u|^2}{2}} \sum_{n_1,\ldots,n_m} \frac{(t^1)^{n_1} \cdots (t^m)^{n_m}}{n_1! \cdots n_m!} \mathcal{H}_{n_1\cdots n_m}(u),$$
$$u = (u^1, \ldots, u^m),\ t = (t^1, \ldots t^m) \in \mathbb{R}^m, \tag{6.2}$$

or equivalently

$$e^{-\frac{|t|^2}{2} + \langle t, u\rangle} = \sum_{n_1,\ldots,n_m} \frac{(t^1)^{n_1} \cdots (t^m)^{n_m}}{n_1! \cdots n_m!} \mathcal{H}_{n_1\cdots n_m}(u),$$
$$u = (u^1, \ldots, u^m),\ t = (t^1, \ldots t^m) \in \mathbb{R}^m. \tag{6.3}$$

It is clear that

$$\mathcal{H}_{n_1\cdots n_m}(u) = \mathcal{H}_{n_1}(u^1) \ldots \mathcal{H}_{n_m}(u^m), \quad \forall u = (u^1, \ldots, u^m) \in \mathbb{R}^m \tag{6.4}$$

where, for each $i = 1, \ldots, m$, $\mathcal{H}_{n_i}(u^i)$ is the Hermitian Polynomial of one variable u^i defined by

$$e^{-\frac{1}{2}|t^i|^2 + u^i t^i} = \sum_{n_i} \frac{(t^i)^{n_i}}{n_i!} \mathcal{H}_{n_i}(u^i), \quad u^i, t^i \in \mathbb{R}. \tag{6.5}$$

By (6.5), we easily find that

$$\mathcal{H}_{n_i+1}(u^i) = u^i\mathcal{H}_{n_i} - n_i\mathcal{H}_{n_i-1}, \quad \frac{\mathrm{d}}{\mathrm{d}u^i}\mathcal{H}_{n_i}(u^i) = n_i\mathcal{H}_{n_i-1}, \ i = 1, \dots, m \tag{6.6}$$

implying that

$$\left(-\frac{d^2}{(du^i)^2} + u^i\frac{\mathrm{d}}{\mathrm{d}u^i}\right)\mathcal{H}_{n_i}(u^i) = n_i\mathcal{H}_{n_i}(u^i), \quad i = 1, \dots, m. \tag{6.7}$$

Consequently, by (6.4), we have

$$(-\triangle_{\mathbb{R}^m} + \nabla_u)\,\mathcal{H}_{n_1\cdots n_m}(u) = \left(\sum_{i=1}^m n_i\right)\mathcal{H}_{n_1\cdots n_m}(u), \quad \forall n_1, \dots, n_m \geq 0. \tag{6.8}$$

It is known that all these multivariable Hermitian polynomials are weighted square integrable with the weight $e^{-\frac{|u|^2}{2}}$, that is

$$\mathcal{H}_{n_1\cdots n_m} \in L^2_w(\mathbb{R}^m) := \overline{\{\varphi \in C^\infty(\mathbb{R}^m);\ \int_{\mathbb{R}^m}\varphi^2 e^{-f}\mathrm{d}V_{\mathbb{R}^m} < +\infty\}}.$$

Consequently, integers $\sum_{i=1}^m n_i$, for all $n_1, \dots, n_m \geq 0$, are eigenvalues of the operator $-\triangle_{\mathbb{R}^m} + \nabla_u$ acting on $L^2_w(\mathbb{R}^m)$. By making a change of coordinates on $\mathbb{R}^{m+p}$ we can assume $x^i - \xi^i = u^i$, $i = 1, 2, \dots, m$, for $x \in P^m$. Thus, (6.8) shows that $-\tilde{L} + 1$ has $n = 0, 1, \dots$ as its eigenvalues, or equivalently, $n - 1 = -1, 0, 1, \dots$ are eigenvalues of $-\tilde{L}$ where constants are those eigenfunctions corresponding to -1.

To complete the claim, we also have to show that $\{\mathcal{H}_{n_1\cdots n_m};\ n_1, \dots, n_m \geq 0\}$ is a complete basis for the space of smooth and weighted square integrable functions on $\mathbb{R}^m$. For doing this, we let E be the orthogonal complement in $L^2_w(\mathbb{R}^m)$ of the closure of the linear span of all $\mathcal{H}_{n_1\dots n_m}$, that is,

$$E := (\overline{\mathrm{Span}\,\{\mathcal{H}_{n_1\dots n_m},\ n_1, \dots, n_m = 0, 1, \dots\}})^\perp.$$

For any $\varphi \in E$, we have

$$0 = (\varphi, \mathcal{H}_{n_1\dots n_m})_w := \int_{\mathbb{R}^m}\varphi(u)\mathcal{H}_{n_1\dots n_m}(u)e^{-f}\mathrm{d}V_{\mathbb{R}^m}, \quad n_1, \dots, n_m = 0, 1, \dots.$$

It then easily follows from (6.3) that $\mathcal{F}(\varphi e^{-f}) = 0$ where $\mathcal{F}$ is the usual multivariable Fourier transformation. Since $\mathcal{F}$ is injective, we obtain that $\varphi e^{-f} = 0$ implying $\varphi \equiv 0$. This shows that $E = 0$ and thus

$$L^2_w(\mathbb{R}^m) = \overline{\mathrm{Span}\,\{\mathcal{H}_{n_1\cdots n_m},\ n_1, \dots, n_m = 0, 1, \dots\}}. \tag{6.9}$$

