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The elliptic modular surface of level 4 and its reduction modulo 3

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Abstract
The elliptic modular surface of level 4 is a complex $K3$ surface with Picard number 20. This surface has a model over a number field such that its reduction modulo 3 yields a surface isomorphic to the Fermat quartic surface in characteristic 3, which is supersingular. The specialization induces an embedding of the Néron–Severi lattices. Using this embedding, we determine the automorphism group of this $K3$ surface over a discrete valuation ring of mixed characteristic whose residue field is of characteristic 3. The elliptic modular surface of level 4 has a fixed-point-free involution that gives rise to the Enriques surface of type IV in Nikulin–Kondo–Martin's classification of Enriques surfaces with finite automorphism group. We investigate the specialization of this involution to characteristic 3.

Keywords K3 surface · Enriques surface · Automorphism group · Petersen graph

Mathematics Subject Classification 14J28 · 14Q10

1 Introduction

Let R be a discrete valuation ring, and let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth proper family of varieties over R. We denote by $X_{\bar{\eta}}$ the geometric generic fiber and by $X_{\bar{s}}$ the geometric special fiber. Let $\operatorname{Aut}(\mathcal{X}/R)$ denote the group of automorphisms of $\mathcal{X}$ over $\operatorname{Spec} R$. Then we have natural homomorphisms

$$\operatorname{Aut}(X_{\bar{s}}) \leftarrow \operatorname{Aut}(\mathcal{X}/R) \rightarrow \operatorname{Aut}(X_{\bar{\eta}}).$$

In this paper, we calculate the group $\operatorname{Aut}(\mathcal{X}/R)$ in the case where $\mathcal{X}$ is a certain natural model of the elliptic modular surface of level 4, and the special fiber $X_{\bar{s}}$ is its reduction modulo 3. In this case, the surfaces $X_{\bar{\eta}}$ and $X_{\bar{s}}$ are $K3$ surfaces, and their automorphism groups have

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been calculated in [1,2], respectively, by Borcherds' method [3,4]. This paper gives the first application of Borcherds' method to the calculation of the automorphism group of a family of $K3$ surfaces.

1.1 Elliptic modular surface of level 4

The *elliptic modular surface of level* N is a natural compactification of the total space of the universal family over $\Gamma(N)\backslash\mathbb{H}$ of complex elliptic curves with level N structure, where $\mathbb{H}\subset\mathbb{C}$ is the upper-half plane and $\Gamma(N)\subset \mathrm{PSL}_2(\mathbb{Z})$ is the congruence subgroup of level N. This important class of surfaces was introduced and studied by Shioda [5].

The elliptic modular surface of level 4 is a $K3$ surface birational to the surface defined by the Weierstrass equation

$$Y^2 = X(X-1)\left(X-\left(\frac{1}{2}\left(\sigma+\frac{1}{\sigma}\right)\right)^2\right), \tag{1.1}$$

where σ is an affine parameter of the base curve $\mathbb{P}^1=\overline{\Gamma(4)\backslash\mathbb{H}}$ (see Section 3 in [6]). Shioda [5,6] studied the reduction of this surface in odd characteristics. On the other hand, Keum and Kondo [1] calculated the automorphism group of the elliptic modular surface of level 4.

To describe the results of Shioda [5,6] and Keum–Kondo [1], we fix some notation. A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle\colon L\times L\to\mathbb{Z}$. The group of isometries of a lattice L is denoted by $\mathrm{O}(L)$, which we let act on L from the *right*. A lattice L of rank n is said to be *hyperbolic* (resp. *negative-definite*) if the signature of $L\otimes\mathbb{R}$ is $(1,n-1)$ (resp. $(0,n)$). For a hyperbolic lattice L, we denote by $\mathrm{O}^+(L)$ the stabilizer subgroup of a connected component of $\{x\in L\otimes\mathbb{R}\mid\langle x,x\rangle>0\}$ in $\mathrm{O}(L)$. Let Z be a smooth projective surface defined over an algebraically closed field. We denote by S_Z the lattice of numerical equivalence classes $[D]$ of divisors D on Z and call it the *Néron–Severi lattice* of Z. Then S_Z is hyperbolic by the Hodge index theorem. We denote by $\mathcal{P}_Z$ the connected component of $\{x\in S_Z\otimes\mathbb{R}\mid\langle x,x\rangle>0\}$ that contains an ample class. We then put

$$N_Z := \{\, x\in\mathcal{P}_Z \mid \langle x,[C]\rangle\ge 0 \ \text{ for all curves } C \text{ on } Z\,\}.$$

We let the automorphism group $\mathrm{Aut}(Z)$ of Z act on S_Z from the *right* by pullback of divisors. Then we have a natural homomorphism

$$\mathrm{Aut}(Z)\to\mathrm{Aut}(N_Z):=\{\, g\in\mathrm{O}^+(S_Z)\mid N_Z^g=N_Z\,\}.$$

For an ample class $h\in S_Z$, we put

$$\mathrm{Aut}(Z,h):=\{\, g\in\mathrm{Aut}(Z)\mid h^g=h\,\},$$

and call it the *projective automorphism group* of the polarized surface (Z,h).

Let k_p be an algebraically closed field of characteristic $p\ge 0$. From now on, we assume that $p\ne 2$. Let $\sigma\colon X_p\to\mathbb{P}^1$ be the smooth minimal elliptic surface defined over k_p by (1.1). Then X_p is a $K3$ surface. For simplicity, we use the following notation throughout this paper:

$$S_p:=S_{X_p},\quad \mathcal{P}_p:=\mathcal{P}_{X_p},\quad N_p:=N_{X_p}.$$

Shioda [5,6] proved the following:

Theorem 1.1 (Shioda [5,6]) *Suppose that* $p\ne 2$.

(1) *The elliptic surface* $\sigma : X_p \to \mathbb{P}^1$ *has exactly six singular fibers. These singular fibers are located over* $\sigma = 0, \pm 1, \pm i, \infty$*, and each of them is of type* I_4*. The torsion part of the Mordell–Weil group of* $\sigma : X_p \to \mathbb{P}^1$ *is isomorphic to* $(\mathbb{Z}/4\mathbb{Z})^2$.

(2) *The Picard number* $\mathrm{rank}(S_p)$ *of* X_p *is*

$$\begin{cases} 20 & \text{if } p = 0 \text{ or } p \equiv 1 \bmod 4, \\ 22 & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

(3) *If* $k_0 = \mathbb{C}$*, the transcendental lattice of the complex* $K3$ *surface* X_0 *is*

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

(4) *The* $K3$ *surface* X_3 *is isomorphic to the Fermat quartic surface*

$$F_3 : x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0$$

in characteristic 3.

It follows from Theorem 1.1 (3) and the theorem of Shioda–Inose [7] that, over the complex number field, X_0 is isomorphic to the Kummer surface associated with $E_{\sqrt{-1}} \times E_{\sqrt{-1}}$, where $E_{\sqrt{-1}}$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\sqrt{-1})$. (See also Proposition 15 of Barth–Hulek [8].) Therefore the result of Keum–Kondo [1] contains the calculation of $\mathrm{Aut}(X_0)$.

Definition 1.2 Let Z be a $K3$ surface defined over k_p. A *double-plane polarization* is a vector $b = [H] \in N_Z \cap S_Z$ with $\langle b, b \rangle = 2$ such that the corresponding complete linear system $|H|$ is base-point-free, so that $|H|$ induces a surjective morphism $\Phi_b \colon Z \to \mathbb{P}^2$. Let b be a double-plane polarization, and let $Z \to Z_b \to \mathbb{P}^2$ be the Stein factorization of Φ_b. Then we have a *double-plane involution* $g(b) \in \mathrm{Aut}(Z)$ associated with the finite double covering $Z_b \to \mathbb{P}^2$. Let $\mathrm{Sing}(b)$ denote the singularities of the normal $K3$ surface Z_b. Since Z_b has only rational double points as its singularities, we have the *ADE-type* of $\mathrm{Sing}(b)$.

Remark 1.3 Suppose that an ample class $a \in S_Z$ and a vector $b \in S_Z$ with $\langle b, b \rangle = 2$ are given. Then we can determine whether b is a double-plane polarization or not, and if b is a double-plane polarization, we can calculate the set of classes of smooth rational curves contracted by $\Phi_b \colon Z \to \mathbb{P}^2$ and compute the matrix representation of the double-plane involution $g(b) \colon Z \to Z$ on S_Z. These algorithms are described in detail in [9] (and also in [10]). They are the key tools of this paper.

We re-calculated $\mathrm{Aut}(X_0)$ by using these algorithms and obtained a generating set of $\mathrm{Aut}(X_0)$ different from the one given in [1].

Theorem 1.4 (Keum–Kondo [1]) *There exist an ample class* $h_0 \in S_0$ *of degree* $\langle h_0, h_0 \rangle = 40$ *and four double-plane polarizations* $b_{80}, b_{112}, b_{296}, b_{688} \in S_0$ *such that* $\mathrm{Aut}(X_0)$ *is generated by the projective automorphism group* $\mathrm{Aut}(X_0, h_0) \cong (\mathbb{Z}/2\mathbb{Z})^5 : \mathfrak{S}_5$ *and the double-plane involutions* $g(b_{80}), g(b_{112}), g(b_{296}), g(b_{688})$.

See Table 1 for the properties of the double-plane polarizations b_d. See Proposition 4.2 for the geometric meaning of these generators of $\mathrm{Aut}(X_0)$ with respect to the action of $\mathrm{Aut}(X_0)$ on N_0. In Sect. 4.3, we also give a detailed description of the finite group $\mathrm{Aut}(X_0, h_0)$ in terms of a certain graph $\mathcal{L}_{40}$.

Remark 1.5 In [2], the automorphism group $\mathrm{Aut}(X_3) \cong \mathrm{Aut}(F_3)$ of the Fermat quartic surface F_3 in characteristic 3 was calculated (see Theorem 4.1). This calculation also plays an important role in the proof of our main results.

Table 1 Four double-plane involutions on X_0

$\langle h_0, b_d \rangle$	ADE type of $\mathrm{Sing}(b_d)$	$d = \langle h_0, h_0^{g(b_d)} \rangle$
16	$2A_3 + 3A_2 + 2A_1$	80
18	$4A_3 + 3A_1$	112
26	$A_5 + 2A_4 + A_3$	296
38	$2A_7 + A_3 + A_1$	688

1.2 Main results

In [1,2], the following was proved, and hence, from now on, we regard $\mathrm{Aut}(X_0)$ as a subgroup of $\mathrm{O}^+(S_0)$ and $\mathrm{Aut}(X_3)$ as a subgroup of $\mathrm{O}^+(S_3)$.

Proposition 1.6 *In each case of X_0 and X_3, the action of the automorphism group on the Néron–Severi lattice is faithful.* □

Let R be a discrete valuation ring whose fraction field K is of characteristic 0 and whose residue field k is of characteristic 3. Suppose that $\sqrt{-1} \in R$. In Sect. 2.5, we construct explicitly a smooth family of $K3$ surfaces $\mathcal{X} \to \mathrm{Spec}\, R$ over R such that the geometric generic fiber $\mathcal{X} \otimes_R \bar{K}$ is isomorphic to X_0 and the geometric special fiber $\mathcal{X} \otimes_R \bar{k}$ is isomorphic to X_3. The construction of this model $\mathcal{X}$ is natural in the sense that it uses the inherent elliptic fibration of X_0. Note that the model of X_0 over R is not unique and that the main results on $\mathrm{Aut}(\mathcal{X}/R)$ below may depend on the choice of the model.

By Proposition 3.3 of Maulik and Poonen [11], the specialization from $\mathcal{X} \otimes_R K$ to $\mathcal{X} \otimes_R k$ gives rise to a homomorphism

$$\rho \colon S_0 \to S_3.$$

In Sect. 2.3, we give an explicit description of ρ. It turns out that ρ is a primitive embedding of lattices. We regard S_0 as a sublattice of S_3 by ρ and put

$$\mathrm{O}^+(S_3, S_0) := \{\, g \in \mathrm{O}^+(S_3) \mid S_0^g = S_0 \,\}.$$

Then we have a natural restriction homomorphism

$$\tilde{\rho} \colon \mathrm{O}^+(S_3, S_0) \to \mathrm{O}^+(S_0).$$

The main results of this paper are as follows:

Theorem 1.7 *The restriction of $\tilde{\rho}$ to $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ induces an injective homomorphism*

$$\tilde{\rho}|_{\mathrm{Aut}} \colon \mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3) \hookrightarrow \mathrm{Aut}(X_0).$$

The image of $\tilde{\rho}|_{\mathrm{Aut}}$ is generated by the finite subgroup $\mathrm{Aut}(X_0, h_0)$ and the two double-plane involutions $g(b_{112})$, $g(b_{688})$. The other double-plane involutions $g(b_{80})$ and $g(b_{296})$ do not belong to the image of $\tilde{\rho}|_{\mathrm{Aut}}$.

Let R' be a finite extension of R, and let $\mathcal{X}' := \mathcal{X} \otimes_R R' \to \mathrm{Spec}\, R'$ be the pullback of $\mathcal{X} \to \mathrm{Spec}\, R$. We have a natural embedding $\mathrm{Aut}(\mathcal{X}/R) \hookrightarrow \mathrm{Aut}(\mathcal{X}'/R')$. We put

$$\mathrm{Aut}(\overline{\mathcal{X}/R}) := \mathrm{colim}_{R'}\, \mathrm{Aut}(\mathcal{X}'/R').$$

Let $\mathrm{res}_3\colon \mathrm{Aut}(\overline{\mathcal{X}/R}) \to \mathrm{Aut}(X_3)$ and $\mathrm{res}_0\colon \mathrm{Aut}(\overline{\mathcal{X}/R}) \to \mathrm{Aut}(X_0)$ denote the restriction homomorphisms. It is obvious that res_0 is injective and that the following diagram commutes.

$$\begin{array}{ccc} & \mathrm{Aut}(\overline{\mathcal{X}/R}) & \\ {}^{\mathrm{res}_3}\swarrow & & \searrow^{\mathrm{res}_0} \\ \mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3) & \overset{\tilde{\rho}|_{\mathrm{Aut}}}{\hookrightarrow} & \mathrm{Aut}(X_0) \end{array} \tag{1.2}$$

Theorem 1.8 *The image of* res_0 *is equal to the image of* $\tilde{\rho}|_{\mathrm{Aut}}$.

Thus we have obtained a set of generators of $\mathrm{Aut}(\overline{\mathcal{X}/R})$.

1.3 Enriques surfaces

By Nikulin [12] and Kondo [13], the complex Enriques surfaces with finite automorphism group are classified, and this classification is extended to Enriques surfaces in odd characteristics by Martin [14]. The Enriques surfaces in characteristic $\neq 2$ with finite automorphism group are divided into seven classes I–VII. In this paper, we concentrate on the Enriques surface of type IV.

Definition 1.9 A fixed-point-free involution of a $K3$ surface in characteristic $\neq 2$ is called an *Enriques involution*. An Enriques surface Y in characteristic $\neq 2$ is *of type* IV if $\mathrm{Aut}(Y)$ is of order 320. An Enriques involution of a $K3$ surface is *of type* IV if the quotient Enriques surface is of type IV.

Proposition 1.10 (Kondo [13], Martin [14]) *In each characteristic* $\neq 2$*, an Enriques surface of type* IV *exists and is unique up to isomorphism. There exist exactly* 20 *smooth rational curves on an Enriques surface of type* IV. □

Let $Y_{\mathrm{IV},\,p}$ denote an Enriques surface of type IV in characteristic $p \neq 2$. Kondo [13] showed that the covering $K3$ surface of $Y_{\mathrm{IV},\,0}$ is isomorphic to X_0.

Proposition 1.11 *There exist exactly six Enriques involutions in the projective automorphism group* $\mathrm{Aut}(X_0, h_0)$*. These six Enriques involutions are conjugate in* $\mathrm{Aut}(X_0, h_0)$*, and hence, the corresponding Enriques surfaces are isomorphic to each other. All of them are of type* IV.

By Theorem 1.7, these six Enriques involutions in $\mathrm{Aut}(X_0, h_0)$ specialize to involutions of X_3.

Theorem 1.12 *Let* $\varepsilon_3 \in \mathrm{Aut}(X_3)$ *be an involution that is mapped to an Enriques involution in* $\mathrm{Aut}(X_0, h_0)$ *by* $\tilde{\rho}|_{\mathrm{Aut}}$*. Then* ε_3 *is an Enriques involution of type* IV*, and the pullbacks of the* 20 *smooth rational curves on* $X_3/\langle\varepsilon_3\rangle \cong Y_{\mathrm{IV},\,3}$ *by the quotient morphism* $X_3 \to X_3/\langle\varepsilon_3\rangle$ *are lines of the Fermat quartic surface* $F_3 \cong X_3$.

During the investigation, we have come to notice that the geometry of X_p and $Y_{\mathrm{IV},\,p}$ is closely related to the *Petersen graph* (Fig. 1). See Sect. 2 for this relation. As a by-product, we see that the dual graph of the 20 smooth rational curves on $Y_{\mathrm{IV},\,p}$ is as in Fig. 2. Compare Fig. 2 with the picturesque but complicated figure of Kondo (Figure 4.4 of [13]).

It has been observed that the Petersen graph is related to various $K3$/Enriques surfaces. See, for example, Vinberg [15] for the relation with the singular $K3$ surface with the transcendental lattice of discriminant 4. See also Dolgachev–Keum [16] and Dolgachev [17] for the relation with Hessian quartic surfaces and associated Enriques surfaces.

