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Holomorphic extension of meromorphic mappings along real analytic hypersurfaces

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Abstract
Let $M \subset \mathbb{C}^n$ be a real analytic hypersurface, $M' \subset \mathbb{C}^N$ $(N \geq n)$ be a strongly pseudoconvex real algebraic hypersurface of the special form, and F be a meromorphic mapping in a neighborhood of a point $p \in M$ which is holomorphic in one side of M. Assuming some additional conditions for the mapping F on the hypersurface M, we proved that F has a holomorphic extension to p. This result may be used to show the regularity of CR mappings between real hypersurfaces of different dimensions.

Keywords Meromorphic mappings · Real analytic hypersurfaces · Holomorphic extension

Mathematics Subject Classification 32D15 · 32H04 · 32H40

1 Introduction

The remarkable result of Forstnerič [5] on the classification problem of proper holomorphic mappings between unit balls states that if f is proper, holomorphic map from a ball in $\mathbb{C}^n$ to a ball in $\mathbb{C}^N$ and smooth of class C^{N-n+1} on the closure then f is a rational mapping. He posed the question of the holomorphic extendibility of such a rational mapping to any boundary point. In [4], Cima and Suffridge proved that every such mapping extends holomorphically to a neighborhood of the closed ball. This result was extended by Chiappari [3] by replacing the unit ball in domain with an arbitrary real analytic hypersurface in $\mathbb{C}^n$.

These results are also related to regularity of CR mappings between real hypersurfaces. When the real hypersurfaces lie in the complex spaces of same dimension, CR mappings of given smoothness must be real analytic (see for example [1]). In the case of real hypersurfaces of different dimensions, analyticity of CR mappings with given smoothness on the boundary was shown provided that the target is a real sphere (see for example [2,7]). In the proof, they first show that the CR mappings extend meromorphically. Then using the results of Chiappari and Cima–Suffridge, this meromorphic extension defines an analytic extension.

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In this work, we obtain a holomorphic extension result for meromorphic mappings with more general target spaces. More precisely, we prove the following theorem.

Theorem 1.1 *Let $M \subset \mathbb{C}^n$ be a real analytic hypersurface and $M' \subset \mathbb{C}^N$ be a strongly pseudoconvex real algebraic hypersurface which is locally equivalent to $\operatorname{Im} z'_N = p(z', \bar{z}')$ by a birational holomorphic change of coordinates at a point $q \in M'$, where $(z', z'_N) \in \mathbb{C}^N$, $N \geq n$ and $p(z', \bar{z}')$ is a real-valued polynomial. Let $U \subset \mathbb{C}^n$ be a neighborhood of a point $p \in M$ and Ω be the portion of U lying on one side of M. If $F : U \to \mathbb{C}^N$ is a meromorphic mapping which maps Ω holomorphically to one side of M', extends continuously on $\overline{\Omega}$, $F(M \cap U) \subset M'$ and $F(p) = q$, then F extends holomorphically to a neighborhood of p.*

In the statement of Theorem 1.1, by $F(M \cap U) \subset M'$, we mean that $\lim_{\Omega \ni z \to p} F(z) \in M'$ and $F(p) := \lim_{\Omega \ni z \to p} F(z)$ for all $p \in M \cap U$. Note that Theorem 1.1 improves the result of Chiappari [3] by replacing the sphere in the target with a special type of real algebraic hypersurface. One cannot expect to have extension for mappings with arbitrary targets. There are examples of proper rational mappings from the unit ball to a compact set that cannot be extended holomorphically through the boundary, (see [4,6]).

2 Proof of Theorem 1.1

Proof For simplicity, we will take $p = (0, \ldots, 0)$. Since the ring of germs of holomorphic functions is a unique factorization domain, we may assume that $F = \frac{f}{g}$ where $f = (f_j)_{1 \leq j \leq N}$ is a holomorphic mapping and g is a holomorphic function near $0 \in \mathbb{C}^n$ which has no common factor with f. If $g(0) \neq 0$, then F defines a holomorphic mapping near 0. Hence, we may assume that $g(0) = 0$.