Now suppose $\eta = \sum \eta^\alpha e_\alpha$ is a compact normal vector field that can be taken as a VP-variation vector field. Then for each α, we have

$$\eta^\alpha \in S^{\infty,2}_w(P^m) := \left\{\varphi \in C^\infty(P^m);\ \int_{P^m}\varphi^2 e^{-f}\mathrm{d}V_{P^m} < +\infty\right\}.$$

Since $\tilde{L}$ is self-adjoint with respect to the weighted measure $e^{-f}\mathrm{d}V$, we know that it is diagonalizable, that is, any compactly supported smooth function can be decomposed into a sum of some eigenfunctions of $\tilde{L}$. In particular, we can write for each $\alpha = m+1, \dots, m+p$,

$$\eta^\alpha = \eta^\alpha_0 + \sum_{k\geq 1}\eta^\alpha_k, \quad \eta^\alpha_0 \in \mathbb{R}, \tag{6.10}$$

where $\eta_k^\alpha \in S_w^{\infty,2}(P^m)$ satisfying $\tilde{L}(\eta_k^\alpha) = -\lambda_k\eta_k^\alpha$, $k \geq 0$. Furthermore, the self-adjointness of $\tilde{L}$ also implies that, for each pair of $k \neq l$, η_k^α and η_l^α are orthogonal, that is

$$\int_{P^m} \sum_\alpha \eta_k^\alpha \eta_l^\alpha e^{-f} \mathrm{d}V = 0, \quad k \neq l. \tag{6.11}$$

Since η is a VP-variation vector field, we have by (6.10) and (6.1) that $\int_{P^m} \eta^\alpha e^{-f} \mathrm{d}V = 0$ for all $\alpha = m+1, \ldots, m+p$. It then follows from (6.11) that $\eta_0^\alpha = 0, \alpha = m+1, \ldots, m+p$. Therefore,

$$\int_{P^m} \sum_\alpha |\eta^\alpha|^2 e^{-f} \mathrm{d}V = \int_{P^m} \sum_\alpha \sum_{k,l\geq 1} \eta_k^\alpha \eta_l^\alpha e^{-f} \mathrm{d}V = \sum_\alpha \sum_{k\geq 1} \int_{P^m} |\eta_k^\alpha|^2 e^{-f} \mathrm{d}V.$$

Consequently, we have

$$\begin{aligned}
\int_{P^m} \sum_\alpha \eta^\alpha(-\tilde{L}\eta^\alpha) e^{-f} \mathrm{d}V &= \int_{P^m} \sum_\alpha \sum_{k\geq 1} \eta_k^\alpha \sum_{l\geq 1} (-\tilde{L}\eta_l^\alpha) e^{-f} \mathrm{d}V \\
&= \sum_\alpha \sum_{k,l\geq 1} \int_{P^m} \lambda_l \eta_k^\alpha \eta_l^\alpha e^{-f} \mathrm{d}V = \sum_\alpha \sum_{k\geq 1} \lambda_k \int_{P^m} |\eta_k^\alpha|^2 e^{-f} \mathrm{d}V \\
&\geq \lambda_1 \sum_\alpha \sum_k \int_{P^m} |\eta_k^\alpha|^2 e^{-f} \mathrm{d}V = \lambda_1 \sum_\alpha \int_{P^m} |\eta^\alpha|^2 e^{-f} \mathrm{d}V \geq 0
\end{aligned}$$

implying that

$$\begin{aligned}
Q(\eta, \eta) = -\int_{P^m} \langle \eta, L\eta \rangle e^{-f} \mathrm{d}V &= \int_{P^m} \sum_\alpha \eta^\alpha(-\tilde{L}\eta^\alpha) e^{-f} \mathrm{d}V \\
&= \sum_\alpha \int_{P^m} \eta^\alpha(-\tilde{L}\eta^\alpha) e^{-f} \mathrm{d}V \geq 0.
\end{aligned}$$

□

Theorem 6.2 *As a ξ-submanifold, the index* $\mathrm{ind}(S^m(r))$ *of the standard m-sphere $S^m(r)$ with respect to VP-variations is no less than $m+1$. Furthermore,* $\mathrm{ind}(S^m(r)) = m+1$ *if and only if $r^2 \leq m$. In particular, all of these spheres are not W_ξ-stable.*

Proof For the standard sphere $S^m(r) \subset \mathbb{R}^{m+1} \subset \mathbb{R}^{m+p}$, we have $x^\top = 0$, $h = -\frac{1}{r^2} g x$ and hence $\xi = \left(-\frac{m}{r^2} + 1\right) x$. It follows that $x - \xi = \frac{m}{r^2} x$ and

$$L = \Delta^\perp_{S^m(r)} + \langle h_{ij}, \cdot \rangle h_{ij} + 1 = \Delta^\perp_{S^m(r)} + \frac{m}{r^4} \langle x, \cdot \rangle x + 1, \quad \tilde{\mathcal{L}} = \Delta_{S^m(r)}.$$

In particular, $L(x) = \frac{1}{r^2}(m + r^2)x$ and, for any parallel normal vector field N orthogonal to x, $L(N) = N$. Let $e_{m+2}, \ldots, e_{m+p}$ be an orthonormal constant basis of the subspace $(\mathrm{Span}\,\{TS^m(r), x\})^\perp \subset \mathbb{R}^{m+p}$. Then, $e_{m+1} :\equiv \frac{1}{r}x, e_{m+2}, \ldots, e_{m+p}$ is an orthonormal normal frame field of $S^m(r)$ and

$$L(e_{m+1}) = \frac{1}{r^2}(m + r^2) e_{m+1}, \quad L(e_\alpha) = e_\alpha, \quad \alpha = m+2, \ldots, m+p. \tag{6.12}$$