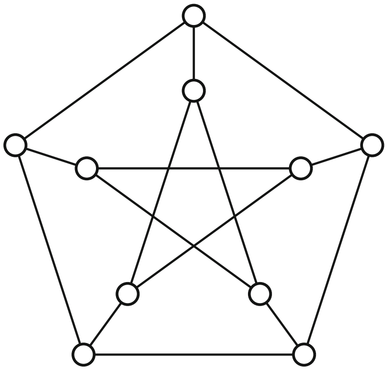
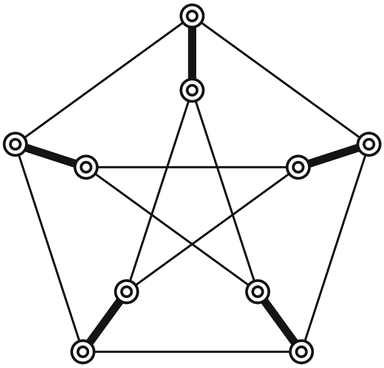
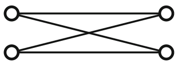


Fig. 1 Petersen graph



Here ◎ is a pair of disjoint smooth rational curves, and ◎—◎ means that the two pairs of smooth rational curves intersect as



whereas ◎━◎ means that the two pairs intersect as

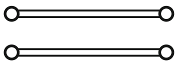


Fig. 2 Smooth rational curves on $Y_{\mathrm{IV},\,p}$

1.4 Plan of the paper

In Sect. 2, we present a precise description of the embedding $\rho\colon S_0 \hookrightarrow S_3$. First we introduce the notion of QP-graphs. Then, using an isomorphism $X_3 \cong F_3$ given by Shioda [6], we show that S_0 is a lattice obtained from a QP-graph and calculate the embedding $\rho\colon S_0 \hookrightarrow S_3$ explicitly. An elliptic modular surface of level 4 over a discrete valuation ring is constructed, and the relation with the Petersen graph is explained geometrically. In Sect. 3, we review the method of Borcherds [3,4] to calculate the orthogonal group of an even hyperbolic lattice and fix terminologies about *chambers*. The application of this method to $K3$ surfaces is also explained. In Sect. 4, we review the results of [1] for $\mathrm{Aut}(X_0)$ and of [2] for $\mathrm{Aut}(X_3)$. Using the chamber tessellations of N_0 and N_3 obtained in these works, we give a proof of Theorems 1.7 and 1.8 in Sect. 5. In Sect. 6, we investigate Enriques involutions of X_0 and X_3.

In this paper, we fix bases of lattices and reduce proofs of our results to simple computations of vectors and matrices. Unfortunately, these vectors and matrices are too large to be presented in the paper. We refer the reader to the author's web site [18] for this data. In the computation, we used `GAP` [19].

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2 The lattices S_0 and S_3

2.1 Graphs and lattices

First we fix terminologies and notation about graphs and lattices.

A *graph* (or more precisely, a *weighted graph*) is a pair (V, η), where V is a set of *vertices* and η is a map from the set $\binom{V}{2}$ of non-ordered pairs of distinct elements of V to $\mathbb{Z}_{\geq 0}$. When the image of η is contained in $\{0, 1\}$, we say that (V, η) is *simple* and denote it by (V, E), where $E = \eta^{-1}(1)$ is the set of *edges*. Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be simple graphs. A *map* $\gamma\colon \Gamma \to \Gamma'$ *of simple graphs* is a pair of maps $\gamma_V\colon V \to V'$ and $\gamma_E\colon E \to E'$ such that, for all $\{v, v'\} \in E$, we have $\gamma_E(\{v, v'\}) = \{\gamma_V(v), \gamma_V(v')\} \in E'$. A graph is depicted by indicating each vertex by $\circ$ and $\eta(\{v, v'\})$ by the number of line segments connecting v and v'. The *Petersen graph* $\mathcal{P} = (V_{\mathcal{P}}, E_{\mathcal{P}})$ is the simple graph given in Fig. 1. It is well known that the automorphism group $\mathrm{Aut}(\mathcal{P})$ of $\mathcal{P}$ is isomorphic to the symmetric group $\mathfrak{S}_5$.

A submodule M of a free $\mathbb{Z}$-module L is *primitive* if L/M is torsion-free. A nonzero vector v of L is *primitive* if $\mathbb{Z}v \subset L$ is primitive.

Let L be a lattice. We say that L is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for all $x \in L$. The *dual lattice* of L is the free $\mathbb{Z}$-module $L^\vee := \mathrm{Hom}(L, \mathbb{Z})$, into which L is embedded by $\langle\ ,\ \rangle$. Hence we have $L^\vee \subset L \otimes \mathbb{Q}$. The *discriminant group* $A(L)$ is the finite abelian group $L^\vee/L$. We say that L is *unimodular* if $A(L)$ is trivial.

With a graph $\Gamma = (V, \eta)$ with $|V| < \infty$, we associate an even lattice $\langle\Gamma\rangle$ as follows. Let $\mathbb{Z}^V$ be the $\mathbb{Z}$-module freely generated by the elements of V. We define a symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbb{Z}^V$ by

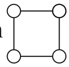
$$\langle v, v'\rangle = \begin{cases} -2 & \text{if } v = v', \\ \eta(\{v, v'\}) & \text{if } v \neq v'. \end{cases}$$

Let $\mathrm{Ker}\langle\ ,\ \rangle \subset \mathbb{Z}^V$ denote the submodule $\{x \in \mathbb{Z}^V \mid \langle x, y\rangle = 0 \text{ for all } y \in \mathbb{Z}^V\}$. Then the quotient module $\langle\Gamma\rangle := \mathbb{Z}^V/\mathrm{Ker}\langle\ ,\ \rangle$ has a natural structure of an even lattice.

Suppose that Z is a $K3$ surface or an Enriques surface defined over an algebraically closed field. Let $\mathcal{L}$ be a set of smooth rational curves on Z. Then the mapping $C \mapsto [C]$ embeds $\mathcal{L}$ into the Néron–Severi lattice S_Z of Z. The *dual graph* of $\mathcal{L}$ is the graph $(\mathcal{L}, \eta)$, where $\eta(\{C_1, C_2\})$ is the intersection number of two distinct curves $C_1, C_2 \in \mathcal{L}$. By abuse of notation, we sometimes use $\mathcal{L}$ to denote the dual graph $(\mathcal{L}, \eta)$ or the image of the embedding $\mathcal{L} \hookrightarrow S_Z$. Then the even lattice $\langle\mathcal{L}\rangle$ constructed from the dual graph of $\mathcal{L}$ is canonically identified with the sublattice of S_Z generated by $\mathcal{L} \subset S_Z$, because every smooth rational curve on Z has self-intersection number -2.

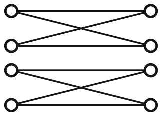
Example 2.1 Let Γ be the graph given in Fig. 2. Then $\langle\Gamma\rangle$ is an even hyperbolic lattice of rank 10 with $A(\langle\Gamma\rangle) \cong (\mathbb{Z}/2\mathbb{Z})^2$. Since the Néron–Severi lattice of an Enriques surface is unimodular of rank 10, the classes of 20 smooth rational curves on $Y_{\mathrm{IV},\,p}$ generate a sublattice of index 2 in the Néron–Severi lattice.

2.2 QP-graph

We introduce the notion of QP-*graphs*, where QP stands for a *quadruple covering of the Petersen graph*. In the following, a *quadrangle* means the simple graph  .

Definition 2.2 A QP-*graph* is a pair $(\mathcal{Q}, \gamma)$ of a simple graph $\mathcal{Q} = (V_{\mathcal{Q}}, E_{\mathcal{Q}})$ and a map $\gamma\colon \mathcal{Q} \to \mathcal{P}$ to the Petersen graph with the following properties.

(i) The map $\gamma_V\colon V_{\mathcal{Q}} \to V_{\mathcal{P}}$ is surjective, and every fiber of γ_V is of size 4.
(ii) For any edge e of $\mathcal{P}$, the subgraph $(\gamma_V^{-1}(e), \gamma_E^{-1}(\{e\}))$ of $\mathcal{Q}$ is isomorphic to the disjoint union of two quadrangles.



(iii) Any two distinct quadrangles in $\mathcal{Q}$ have at most one common vertex.

A map $\gamma\colon \mathcal{Q} \to \mathcal{P}$ satisfying conditions (i)–(iii) is called a QP-*covering map*. Two QP-graphs $(\mathcal{Q}, \gamma)$ and $(\mathcal{Q}', \gamma')$ are said to be *isomorphic* if there exists an isomorphism $h\colon \mathcal{Q} \to \mathcal{Q}'$ such that $\gamma' \circ h = \gamma$.

Proposition 2.3 *Up to isomorphism, there exist exactly two* QP-*graphs* $(\mathcal{Q}_0, \gamma_0)$ *and* $(\mathcal{Q}_1, \gamma_1)$. *The even lattices* $\langle \mathcal{Q}_0 \rangle$ *and* $\langle \mathcal{Q}_1 \rangle$ *are hyperbolic of rank* 20. *The discriminant group* $A(\langle \mathcal{Q}_0 \rangle)$ *of* $\langle \mathcal{Q}_0 \rangle$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$, *whereas* $A(\langle \mathcal{Q}_1 \rangle)$ *is isomorphic to* $(\mathbb{Z}/4\mathbb{Z})^2$.

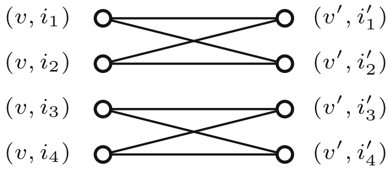
Proof We enumerate all isomorphism classes of QP-graphs. Let Δ be the set of ordered pairs $[\{i_1, i_2\}, \{i_3, i_4\}]$ of non-ordered pairs of elements of $\{1, 2, 3, 4\}$ such that $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. We have $|\Delta| = 6$. Let $\mathcal{T}(\Delta)$ be the set of ordered triples $[\delta_1, \delta_2, \delta_3]$ of elements of Δ such that, if $\mu \neq \nu$, then $\delta_\mu = [\{i_1, i_2\}, \{i_3, i_4\}]$ and $\delta_\nu = [\{i'_1, i'_2\}, \{i'_3, i'_4\}]$ satisfy $|\{i_1, i_2\} \cap \{i'_1, i'_2\}| = 1$. Then we have $|\mathcal{T}(\Delta)| = 48$. The following facts can be easily verified.

(a) The natural action on $\mathcal{T}(\Delta)$ of the full permutation group $\mathfrak{S}_4$ of $\{1, 2, 3, 4\}$ decomposes $\mathcal{T}(\Delta)$ into two orbits o_1 and o_2 of size 24.
(b) For any triple $[\delta_1, \delta_2, \delta_3] \in \mathcal{T}(\Delta)$ and any permutation μ, ν, ρ of $1, 2, 3$, the triple $[\delta_\mu, \delta_\nu, \delta_\rho]$ belongs to the same orbit as $[\delta_1, \delta_2, \delta_3]$.
(c) For $\delta = [\{i_1, i_2\}, \{i_3, i_4\}] \in \Delta$, we put $\bar{\delta} := [\{i_3, i_4\}, \{i_1, i_2\}] \in \Delta$. Then $[\delta_1, \delta_2, \delta_3] \in \mathcal{T}(\Delta)$ and $[\delta_1, \delta_2, \bar{\delta}_3] \in \mathcal{T}(\Delta)$ belong to different orbits.

Let ψ be a map from the set $V_{\mathcal{P}}$ of vertices of $\mathcal{P}$ to the set $\{o_1, o_2\}$ of the orbits. We construct a QP-graph $(\mathcal{Q}_\psi, \gamma_\psi)$ with the set of vertices

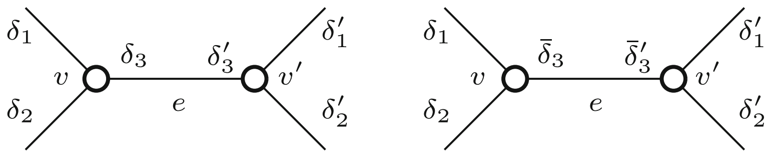
$$V_{\mathcal{Q}} := V_{\mathcal{P}} \times \{1, 2, 3, 4\}$$

as follows. For each vertex $v \in V_{\mathcal{P}}$, we choose an element $[\delta_1, \delta_2, \delta_3]$ from the orbit $\psi(v)$, choose an ordering e_1, e_2, e_3 on the three edges of $\mathcal{P}$ emitting from v, and assign δ_i to the pair (v, e_i) for $i = 1, 2, 3$. Let $e = \{v, v'\}$ be an edge of $\mathcal{P}$. Suppose that $\delta = [\{i_1, i_2\}, \{i_3, i_4\}]$ is assigned to (v, e) and $\delta' = [\{i'_1, i'_2\}, \{i'_3, i'_4\}]$ is assigned to (v', e). Then the edges of $\mathcal{Q}_\psi$ lying over the edge e of $\mathcal{P}$ are the following eight edges.



Let $\gamma_\psi : \mathcal{Q}_\psi \to \mathcal{P}$ be obtained from the first projection $V_\mathcal{Q} \to V_\mathcal{P}$. Then $(\mathcal{Q}_\psi, \gamma_\psi)$ is a QP-graph. The isomorphism class of $(\mathcal{Q}_\psi, \gamma_\psi)$ is independent of the choice of a representative $[\delta_1, \delta_2, \delta_3]$ of each orbit $\psi(v)$ and the choice of the ordering of the edges emitting from each vertex of $\mathcal{P}$. Indeed, changing these choices merely amounts to relabeling the vertices in each fiber of the first projection $V_\mathcal{Q} \to V_\mathcal{P}$ (see fact (b)). It is also obvious that every QP-graph is isomorphic to $(\mathcal{Q}_\psi, \gamma_\psi)$ for some $\psi : V_\mathcal{P} \to \{o_1, o_2\}$.

For an orbit $o \in \{o_1, o_2\}$, let $\bar{o}$ denote the other orbit; $\{o_1, o_2\} = \{o, \bar{o}\}$. Let $\psi : V_\mathcal{P} \to \{o_1, o_2\}$ be a map, and let $e = \{v, v'\}$ be an edge of $\mathcal{P}$. We define $\psi' : V_\mathcal{P} \to \{o_1, o_2\}$ by $\psi'(v) := \overline{\psi(v)}$, $\psi'(v') := \overline{\psi(v')}$ and $\psi'(v'') := \psi(v'')$ for all $v'' \in V_\mathcal{P} \backslash \{v, v'\}$. Then $(\mathcal{Q}_\psi, \gamma_\psi)$ and $(\mathcal{Q}_{\psi'}, \gamma_{\psi'})$ are isomorphic. (See the picture below and fact (c).)



Hence the isomorphism class of $(\mathcal{Q}_\psi, \gamma_\psi)$ depends only on $|\psi^{-1}(o_1)| \bmod 2$. We denote by $(\mathcal{Q}_0, \gamma_0)$ the QP-graph $(\mathcal{Q}_\psi, \gamma_\psi)$ with $|\psi^{-1}(o_1)| \equiv 0 \bmod 2$ and by $(\mathcal{Q}_1, \gamma_1)$ the QP-graph $(\mathcal{Q}_\psi, \gamma_\psi)$ with $|\psi^{-1}(o_1)| \equiv 1 \bmod 2$. Since we have constructed $\mathcal{Q}_0$ and $\mathcal{Q}_1$ explicitly, the assertions on $\langle \mathcal{Q}_0 \rangle$ and $\langle \mathcal{Q}_1 \rangle$ can be proved by direct computation. □

Proposition 2.4 *Let $(\mathcal{Q}, \gamma)$ be a* QP*-graph. Each automorphism $g \in$* Aut$(\mathcal{Q})$ *maps every fiber of $\gamma_V : V_\mathcal{Q} \to V_\mathcal{P}$ to a fiber of γ_V, and hence induces $\bar{g} \in$* Aut$(\mathcal{P})$ *such that $\bar{g} \circ \gamma = \gamma \circ g$. The mapping $g \mapsto \bar{g}$ gives a surjective homomorphism*

$$\mathrm{Aut}(\mathcal{Q}) \to \mathrm{Aut}(\mathcal{P}) \cong \mathfrak{S}_5,$$

and its kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^6$.

Proof Since $\mathcal{P}$ does not contain a quadrangle, every quadrangle of $\mathcal{Q}$ is mapped to an edge of $\mathcal{P}$ by γ. Hence two distinct vertices v, v' of $\mathcal{Q}$ are mapped to the same vertex of $\mathcal{P}$ by γ if and only if $\{v, v'\}$ is not an edge of $\mathcal{Q}$ and there exists a quadrangle of $\mathcal{Q}$ containing v and v'. Thus the first assertion follows. We make the complete list of elements of Aut$(\mathcal{Q})$ by computer and verify the assertion on Aut$(\mathcal{Q}) \to$ Aut$(\mathcal{P})$. □

Corollary 2.5 *A* QP*-covering map $\gamma : \mathcal{Q} \to \mathcal{P}$ from the graph $\mathcal{Q}$ is unique up to the action of* Aut$(\mathcal{P})$. □

2.3 The configurations $\mathcal{L}_{40}$ and $\mathcal{L}_{112}$

In this section, following the argument of Shioda [6], we describe the Néron–Severi lattices S_0 of X_0 and S_3 of X_3 and investigate the embedding $\rho : S_0 \hookrightarrow S_3$ induced by the specialization of X_0 to X_3.

By Theorem 1.1 (1), we have a distinguished set of

$$6 \times 4 + 4^2 = 40$$

smooth rational curves on X_p, where the 6×4 curves are the irreducible components of the six singular fibers of $\sigma : X_p \to \mathbb{P}^1$ and the 4^2 curves are the torsion sections of the Mordell–Weil group. We denote the configuration of these smooth rational curves by $\mathcal{L}_{40,p}$, or simply by $\mathcal{L}_{40}$. The specialization of X_0 to X_p gives a bijection from $\mathcal{L}_{40,0}$ to $\mathcal{L}_{40,p}$, because the specialization preserves the elliptic fibration $\sigma : X_p \to \mathbb{P}^1$ and its zero section. This bijection is obviously compatible with the specialization homomorphism $S_0 \to S_p$.