Now for any $\eta \in \Gamma(T^\perp S^m(r))$ we can write

$$\eta = \sum_\alpha \eta^\alpha e_\alpha \text{ with } \eta^\alpha \in C^\infty(S^m(r)),\ m+1 \leq \alpha \leq m+p.$$

Then by (5.4) and (6.12)

$$\begin{aligned} L(\eta) &= \sum_{\alpha} (\tilde{\mathcal{L}}(\eta^\alpha)) e_\alpha + \eta^\alpha L(e_\alpha) \\ &= (\triangle_{S^m(r)} \eta^{m+1}) e_{m+1} + \eta^{m+1} L(e_{m+1}) + \sum_{\alpha \geq m+2} ((\triangle_{S^m(r)} \eta^\alpha) e_\alpha + \eta^\alpha L(e_\alpha)) \\ &= \left(\tilde{L} + \frac{m}{r^2}\right) \eta^{m+1} e_{m+1} + \sum_{\alpha \geq m+2} \tilde{L}(\eta^\alpha) e_\alpha \end{aligned}$$

where $\tilde{L} = \triangle_{S^m(r)} + 1$. Furthermore, let λ_k, $k \geq 0$ be the eigenvalues of $\tilde{L}$ and write $\eta^\alpha = \sum_{k \geq 0} \eta_k^\alpha$ for some eigenfunctions η_k^α satisfying $\tilde{L}(\eta_k^\alpha) = -\lambda_k \eta_k^\alpha$, $k \geq 0$.

It is well known that the eigenvalues of $-\triangle_{S^m(r)}$ is $\frac{k(m+k-1)}{r^2}$, $k \geq 0$, so that

$$\lambda_k = \frac{k(m+k-1)}{r^2} - 1, \text{ for } k = 0, 1, 2, \ldots,$$

with constants being the eigenfunctions corresponding to $k = 0$. But by (6.1), $\int_{S^m(r)} \eta^\alpha e^{-f} \mathrm{d}V_{S^m(r)} = 0$ which implies that $\eta_0^\alpha = 0$. Therefore,

$$\begin{aligned} Q(\eta, \eta) &= -\int_{S^m(r)} \langle \eta, L(\eta) \rangle \, e^{-f} \mathrm{d}V_{S^m(r)} \\ &= -\int_{S^m(r)} \eta^{m+1} \left(\tilde{L} + \frac{m}{r^2}\right) \eta^{m+1} e^{-f} \mathrm{d}V_{S^m(r)} + \sum_{\alpha \geq m+2} \int_{S^m(r)} \eta^\alpha (-\tilde{L} \eta^\alpha) e^{-f} \mathrm{d}V_{S^m(r)} \\ &= \sum_{k \geq 1} \int_{S^m(r)} \left(\frac{k(m+k-1)}{r^2} - \frac{1}{r^2}(m + r^2)\right) |\eta_k^{m+1}|^2 e^{-f} \mathrm{d}V_{S^m(r)} \\ &\quad + \sum_{\alpha \geq m+2, k \geq 1} \int_{S^m(r)} \left(\frac{k(m+k-1)}{r^2} - 1\right) |\eta_k^\alpha|^2 e^{-f} \mathrm{d}V_{S^m(r)} \\ &= -\int_{S^m(r)} |\eta_1^{m+1}|^2 e^{-f} \mathrm{d}V_{S^m(r)} \\ &\quad + \sum_{k \geq 2} \int_{S^m(r)} \left(\frac{k(m+k-1)}{r^2} - \frac{1}{r^2}(m + r^2)\right) |\eta_k^{m+1}|^2 e^{-f} \mathrm{d}V_{S^m(r)} \\ &\quad + \sum_{\alpha \geq m+2, k \geq 1} \int_{S^m(r)} \left(\frac{k(m+k-1)}{r^2} - 1\right) |\eta_k^\alpha|^2 e^{-f} \mathrm{d}V_{S^m(r)} \\ &\geq -\int_{S^m(r)} |\eta_1^{m+1}|^2 e^{-f} \mathrm{d}V_{S^m(r)} + \left(\frac{m+2}{r^2} - 1\right) \sum_{k \geq 2} \int_{S^m(r)} |\eta_k^{m+1}|^2 e^{-f} \mathrm{d}V_{S^m(r)} \\ &\quad + \left(\frac{m}{r^2} - 1\right) \sum_{\alpha \geq m+2, k \geq 1} \int_{S^m(r)} |\eta_k^\alpha|^2 e^{-f} \mathrm{d}V_{S^m(r)}. \end{aligned} \tag{6.13}$$

Define

$$V_{\lambda_1} = \{\varphi \in C^\infty(S^m(r));\ \triangle_{S^m(r)} \varphi = -\frac{m}{r^2} \varphi\}, \quad \tilde{V}_{\lambda_1} = \{\varphi e_{m+1};\ \varphi \in V_{\lambda_1}\}.$$

Then $\dim \tilde{V}_{\lambda_1} = \dim V_{\lambda_1}$ and the left side is well known to be $m+1$. It is not hard to see from (6.13) that Q is negative definite on $\tilde{V}_{\lambda_1}$, and thus, $\mathrm{ind}(S^m(r)) \geq m+1$ with the equality holding if and only if $\frac{m}{r^2} - 1 \geq 0$, that is, $r^2 \leq m$. □

7 The uniqueness problem for complete W-stable ξ-submanifolds

It is interesting to know whether or not m-planes are the only W_ξ-stable ξ-submanifolds. We shall start to deal with this problem in this section. To make things more clear, we would better use the standard weighted volume functional V_w for immersed submanifolds, which is a special case of either V_ξ or $\bar{V}_\xi$ with $\xi \equiv 0$:

$$V_w(x) \equiv V_0(x) = \int_M e^{-\frac{1}{2}|x|^2} \mathrm{d}V_x, \quad x \in \mathcal{M}.$$

Then, the same argument as in the proof of Theorem 4.1, Theorem 4.3 and Corollary 4.4 easily lead to the following

Proposition 7.1 *Let $x : M^m \to \mathbb{R}^{m+p}$ be a ξ-submanifold. Then for any VP-variation of x, we have*

$$V_w'(t) = -\int_M \left\langle H_t + x_t^\perp, \frac{\partial F}{\partial t} \right\rangle e^{-\frac{1}{2}|x_t|^2} \mathrm{d}V_t, \quad x_t \in \mathcal{M}, \tag{7.1}$$

$$V_w''(0) = -\int_M \left(\left\langle \Delta_M^\perp(\eta) - D_{x^\top}^\perp \eta + g^{ik} g^{jl} \left\langle h_{ij}, \eta \right\rangle h_{kl} + \eta, \eta \right\rangle \right) e^{-\frac{1}{2}|x|^2} \mathrm{d}V. \tag{7.2}$$

By making applications of (7.1) and (7.2), we can generalize the conventional extreme value problem with conditions to our situation. For example, we have by Definition 6.1 and (7.1):

Corollary 7.2 (see [27] for the hypersurface case). *An immersion $x \in \mathcal{M}$ is a ξ-submanifold if and only if it is a critical point of V_w under the VP-variations (the "critical point with condition").*

Now we introduce the concept of W-stability for ξ-submanifolds, which can be viewed as the "*conditional*" critical points of V_w.

Definition 7.1 A ξ-submanifold $x : M^m \to \mathbb{R}^{m+p}$ is called W-stable if it has a finite standard weighted volume $V_w(x)$ and $V_w''(0) \geq 0$ for all VP-variations of x.

In other words, the W-stability is exactly the W_0-stability, a typical one to the W_ξ-stability: just put $\xi = 0$ in the functional V_ξ. In this sense our main theorem can be stated as follows:

Theorem 7.3 *Let $x : M^m \to \mathbb{R}^{m+p}$ be a complete and properly immersed ξ-submanifold with flat normal bundle. Then, x is W-stable if and only if $x(M^m)$ is an m-plane.*

To prove this theorem, we shall extend the main idea in [27], originally applied for the hypersurface case, to our higher codimension case here by solving some certain technical problems. Clearly, we only need to prove the necessity part of Theorem 7.3. For this, we can first make use of the universal covering if necessary to assume that M^m is simply connected. Then that x has a flat normal bundle implies the existence of a parallel orthonormal normal frame $\{e_\alpha;\ m+1 \leq \alpha \leq m+p\}$. Furthermore, from (5.1) or (7.2) we have

$$\mathcal{L} = \Delta_{M^m}^\perp - D_{x^\top}^\perp, \quad L = \mathcal{L} + \left\langle h_{ij}, \cdot \right\rangle h_{ij} + 1, \quad \tilde{\mathcal{L}} = \Delta_{M^m} - \nabla_{x^\top}. \tag{7.3}$$

Lemma 7.4 *Let x be a ξ-submanifold. Then for any constant vector $v \in \mathbb{R}^{m+p}$ and any parallel normal vector field N, we have*

$$\tilde{\mathcal{L}} \left\langle v, N \right\rangle = -\left\langle A_N, A_{v^\perp} \right\rangle \equiv -\left\langle h_{ij}, N \right\rangle \left\langle h_{ij}, v \right\rangle, \quad L(v^\perp) = v^\perp \tag{7.4}$$

where $v^\top$ and $v^\perp$ are the orthogonal projections of the vector v on TM^m and $T^\perp M^m$, respectively.

Proof By using Weingarten formula and the equality that $D^{\perp}(H + x^{\perp}) \equiv 0$, we find

$$\begin{aligned}
\tilde{\mathcal{L}}\langle v, N\rangle &= \triangle_{M^m}\langle v, N\rangle - \nabla_{x^\top}\langle v, N\rangle \\
&= (\langle v, -A_N(e_i)\rangle)_{,i} - \left\langle v, -A_N(x^\top)\right\rangle \\
&= -\left\langle h_{iji}, N\right\rangle\left\langle v, e_j\right\rangle - \left\langle h_{ij}, N\right\rangle\left\langle v, e_j\right\rangle_{,i} + \left\langle v, A_N(x^\top)\right\rangle \\
&= \langle x, N\rangle_{j}\left\langle v, e_j\right\rangle - \left\langle h_{ij}, N\right\rangle\left\langle v, h_{ji}\right\rangle + \left\langle v, A_N(x^\top)\right\rangle \\
&= -\left\langle x, A_N(e_j)\right\rangle\left\langle v, e_j\right\rangle - \left\langle A_N, A_{v^\perp}\right\rangle + \left\langle v, A_N(x^\top)\right\rangle \\
&= -\left\langle x^\top, A_N(v^\top)\right\rangle - \left\langle A_N, A_{v^\perp}\right\rangle + \left\langle A_N(v^\top), x^\top\right\rangle \\
&= -\left\langle A_N, A_{v^\perp}\right\rangle.
\end{aligned}$$

The second equality follows directly from (5.3), (5.4) and the first equality in (7.4). □

Lemma 7.5 *For any $\eta = e_\alpha + v^\perp$, $v \in \mathbb{R}^{m+p}$, it holds that*

$$Q(\phi\eta, \phi\eta) \le -\int_M \phi^2|\eta|^2 e^{-f}\,\mathrm{d}V + \int_M |\nabla\phi|^2(|\eta|^2 + |v^\top|^2)e^{-f}\,\mathrm{d}V, \quad \forall \phi \in C_0^\infty(M^m), \tag{7.5}$$

where and hereafter we denote $f = \frac{1}{2}|x|^2$.