The set of lines on the Fermat quartic surface F_3 in characteristic 3 has been studied classically by Segre [20]. The surface $F_3 \subset \mathbb{P}^3$ contains exactly 112 lines, and every line on F_3 is defined over the finite field $\mathbb{F}_9$. We denote by $\mathcal{L}_{112}$ the set of these lines. We can easily make the list of defining equations of all lines on F_3 and calculate the dual graph of $\mathcal{L}_{112}$. It is also known ([2]) that the classes of 22 lines appropriately chosen from $\mathcal{L}_{112}$ form a basis of $S_{F_3} \cong S_3$. Fixing a basis of S_3, we can express all classes of lines as integer vectors of length 22 (see [18]).

We show that the specialization of X_0 to $X_3 \cong F_3$ induces an embedding

$$\rho_{\mathcal{L}} : \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$$

of configurations. We recall the construction of the isomorphism $X_3 \cong F_3$ by Shioda [6]. Let $\sigma_F : F_3 \to \mathbb{P}^1$ be the morphism defined by

$$\sigma_F \ : \ [x_1 : x_2 : x_3 : x_4] \ \mapsto \ [x_3^2 - i\, x_4^2 : x_1^2 + i\, x_2^2] = [-x_1^2 + i\, x_2^2 : x_3^2 + i\, x_4^2], \quad (2.1)$$

where $i = \sqrt{-1} \in \mathbb{F}_9$. The generic fiber of σ_F is a curve of genus 1, and σ_F has a section (see the next paragraph). Hence the generic fiber of σ_F is isomorphic to its Jacobian, which is defined by Eq. (1.1) by the result of Bašmakov and Faddeev [21]. Therefore $\sigma_F : F_3 \to \mathbb{P}^1$ is isomorphic to $\sigma : X_3 \to \mathbb{P}^1$ over $\mathbb{P}^1$.

Remark 2.6 In characteristic 0, morphism (2.1) with $i \in \mathbb{C}$ from the Fermat quartic surface to $\mathbb{P}^1$ has no sections.

Using the defining equations of lines and the vector representations of their classes, we confirm the following facts. These facts make the isomorphism between $\sigma_F : F_3 \to \mathbb{P}^1$ and $\sigma : X_3 \to \mathbb{P}^1$ over $\mathbb{P}^1$ more explicit. There exist exactly 6×4 lines on F_3 that are contracted to points by σ_F. These 24 lines form, of course, a configuration of six disjoint quadrangles. Moreover, there exist exactly 64 lines on F_3 that are mapped to $\mathbb{P}^1$ isomorphically by σ_F. Let $z_F \in \mathcal{L}_{112}$ be one of these 64 sections of σ_F. To be explicit, we choose the following line as z_F. (See Remark in Section 4 of [6]):

$$x_1 + i\, x_3 - x_4 = x_2 + x_3 - i\, x_4 = 0. \quad (2.2)$$

Let $\mathrm{MW}(\sigma_F, z_F)$ denote the Mordell–Weil group of $\sigma_F : F_3 \to \mathbb{P}^1$ with the zero section z_F, and let $\mathrm{Triv}(\sigma_F, z_F)$ be the sublattice of S_3 generated by the classes of the zero section z_F and the 24 lines in the singular fibers of σ_F. (This lattice is called the *trivial sublattice* of the Jacobian fibration (σ_F, z_F) in the theory of Mordell–Weil lattices [22].) Let $\mathrm{Triv}^-(\sigma_F, z_F)$ denote the primitive closure of $\mathrm{Triv}(\sigma_F, z_F)$ in S_3. By [22], we have a canonical isomorphism

$$\mathrm{Triv}^-(\sigma_F, z_F)/\mathrm{Triv}(\sigma_F, z_F) \ \cong \ \text{the torsion part of } \mathrm{MW}(\sigma_F, z_F). \quad (2.3)$$

Therefore a section $s : \mathbb{P}^1 \to F_3$ of σ_F is a torsion element of $\mathrm{MW}(\sigma_F, z_F)$ if the class of s belongs to $\mathrm{Triv}^-(\sigma_F, z_F)$. By this criterion, we find 16 lines among the 64 sections of σ_F

that form the torsion part of $\mathrm{MW}(\sigma_F, z_F)$. Thus we obtain the configuration $\mathcal{L}_{40,3}$ on X_3 as a sub-configuration of $\mathcal{L}_{112}$. Combining this embedding $\mathcal{L}_{40,3} \hookrightarrow \mathcal{L}_{112}$ with the bijection $\mathcal{L}_{40} = \mathcal{L}_{40,0} \cong \mathcal{L}_{40,3}$ induced by specialization of X_0 to X_3, we obtain the embedding $\rho_{\mathcal{L}}\colon \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$ induced by the specialization of X_0 to X_3.

The dual graph of $\mathcal{L}_{40}$ is now calculated explicitly. Hence we can prove the following by a direct computation.

Proposition 2.7 *The dual graph of $\mathcal{L}_{40}$ is isomorphic to the* QP*-graph $\mathcal{Q}_1$.* □

Comparing the ranks and the discriminants of $\langle \mathcal{L}_{40} \rangle \cong \langle \mathcal{Q}_1 \rangle$ and S_0, we obtain the following:

Corollary 2.8 *The lattice S_0 is generated by the classes of curves in $\mathcal{L}_{40}$.* □

Corollary 2.9 *The embedding $\rho_{\mathcal{L}}\colon \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$ induces the embedding $\rho\colon S_0 \hookrightarrow S_3$ induced by the specialization of X_0 to X_3. This embedding ρ is primitive.* □

The last assertion follows from the explicit matrix form of the embedding ρ with respect to some bases of S_0 and S_3 (see [18]).

Remark 2.10 The existence of an isomorphism $X_3 \cong F_3$ can be easily seen by the following argument. By [23], we know that X_3 is a supersingular $K3$ surface with Artin invariant 1, and hence is isomorphic to F_3 by the uniqueness of a supersingular $K3$ surface with Artin invariant 1.

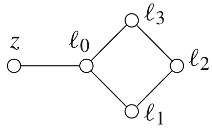
2.4 All embeddings of $\mathcal{L}_{40}$ into $\mathcal{L}_{112}$

The embedding $\rho_{\mathcal{L}}\colon \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$ constructed in the preceding section depends on the choice of σ_F and z_F. In this section, we make the complete list of all embeddings $\mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$.

Let $a \mapsto \bar{a} := a^3$ denote the Frobenius automorphism of the base field k_3. Then the projective automorphism group of $F_3 \subset \mathbb{P}^3$ is equal to

$$\mathrm{PGU}_4(\mathbb{F}_9) := \{\, g \in \mathrm{GL}_4(k_3) \mid {}^{\mathrm{T}}g \cdot \bar{g} \text{ is a scalar matrix} \,\}/k_3^{\times},$$

which is of order 13063680. We can calculate the action of $\mathrm{PGU}_4(\mathbb{F}_9)$ on $\mathcal{L}_{112}$ and on $S_3 = \langle \mathcal{L}_{112} \rangle$. Let $\mathcal{A}$ denote the set of all ordered five tuples $[z, \ell_0, \dots, \ell_3]$ of lines on F_3 that form the configuration whose dual graph is as follows:



(2.4)

Note that $\mathrm{PGU}_4(\mathbb{F}_9)$ acts on $\mathcal{A}$ naturally. We have the following:

Proposition 2.11 *The action of* $\mathrm{PGU}_4(\mathbb{F}_9)$ *on $\mathcal{A}$ is simply transitive.*

Proof By [24], we have the following facts.

(1) Since every line on F_3 is defined over $\mathbb{F}_9$, the intersection points of $\ell \in \mathcal{L}_{112}$ with other lines in $\mathcal{L}_{112}$ are $\mathbb{F}_9$-rational. For each $\mathbb{F}_9$-rational point P of ℓ, there exist exactly three lines in $\mathcal{L}_{112} \backslash \{\ell\}$ that intersect ℓ at P. Hence there exist exactly $112 - 3 \times 10 - 1 = 81$ lines in $\mathcal{L}_{112}$ that are disjoint from ℓ. The group $\mathrm{PGU}_4(\mathbb{F}_9)$ acts on the set of ordered pairs of disjoint lines in $\mathcal{L}_{112}$.

(2) If $\ell_1, \ell_2, \ell_3 \in \mathcal{L}_{112}$ satisfy $\langle \ell_1, \ell_2 \rangle = \langle \ell_2, \ell_3 \rangle = \langle \ell_3, \ell_1 \rangle = 1$, then there exist a plane $\Pi \subset \mathbb{P}^3$ containing ℓ_1, ℓ_2, ℓ_3 and a point $P \in \Pi$ contained in ℓ_1, ℓ_2, ℓ_3. The residual line $\ell_4 = (F_3 \cap \Pi) - (\ell_1 + \ell_2 + \ell_3)$ also passes through P.

(3) Let $[\ell_1, \ell_2]$ be an ordered pair of disjoint lines in $\mathcal{L}_{112}$. Then there exist exactly ten lines that intersect both ℓ_1 and ℓ_2. Let $\mathrm{Stab}([\ell_1, \ell_2])$ denote the stabilizer subgroup of $[\ell_1, \ell_2]$ in $\mathrm{PGU}_4(\mathbb{F}_9)$. Then the restriction homomorphism

$$\mathrm{res}_\ell \colon \mathrm{Stab}([\ell_1, \ell_2]) \to \mathrm{PGL}(\ell_1, \mathbb{F}_9)$$

to the group of linear automorphisms of $\ell_1 \cong \mathbb{P}^1$ over $\mathbb{F}_9$ is surjective, and its kernel is of order 2. Let P be an $\mathbb{F}_9$-rational point of ℓ_1, and let $m_P, m'_P \in \mathcal{L}_{112}$ be the lines that intersect ℓ_1 at P but are disjoint from ℓ_2. Then the non-trivial element of $\mathrm{Ker}(\mathrm{res}_\ell)$ exchanges m_P and m'_P.

The transitivity of the action of $\mathrm{PGU}_4(\mathbb{F}_9)$ on $\mathcal{A}$ follows from these facts. Moreover, we have

$$|\mathcal{A}| = 112 \cdot 81 \cdot 10 \cdot 9 \cdot 16 = 13063680 = |\mathrm{PGU}_4(\mathbb{F}_9)|,$$

where the factor 112 is the number of choices of ℓ_0 in $[z, \ell_0, \dots, \ell_3] \in \mathcal{A}$, the factor 81 is the number of choices of ℓ_2 when ℓ_0 is given, the factor $10 \cdot 9$ is the number of choices of ℓ_1 and ℓ_3 when ℓ_0 and ℓ_2 are given, and the factor 16 is the number of choices of z for a given quadrangle $[\ell_0, \dots, \ell_3]$. Therefore the action of $\mathrm{PGU}_4(\mathbb{F}_9)$ on $\mathcal{A}$ is simply transitive. □

Let $\mathcal{F}$ denote the set of sub-configurations of $\mathcal{L}_{112}$ isomorphic to $\mathcal{L}_{40}$. Let $\alpha = [z_\alpha, \ell_0, \dots, \ell_3]$ be an element of $\mathcal{A}$. Then there exists a unique Jacobian fibration

$$\sigma_\alpha \colon F_3 \to \mathbb{P}^1$$

with the zero section z_α such that $\ell_0 + \ell_1 + \ell_2 + \ell_3$ is a singular fiber of σ_α. The Jacobian fibration (σ_F, z_F) that was used in the construction of $\rho_{\mathcal{L}}$ is obtained as one of the $(\sigma_\alpha, z_\alpha)$. By Proposition 2.11, all Jacobian fibrations $(\sigma_\alpha, z_\alpha)$ are conjugate under the action of $\mathrm{PGU}_4(\mathbb{F}_9)$. Therefore $(\sigma_\alpha, z_\alpha)$ yields a sub-configuration $\mathcal{L}_\alpha$ of $\mathcal{L}_{112}$ isomorphic to $\mathcal{L}_{40}$, and the map $\alpha \mapsto \mathcal{L}_\alpha$ gives a surjection $\lambda \colon \mathcal{A} \to \mathcal{F}$ compatible with the action of $\mathrm{PGU}_4(\mathbb{F}_9)$. The size of a fiber of λ over $\mathcal{L}' \in \mathcal{F}$ is

$$30 \times 2 \times 16 = 960,$$

where the factor 30 is the number of quadrangles in $\mathcal{L}' \cong \mathcal{L}_{40}$, the factor 2 counts the flipping $\ell_1 \leftrightarrow \ell_3$, and the factor 16 is the number of choices of the zero section z_α. Thus we obtain the following:

Corollary 2.12 *The number of sub-configurations of $\mathcal{L}_{112}$ isomorphic to $\mathcal{L}_{40}$ is* $|\mathrm{PGU}_4(\mathbb{F}_9)|/960 = 13608$*, and* $\mathrm{PGU}_4(\mathbb{F}_9)$ *acts on the set of these sub-configurations transitively.* □

2.5 An elliptic modular surface of level 4 over a discrete valuation ring

Let R be a discrete valuation ring such that $2 \in R^\times$ and $i = \sqrt{-1} \in R$. We construct a model of the elliptic modular surface of level 4 over R, that is, we perform over R the resolution of the completion of the affine surface defined by (1.1). This construction explains the isomorphism $\mathcal{L}_{40} \cong \mathcal{Q}_1$ of graphs geometrically.

In this paragraph, all schemes and morphisms are defined over R. We consider the complete quadrangle on $\mathbb{P}^2$ (Fig. 3) such that each of the triple points $t_1, \dots, t_4$ is an R-valued point. Let $M \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $t_1, \dots, t_4$. Let $\bar{l}_1, \dots, \bar{l}_6$ be the strict transforms of the

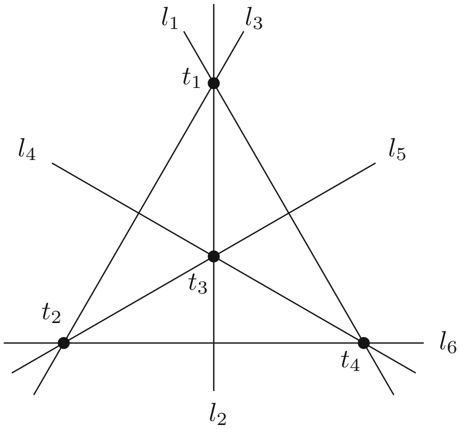


Fig. 3 Complete quadrangle

lines $l_1, \dots, l_6$, and let $\bar{t}_1, \dots, \bar{t}_4$ be the exceptional divisors over $t_1, \dots, t_4$. It is well known that these $6 + 4 = 10$ smooth rational curves on M form a configuration whose dual graph is the Petersen graph $\mathcal{P}$. Let

$$\varphi_M : M \to \mathbb{P}^1 \tag{2.5}$$

be the fibration induced by the pencil of lines on $\mathbb{P}^2$ passing through t_1. (The dependence of the construction on the choice of this $\mathbb{P}^1$-fibration φ_M will be discussed in Sect. 4.3. See Remark 4.5.) Then φ_M has exactly three singular fibers $\bar{l}_1 + \bar{t}_4$, $\bar{l}_2 + \bar{t}_3$, $\bar{l}_3 + \bar{t}_2$, and four sections $\bar{t}_1, \bar{l}_4, \bar{l}_5, \bar{l}_6$. Let $M' \to M$ be the blowup at the nodes on $\bar{l}_1 + \bar{t}_4, \bar{l}_2 + \bar{t}_3, \bar{l}_3 + \bar{t}_2$, and let $\varphi'_M : M' \to \mathbb{P}^1$ be the composite of φ_M and $M' \to M$. We choose an affine parameter λ on the base curve $\mathbb{P}^1$ of φ'_M such that the singular fibers are located over $\lambda = 0, 1, \infty$. Let $\tilde{M}' \to \mathbb{P}^1$ be the pullback of $\varphi'_M : M' \to \mathbb{P}^1$ by the covering $\mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\sigma \mapsto \lambda = ((\sigma + \sigma^{-1})/2)^2, \tag{2.6}$$

and let $\tilde{M} \to \tilde{M}'$ be the normalization of $\tilde{M}'$. Then $\tilde{M}$ is smooth over R, and the natural morphism $\tilde{\varphi}_M : \tilde{M} \to \mathbb{P}^1$ to the σ-line has exactly 6 singular fibers over $\sigma = 0, \pm 1, \pm i, \infty$. Each singular fiber is a union of three smooth rational curves forming the configuration ○—○—○ , the middle of which is with multiplicity 2. Let $\tilde{t}_1, \tilde{l}_4, \tilde{l}_5, \tilde{l}_6$ be the pullbacks of the sections $\bar{t}_1, \bar{l}_4, \bar{l}_5, \bar{l}_6$ of φ_M by $\tilde{M} \to M$. For a divisor D on $\tilde{M}$, let $[D]$ denote the class of D in the Picard group Pic $\tilde{M}$. Note that, via $\tilde{M} \to M$, a fiber F of $\varphi_M : M \to \mathbb{P}^1$ is pulled back to a sum of two fibers of $\tilde{\varphi}_M : \tilde{M} \to \mathbb{P}^1$, and hence the class $[\tilde{F}]$ of the pullback $\tilde{F}$ of F is divisible by 2 in Pic $\tilde{M}$. Let $[H] \in$ Pic $\tilde{M}$ denote the class of the pullback of a general line of $\mathbb{P}^2$. We put $B := \tilde{t}_1 + \tilde{l}_4 + \tilde{l}_5 + \tilde{l}_6$. Since $[\tilde{F}] = [H] - [\tilde{t}_1]$ and $[\tilde{l}_i] = [H] - [\tilde{t}_j] - [\tilde{t}_k]$ for $(i, j, k) = (4, 3, 4), (5, 2, 3), (6, 2, 4)$, we have

$$[B] = 3[\tilde{F}] + 2[2\tilde{t}_1 - \tilde{t}_2 - \tilde{t}_3 - \tilde{t}_4].$$

Therefore $[B]$ is divisible by 2 in Pic $\tilde{M}$, and we can construct a double covering $\mathcal{X} \to \tilde{M}$ branched along B. Then $\mathcal{X}$ is a model of the elliptic modular surface of level 4 over R, and the Jacobian fibration $\sigma : \mathcal{X} \to \mathbb{P}^1$ is obtained as the composite of the double covering $\mathcal{X} \to \tilde{M}$ and $\tilde{\varphi}_M : \tilde{M} \to \mathbb{P}^1$.