Proof By (5.3) and (7.4),

$$L(\eta) = L(e_\alpha + v^\perp) = e_\alpha + h^\alpha_{ij}h_{ij} + v^\perp = \eta + h^\alpha_{ij}h_{ij}.$$

It follows from (5.9) that

$$\begin{aligned}
Q(\phi\eta, \phi\eta) &= -\int_M \langle\phi\eta, L(\phi\eta)\rangle\, e^{-f}\,\mathrm{d}V \\
&= -\int_M \phi^2\langle\eta, L(\eta)\rangle\, e^{-f}\,\mathrm{d}V + \int_M |\nabla\phi|^2|\eta|^2 e^{-f}\,\mathrm{d}V \\
&= -\int_M \phi^2\left\langle\eta, \eta + h^\alpha_{ij}h_{ij}\right\rangle e^{-f}\,\mathrm{d}V + \int_M |\nabla\phi|^2|\eta|^2 e^{-f}\,\mathrm{d}V \\
&= -\int_M \phi^2|\eta|^2 e^{-f}\,\mathrm{d}V - \int_M \phi^2 h^\alpha_{ij}\left\langle h_{ij}, e_\alpha + v^\perp\right\rangle e^{-f}\,\mathrm{d}V + \int_M |\nabla\phi|^2|\eta|^2 e^{-f}\,\mathrm{d}V.
\end{aligned} \tag{7.6}$$

On the other hand, by (5.1) and (5.5)

$$\begin{aligned}
&\int_M \phi^2\left\langle e_\alpha, v^\perp\right\rangle e^{-f}\,\mathrm{d}V = \int_M \phi^2\left\langle e_\alpha, L(v^\perp)\right\rangle e^{-f}\,\mathrm{d}V \\
&= \int_M \phi^2\left\langle e_\alpha, \left\langle h_{ij}, v^\perp\right\rangle h_{ij} + v^\perp\right\rangle e^{-f}\,\mathrm{d}V + \int_M \left\langle\phi^2 e_\alpha, \mathcal{L}v^\perp\right\rangle e^{-f}\,\mathrm{d}V \\
&= \int_M \phi^2\left\langle e_\alpha, v^\perp\right\rangle e^{-f}\,\mathrm{d}V + \int_M \phi^2 h^\alpha_{ij}\left\langle h_{ij}, v^\perp\right\rangle e^{-f}\,\mathrm{d}V - \int_M \left\langle D^\perp(\phi^2 e_\alpha), D^\perp v^\perp\right\rangle e^{-f}\,\mathrm{d}V \\
&= \int_M \phi^2\left\langle e_\alpha, v^\perp\right\rangle e^{-f}\,\mathrm{d}V + \int_M \phi^2 h^\alpha_{ij}\left\langle h_{ij}, v^\perp\right\rangle e^{-f}\,\mathrm{d}V - 2\int_M \phi\left\langle(\nabla\phi)e_\alpha, -d(v^\top)\right\rangle e^{-f}\,\mathrm{d}V \\
&= \int_M \phi^2\left\langle e_\alpha, v^\perp\right\rangle e^{-f}\,\mathrm{d}V + \int_M \phi^2 h^\alpha_{ij}\left\langle h_{ij}, v^\perp\right\rangle e^{-f}\,\mathrm{d}V + 2\int_M \phi h^\alpha(\nabla\phi, v^\top)e^{-f}\,\mathrm{d}V,
\end{aligned}$$

implying that

$$\left|\int_M \phi^2 h^\alpha_{ij} \left\langle h_{ij}, v^\perp \right\rangle e^{-f} \mathrm{d}V\right| = \left|2\int_M \phi h^\alpha (\nabla\phi, v^\top) e^{-f} \mathrm{d}V\right|$$
$$\leq 2\int_M |\phi||h^\alpha||\nabla\phi||v^\top| e^{-f} \mathrm{d}V \leq \int_M \phi^2 |h^\alpha|^2 e^{-f} \mathrm{d}V + \int_M |\nabla\phi|^2 |v^\top|^2 e^{-f} \mathrm{d}V.$$

Inserting this into (7.6) we complete the proof. □

Define

$$W = \mathrm{Span}_{\mathbb{R}}\{e_\alpha\}, \quad V^\top = \{v^\top; v \in \mathbb{R}^{m+p}\}, \quad V^\perp = \{v^\perp; v \in \mathbb{R}^{m+p}\}. \tag{7.7}$$

Then W is the space of parallel normal fields of x and $p \leq \dim V^\perp \leq m + p$.

Lemma 7.6 *Denote*

$$V_0^\perp = \{v^\perp = \mathrm{const};\ v \in \mathbb{R}^{m+p}\}. \tag{7.8}$$

Then $W \cap V^\perp = V_0^\perp$.

Proof For any $\eta \in W \cap V^\perp$, we have $\eta = v^\perp = c^\alpha e_\alpha$ for some $v \in \mathbb{R}^{m+p}$ and $c^\alpha \in \mathbb{R}$. Then it follows from (5.3) and (5.4) that

$$v^\perp = L(v^\perp) = c^\alpha L(e_\alpha) = c^\alpha (e_\alpha + h^\alpha_{ij} h_{ij}) = v^\perp + c^\alpha h^\alpha_{ij} h_{ij}$$

implying that $c^\alpha h^\alpha_{ij} h_{ij} = 0$. Multiplying this with $v^\perp = c^\alpha e_\alpha$ it follows that

$$\left\langle h, v^\perp \right\rangle^2 = \sum_{i,j,\alpha,\beta} c^\alpha c^\beta h^\alpha_{ij} h^\beta_{ij} = 0.$$

Thus $\left\langle h, v^\perp \right\rangle = 0$ or equivalently $A_{v^\perp} = 0$ which with the fact that $v^\perp$ is parallel in the normal bundle shows that $v^\perp$ must be a constant vector.

The inverse part is trivial. □

Define

$$\Gamma^{\infty,2}_w(T^\perp M^m) := \{\eta \in \Gamma(T^\perp M);\ \int_M |\eta|^2 e^{-f} \mathrm{d}V < +\infty\},$$

on which there is a standard L^2_w inner product $(\cdot,\cdot)$ by

$$(\eta_1, \eta_2) := \int_M \langle \eta_1, \eta_2 \rangle e^{-f} \mathrm{d}V, \quad \forall \eta_1, \eta_2 \in \Gamma^{\infty,2}_w(T^\perp M^m),$$

giving the corresponding L^2_w-norm $\|\cdot\|_{2,w}$. The L^2_w inner product $(\cdot,\cdot)$ and L^2_w-norm $\|\cdot\|_{2,w}$ for all weighted square integrable tangent vector fields and functions on M^m are defined in the same way. In particular, for a constant c, we have $\|c\|^2_{2,w} = c^2 \int_M e^{-f} \mathrm{d}V$.

Let $V_1^\perp$ be the orthogonal complement of $V_0^\perp$ in $V^\perp$ with respect to the L^2_w inner product, and define $\tilde{V} = W \oplus V_1^\perp$ as subspaces of $\Gamma^{\infty,2}_w(T^\perp M^m)$. So for any $\eta \in \tilde{V}$ we can write $\eta = w + v^\perp$ for a unique $w \in W$ and some $v \in \mathbb{R}^{m+p}$ such that $\eta = w + v^\perp$ where v may not be unique. Since $\dim W = p$ and $\dim V_1^\perp \leq \dim V^\perp \leq m + p$, we have $\dim \tilde{V} < +\infty$. Fix a basis $\{w_a + v_a^\perp;\ 1 \leq a \leq \dim \tilde{V}\}$ for $\tilde{V}$ such that $\|w_a\|^2_{2,w} + \|v_a\|^2_{2,w} = 1$ for $1 \leq a \leq \dim \tilde{V}$. Define

$$\mathbb{S} = \{\eta = \sum_a \eta^a (w_a + v_a^\perp);\ \sum_a (\eta^a)^2 = 1\} \subset \tilde{V}.$$

Then the finiteness of dim $\tilde{V}$ implies that $\mathbb{S}$ is compact. Note that for any $\eta \in \mathbb{S}$, η can not be zero.

Now we consider the compact case and prove the following

Proposition 7.7 *Any compact ξ-submanifold with parallel normal bundle can not be W-stable.*

Proof It suffices to show that both of the following two are true:

(1) Q is negative definite on $\tilde{V}$ and, consequently, is negative definite on $V_1^\perp$;
(2) $\dim V_1^\perp > 0$.

In fact, the conclusion (1) follows directly from Lemma 7.5 by choosing $\phi \equiv 1$; while conclusion (2) follows from the fact that the converse of (2) would imply that $M^m = \mathbb{R}^m$, by the argument at the end of this paper, which contradicts the compactness assumption. □

Next we consider the non-compact case and thus assume that $x : M^m \to \mathbb{R}^{m+p}$ is a complete and non-compact ξ-submanifold.

Let o be a fixed point of M and $\bar{o} = x(o)$. For any $R > 0$, we define $\bar{B}_R(\bar{o}) = \{x \in \mathbb{R}^{m+p};\ |x - \bar{o}| \le R\}$ and introduce a cut-off function $\bar{\phi}_R$ as follows (cf. [27]):

$$\bar{\phi}_R(x) = \begin{cases} 1, & x \in \bar{B}_R(\bar{o}); \\ 1 - \frac{1}{R}(|x - \bar{o}| - R), & x \in \bar{B}_{2R}(\bar{o}) \backslash \bar{B}_R(\bar{o}); \\ 0, & x \in \mathbb{R}^{m+p} \backslash \bar{B}_{2R}(\bar{o}). \end{cases} \tag{7.9}$$

For the given immersion $x : M^m \to \mathbb{R}^{m+p}$, let $\phi_R = \bar{\phi}_R \circ x \in C^\infty(M^m)$ and $B_R(o) = x^{-1}(\bar{B}_R(\bar{o}))$. Then, $B_R(o)$ is compact since x is properly immersed. In particular, ϕ_R is compactly supported. Furthermore, it is easily seen that $|\nabla \phi_R| \le |D\bar{\phi}_R| \le \frac{1}{R}$.