The QP-covering map $\mathcal{L}_{40} \to \mathcal{P}$ (see Corollary 2.5) is constructed as follows. We consider an F-valued point of Spec R, where F is a field. We put $X_F := \mathcal{X} \otimes_R F$, and $\tilde{M}_F := \tilde{M} \otimes_R F$,

$M_F := M \otimes_R F$. Let $\mathcal{E}_F$ be the generic fiber of $\sigma \otimes F: X_F \to \mathbb{P}^1_F$, which is an elliptic curve over the function field $F(\sigma)$ defined by (1.1). Let $m_2: X_F \to X_F$ be the rational map induced by multiplication by 2 on $\mathcal{E}_F$. Then the rational map

$$\mu_F \;:\; X_F \xrightarrow{m_2} X_F \longrightarrow \tilde{M}_F \longrightarrow M_F \tag{2.7}$$

gives a map from $\mathcal{L}_{40}$ to the Petersen graph $\mathcal{P}$ formed by $\{\bar{t}_1, \dots, \bar{t}_4, \bar{\ell}_1, \dots, \bar{\ell}_6\}$.

Proposition 2.13 *The rational map μ_F induces a Galois extension of the function fields. Its Galois group* $\mathrm{Gal}(\mu)$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^5$ *and is generated by the inversion* $\iota: (X, Y, \sigma) \mapsto (X, -Y, \sigma)$ *of the elliptic curve $\mathcal{E}_F$, two involutions*

$$(X, Y, \sigma) \mapsto (X, Y, -\sigma), \quad (X, Y, \sigma) \mapsto (X, Y, 1/\sigma), \tag{2.8}$$

and the translations by the 2-torsion points of $\mathcal{E}_F$.

Proof The inversion ι and the involutions in (2.8) fix each 2-torsion point of $\mathcal{E}_F$. Hence the involutions in the statement of Proposition 2.13 generate a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$. By (2.6), the function field $F(\sigma)$ is a Galois extension of $F(\lambda)$ with Galois group generated by $\sigma \mapsto -\sigma$ and $\sigma \mapsto 1/\sigma$. Hence the covering $\tilde{M}_F \to M_F$ in (2.7) is the quotient by the involutions in (2.8). The covering $X_F \to \tilde{M}_F$ in (2.7) is the quotient by ι, and the map m_2 is the quotient by the group of translations by the 2-torsion points of $\mathcal{E}_F$. Thus the proof is completed. □

2.6 Another model of the elliptic modular surface of level 4

We give a much simpler construction of a $(\mathbb{Z}/2\mathbb{Z})^5$-covering $X_0 \to M_{\mathbb{C}}$ over the complex numbers by means of a Hirzebruch covering (see Hironaka [25]). This section is due to a suggestion by one of the referees of the first version of the paper. Let $M_{\mathbb{C}}$ be the complex surface obtained by blowing up $\mathbb{P}^2_{\mathbb{C}}$ at the triple points of the complete quadrangle on $\mathbb{P}^2_{\mathbb{C}}$, and let $M^\circ_{\mathbb{C}}$ be the complement of the ten (-1)-curves on $M_{\mathbb{C}}$. We have a canonical surjective homomorphism $\pi_1(M^\circ_{\mathbb{C}}) \twoheadrightarrow H_1(M^\circ_{\mathbb{C}}, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^5$. It is known (see [25]) that the corresponding étale covering $W^\circ \to M^\circ_{\mathbb{C}}$ extends to a finite morphism $W \to M_{\mathbb{C}}$ from a smooth surface W and that W is a $K3$ surface.

Proposition 2.14 *The surface W has a Jacobian fibration $\sigma_W: W \to \mathbb{P}^1$ that is isomorphic to $\sigma: X_0 \to \mathbb{P}^1$.*

Proof Consider the $(\mathbb{Z}/2\mathbb{Z})^5$-covering $\gamma: \mathbb{P}^5 \to \mathbf{P}^5$ defined by

$$[x_0 : x_1 : \cdots : x_6] \mapsto [X_0 : X_1 : \cdots : X_6] = [x_0^2 : x_1^2 : \cdots : x_6^2].$$

Let $P \subset \mathbf{P}^5$ be the linear plane defined by

$$X_1 - X_2 + X_3 = -X_3 + X_5 + X_6 = X_2 + X_4 - X_5 = 0,$$

and, for $i = 1, \dots, 6$, let $l_i \subset P$ denote the intersection of P and the coordinate hyperplane $X_i = 0$. Then the six lines $l_1, \dots, l_6$ form the complete quadrangle in Fig. 3. The surface $\overline{W} := \gamma^{-1}(P) \subset \mathbb{P}^5$ is the complete intersection of three quadratic hypersurfaces

$$x_1^2 - x_2^2 + x_3^2 = -x_3^2 + x_5^2 + x_6^2 = x_2^2 + x_4^2 - x_5^2 = 0. \tag{2.9}$$

The finite covering $\gamma|\overline{W}: \overline{W} \to P$ extends to the covering $\gamma_W: W \to M_{\mathbb{C}}$ by the blowing up of $M_{\mathbb{C}} \to P$ at the triple points $t_1, \dots, t_4$ of the complete quadrangle on P. The pullback of each line l_i by $\gamma|\overline{W}$ is a union of four conics, and $\overline{W}$ has 4×4 nodes over $t_1, \dots, t_4$. Thus we obtain a configuration $\mathcal{L}_W$ of 40 smooth rational curves on W consisting of 4×6 pullbacks of conics on $\overline{W}$ and 4×4 exceptional curves over the nodes of $\overline{W}$. By computing the intersection numbers of the 24 conics and the incidence relation between the conics and the 16 nodes, we can write the intersection matrix of the configuration $\mathcal{L}_W$ explicitly. Then we confirm that this configuration $\mathcal{L}_W$ is isomorphic to $\mathcal{L}_{40}$. In fact, by Proposition 2.4, there exist 7680 isomorphisms between $\mathcal{L}_W$ and $\mathcal{L}_{40}$. Among these isomorphisms, we have 1536 isomorphisms such that the 16 smooth rational curves corresponding to the nodes of $\overline{W}$ are mapped to the sections of $\sigma\colon X_0 \to \mathbb{P}^1$ and the 24 smooth rational curves over the lines l_i are mapped to the irreducible components of singular fibers of σ. Hence W has an elliptic fibration $\sigma_W\colon W \to \mathbb{P}^1$ with a section and 6 singular fibers of type I_4. By [26], such an elliptic $K3$ surface is unique up to isomorphism. Hence $\sigma_W\colon W \to \mathbb{P}^1$ is isomorphic to $\sigma\colon X_0 \to \mathbb{P}^1$. □

Remark 2.15 The Jacobian fibration $\sigma_W\colon W \to \mathbb{P}^1$ is obtained from the elliptic fibration $M_{\mathbb{C}} \to \mathbb{P}^1$ induced by the pencil of conics passing through all the triple points $t_1, \dots, t_4$. See Remark 4.5, which also explains the number $1536 = 7680/5$ of the special isomorphisms $\mathcal{L}_W \cong \mathcal{L}_{40}$ in the proof.

For $J \subset \{1, \dots, 6\}$, let $\tilde{\tau}_J$ denote the involution of $\mathbb{P}^5$ given by

$$x_m \mapsto -x_m \;\text{ if }\; m \in J, \quad x_n \mapsto x_n \;\text{ if }\; n \notin J.$$

Note that $\tilde{\tau}_J = \tilde{\tau}_{J'}$ if $J \cap J' = \emptyset$ and $J \cup J' = \{1, \dots, 6\}$. The Galois group $\mathrm{Gal}(\gamma_W)$ of the covering $\gamma_W\colon W \to M_{\mathbb{C}}$ consists of the restrictions $\tau_J := \tilde{\tau}_J|\overline{W}$ of these involutions $\tilde{\tau}_J$ to $\overline{W}$. Let S_W denote the Néron–Severi lattice of W, which is equal to $\langle \mathcal{L}_W \rangle$. We can calculate the action of $\mathrm{Gal}(\gamma_W)$ on S_W explicitly.

For an isomorphism $\varphi\colon \mathcal{L}_W \cong \mathcal{L}_{40}$ of graphs, let $\langle\varphi\rangle\colon S_W \cong S_0$ denote the induced isometry of lattices, and let $\mathrm{O}(\langle\varphi\rangle)\colon \mathrm{O}(S_W) \cong \mathrm{O}(S_0)$ denote the induced isomorphism of the automorphism groups of lattices. By checking all the 7680 isomorphisms $\varphi\colon \mathcal{L}_W \cong \mathcal{L}_{40}$, we confirmed the following fact. See Remark 4.5 for a geometric reason of this result.

Proposition 2.16 *For each isomorphism* $\varphi\colon \mathcal{L}_W \cong \mathcal{L}_{40}$ *of graphs, the isomorphism* $\mathrm{O}(\langle\varphi\rangle)$ *maps* $\mathrm{Gal}(\gamma_W) \subset \mathrm{O}^+(S_W)$ *to* $\mathrm{Gal}(\mu) \subset \mathrm{O}^+(S_0)$ *isomorphically.* □

By Barth–Hulek [8], we know that the sum I of the classes of sections of $\sigma\colon X_0 \to \mathbb{P}^1$ is divisible by 2 in $\mathrm{Pic}\, X_0$. We put $h_8 := (1/2)I + F$, where $F \in \mathrm{Pic}\, X_0$ is a fiber of σ. Then h_8 is primitive in $\mathrm{Pic}\, X_0$ and nef of degree 8. The complete linear system $|h_8|$ is base-point-free, because there exist no vectors $f \in S_0$ such that $\langle f, f\rangle = 0$ and $\langle f, h_8\rangle = 1$ (see Nikulin [27] and Proposition 12 of [8]). Let $\Phi_8\colon X_0 \to \mathbb{P}^5$ be the morphism induced by $|h_8|$. The curves contracted by Φ_8 are exactly the sections of $\sigma\colon X_0 \to \mathbb{P}^1$, and Φ_8 maps each irreducible component of singular fibers of σ to a conic. Hence the image of Φ_8 is equal to $\overline{W}$. We consider the involutions τ_J of $\overline{W}$ as elements of $\mathrm{Aut}(X_0)$ via the birational morphism Φ_8. By Proposition 2.16, we have the following description of $\mathrm{Gal}(\mu)$ simpler than the one given in Proposition 2.13.

Proposition 2.17 *The Galois group* $\mathrm{Gal}(\mu)$ *consists of* 32 *involutions* τ_J. □

Remark 2.18 In [28], Abo–Sasakura–Terasoma studied X_p, where $p \equiv 1 \bmod 4$, and obtained an isomorphism from X_p to the reduction of the complete intersection (2.9) modulo p.

3 Borcherds' method

3.1 Chambers

We fix notions about tessellation of a positive cone of an even hyperbolic lattice by chambers.

Let L be an even lattice. A vector $r \in L$ is called a *root* if $\langle r, r\rangle = -2$. The set of roots of L is denoted by $\mathcal{R}(L)$.

Let L be an even hyperbolic lattice. Let $\mathcal{P}(L)$ be one of the two connected components of $\{x \in L \otimes \mathbb{R} \mid \langle x, x\rangle > 0\}$. Then $\mathrm{O}^+(L)$ acts on $\mathcal{P}(L)$. For $v \in L \otimes \mathbb{Q}$ with $\langle v, v\rangle < 0$, let $(v)^\perp$ denote the hyperplane of $\mathcal{P}(L)$ defined by $\langle x, v\rangle = 0$. Let $\mathcal{V}$ be a set of vectors of $L \otimes \mathbb{Q}$ such that $\langle v, v\rangle < 0$ for all $v \in \mathcal{V}$. We assume that *the family* $\{(v)^\perp \mid v \in \mathcal{V}\}$ *of hyperplanes is locally finite in* $\mathcal{P}(L)$. A $\mathcal{V}$*-chamber* is the closure in $\mathcal{P}(L)$ of a connected component of

$$\mathcal{P}(L) \setminus \bigcup_{v \in \mathcal{V}} (v)^\perp.$$

Typical examples are $\mathcal{R}(L)$-chambers defined by the set $\mathcal{R}(L)$ of roots of L.

Definition 3.1 Let N be a closed subset of $\mathcal{P}(L)$. We say that N *is tessellated by* $\mathcal{V}$*-chambers* if N is a union of $\mathcal{V}$-chambers. Suppose that N is tessellated by $\mathcal{V}$-chambers, and let H be a subgroup of $\mathrm{O}^+(L)$ that preserves N. We say that H *preserves the tessellation of* N *by* $\mathcal{V}$*-chambers* if any $g \in H$ maps each $\mathcal{V}$-chamber in N to a $\mathcal{V}$-chamber. Suppose that this is the case. We say that the tessellation of N is H*-transitive* if H acts transitively on the set of $\mathcal{V}$-chambers in N.

Remark 3.2 Let U be a subset of $\mathcal{V}$ such that the closed subset

$$N_U := \{\, x \in \mathcal{P}(L) \mid \langle x, v\rangle \ge 0 \ \text{ for all } v \in U \,\}$$

of $\mathcal{P}(L)$ contains an interior point. Then N_U is tessellated by $\mathcal{V}$-chambers. In particular, if $\mathcal{V}'$ is a subset of $\mathcal{V}$, then each $\mathcal{V}'$-chamber is tessellated by $\mathcal{V}$-chambers.

Let D be a $\mathcal{V}$-chamber. We put

$$\mathrm{Aut}(D) := \{\, g \in \mathrm{O}^+(L) \mid D^g = D \,\}.$$

A *wall of* D is a closed subset of D of the form $(v)^\perp \cap D$ such that the hyperplane $(v)^\perp$ of $\mathcal{P}(L)$ is disjoint from the interior of D and $(v)^\perp \cap D$ contains a non-empty open subset of $(v)^\perp$. We say that a hyperplane $(v)^\perp$ of $\mathcal{P}(L)$ *defines a wall of* D if $(v)^\perp \cap D$ is a wall of D. We say that a vector $v \in L \otimes \mathbb{Q}$ with $\langle v, v\rangle < 0$ *defines a wall of* D if $(v)^\perp$ defines a wall of D and $\langle v, x\rangle \ge 0$ for all $x \in D$. Note that, for each wall of D, there exists a unique *primitive* vector in $L^\vee$ defining the wall. Let $(v)^\perp \cap D$ be a wall of D. Then there exists a unique $\mathcal{V}$-chamber D' such that the interiors of D and D' are disjoint and that $(v)^\perp \cap D$ is equal to $(v)^\perp \cap D'$. (Hence $(v)^\perp \cap D'$ is a wall of D'.) We say that D' is a $\mathcal{V}$-chamber *adjacent to* D *across the wall* $(v)^\perp \cap D$. A *face* of D is a closed subset of D of the form $F \cap D$ such that

$$F = (v_1)^\perp \cap \cdots \cap (v_m)^\perp, \quad \text{where } (v_1)^\perp, \ldots, (v_m)^\perp \text{ define walls of } D,$$

and that $F \cap D$ contains a non-empty open subset of F.

Example 3.3 We consider the tessellation of $\mathcal{P}(L)$ by $\mathcal{R}(L)$-chambers. Each root r of L defines a *reflection* $s_r \in \mathrm{O}^+(L)$ via $x \mapsto x + \langle x, r\rangle r$. Let $W(L)$ denote the subgroup of $\mathrm{O}^+(L)$ generated by all the reflections with respect to the roots. Then the tessellation of $\mathcal{P}(L)$ by $\mathcal{R}(L)$-chambers is $W(L)$-transitive. An $\mathcal{R}(L)$-chamber N is a fundamental domain of the action of $W(L)$ on $\mathcal{P}(L)$, and $\mathrm{O}^+(L)$ is equal to $W(L) \rtimes \mathrm{Aut}(N)$. Moreover, $W(L)$ is generated by the reflections s_r associated with the roots r of L defining the walls of N, and the faces of codimension 2 of N give the defining relations of $W(L)$ with respect to this set of generators.

Let L_{26} be an even *unimodular* hyperbolic lattice of rank 26, which is unique up to isomorphism. The shape of an $\mathcal{R}(L_{26})$-chamber was determined by Conway [29], and hence we call an $\mathcal{R}(L_{26})$-chamber a *Conway chamber*. Let w be a nonzero primitive vector of L_{26} with $\langle w, w\rangle = 0$ such that w is contained in the closure of $\mathcal{P}(L_{26})$ in $L_{26} \otimes \mathbb{R}$. We say that w is a *Weyl vector* if the lattice $\langle w\rangle^\perp/\langle w\rangle$ is isomorphic to the negative-definite Leech lattice, where $\langle w\rangle^\perp$ is the orthogonal complement in L_{26} of $\langle w\rangle := \mathbb{Z}w \subset L_{26}$. Let $w \in L_{26}$ be a Weyl vector. Then a root r of L_{26} is called a *Leech root with respect to* w if $\langle w, r\rangle = 1$. We put

$$\mathcal{C}(w) := \{\, x \in \mathcal{P}(L_{26}) \mid \langle x, r\rangle \geq 0 \;\text{ for all Leech roots } r \text{ with respect to } w \,\}.$$

Theorem 3.4 (Conway [29]) *The mapping $w \mapsto \mathcal{C}(w)$ gives a bijection from the set of Weyl vectors to the set of Conway chambers.* □

3.2 Borcherds' method

Borcherds [3,4] developed a method to analyze $\mathcal{R}(S)$-chambers of an even hyperbolic lattice S by means of Conway chambers. We briefly review this method, and fix some terminologies. See [30] for details of the algorithms.