Lemma 7.8 *There is a large $R_0 > 0$ such that*

$$\int_{B_R(o)} |\eta|^2 e^{-f} \mathrm{d}V \ge \int_{B_{R_0}(o)} |\eta|^2 e^{-f} \mathrm{d}V > 0, \quad \forall \, \eta \in \mathbb{S}, \quad \forall \, R \ge R_0.$$

Proof If the lemma is not true, then one can find a sequence $\{\eta_j\} \subset \mathbb{S}$ such that

$$\int_{B_j(o)} |\eta_j|^2 e^{-f} \mathrm{d}V = 0, \quad j = 1, 2, \ldots.$$

By the compactness of $\mathbb{S}$, there exists a subsequence $\{\eta_{j_k}\}$ which is convergent to some $\eta_0 \in \mathbb{S}$. For any $R > 0$, there exists some $K > 0$ such that $j_k > R$ for all $k > K$. It follows that

$$\int_{B_R(o)} |\eta_0|^2 e^{-f} \mathrm{d}V = \lim_{k \to +\infty} \int_{B_R(o)} |\eta_{j_k}|^2 e^{-f} \mathrm{d}V = 0$$

which implies that

$$\int_M |\eta_0|^2 e^{-f} \mathrm{d}V = \lim_{R \to +\infty} \int_{B_R(o)} |\eta_0|^2 e^{-f} \mathrm{d}V = 0.$$

Thus we have $\eta_0 = 0$ contradicting to the fact that $\eta_0 \in \mathbb{S}$. □

For each $R > 0$, define

$$m_R := \min_{\eta\in\mathbb{S}}\{\int_M \phi_R^2|\eta|^2 e^{-f}\mathrm{d}V\},\quad M_R = \max_{\eta\in\mathbb{S}}\{\int_M \phi_R^2|\eta|^2 e^{-f}\mathrm{d}V\}. \tag{7.10}$$

Clearly,

$$M_R \le C :\equiv \max_{\eta\in\mathbb{S}}\int_M |\eta|^2 e^{-f}\mathrm{d}V < +\infty. \tag{7.11}$$

Moreover, m_R is increasing with respect to R which together with Lemma 7.8 gives that

$$m_R \ge m_{R_0} > 0,\quad \forall R \ge R_0. \tag{7.12}$$

Lemma 7.9 *There exists a large R_0, such that*

$$\dim \phi_R\tilde V = \dim \tilde V,\quad \dim \phi_R V_1^\perp = \dim V_1^\perp,\quad R \ge R_0; \tag{7.13}$$

Furthermore, Q is negative definite on $\phi_R\tilde V \supset \phi_R V_1^\perp$.

Proof First, we prove $\dim \phi_R\tilde V = \dim \tilde V$ for all $R \ge R_0$ if R_0 is large enough. For a given $R > 0$, consider the surjective linear map

$$\Phi_R : \tilde V \to \phi_R\tilde V,\quad \eta \mapsto \Phi_R(\eta) := \phi_R\eta,\quad \forall \eta \in \tilde V.$$

We claim that, when R_0 is large enough, the kernel $\ker\Phi_{R_0}$ of Φ_{R_0} must be trivial. In fact, if it is not the case, there should be a nonzero sequence $\{\eta_j \in \tilde V\}$ such that $\phi_j\eta_j = 0$. By writing $\eta_j = \sum_a \eta_j^a(w_a + v_a^\perp)$, we can define $\tilde\eta_j = \frac{\eta_j}{\sqrt{\sum_a(\eta_j^a)^2}}$. Then $\phi_j\tilde\eta_j = 0$, and $\{\tilde\eta_j\}$ is contained in $\mathbb{S}$. Then the compactness of $\mathbb{S}$ assures that, by passing to a subsequence if possible, we can assume that $\tilde\eta_j \to \tilde\eta_0 \in \mathbb{S}$. Consequently, we have $\tilde\eta_0 = \lim_{j\to+\infty}\phi_j\tilde\eta_j = 0$ which is not possible. So there must be a large $R_0 > 0$ such that $\ker\Phi_{R_0} = 0$ and the claim is proved.

For any $R \ge R_0$, it is easily seen that $\ker\Phi_R \subset \ker\Phi_{R_0}$ which implies that $\ker\Phi_R = 0$ and $\phi_R\tilde V \cong \tilde V$. In particular, $\dim\phi_R\tilde V = \dim\tilde V$.

That $\dim\phi_R V_1^\perp = \dim V_1^\perp$ follows in the same way.

Next we are to find a larger $R \ge R_0$ such that Q is negative definite on $\phi_R\tilde V$. For this, we first note that $|\nabla\phi_R|$ supports in $B_{2R}(o)\backslash B_R(o)$ and $|\nabla\phi_R| \le \frac{1}{R}$, and then use Lemma 7.5 to conclude that, for all $\eta \in \mathbb{S}$

$$\begin{aligned}
Q(\phi_R\eta,\phi_R\eta) &\le -\int_M \phi_R^2|\eta|^2e^{-f}\mathrm{d}V + \int_M |\nabla\phi_R|^2(|\eta|^2 + |v^\top|^2)e^{-f}\mathrm{d}V\\
&\le -\int_M \phi_R^2|\eta|^2e^{-f}\mathrm{d}V + \frac{1}{R^2}\int_{B_{2R}(0)\backslash B_R(0)}(|\eta|^2 + |v^\top|^2)e^{-f}\mathrm{d}V\\
&\le -\int_M \phi_R^2|\eta|^2e^{-f}\mathrm{d}V + \frac{3}{R^2}\dim\tilde V.
\end{aligned}$$

Therefore, by (7.10)–(7.12) and Lemma 7.8, there must be an R_0 large enough such that $Q(\phi_R\eta,\phi_R\eta) < 0$ for all $\eta \in \mathbb{S}$, $R \ge R_0$. Then the conclusion that Q is negative definite on $\phi_R\tilde V$ follows directly from the bilinearity of Q. □

Lemma 7.10 *Under the complete and non-compact assumption, we have*

$$V_1^\perp = 0 \textit{ or equivalently, } V^\perp = V_0^\perp. \tag{7.14}$$

Proof Let $W^\perp$ be the orthogonal complement of W in the space $\Gamma_w^{\infty,2}(T^\perp M^m)$ of L_w^2-smooth normal sections. For any given $R > 0$, define a subspace

$$W^\perp(\phi_R\tilde{V}) := W^\perp \cap (\phi_R\tilde{V})$$

of $W^\perp$ and a linear map $\Psi_R : \phi_R V_1^\perp \to W^\perp(\phi_R\tilde{V})$ by

$$\phi_R v^\perp \mapsto \Psi_R(\phi_R v^\perp) := \phi_R v^\perp - \frac{\int_M \langle \phi_R v^\perp, e_\alpha\rangle e^{-f}\mathrm{d}V}{\int_M \phi_R e^{-f}\mathrm{d}V}\phi_R e_\alpha, \quad \forall v^\perp \in V_1^\perp.$$

Claim: There must be a large $R > 0$ such that $\ker \Psi_R = 0$.

In fact, if this is not true, then we can find a sequence $\{v_j^\perp\} \subset V_1^\perp$ with $\phi_j v_j^\perp \neq 0$ and $\Psi_j(\phi_j v_j^\perp) = 0$ for each $j = 1, 2, \ldots$. It follows that $v_j^\perp \neq 0$, $j = 1, 2, \ldots$. Define

$$\tilde{v}_j^\perp := \frac{v_j^\perp}{\|v_j^\perp\|_{2,w}}, \quad j = 1, 2, \ldots.$$

Then $\Psi_j(\phi_j\tilde{v}_j^\perp) = 0$, $j = 1, 2, \ldots$. Without loss of generality, we can assume that $\tilde{v}_j^\perp \to \tilde{v}_0^\perp$. Then $\tilde{v}_0^\perp \in V_1^\perp$ and $\|\tilde{v}_0^\perp\|_{2,w} = 1$.

On the other hand, from $\Psi_j(\phi_j\tilde{v}_j^\perp) = 0$ $(j = 1, 2, \ldots)$ it follows that

$$\phi_j\tilde{v}_j^\perp = \frac{\int_M \left\langle \phi_j\tilde{v}_j^\perp, e_\alpha\right\rangle e^{-f}\mathrm{d}V}{\int_M \phi_j e^{-f}\mathrm{d}V}\phi_j e_\alpha, \quad j = 1, 2, \ldots,$$

implying that

$$\|\phi_j\tilde{v}_j^\perp\|_{2,w}^2 = \frac{\int_M \left\langle \phi_j\tilde{v}_j^\perp, e_\alpha\right\rangle e^{-f}\mathrm{d}V}{\int_M \phi_j e^{-f}\mathrm{d}V}(\phi_j e_\alpha, \phi_j\tilde{v}_j^\perp), \quad j = 1, 2, \ldots. \tag{7.15}$$

But it is clear that $\phi_j\tilde{v}_j^\perp \to \tilde{v}_0^\perp$ when $j \to +\infty$ since

$$\begin{aligned}\|\phi_j\tilde{v}_j^\perp - \tilde{v}_0^\perp\|_{2,w} &\le \|\phi_j(\tilde{v}_j^\perp - \tilde{v}_0^\perp)\|_{2,w} + \|(\phi_j - 1)\tilde{v}_0^\perp\|_{2,w}\\ &\le \|\tilde{v}_j^\perp - \tilde{v}_0^\perp\|_{2,w} + \|\phi_j - 1\|_{2,w} \to 0, \quad j \to +\infty.\end{aligned}$$

Let $j \to +\infty$ in (7.15) then we obtain

$$\|\tilde{v}_0^\perp\|_{2,w}^2 = \frac{\int_M \langle \tilde{v}_0^\perp, e_\alpha\rangle e^{-f}\mathrm{d}V}{\int_M e^{-f}\mathrm{d}V}(e_\alpha, \tilde{v}_0^\perp) = 0$$

because $\tilde{v}_0^\perp \in V_1^\perp$ is orthogonal to W, contradicting to the fact that $\|\tilde{v}_0^\perp\|_{2,w} = 1$. So the claim is proved.

Thus by (7.13), when R is large enough it holds that

$$\dim V_1^\perp = \dim \phi_R V_1^\perp \le \dim W^\perp(\phi_R\tilde{V}) \le \mathrm{ind}_W(Q)$$

where $\mathrm{ind}_W(Q)$ denotes the W-stability index of Q. By the W-stability of x we have $\mathrm{ind}_W(Q) = 0$, implying that $\dim V_1^\perp = 0$ and thus $V_1^\perp = 0$, which is equivalent to $V^\perp = V_0^\perp$. □

Proof of Theorem 7.3 Using Proposition 7.7, we conclude that $x : M^m \to \mathbb{R}^{m+p}$ must be non-compact. Then by Lemma 7.10, we have a direct decomposition

$$\mathbb{R}^{m+p} = V^\top \oplus V^\perp$$

where $V^\top$ now consists of all constant vectors in $\mathbb{R}^{m+p}$ that are tangent to $x_* TM^m$ at each point of M^m, while $V^\perp$ consists of all constant vectors in $\mathbb{R}^{m+p}$ that are normal to $x_* TM^m$ at each point of M^m. It then follows that dim $V^\top \le m$ and dim $V^\perp \le p$. Consequently,

$$m + p = \dim \mathbb{R}^{m+p} = \dim V^\top + \dim V^\perp \le m + p$$

which implies that dim $V^\top = m$ and dim $V^\perp = p$. This is true only if $x(M^m) \equiv P^m$.

Theorem 7.3 is proved. $\square$

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