Let S be an even hyperbolic lattice. Suppose that we have a primitive embedding $i\colon S \hookrightarrow L_{26}$ such that the orthogonal complement R of S in L_{26} satisfies the following condition:

$$R \text{ cannot be embedded into the negative-definite Leech lattice.} \tag{3.1}$$

(This condition is fulfilled, for example, if R contains a root.) We choose $\mathcal{P}(S)$ so that the embedding $i\colon S \hookrightarrow L_{26}$ induces an embedding $i_{\mathcal{P}}\colon \mathcal{P}(S) \hookrightarrow \mathcal{P}(L_{26})$. Let

$$\mathrm{pr}_S\colon L_{26} \otimes \mathbb{Q} \to S \otimes \mathbb{Q}$$

denote the orthogonal projection. A hyperplane $(v)^\perp$ of $\mathcal{P}(L_{26})$ intersects $\mathcal{P}(S)$ in a hyperplane if and only if $\langle \mathrm{pr}_S(v), \mathrm{pr}_S(v)\rangle < 0$, and, if this is the case, we have $\mathcal{P}(S) \cap (v)^\perp = (\mathrm{pr}_S(v))^\perp$. We put

$$\mathcal{V}(i) := \{\, \mathrm{pr}_S(r) \mid r \in \mathcal{R}(L_{26}), \;\; \langle \mathrm{pr}_S(r), \mathrm{pr}_S(r)\rangle < 0 \,\}. \tag{3.2}$$

The tessellation of $\mathcal{P}(L_{26})$ by Conway chambers induces a tessellation of $\mathcal{P}(S)$ by $\mathcal{V}(i)$-chambers. Each $\mathcal{V}(i)$-chamber is of the form $i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$. It is easily seen (see [30]) that assumption (3.1) implies that each $\mathcal{V}(i)$-chamber has only a finite number of walls. The defining vectors of walls of a $\mathcal{V}(i)$-chamber $i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$ can be calculated from the Weyl vector $w \in L_{26}$ of the Conway chamber $\mathcal{C}(w)$. From this set of walls of $i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$, we can calculate the finite group $\mathrm{Aut}(i_{\mathcal{P}}^{-1}(\mathcal{C}(w))) \subset \mathrm{O}^+(S)$. Moreover, for each wall $(v)^\perp \cap i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$ of a $\mathcal{V}(i)$-chamber $i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$, we can calculate a Weyl vector w' such that $i_{\mathcal{P}}^{-1}(\mathcal{C}(w'))$ is the $\mathcal{V}(i)$-chamber adjacent to $i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$ across the wall $(v)^\perp \cap i_{\mathcal{P}}^{-1}(\mathcal{C}(w))$.

Since $\mathcal{R}(S) \subset \mathcal{V}(i)$, Remark 3.2 implies the following:

Proposition 3.5 *An $\mathcal{R}(S)$-chamber is tessellated by $\mathcal{V}(i)$-chambers.* □

3.3 Discriminant forms

For the application of Borcherds' method to $K3$ surfaces, we need the notion of discriminant forms due to Nikulin [31].

Let $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ be a non-degenerate quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$ on a finite abelian group A. We denote by $\mathrm{O}(q)$ the automorphism group of (A, q). For a prime p, we denote by A_p the p-part of A and by $q_p\colon A_p \to \mathbb{Q}/2\mathbb{Z}$ the restriction of q to A_p. Then we have a canonical orthogonal direct-sum decomposition

$$(A, q) = \bigoplus (A_p, q_p).$$

Hence $\mathrm{O}(q)$ is canonically isomorphic to the direct product of $\mathrm{O}(q_p)$.

Let L be an even lattice, and let $A(L) = L^\vee/L$ denote the discriminant group of L. We define the *discriminant form of* L

$$q(L)\colon A(L) \to \mathbb{Q}/2\mathbb{Z}$$

by $q(L)(\bar{x}) := \langle x, x\rangle \bmod 2\mathbb{Z}$, where $x \mapsto \bar{x}$ is the natural projection $L^\vee \to A(L)$. Then we have a natural homomorphism

$$\eta_L\colon \mathrm{O}(L) \to \mathrm{O}(q(L)).$$

Let M be a primitive sublattice of an even lattice L, and N the orthogonal complement of M in L. Let $\mathrm{O}(L, M)$ denote the subgroup $\{g \in \mathrm{O}(L) \mid M^g = M\}$ of $\mathrm{O}(L)$. Then we have a canonical embedding $\mathrm{O}(L, M) \hookrightarrow \mathrm{O}(M) \times \mathrm{O}(N)$. The submodule $L \subset M^\vee \oplus N^\vee$ defines a subgroup $\Gamma_L := L/(M \oplus N) \subset A(M) \times A(N)$. By Nikulin [31], we have the following:

Proposition 3.6 *Let p be a prime that does not divide $|A(M)|$. Then $N \hookrightarrow L$ induces an isomorphism $q(L)_p \cong q(N)_p$, which is compatible with the actions of $\mathrm{O}(L, M)$ on L and on N.* □

Proposition 3.7 *Let p be a prime that does not divide $|A(L)|$. Then the p-part of Γ_L is the graph of an isomorphism $q(M)_p \cong -q(N)_p$, which is compatible with the actions of $\mathrm{O}(L, M)$ on M and on N.* □

Proposition 3.8 *Suppose that L is unimodular, and let $\gamma_L\colon q(M) \cong -q(N)$ be the isomorphism with the graph Γ_L. Let H be a subgroup of $\mathrm{O}(N)$. Then $g \in \mathrm{O}(M)$ extends to $\tilde{g} \in \mathrm{O}(L, M)$ with $\tilde{g}|_N \in H$ if and only if the isomorphism $\mathrm{O}(q(M)) \cong \mathrm{O}(q(N))$ induced by γ_L maps $\eta_M(g) \in \mathrm{O}(q(M))$ into $\eta_N(H) \subset \mathrm{O}(q(N))$.* □

3.4 Geometric application of Borcherds' method

Let Z be a $K3$ surface defined over an algebraically closed field. We use the notation S_Z, $\mathcal{P}_Z$ and N_Z defined in Sect. 1.1. The following is well known.

Proposition 3.9 *The closed subset N_Z of $\mathcal{P}_Z$ is an $\mathcal{R}(S_Z)$-chamber. The mapping $C \mapsto ([C])^\perp \cap N_Z$ gives a one-to-one correspondence between the set of smooth rational curves on Z and the set of walls of N_Z.* □

Since the action of $\mathrm{O}^+(S_Z)$ on $\mathcal{P}_Z$ preserves the tessellation by $\mathcal{R}(S_Z)$-chambers and an ample class is an interior point of $N_Z \subset \mathcal{P}_Z$, we obtain the following.

Corollary 3.10 *Let $a \in S_Z$ be an ample class. Then the following three conditions on $g \in \mathrm{O}^+(S_Z)$ are equivalent:* (i) $N_Z = N_Z^g$. (ii) $N_Z \cap N_Z^g$ *contains an interior point of* N_Z. (iii) *There exist no roots r of S_Z such that $\langle r, a\rangle$ and $\langle r, a^g\rangle$ have different signs.* □

Let Z be a complex $K3$ surface. Let T_Z denote the orthogonal complement of $S_Z = H^2(Z, \mathbb{Z}) \cap H^{1,1}(Z)$ in the even unimodular lattice $H^2(Z, \mathbb{Z})$ with the cup-product. Then $T_Z \otimes \mathbb{C}$ contains a one-dimensional subspace $H^{2,0}(Z) = \mathbb{C}\,\omega$, where ω is a nonzero holomorphic 2-form on Z. We put

$$\mathrm{O}(T_Z, \omega) := \{\, g \in \mathrm{O}(T_Z) \mid \mathbb{C}\,\omega^g = \mathbb{C}\,\omega\,\}.$$

Recall that we have a natural homomorphism $\eta_{T_Z}\colon \mathrm{O}(T_Z) \to \mathrm{O}(q(T_Z))$. We put

$$\mathrm{O}(q(T_Z), \omega) := \text{the image of } \mathrm{O}(T_Z, \omega) \text{ under } \eta_{T_Z}.$$

The even unimodular overlattice $H^2(Z, \mathbb{Z})$ of $S_Z \oplus T_Z$ induces an isomorphism γ_H between $q(S_Z)$ and $-q(T_Z)$. Let $\mathrm{O}(q(S_Z), \omega)$ denote the subgroup of $\mathrm{O}(q(S_Z))$ corresponding to $\mathrm{O}(q(T_Z), \omega)$ via the isomorphism $\mathrm{O}(q(T_Z)) \cong \mathrm{O}(q(S_Z))$ induced by γ_H. By Proposition 3.8, an isometry $g \in \mathrm{O}(S_Z)$ extends to an isometry $\tilde{g}$ of $H^2(Z, \mathbb{Z})$ that preserves $H^{2,0}(Z)$ if and only if $\eta_{S_Z}(g) \in \mathrm{O}(q(S_Z), \omega)$.

Let Z be a supersingular $K3$ surface defined over an algebraically closed field k_p of odd characteristic p. Then $A(S_Z)$ is an $\mathbb{F}_p$-vector space, and we have the *period* of Z, which is a subspace of $A(S_Z) \otimes k_p$. (See Ogus [32,33].) Let $\mathrm{O}(q(S_Z), \omega)$ denote the subgroup of $\mathrm{O}(q(S_Z))$ consisting of automorphisms that preserve the period.

In the two cases where Z is defined over $\mathbb{C}$ or supersingular in odd characteristic, we call the condition

$$\eta_{S_Z}(g) \in \mathrm{O}(q(S_Z), \omega) \tag{3.3}$$

on $g \in \mathrm{O}^+(S_Z)$ the *period condition*. In these two cases, we have the Torelli theorem. (See Piatetski-Shapiro and Shafarevich [34], Ogus [32,33] for $p > 3$ and Bragg and Lieblich [35] for $p \geq 3$.) By virtue of this theorem, we have the following:

Theorem 3.11 *Let Z be a complex $K3$ surface or a supersingular $K3$ surface in odd characteristic, and let $\psi_Z\colon \mathrm{Aut}(Z) \to \mathrm{O}^+(S_Z)$ be the natural representation of $\mathrm{Aut}(Z)$ on S_Z. Then an isometry $g \in \mathrm{O}^+(S_Z)$ belongs to the image of ψ_Z if and only if g preserves N_Z and satisfies the period condition* (3.3). □

We explain the procedure of Borcherds' method in the simplest case. See [30] for more general cases. In the following, we assume that Z is a complex $K3$ surface or a supersingular $K3$ surface in odd characteristic. We also assume that *ψ_Z is injective, and regard $\mathrm{Aut}(Z)$ as a subgroup of $\mathrm{O}^+(S_Z)$*. We search for a primitive embedding $i\colon S_Z \hookrightarrow L_{26}$ inducing $i_{\mathcal{P}}\colon \mathcal{P}_Z \hookrightarrow \mathcal{P}(L_{26})$ and a Weyl vector $w_0 \in L_{26}$ with the following properties, and look at the tessellation of the $\mathcal{R}(S_Z)$-chamber N_Z by $\mathcal{V}(i)$-chambers, where $\mathcal{V}(i)$ is defined by (3.2).

(I) Let R denote the orthogonal complement of S_Z in L_{26}. We require that R satisfies (3.1), so that each $\mathcal{V}(i)$-chamber has only a finite number of walls. We also require that $\eta_R\colon \mathrm{O}(R) \to \mathrm{O}(q(R))$ is surjective. By Proposition 3.8, every isometry $g \in \mathrm{O}^+(S_Z)$ extends to an isometry of L_{26}. Hence the action of $\mathrm{O}^+(S_Z)$ preserves the tessellation of $\mathcal{P}_Z$ by $\mathcal{V}(i)$-chambers. In particular, the action of $\mathrm{Aut}(Z)$ on N_Z preserves the tessellation of N_Z by $\mathcal{V}(i)$-chambers.

(II) Let D be the closed subset $i_{\mathcal{P}}^{-1}(\mathcal{C}(w_0))$ of $\mathcal{P}_Z$. We require that D contains an ample class in its interior. Then D is a $\mathcal{V}(i)$-chamber contained in N_Z.

Definition 3.12 The $\mathcal{V}(i)$-chamber D is called the *initial chamber* of this procedure. A wall $(v)^\perp \cap D$ of D is called an *outer wall* if $(v)^\perp$ defines a wall of the $\mathcal{R}(S_Z)$-chamber N_Z, that is, if there exists a root r of S_Z such that $(v)^\perp = (r)^\perp$. We call the wall $(v)^\perp \cap D$ an *inner wall* otherwise. Let $\mathcal{W}_{\text{out}}(D)$ and $\mathcal{W}_{\text{inn}}(D)$ denote the set of outer walls and inner walls, respectively.

We calculate the set of walls of the initial chamber D. Since each outer wall corresponds to a smooth rational curve on Z by Proposition 3.9, we obtain a configuration of smooth rational curves on Z from $\mathcal{W}_{\text{out}}(D)$.

(III) We calculate $\text{Aut}(D) := \{g \in \text{O}^+(S_Z) \mid D^g = D\}$. By Corollary 3.10, any element of $\text{Aut}(D)$ preserves N_Z. Therefore the group

$$\text{Aut}(Z, D) := \{\, g \in \text{Aut}(D) \mid g \text{ satisfies the period condition (3.3)} \,\} \tag{3.4}$$

is contained in $\text{Aut}(Z)$. We find an ample class h in the interior of D such that $h^g = h$ for all $g \in \text{Aut}(Z, D)$. Then $\text{Aut}(Z, D)$ is equal to the projective automorphism group $\text{Aut}(Z, h)$.

(IV) Note that $\text{Aut}(Z, D) = \text{Aut}(Z, h)$ acts on $\mathcal{W}_{\text{out}}(D)$ and $\mathcal{W}_{\text{inn}}(D)$. We decompose $\mathcal{W}_{\text{inn}}(D)$ into the orbits under the action of $\text{Aut}(Z, h)$:

$$\mathcal{W}_{\text{inn}}(D) = O_1 \cup \cdots \cup O_J.$$

From each orbit O_j, we choose a wall $(v_j)^\perp \cap D$ and calculate a Weyl vector $w_j \in L_{26}$ such that $D_j := i_{\mathcal{P}}^{-1}(\mathcal{C}(w_j))$ is the $\mathcal{V}(i)$-chamber adjacent to D across $(v_j)^\perp \cap D$. Since $(v_j)^\perp \cap N_Z$ is not a wall of N_Z, the $\mathcal{V}(i)$-chamber D_j is contained in N_Z. For each $j = 1, \ldots, J$, we find an isometry g_j of $\text{O}^+(S_Z)$ that satisfies the period condition (3.3) and $D^{g_j} = D_j$. Note that each g_j preserves N_Z by Corollary 3.10, and hence $g_j \in \text{Aut}(Z)$. Note also that, for each inner wall $(v')^\perp \cap D \in O_j$, there exists a conjugate $g' \in \text{Aut}(Z)$ of g_j by $\text{Aut}(Z, h)$ that maps D to the $\mathcal{V}(i)$-chamber adjacent to D across the wall $(v')^\perp \cap D$.

(V) Under the assumptions given in (I)–(IV), the group $\text{Aut}(Z)$ is generated by $\text{Aut}(Z, h)$ and the automorphisms $g_1, \ldots, g_J$. Moreover, the tessellation of N_Z by $\mathcal{V}(i)$-chambers is $\text{Aut}(Z)$-transitive, and the mappings $g \mapsto h^g$ and $g \mapsto D^g$ give one-to-one correspondences between the following sets:

- The set of cosets $\text{Aut}(Z, h) \backslash \text{Aut}(Z)$.
- The set of $\mathcal{V}(i)$-chambers contained in N_Z.
- The subset $\{h^g \mid g \in \text{Aut}(Z)\}$ of S_Z.

Moreover, considering the reflections with respect to the roots r defining the outer walls $(r)^\perp \cap D$ of D, we see that, under the assumptions given in (I)–(IV), the tessellation of $\mathcal{P}_Z$ by $\mathcal{V}(i)$-chambers is $\text{O}^+(S_Z)$-transitive.

The method described in this section was applied by Kondo [36] to the calculation of the automorphism group of a generic Jacobian Kummer surface, and since then, many studies have been done on the automorphism groups of various $K3$ surfaces (see the references of [30]). This method was also applied to the study of automorphism group of an Enriques surface in [37,38].

4 Borcherds' method for X_0 and X_3

Recall from Sect. 1.1 that we use the following notation:

$$S_3 := S_{X_3},\ \mathcal{P}_3 := \mathcal{P}_{X_3},\ N_3 := N_{X_3}, \quad S_0 := S_{X_0},\ \mathcal{P}_0 := \mathcal{P}_{X_0},\ N_0 := N_{X_0}.$$

Table 2 Inner walls of D_3

Orbit	$\langle v, v\rangle$	$\langle v, h_3\rangle$	$\langle h_3, b'_d\rangle$	$\mathrm{Sing}(b'_d)$	$d = \langle h_3, h_3^{g(b'_d)}\rangle$
O'_{648}	$-4/3$	2	6	$4A_2 + 6A_1$	10
O'_{5184}	$-2/3$	3	9	$4A_3 + 6A_1$	31

4.1 Borcherds' method for X_3

We identify X_3 and F_3 via Shioda's isomorphism explained in Sect. 2.3. Hence S_3 is the Néron–Severi lattice of F_3. In [2], we have obtained a generating set of $\mathrm{Aut}(X_3)$ by finding a primitive embedding $i_3\colon S_3 \hookrightarrow L_{26}$ inducing $i_{3,\mathcal{P}}\colon \mathcal{P}_3 \hookrightarrow \mathcal{P}(L_{26})$ and a Weyl vector $w_0 \in L_{26}$ that satisfy the requirements in Sect. 3.4. The result is as follows. See [18] or [2] for the explicit descriptions of i_3, w_0, and other computational data.

We have $A(S_3) \cong (\mathbb{Z}/3\mathbb{Z})^2$. The group $\mathrm{O}(q(S_3))$ is a dihedral group of order 8, and $\mathrm{O}(q(S_3), \omega)$ is a cyclic subgroup of order 4. The orthogonal complement R_3 of S_3 in L_{26} is a negative-definite root lattice of type $2A_2$. The order of $\mathrm{O}(R_3)$ is 288, the order of $\mathrm{O}(q(R_3))$ is 8, and the natural homomorphism $\mathrm{O}(R_3) \to \mathrm{O}(q(R_3))$ is surjective. We put

$$D_3 := i_{3,\mathcal{P}}^{-1}(\mathcal{C}(w_0)).$$

Then D_3 contains the class $h_3 \in S_3$ of a hyperplane section of $X_3 = F_3 \subset \mathbb{P}^3$ in its interior. Hence D_3 is a $\mathcal{V}(i_3)$-chamber. The set $\mathcal{W}_{\mathrm{out}}(D_3)$ of outer walls of the initial chamber D_3 is equal to $\{(\ell)^{\perp} \cap D_3 \mid \ell \in \mathcal{L}_{112}\}$. Because

$$h_3 = \frac{1}{28} \sum_{\ell \in \mathcal{L}_{112}} [\ell],$$

the group $\mathrm{Aut}(X_3, D_3)$ defined by (3.4) is equal to $\mathrm{Aut}(X_3, h_3)$, which is the projective automorphism group $\{g \in \mathrm{PGL}_4(k_3) \mid g(F_3) = F_3\} = \mathrm{PGU}_4(\mathbb{F}_9)$ of $F_3 \subset \mathbb{P}^3$. Hence $\mathrm{Aut}(X_3, D_3)$ is of order 13063680. The class h_3 is in fact the image of w_0 under the orthogonal projection $L_{26} \otimes \mathbb{Q} \to S_3 \otimes \mathbb{Q}$. Under the action of $\mathrm{Aut}(X_3, h_3) = \mathrm{PGU}_4(\mathbb{F}_9)$, the set $\mathcal{W}_{\mathrm{inn}}(D_3)$ of inner walls of D_3 is decomposed into two orbits O'_{648} and O'_{5184} of size 648 and 5184, respectively. Each inner wall $(v)^{\perp} \cap D_3$ in the orbit O'_s is defined by a primitive vector v of $S_3^{\vee}$ with the properties given in Table 2, and there exists a double-plane polarization $b'_d \in S_3$ such that the corresponding double-plane involution $g(b'_d) \in \mathrm{Aut}(X_3)$ maps D_3 to the $\mathcal{V}(i_3)$-chamber adjacent to D_3 across the wall $(v)^{\perp} \cap D_3$. These results prove the following:

Theorem 4.1 (Kondo–Shimada [2]) *The automorphism group* $\mathrm{Aut}(X_3)$ *is generated by the projective automorphism group* $\mathrm{Aut}(X_3, h_3) = \mathrm{PGU}_4(\mathbb{F}_9)$ *and two double-plane involutions* $g(b'_{10})$, $g(b'_{31})$ *corresponding the orbits* O'_{648}, O'_{5184} *of the action of* $\mathrm{PGU}_4(\mathbb{F}_9)$ *on the set* $\mathcal{W}_{\mathrm{inn}}(D_3)$ *of inner walls of the initial chamber* D_3. □

4.2 Borcherds' method for X_0

We define an embedding $i_0\colon S_0 \hookrightarrow L_{26}$ by

$$i_0 := i_3 \circ \rho, \tag{4.1}$$

Table 3 Inner walls of D_0

Orbit	$\langle v, v\rangle$	$\langle v, h_0\rangle$	$\langle h_0, b_d\rangle$	$\mathrm{Sing}(b_d)$	$d = \langle h_0, h_0^{g(b_d)}\rangle$
O_{64}	– 5/4	5	16	$2A_3 + 3A_2 + 2A_1$	80
O_{40}	– 1	6	18	$4A_3 + 3A_1$	112
O_{160}	– 1/2	8	26	$A_5 + 2A_4 + A_3$	296
O_{320}	– 1/4	9	38	$2A_7 + A_3 + A_1$	688

where $i_3 \colon S_3 \hookrightarrow L_{26}$ is the embedding used in Sect. 4.1, and $\rho \colon S_0 \hookrightarrow S_3$ is the embedding given by the specialization of X_0 to X_3. The key observation of this article is that i_0 is equal to the embedding used by Keum–Kondo [1] for the calculation of $\mathrm{Aut}(X_0)$.

We have $A(S_0) \cong (\mathbb{Z}/4\mathbb{Z})^2$. The group $\mathrm{O}(q(S_0))$ is isomorphic to the dihedral group of order 8, and the subgroup $\mathrm{O}(q(S_0), \omega)$ is cyclic of order 4. The embedding i_0 is primitive and induces $i_{0,\mathcal{P}} \colon \mathcal{P}_0 \hookrightarrow \mathcal{P}(L_{26})$. The orthogonal complement R_0 of S_0 in L_{26} is a negative-definite root lattice of type $2A_3$. The order of $\mathrm{O}(R_0)$ is 4608, the order of $\mathrm{O}(q(R_0))$ is 8, and the natural homomorphism $\mathrm{O}(R_0) \to \mathrm{O}(q(R_0))$ is surjective. The vector

$$h_0 := \frac{1}{2} \sum_{\ell \in \mathcal{L}_{40}} [\ell] \quad \in \quad S_0 \otimes \mathbb{Q} \tag{4.2}$$

is in fact in S_0, and we have $\langle h_0, h_0\rangle = 40$. Since $\langle h_0, \ell\rangle = 2$ for all $\ell \in \mathcal{L}_{40}$, the class h_0 is nef. Since there exist no roots r of S_0 such that $h_0 \in (r)^\perp$, the class h_0 is ample. *Let $w_0 \in L_{26}$ be the same Weyl vector that was used in* Sect. 4.1. The orthogonal projection of w_0 to $S_0 \otimes \mathbb{Q}$ is equal to $h_0/2$. (In [1], the vector $h_0/2$ is used instead of h_0.) We put

$$D_0 := i_{0,\mathcal{P}}^{-1}(\mathcal{C}(w_0)).$$

Then D_0 contains h_0 in its interior, and hence D_0 is a $\mathcal{V}(i_0)$-chamber. The set $\mathcal{W}_{\mathrm{out}}(D_0)$ of outer walls of the initial chamber D_0 is equal to $\{(\ell)^\perp \cap D_0 \mid \ell \in \mathcal{L}_{40}\}$. We have

$$\mathrm{Aut}(X_0, D_0) = \mathrm{Aut}(X_0, h_0), \tag{4.3}$$

which is of order 3840 and acts on $\mathcal{W}_{\mathrm{out}}(D_0)$ transitively. Using the algorithms in Remark 1.3, we search for double-plane polarizations in S_0 and obtain the following proposition, which proves Theorem 1.4.

Proposition 4.2 *The action of* $\mathrm{Aut}(X_0, h_0)$ *decomposes the set* $\mathcal{W}_{\mathrm{inn}}(D_0)$ *of inner walls of the initial chamber* D_0 *into four orbits* O_{64}, O_{40}, O_{160}, O_{320}, *where* $|O_s| = s$. *For each inner wall* $(v)^\perp \cap D_0 \in O_s$, *there exists a double-plane polarization* $b_d \in S_0$ *such that the corresponding double-plane involution* $g(b_d) \in \mathrm{Aut}(X_0)$ *maps* D_0 *to the* $\mathcal{V}(i_0)$*-chamber adjacent to* D_0 *across the wall* $(v)^\perp \cap D_0$. □

Each inner wall $(v)^\perp \cap D_0 \in O_s$ is defined by a primitive vector $v \in S_0^\vee$ with the properties given in Table 3. See [18] for the matrix representations of double-plane involutions $g(b_d)$.

4.3 The group $\mathrm{Aut}(X_0, h_0)$

We investigate the finite group $\mathrm{Aut}(X_0, h_0)$ more closely. Note that the order 3840 of this group is the maximum among all finite subgroups of automorphisms of complex $K3$ surfaces (see Kondo [39]). There exists a natural identification between $\mathcal{W}_{\mathrm{out}}(D_0)$ and $\mathcal{L}_{40}$. Therefore by (4.3), the group $\mathrm{Aut}(X_0, h_0)$ acts on $\mathcal{L}_{40}$ faithfully, and hence $\mathrm{Aut}(X_0, h_0)$ is embedded into the automorphism group $\mathrm{Aut}(\mathcal{L}_{40})$ of the dual graph of $\mathcal{L}_{40}$. On the other hand, since $\langle \mathcal{L}_{40} \rangle = S_0$ (Corollary 2.8), we have an embedding $\mathrm{Aut}(\mathcal{L}_{40}) \hookrightarrow \mathrm{O}^+(S_0)$. In fact, we confirm by direct calculation the following:

$$\mathrm{Aut}(X_0, h_0) = \left\{ g \in \mathrm{Aut}(\mathcal{L}_{40}) \;\middle|\; \begin{array}{l} g\text{, as an element of } \mathrm{O}^+(S_0)\text{, satisfies the} \\ \text{period condition (3.3)} \end{array} \right\},$$

and $\mathrm{Aut}(X_0, h_0)$ is of index 2 in $\mathrm{Aut}(\mathcal{L}_{40})$. By Propositions 2.4 and 2.7, we have a natural homomorphism $\mathrm{Aut}(\mathcal{L}_{40}) \to \mathrm{Aut}(\mathcal{P})$ to the automorphism group of the Petersen graph $\mathcal{P}$. Recall that, in Sects. 2.5 and 2.6, we have constructed a morphism $\mu_{\mathbb{C}} \colon X_0 \to M_{\mathbb{C}}$ that induces the QP-covering map $\mathcal{L}_{40} \to \mathcal{P}$, and calculated the Galois group $\mathrm{Gal}(\mu)$ in Propositions 2.13 and 2.17.

Proposition 4.3 *The homomorphism*

$$\mathrm{Aut}(X_0, h_0) \hookrightarrow \mathrm{Aut}(\mathcal{L}_{40}) \to \mathrm{Aut}(\mathcal{P}) \tag{4.4}$$

is surjective, and its kernel is equal to the Galois group $\mathrm{Gal}(\mu) \cong (\mathbb{Z}/2\mathbb{Z})^5$.

Proof By the list of elements of $\mathrm{Aut}(X_0, h_0)$ (see [18]), we see that the homomorphism (4.4) is surjective, and its kernel is of order 32. Each generator of $\mathrm{Gal}(\mu)$ given in Propositions 2.13 or 2.17 preserves $\mathcal{L}_{40}$, and hence $\mathrm{Gal}(\mu)$ is contained in $\mathrm{Aut}(X_0, h_0)$. Since μ induces the QP-covering map $\mathcal{L}_{40} \to \mathcal{P}$, it follows that $\mathrm{Gal}(\mu)$ is contained in the kernel of (4.4). Comparing the order, we complete the proof. □

For $v \in S_0$, we put

$$\mathrm{Aut}(X_0, v) := \{\, g \in \mathrm{Aut}(X_0) \mid v^g = v \,\}.$$

Let $f \in S_0$ be the class of a fiber of the Jacobian fibration $\sigma \colon X_0 \to \mathbb{P}^1$ defined by (1.1). For each element g of $\mathrm{Aut}(X_0, f)$, there exists an automorphism $\bar{g} \in \mathrm{Aut}(\mathbb{P}^1)$ such that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{g} & X_0 \\ \sigma \downarrow & & \downarrow \sigma \\ \mathbb{P}^1 & \xrightarrow{\bar{g}} & \mathbb{P}^1 \end{array} \tag{4.5}$$

commutes and hence g preserves $\mathcal{L}_{40}$. Therefore $\mathrm{Aut}(X_0, f)$ is contained in $\mathrm{Aut}(X_0, h_0)$, and we have a homomorphism

$$\beta \colon \mathrm{Aut}(X_0, f) \to \mathrm{Stab}(\mathrm{Cr}(\sigma)),$$

where $\mathrm{Cr}(\sigma) := \{0, \infty, \pm 1, \pm i\}$ is the set of critical values of σ and $\mathrm{Stab}(\mathrm{Cr}(\sigma))$ is the stabilizer subgroup of $\mathrm{Cr}(\sigma)$ in $\mathrm{Aut}(\mathbb{P}^1)$.

We have the inversion $\iota_\sigma \colon X_0 \to X_0$ of the Jacobian fibration σ. We also have a subgroup T_σ of $\mathrm{Aut}(X_0, f)$ consisting of translations by the 16 sections of σ.

Proposition 4.4 *The order of* $\mathrm{Aut}(X_0, f)$ *is* 768. *The image of* β *is isomorphic to* $\mathfrak{S}_4$, *and the kernel of* β *is equal to the subgroup* $T_\sigma \rtimes \langle \iota_\sigma \rangle$ *of* $\mathrm{Aut}(X_0, f)$.

Proof By means of $\rho_{\mathcal{L}} \colon \mathcal{L}_{40} \hookrightarrow \mathcal{L}_{112}$ and (2.1), we can calculate the quadrangle F_c in $\mathcal{L}_{40}$ consisting of the classes of irreducible components of the singular fiber $\sigma^{-1}(c)$ for each $c \in \mathrm{Cr}(\sigma)$. Then f is the sum of vectors in one of these F_c, and hence we can calculate

$\mathrm{Aut}(X_0, f)$ from the list of elements of $\mathrm{Aut}(X_0, h_0)$. Looking at the action of $\mathrm{Aut}(X_0, f)$ on the set of the quadrangles F_c, we see that the image of β is isomorphic to $\mathfrak{S}_4$ generated by permutations $(0, -1, -i)(\infty, 1, i)$ and $(0, -i)(\infty, i)(1, -1)$ of $\mathrm{Cr}(\sigma)$. Therefore the kernel is of order 32. Since $T_\sigma \rtimes \langle \iota_{\sigma,z} \rangle$ is of order 32 and contained in the kernel, we complete the proof. □

Remark 4.5 Since $|\mathrm{Aut}(X_0, h_0)|/|\mathrm{Aut}(X_0, f)| = 5$, the orbit of f under the action of $\mathrm{Aut}(X_0, h_0)$ consists of five elements $f = f^{(1)}, f^{(2)}, \dots, f^{(5)}$. We can easily confirm that

$$\mathrm{Gal}(\mu) = \bigcap_{\nu=1}^{5} \mathrm{Aut}(X_0, f^{(\nu)}).$$

The five classes $f^{(\nu)}$ give rise to five elliptic fibrations $\sigma^{(\nu)}\colon X_0 \to \mathbb{P}^1$. These elliptic fibrations correspond to the choices of the $\mathbb{P}^1$-fibration $\varphi_M\colon M \to \mathbb{P}^1$ in (2.5): for $\nu = 1, \dots, 4$, the class $f^{(\nu)}$ is induced by the pencil of lines passing through the triple point t_ν, and $f^{(5)}$ is induced by the pencil of conics passing through all the triple points (see Remark 2.15). Let $h_8^{(\nu)} \in S_0$ be the polarization of degree 8 constructed from $\sigma^{(\nu)}\colon X_0 \to \mathbb{P}^1$ via the recipe of Barth–Hulek explained in Sect. 2.6. Then we have $\mathrm{Aut}(X_0, f^{(\nu)}) = \mathrm{Aut}(X_0, h_8^{(\nu)})$.

5 Proof of Theorems 1.7 and 1.8

We use the same notation as in Sect. 4. The following fact has been established.

Proposition 5.1 (1) *The tessellation of N_3 by $\mathcal{V}(i_3)$-chambers is* $\mathrm{Aut}(X_3)$*-transitive, and the tessellation of $\mathcal{P}_3$ by $\mathcal{V}(i_3)$-chambers is* $\mathrm{O}^+(S_3)$*-transitive.*

(2) *The tessellation of N_0 by $\mathcal{V}(i_0)$-chambers is* $\mathrm{Aut}(X_0)$*-transitive, and the tessellation of $\mathcal{P}_0$ by $\mathcal{V}(i_0)$-chambers is* $\mathrm{O}^+(S_0)$*-transitive.* □

From now on, we consider S_0 as a sublattice of S_3 via $\rho\colon S_0 \hookrightarrow S_3$ and $\mathcal{P}_0$ as a subspace of $\mathcal{P}_3$. For example, we use notation such as $h_0 \in S_3, D_0 \subset \mathcal{P}_3, \mathcal{P}_0 \subset \mathcal{P}_3, \dots$. By definition (4.1) of i_0, we have the following:

Proposition 5.2 *The tessellation of $\mathcal{P}_0$ by $\mathcal{V}(i_0)$-chambers is obtained as the restriction to $\mathcal{P}_0$ of the tessellation of $\mathcal{P}_3$ by $\mathcal{V}(i_3)$-chambers.* □

5.1 Proof of Theorem 1.7

First, we show that the restriction homomorphism $\tilde{\rho}$ from $\mathrm{O}^+(S_3, S_0)$ to $\mathrm{O}^+(S_0)$ maps $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ to $\mathrm{Aut}(X_0)$. By Theorem 3.11, it suffices to show that, for each $g \in \mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$, the restriction $g|_{S_0} \in \mathrm{O}^+(S_0)$ satisfies the period condition (3.3) and preserves N_0.

Lemma 5.3 *If $g \in \mathrm{O}^+(S_3, S_0)$ satisfies the period condition $\eta_{S_3}(g) \in \mathrm{O}(q(S_3), \omega)$ for X_3, then $g|_{S_0} \in \mathrm{O}^+(S_0)$ satisfies the period condition $\eta_{S_0}(g|_{S_0}) \in \mathrm{O}(q(S_0), \omega)$ for X_0.*

Proof Let Q denote the orthogonal complement of S_0 in S_3. Then Q is an even negative-definite lattice of rank 2 with discriminant group isomorphic to $(\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})^2$. By the

$$\begin{array}{ccccc}
\mathrm{O}^+(S_3, S_0) & \hookrightarrow & \mathrm{O}(Q) \times \mathrm{O}^+(S_0) & & \\
\downarrow & & \downarrow \mathrm{pr}_1 & \searrow \mathrm{pr}_2 & \\
\mathrm{O}^+(S_3) & & \mathrm{O}(Q) & & \mathrm{O}^+(S_0) \\
\downarrow \eta_{S_3} & & \downarrow \eta_Q & & \downarrow \eta_{S_0} \\
 & & \mathrm{O}(q(Q)) & & \\
 & \swarrow p_3 & & \searrow p_2 & \\
\mathrm{O}(q(S_3)) \cong \mathrm{O}(q(Q)_3) & & & & \mathrm{O}(q(Q)_2) \cong \mathrm{O}(q(S_0))
\end{array}$$

Fig. 4 Commutative diagram for the period condition

classical theory of Gauss, such a lattice is unique up to isomorphism, and the lattice Q is given by a Gram matrix

$$\begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix}.$$

We consider the commutative diagram in Fig. 4. The two isomorphisms in the bottom line of this diagram are derived from the isomorphism $q(S_3) \cong q(Q)_3$ given in Proposition 3.6 and the isomorphism $q(Q)_2 \cong -q(S_0)$ given in Proposition 3.7. It is easy to verify that $\mathrm{O}(Q)$ is a dihedral group of order 8, and the composites $p_3 \circ \eta_Q \colon \mathrm{O}(Q) \to \mathrm{O}(q(Q)_3)$ and $p_2 \circ \eta_Q \colon \mathrm{O}(Q) \to \mathrm{O}(q(Q)_2)$ are isomorphisms, where p_2 and p_3 are projections to the 2-part and the 3-part, respectively. Using the image of $\eta_Q \colon \mathrm{O}(Q) \to \mathrm{O}(q(Q))$ as the graph of an isomorphism between $\mathrm{O}(q(Q)_3)$ and $\mathrm{O}(q(Q)_2)$, we obtain an isomorphism $\mathrm{O}(q(S_3)) \cong \mathrm{O}(q(S_0))$ that is compatible with the homomorphisms from $\mathrm{O}^+(S_3, S_0)$. Recall that $\mathrm{O}(q(S_3), \omega)$ and $\mathrm{O}(q(S_0), \omega)$ are cyclic of order 4. Since the cyclic subgroup of order 4 is a characteristic subgroup of the dihedral group of order 8, the isomorphism $\mathrm{O}(q(S_3)) \cong \mathrm{O}(q(S_0))$ maps $\mathrm{O}(q(S_3), \omega)$ to $\mathrm{O}(q(S_0), \omega)$. □

Since we have calculated the embedding $\rho \colon S_0 \hookrightarrow S_3$ in the form of a matrix and the set $\mathcal{W}_{\mathrm{out}}(D_3) \cup \mathcal{W}_{\mathrm{inn}}(D_3)$ of walls of the initial chamber D_3 for X_3 in the form of a list of vectors (see [18]), we can easily prove the following:

Lemma 5.4 (1) *The ample class h_0 of X_0 is contained in D_3, and no outer walls of D_3 pass through h_0. In particular, h_0 belongs to the interior of N_3 and hence is ample for X_3.*
(2) *Among the walls $(v)^\perp \cap D_3$ of D_3, there exist exactly two walls such that the hyperplane $(v)^\perp$ of $\mathcal{P}_3$ contains $\mathcal{P}_0$. These two walls $(v_1)^\perp \cap D_3$ and $(v_2)^\perp \cap D_3$ belong to the orbit $O'_{648} \subset \mathcal{W}_{\mathrm{inn}}(D_3)$. Moreover, we have $\langle v_1, v_2 \rangle = 0$.* □

Combining Lemma 5.4 with Propositions 5.1 and 5.2, we obtain the following:

Corollary 5.5 (1) *We have $\mathcal{P}_0 = (v_1)^\perp \cap (v_2)^\perp$, where $(v_1)^\perp$ and $(v_2)^\perp$ are the hyperplanes of $\mathcal{P}_3$ given in Lemma 5.4.*
(2) *For each $\mathcal{V}(i_0)$-chamber $D'_0 \subset \mathcal{P}_0$, there exist exactly four $\mathcal{V}(i_3)$-chambers that contain D'_0.*
(3) *The initial chamber D_0 for X_0 is a face $(v_1)^\perp \cap (v_2)^\perp \cap D_3$ of the initial chamber D_3 for X_3, and the interior of $D_0 \subset \mathcal{P}_0$ is contained in the interior of $N_3 \subset \mathcal{P}_3$.*
(4) *The four $\mathcal{V}(i_3)$-chambers containing D_0 are contained in N_3. In particular, we have $\gamma_1, \gamma_2, \varepsilon \in \mathrm{Aut}(X_3)$ such that the four $\mathcal{V}(i_3)$-chambers containing D_0 are D_3 and $D_3^{\gamma_1}$, $D_3^{\gamma_2}$, D_3^{ε}. See Fig. 5.* □

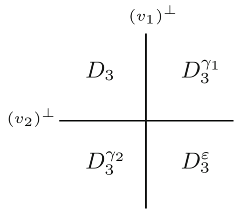


Fig. 5 $\mathcal{V}(i_3)$-chambers containing D_0

Remark 5.6 The automorphisms γ_1 and γ_2 of X_3 in Corollary 5.5 (4) can be obtained as conjugates of the double-plane involution $g(b'_{10})$ by $\mathrm{PGU}_4(\mathbb{F}_9)$. Let $(v'')^\perp \cap D_3$ be the wall of D_3 that is mapped to the wall $(v_2)^\perp \cap D_3^{\gamma_1}$ of $D_3^{\gamma_1}$ by γ_1. Then $(v'')^\perp \cap D_3$ is an inner wall belonging to O'_{648}, and hence we have a conjugate γ'' of $g(b'_{10})$ by $\mathrm{PGU}_4(\mathbb{F}_9)$ that maps D_3 to the $\mathcal{V}(i_3)$-chamber adjacent to D_3 across $(v'')^\perp \cap D_3$. Then, as the automorphism ε, we can take $\gamma''\gamma_1$. See Sect. 6.2 for another construction of ε.

Let $\mathrm{pr}_3 \colon L_{26} \otimes \mathbb{Q} \to S_3 \otimes \mathbb{Q}$, $\mathrm{pr}_0 \colon L_{26} \otimes \mathbb{Q} \to S_0 \otimes \mathbb{Q}$ and $\mathrm{pr}_{30} \colon S_3 \otimes \mathbb{Q} \to S_0 \otimes \mathbb{Q}$ be the orthogonal projections. Then we have $\mathrm{pr}_{30} \circ \mathrm{pr}_3 = \mathrm{pr}_0$. We put

$$\mathcal{V}(\rho) := \{\, \mathrm{pr}_{30}(r) \mid r \in \mathcal{R}(S_3), \quad \langle \mathrm{pr}_{30}(r), \mathrm{pr}_{30}(r)\rangle < 0 \,\}.$$

The restriction to $\mathcal{P}_0$ of the tessellation of $\mathcal{P}_3$ by $\mathcal{R}(S_3)$-chambers is the tessellation of $\mathcal{P}_0$ by $\mathcal{V}(\rho)$-chambers. The closed subset

$$N_{30} := N_3 \cap \mathcal{P}_0$$

of $\mathcal{P}_0$ contains D_0 by Corollary 5.5 (3), and hence its interior is non-empty. Therefore N_{30} is a $\mathcal{V}(\rho)$-chamber. We have

$$\mathcal{R}(S_0) \subset \mathcal{V}(\rho) \subset \mathcal{V}(i_0),$$

where the second inclusion follows from $\mathcal{R}(S_3) \subset \mathcal{R}(L_{26})$ and $\mathrm{pr}_{30} \circ \mathrm{pr}_3 = \mathrm{pr}_0$. It follows from Remark 3.2 that

$$D_0 \subset N_{30} \subset N_0, \tag{5.1}$$

and that the $\mathcal{V}(\rho)$-chamber N_{30} is tessellated by $\mathcal{V}(i_0)$-chambers. If $g \in \mathrm{O}^+(S_3, S_0)$ preserves N_3, then $g|_{S_0} \in \mathrm{O}^+(S_0)$ preserves N_{30} and hence preserves N_0 by Corollary 3.10. Combining this fact with Lemma 5.3, we conclude that every element of the image of $\tilde{\rho}|_{\mathrm{Aut}}$ belongs to $\mathrm{Aut}(X_0)$.

Next we calculate a generating set of the image of $\tilde{\rho}|_{\mathrm{Aut}}$.

Lemma 5.7 *The group* $\mathrm{PGU}_4(\mathbb{F}_9) = \mathrm{Aut}(X_3, h_3)$ *acts transitively on the set of non-ordered pairs* $\{(v)^\perp, (v')^\perp\}$ *of hyperplanes of* $\mathcal{P}_3$ *such that* $(v)^\perp \cap D_3$ *and* $(v')^\perp \cap D_3$ *are inner walls of* D_3 *belonging to* O'_{648}*, and such that* $\langle v, v'\rangle = 0$.

Proof As can be seen from the list [18] of walls of D_3, for each inner wall $(v)^\perp \cap D_3$ in O'_{648}, the number of inner walls $(v')^\perp \cap D_3$ in O'_{648} satisfying $\langle v, v'\rangle = 0$ is 42. Comparing $42 \times 648/2 = 13608$ with Corollary 2.12, we obtain the proof. □

Corollary 5.8 *Let g be an element of* $\mathrm{Aut}(X_3)$ *such that* $D'_0 := \mathcal{P}_0 \cap D_3^g$ *is a* $\mathcal{V}(i_0)$*-chamber, that is, D'_0 has an interior point as a subset of $\mathcal{P}_0$. Then there exists an element* $\gamma \in \mathrm{PGU}_4(\mathbb{F}_9)$ *such that* $\gamma g \in \mathrm{Aut}(X_3)$ *maps the face D_0 of D_3 to the face D'_0 of* $D_3^g = D_3^{\gamma g}$.

Proof We put $v_1' := v_1^{g^{-1}}$ and $v_2' := v_2^{g^{-1}}$, where v_1 and v_2 are given in Lemma 5.4. Then $D_0'^{g^{-1}} = \mathcal{P}_0^{g^{-1}} \cap D_3 = (v_1')^\perp \cap (v_2')^\perp \cap D_3$ is a face of D_3, which is the intersection of two perpendicular inner walls $(v_1')^\perp \cap D_3$ and $(v_2')^\perp \cap D_3$ in O'_{648}. Hence the existence of $\gamma \in \mathrm{PGU}_4(\mathbb{F}_9)$ follows from Lemma 5.7. □

We put

$$\mathrm{Aut}(X_3, D_0) := \{\, g \in \mathrm{Aut}(X_3) \mid D_0^g = D_0 \,\}, \tag{5.2}$$

and compare it with $\mathrm{Aut}(X_0, D_0) = \mathrm{Aut}(X_0, h_0)$. Note that $\mathrm{Aut}(X_3, D_0)$ is a subgroup of $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ containing the kernel of $\tilde{\rho}|_{\mathrm{Aut}}$.

Lemma 5.9 *The homomorphism $\tilde{\rho}|_{\mathrm{Aut}}$ maps $\mathrm{Aut}(X_3, D_0)$ to $\mathrm{Aut}(X_0, h_0)$ isomorphically. In particular, the homomorphism $\tilde{\rho}|_{\mathrm{Aut}}$ is injective, and the image of $\tilde{\rho}|_{\mathrm{Aut}}$ contains $\mathrm{Aut}(X_0, h_0)$.*

Proof By Corollary 5.5 (4), the subgroup $\mathrm{Aut}(X_3, D_0)$ of $\mathrm{Aut}(X_3)$ is contained in the finite subset

$$\mathrm{PGU}_4(\mathbb{F}_9) \ \sqcup\ \mathrm{PGU}_4(\mathbb{F}_9)\cdot\gamma_1 \ \sqcup\ \mathrm{PGU}_4(\mathbb{F}_9)\cdot\gamma_2 \ \sqcup\ \mathrm{PGU}_4(\mathbb{F}_9)\cdot\varepsilon \tag{5.3}$$

of $\mathrm{Aut}(X_3)$. For each element g of this subset, we determine whether g preserves $\mathcal{P}_0$ or not. We see that, in each coset $\mathrm{PGU}_4(\mathbb{F}_9)\cdot\gamma$ in (5.3), exactly 960 elements g satisfy $\mathcal{P}_0^g = \mathcal{P}_0$, and that the set of restrictions $g|_{S_0}$ of these $960 \times 4 = 3840$ elements g is equal to $\mathrm{Aut}(X_0, h_0)$. □

We discuss the following problem: Let $(v)^\perp$ be a hyperplane of $\mathcal{P}_0$ that defines a wall of D_0. Determine whether $(v)^\perp$ defines a wall of N_{30} or not.

Since $\mathcal{L}_{40} \subset \mathcal{L}_{112}$, it immediately follows that, if $(v)^\perp \cap D_0$ is an outer wall of D_0, then $(v)^\perp \cap N_{30}$ is a wall of N_{30}.

Lemma 5.10 *Let $(v)^\perp \cap D_0$ be an inner wall of D_0. Then $(v)^\perp \cap N_{30}$ is a wall of N_{30} if and only if $(v)^\perp \cap D_0 \in O_{64}$ or $(v)^\perp \cap D_0 \in O_{160}$.*

Proof Let $g \in \mathrm{Aut}(X_0)$ be an automorphism that maps D_0 to the $\mathcal{V}(i_0)$-chamber adjacent to D_0 across the inner wall $(v)^\perp \cap D_0$ (for example, we can take as g a conjugate by $\mathrm{Aut}(X_0, h_0)$ of the double-plane involution $g(b_d)$ corresponding to the orbit O_s containing $(v)^\perp \cap D_0$). Then $(v)^\perp \cap N_{30}$ is a wall of N_{30} if and only if h_0 and h_0^g, regarded as vectors of S_3 via $\rho\colon S_0 \hookrightarrow S_3$, are *separated by a root in* S_3, that is, the set

$$\{\, r \in \mathcal{R}(S_3) \mid \langle h_0, r\rangle \text{ and } \langle h_0^g, r\rangle \text{ have different sign} \,\}$$

is non-empty (see Corollary 3.10). We can calculate this set using the algorithm described in Section 3.3 of [9]. □

Remark 5.11 The 'if'-part of Lemma 5.10 is refined as follows. For each positive integer d, we put

$$\mathcal{C}_d := \{\, [C] \in S_3 \mid C \text{ is a smooth rational curve on } X_3 \text{ such that } \langle h_3, [C]\rangle = d \,\}.$$

The walls of N_3 are in one-to-one correspondence with the union of these sets $\mathcal{C}_d$. We have $\mathcal{C}_1 = \mathcal{L}_{112}$. The set $\mathcal{C}_d$ can be calculated by induction on d. Indeed, a root r of S_3 satisfying $\langle h_3, r\rangle = d$ belongs to $\mathcal{C}_d$ if and only if there exists no class $r' \in \mathcal{C}_{d'}$ with $d' < d$ such that $\langle r, r'\rangle < 0$. By this method, we obtain the following:

Proposition 5.12 *For $d = 2, 3, 5, 6$, the set $\mathcal{C}_d$ is empty. We have*

$$|\mathcal{C}_1| = 112, \quad |\mathcal{C}_4| = 18144, \quad |\mathcal{C}_7| = 2177280 = 1632960 + 544320.$$

The actions of $\mathrm{PGU}_4(\mathbb{F}_9)$ *on $\mathcal{C}_1$ and on $\mathcal{C}_4$ are transitive. The action of* $\mathrm{PGU}_4(\mathbb{F}_9)$ *on $\mathcal{C}_7$ has two orbits of size* 1632960 *and* 544320. □

Then we have the following:

- Among the 64 walls in O_{64}, 32 walls are defined by $(\mathrm{pr}_{30}(r))^{\perp}$ with $r \in \mathcal{C}_1$, and the other 32 walls are defined by $(\mathrm{pr}_{30}(r))^{\perp}$ with $r \in \mathcal{C}_4$.
- Among the 160 walls in O_{160}, 40 walls are defined by $(\mathrm{pr}_{30}(r))^{\perp}$ with $r \in \mathcal{C}_1$, 80 walls are defined by $(\mathrm{pr}_{30}(r))^{\perp}$ with $r \in \mathcal{C}_4$, and 40 walls are defined by $(\mathrm{pr}_{30}(r))^{\perp}$ with $r \in \mathcal{C}_7$.

Note that, if $g \in \mathrm{Aut}(X_0)$ belongs to the image of $\tilde{\rho}|_{\mathrm{Aut}}$, then g preserves $N_{30} \subset N_0$. Hence the double-plane involutions $g(b_{80})$ and $g(b_{296})$ corresponding to the orbits O_{64} and O_{160} are *not* in the image of $\tilde{\rho}|_{\mathrm{Aut}}$.

Lemma 5.13 *Let O be either O_{40} or O_{320}, and let $(v)^{\perp} \cap D_0$ be an element of O. Let D'_0 be the $\mathcal{V}(i_0)$-chamber adjacent to D_0 across $(v)^{\perp} \cap D_0$. Then there exists an element g' of* $\mathrm{O}^+(S_3, S_0) \cap \mathrm{Aut}(X_3)$ *such that $g'|_{S_0}$ maps D_0 to D'_0.*

Proof Let F denote the hyperplane $(v)^{\perp}$ of $\mathcal{P}_0$ considered as a linear subspace of $\mathcal{P}_3$ of codimension 3. Let D'_3 be one of the four $\mathcal{V}(i_3)$-chambers such that $D'_0 = \mathcal{P}_0 \cap D'_3$. (See Corollary 5.5 (2).) We have $F \cap D_0 = F \cap D'_0 = F \cap D_3 = F \cap D'_3$, and this set contains a non-empty open subset of F. Lemma 5.10 implies that there exists no root r of S_3 such that the hyperplane $(r)^{\perp}$ of $\mathcal{P}_3$ contains F. Since $F \cap D_3 = F \cap D'_3$, we see that D_3 and D'_3 are on the same side of $(r)^{\perp}$ for any root r of S_3, and hence D'_3 is contained in N_3. Therefore we have an element g' of $\mathrm{Aut}(X_3)$ such that $D_3^{g'} = D'_3$. By Lemma 5.8, there exists an element γ of $\mathrm{PGU}_4(\mathbb{F}_9)$ such that $\gamma g'$ maps the face D_0 of D_3 to the face D'_0 of $D'_3 = D_3^{g'} = D_3^{\gamma g'}$. Since each of D_0 and D'_0 contains a non-empty open subset of $\mathcal{P}_0$, we see that $\gamma g' \in \mathrm{Aut}(X_3)$ belongs to $\mathrm{O}^+(S_3, S_0)$. Then $\gamma g'|_{S_0}$ maps D_0 to D'_0. □

Lemmas 5.9 and 5.13 imply that $g(b_{112})$ and $g(b_{688})$ are in the image of $\tilde{\rho}|_{\mathrm{Aut}}$. Let G be the subgroup of $\mathrm{Aut}(X_0)$ generated by $\mathrm{Aut}(X_0, h_0)$ and $g(b_{112})$ and $g(b_{688})$. Since G is contained in the image of $\tilde{\rho}|_{\mathrm{Aut}}$, each $g \in G$ preserves N_{30}.

Lemma 5.14 *If a $\mathcal{V}(i_0)$-chamber D' is contained in N_{30}, then there exists an element $g \in G$ such that $D' = D_0^g$.*

Proof Since N_{30} is tessellated by $\mathcal{V}(i_0)$-chambers, there exists a sequence

$$D^{(0)} = D_0, \; D^{(1)}, \; \dots \;, \; D^{(N)} = D'$$

of $\mathcal{V}(i_0)$-chambers such that each $D^{(\nu)}$ is contained in N_{30} and that $D^{(\nu)}$ is adjacent to $D^{(\nu-1)}$ for $\nu = 1, \dots, N$. We prove the existence of $g \in G$ by induction on N. The case $N = 0$ is trivial. Suppose that $N > 0$, and let $g' \in G$ be an element such that $D_0^{g'} = D^{(N-1)}$. Note that g' preserves N_{30}. The $\mathcal{V}(i_0)$-chambers D_0 and $D'^{g'^{-1}}$ are adjacent, and both are contained in N_{30}. Hence, by Lemma 5.10, the wall of D_0 across which $D'^{g'^{-1}}$ is adjacent to D_0 is either in O_{40} or in O_{320}. Therefore we have an element $g'' \in G$ (a conjugate of $g(b_{112})$ or $g(b_{688})$ by $\mathrm{Aut}(X_0, h_0)$) such that $D'^{g'^{-1}} = D_0^{g''}$. Then $g'' g' \in G$ maps D_0 to D'. □

Let g be an arbitrary element of the image of $\tilde{\rho}|_{\mathrm{Aut}}$. Since g preserves N_{30}, there exists an element $g' \in G$ such that $D_0^g = D_0^{g'}$. Then $g'g^{-1} \in \mathrm{Aut}(X_0, h_0)$, and hence $g \in G$. Thus the proof of Theorem 1.7 is completed. □

5.2 Proof of Theorem 1.8

By the commutativity of diagram (1.2) and Theorem 1.7, it suffices to prove that the image of $\mathrm{res}_0\colon \mathrm{Aut}(\overline{\mathcal{X}/R}) \to \mathrm{Aut}(X_0)$ contains $\mathrm{Aut}(X_0, h_0)$ and the double-plane involutions $g(b_{112})$ and $g(b_{688})$. Let $\pi : \mathcal{X} \to \mathrm{Spec}\, R$ be the elliptic modular surface of level 4 over a discrete valuation ring R of mixed characteristic with residue field k of characteristic 3. Let K be the fraction field of R. We put $X_K := \mathcal{X} \otimes_R K$ and $X_k := \mathcal{X} \otimes_R k$ and identify X_0 with $X_K \otimes_K \bar{K}$ and X_3 with $X \otimes_k \bar{k}$, where $\bar{K}$ and $\bar{k}$ are algebraic closures of K and k, respectively.

Replacing R by a finite extension of R, we can assume that h_0 is the class of a line bundle L_K on X_K and that every element of $\mathrm{Aut}(X_0, h_0)$ is defined over K. We can extend L_K to a line bundle $\mathcal{L}$ on $\mathcal{X}$ by (21.6.11) of EGA, IV [40]. Then the class of the line bundle $L_k := \mathcal{L}|X_k$ on X_k is $\rho(h_0) \in S_3$. Hence L_k is ample by Lemma 5.4. Therefore $\mathcal{L}$ is ample relative to $\mathrm{Spec}\, R$ by (4.7.1) of EGA, III [41]. We choose $n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample relative to $\mathrm{Spec}\, R$, embed $\mathcal{X}$ into a projective space $\mathbb{P}^N_R$ over $\mathrm{Spec}\, R$ by $\mathcal{L}^{\otimes n}$, and regard $\mathrm{Aut}(X_0, h_0)$ as the group of projective automorphisms of $X_K \subset \mathbb{P}^N_K$. Since X_3 is not birationally ruled, we can apply the theorem of Matsusaka–Mumford [42] and conclude that every element of $\mathrm{Aut}(X_0, h_0)$ has a lift in $\mathrm{Aut}(\mathcal{X}/R)$.

Remark 5.15 The argument in the preceding paragraph is a special case of Theorem 2.1 of Lieblich and Maulik [43].

Let b be either b_{112} or b_{688}. Replacing R by a finite extension of R, we can assume that b is the class of a line bundle M_K on X_K, and that each smooth rational curve contracted by $\Phi_b\colon X_K \to \mathbb{P}^2_K$ is defined over K. Let $\Sigma(b) \subset S_0$ be the set of classes of smooth rational curves contracted by Φ_b. We extend M_K to a line bundle $\mathcal{M}$ on $\mathcal{X}$. Then the class of the line bundle $M_k := \mathcal{M}|X_k$ on X_k is $\rho(b) \in S_3$. Using the algorithms in Remark 1.3, we can verify that $\rho(b)$ is a double-plane polarization of X_3 and calculate the set $\Sigma(\rho(b)) \subset S_3$ of classes of smooth rational curves contracted by $\Phi_{\rho(b)}\colon X_k \to \mathbb{P}^2_k$. Then we have the following equality:

$$\Sigma(\rho(b)) = \rho(\Sigma(b)). \tag{5.4}$$

Since the complete linear systems $|M_K|$ and $|M_k|$ are of dimension 2 and fixed-point-free, we see that $\pi_*\mathcal{M}$ is free of rank 3 over R and defines a morphism

$$\widetilde{\Phi}\colon \mathcal{X} \to \mathbb{P}^2_R$$

over R. We execute, over R, Horikawa's canonical resolution for double coverings branched along a curve with only ADE-singularities (see Section 2 of [44]). Let $C_{1,K}, \ldots, C_{\mu,K}$ be the smooth rational curves on X_K contracted by Φ_b, where μ is the total Milnor number of the singularities of the branch curve of Φ_b (and hence of $\Phi_{\rho(b)}$). It follows from (5.4) that the closure $\mathcal{C}_j$ of each $C_{j,K}$ in $\mathcal{X}$ is a smooth family of rational curves over $\mathrm{Spec}\, R$, that $\widetilde{\Phi}$ contracts $\mathcal{C}_j$ to an R-valued point q_{0j} of $\mathbb{P}^2_R$ (that is, a section of the structure morphism $\mathbb{P}^2_R \to \mathrm{Spec}\, R$), and that $\widetilde{\Phi}$ is finite of degree 2 over the complement of $\{q_{01}, \ldots, q_{0\mu}\}$ in $\mathbb{P}^2_R$. We put $J_0 := \{1, \ldots, \mu\}$, $P_0 := \mathbb{P}^2_R$, and let $\beta_0\colon P_0 \to \mathbb{P}^2_R$ be the identity. Suppose that we have a morphism $\beta_i \colon P_i \to \mathbb{P}^2_R$ over R from a smooth R-scheme P_i and a subset $J_i \subset J_0$ such that

(i) $\widetilde{\Phi}$ factors as

$$\mathcal{X} \xrightarrow{\alpha_i} P_i \xrightarrow{\beta_i} \mathbb{P}^2_R,$$

(ii) α_i contracts $\mathcal{C}_j$ to an R-valued point q_{ij} of P_i for each $j \in J_i$, and
(iii) α_i is finite of degree 2 over the complement of $\{q_{ij} \mid j \in J_i\}$ in P_i.

Suppose that J_i is non-empty. We choose an index $j_0 \in J_i$, and let $\beta' \colon P_{i+1} \to P_i$ be the blowup at the R-valued point q_{ij_0}. Let $\beta_{i+1} \colon P_{i+1} \to \mathbb{P}^2_R$ be the composite of β' and β_i. Then properties (i)–(iii) are satisfied with i replaced by $i+1$ for some $J_{i+1} \subset J_i$ with $J_{i+1} \neq J_i$. Indeed, α_{i+1} induces a finite morphism from at least one of the $\mathcal{C}_j$ with $j \in J_i$ to the exceptional divisor of β'. Therefore after finitely many steps, we obtain a finite double covering $\mathcal{X} \to P$ that factors $\widetilde{\Phi}$, where P is obtained from $\mathbb{P}^2_R$ by a finite number of blowups at R-valued points. Then the deck-transformation of $\mathcal{X} \to P$ gives a lift of the double-plane involution $g(b) \in \mathrm{Aut}(X_0)$ to $\mathrm{Aut}(\mathcal{X}/R)$. □

Remark 5.16 The double-plane polarizations $\rho(b_{112}), \rho(b_{688}) \in S_3$ have the following properties with respect to h_3:

$$\langle h_3, \rho(b_{112})\rangle = 9, \quad \langle h_3, h_3^{g(b_{\rho(b_{112})})}\rangle = 34,$$
$$\langle h_3, \rho(b_{688})\rangle = 19, \quad \langle h_3, h_3^{g(b_{\rho(b_{688})})}\rangle = 178.$$

6 Enriques surface of type IV

Let Z be a $K3$ surface defined over an algebraically closed field of characteristic $\neq 2$. For an element $g \in \mathrm{O}^+(S_Z)$ of order 2, we put

$$S_Z^{+g} := \{\, v \in S_Z \mid v^g = v \,\}, \quad S_Z^{-g} := \{\, v \in S_Z \mid v^g = -v \,\}.$$

Suppose that $\varepsilon \colon Z \to Z$ is an Enriques involution, and let $\pi \colon Z \to Y := Z/\langle \varepsilon \rangle$ be the quotient morphism. Note that the lattice S_Y of numerical equivalence classes of divisors on the Enriques surface Y is an even unimodular hyperbolic lattice of rank 10, which is unique up to isomorphism. Then the pullback homomorphism $\pi^* \colon S_Y \hookrightarrow S_Z$ induces an isometry of lattices from $S_Y(2)$ to $S_Z^{+\varepsilon}$, where $S_Y(2)$ is the lattice obtained from S_Y by multiplying the intersection form by 2. Hence the following are satisfied: (i) $S_Z^{+\varepsilon}$ is a hyperbolic lattice of rank 10 and (ii) if M is a Gram matrix of $S_Z^{+\varepsilon}$, then $(1/2)M$ is an integer matrix that defines an even unimodular lattice. Moreover, since π is étale, we have that (iii) the orthogonal complement $S_Z^{-\varepsilon}$ of $S_Z^{+\varepsilon}$ in S_Z contains no roots.

6.1 Proof of Proposition 1.11

We check conditions (i), (ii), (iii) for all involutions in the finite group $\mathrm{Aut}(X_0, h_0)$. It turns out that there exist exactly six involutions $\varepsilon^{(1)}, \dots, \varepsilon^{(6)}$ satisfying these conditions. They are conjugate to each other, and they belong to the subgroup $\mathrm{Gal}(\mu)$ of $\mathrm{Aut}(X_0, h_0)$ (see Proposition 4.3). We show that these involutions are Enriques involutions of type IV.

Let ε_0 be one of $\varepsilon^{(1)}, \dots, \varepsilon^{(6)}$. Recall that $\sigma \colon X_0 \to \mathbb{P}^1$ is the Jacobian fibration defined by (1.1), and let $f \in S_0$ be the class of a fiber of σ. Since $\varepsilon_0 \in \mathrm{Gal}(\mu)$, we have $\varepsilon_0 \in$

$\mathrm{Aut}(X_0, f)$ by Remark 4.5. Let $F_c \subset \mathcal{L}_{40}$ be the set of classes of irreducible components of the singular fiber $\sigma^{-1}(c)$ over $c \in \mathrm{Cr}(\sigma)$. Looking at the action of ε_0 on these 6 quadrangles F_c, we see that the element $\bar{\varepsilon}_0 \in \mathrm{Stab}(\mathrm{Cr}(\sigma))$ defined by diagram (4.5) is of order 2 and fixes exactly 2 points of $\mathrm{Cr}(\sigma)$. Suppose that F_c is fixed by ε_0. Then ε_0 acts on F_c as $\ell_0 \leftrightarrow \ell_2$ and $\ell_1 \leftrightarrow \ell_3$, where $\ell_0, \dots, \ell_3$ are labeled as in (2.4). Therefore ε_0 is fixed-point-free, and $Y_0 := X_0/\langle \varepsilon_0 \rangle$ is an Enriques surface.

The Enriques involution ε_0 acts on $\mathcal{L}_{40}$ in such a way that, for any curve $C \in \mathcal{L}_{40}$, we have $C \cap C^{\varepsilon_0} = \emptyset$. Hence we obtain a configuration of 20 smooth rational curves on Y_0. It is easy to check that this configuration is isomorphic to the configuration of type IV. By Theorem 6.1 of [14], we see that Y_0 is an Enriques surface of type IV. □

Using Proposition 2.17, we can describe the six Enriques involutions $\varepsilon^{(\nu)}$ in $\mathrm{Gal}(\mu)$ as follows.

Proposition 6.1 *The involution $\tau_J \in \mathrm{Gal}(\mu)$ is an Enriques involution if and only if $|J| = 3$ and J contains $\{1, 5\}$ or $\{2, 6\}$ or $\{3, 4\}$.* □

6.2 Proof of Theorem 1.12

Let $\varepsilon_0 \in \mathrm{Aut}(X_0, h_0)$ be the image of ε_3 under $\tilde{\rho}|_{\mathrm{Aut}}$, which is one of $\varepsilon^{(1)}, \dots, \varepsilon^{(6)}$. Since $\varepsilon_0 \in \mathrm{Aut}(X_0, h_0)$, the involution ε_3 preserves the face $D_0 = \mathcal{P}_0 \cap D_3$ of D_3. Therefore ε_3 belongs to the finite group $\mathrm{Aut}(X_3, D_0)$ defined by (5.2). We check all involutions in $\mathrm{Aut}(X_3, D_0)$ and find ε_3 in the form of a matrix acting on S_3. We have $\langle h_3, h_3^{\varepsilon_3} \rangle = 16$. Indeed, the $\mathcal{V}(i_3)$-chamber $D_3^{\varepsilon_3}$ is the chamber D_3^{ε} in Fig. 5. The action of ε_3 on the fibers of the Jacobian fibration $\sigma : X_3 \to \mathbb{P}^1$ defined by (1.1) is exactly the same as the action of ε_0 on the fibers of the corresponding fibration of X_0. Hence ε_3 is fixed-point-free. Moreover, the configuration on $Y_3 := X_3/\langle \varepsilon_3 \rangle$ of 20 smooth rational curves obtained from $\mathcal{L}_{40} \subset \mathcal{L}_{112}$ is isomorphic to the configuration of type IV, and hence Y_3 is of type IV by Theorem 6.1 of [14]. The set of pullbacks of the smooth rational curves on Y_3 by π_3 is $\mathcal{L}_{40} \subset \mathcal{L}_{112}$. Hence they are lines on F_3. □

Remark 6.2 Recently, we have shown in [38] that X_0 has exactly nine Enriques involutions modulo conjugation in $\mathrm{Aut}(X_0)$ and that four of the quotient Enriques surfaces have finite automorphism groups (of types I, II, III, IV), whereas the other five have infinite automorphism groups.

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