Let M be given by $\psi(z, \bar{z}) = 0$ for some real analytic function ψ near 0 such that $\frac{\partial \psi}{\partial z_1}(0) \neq 0$. We define a nonzero holomorphic function $m(z) = \sum_{i=1}^n m_i z_i$ where $m_i = \frac{\partial \psi}{\partial z_i}(0)$. Since the zero sets of holomorphic functions are of measure zero, we can find a point $a = (a_1, \ldots, a_n) \neq 0$ such that $m(a) \neq 0$, $g(a) \neq 0$, $f_j(a) \neq 0$ for all $j = 1, \ldots, N$. Here, we have assumed that f'_j-s are not identically equal to 0, otherwise we can replace those f'_j-s with zeros in the rest of the proof. Now, we change the coordinates by

$$z_i = a_i \zeta_1 + \sum_{j=2}^n b_{ij} \zeta_j.$$

Since $a \neq 0$, we can choose b_{ij} so that $\zeta = (\zeta_1, \ldots, \zeta_n)$ gives a non-singular linear change of coordinates. In these new coordinates $\zeta = (\zeta_1, \ldots, \zeta_n)$, we have that $f_j(1, 0, \ldots, 0) = f_j(a) \neq 0$, $g(1, 0 \ldots, 0) = g(a) \neq 0$ and

$$\frac{\partial \psi}{\partial \zeta_1}(0) = \sum_{i=1}^n \frac{\partial \psi}{\partial z_i}(0) a_i = m(a) \neq 0.$$

For the convenience, we will denote the new coordinates by z again. Then, we may assume that $f_j(z_1, 0) \not\equiv 0$, $g(z_1, 0) \not\equiv 0$ and $\frac{\partial \psi}{\partial z_1}(0) \neq 0$. Hence, M can be defined as a graph $z_1 = \rho(\overline{z_1}, \tilde{z}, \bar{\tilde{z}})$ where $\tilde{z} = (z_2, \ldots, z_n)$ and $\rho(z_1, \lambda, \tau)$ is a holomorphic function near 0 in $\mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$. We may also assume that $\rho(\overline{z_1}, 0, 0) = \overline{z_1}$.

By Weierstrass preparation theorem, g can be written as $g(z_1, \tilde{z}) = u(z_1, \tilde{z}) h(z_1, \tilde{z})$ where $u(z_1, \tilde{z}) = \sum_{j=0}^l a_j(\tilde{z}) z_1^j$ is a Weierstrass polynomial so that $a_l(\tilde{z}) \equiv 1$, $a_j(0) = 0$

for $0 \leq j \leq l-1$ and $h(0,0) \neq 0$. Since F is bounded as $z = (z_1, \tilde{z}) \to 0$ in Ω, f_j can be decomposed as $f_j(z_1, \tilde{z}) = u_j(z_1, \tilde{z}) h_j(z_1, \tilde{z})$ where u'_j-s are Weierstrass polynomials in z_1 of degree $k_j \geq l$ and $h_j(0,0) \neq 0$. Using division algorithm, one can find r_j of degree smaller than l in z_1 such that $u_j(z_1, \tilde{z}) = u(z_1, \tilde{z}) d_j(z_1, \tilde{z}) + r_j(z_1, \tilde{z})$. Setting $D_j = d_j h_j$, $R_j = r_j h_j$, $D = (D_j)_{j=1}^N$, $R = (R_j)_{j=1}^N$, we have $f = uD + R$. Our aim is to show that $R \equiv 0$.

Since M' is strongly pseudoconvex, by a holomorphic change of variables, it can be written as

$$M' = \left\{ (z', z'_N) \in \mathbb{C}^N : \operatorname{Im} z'_N - \sum_{j=1}^{N-1} |z'_j|^2 + \phi(z', \overline{z'}) = 0 \right\}$$

where $\phi \equiv 0$ or ϕ is a real-valued polynomial of degree bigger than 2. If $\phi \equiv 0$, then M' is locally equivalent to the real sphere in $\mathbb{C}^N$, and Theorem 1.1 follows from the main result in [3].

Hence, we can assume that $\phi \not\equiv 0$. Let's write ϕ as

$$\phi(z', \overline{z'}) = \sum_{I,J} \alpha_{I,J} z'^I \overline{z'}^J.$$

Since ϕ is real valued, $\overline{\alpha_{IJ}} = \alpha_{JI}$ and hence the highest degrees of z' and $\overline{z'}$ in ϕ are the same, say they are equal to $d \geq 2$.

Since F maps M into M', $\forall z \in M$, we have that

$$\frac{f_N(z)}{g(z)} - \frac{\overline{f_N(z)}}{\overline{g(z)}} - 2i \sum_{j=1}^{N-1} \frac{|f_j(z)|^2}{|g(z)|^2} - 2i\phi\left(\frac{\tilde{f}(z)}{g(z)}, \frac{\overline{\tilde{f}(z)}}{\overline{g(z)}} \right) \equiv 0 \tag{2.1}$$

where $\tilde{f} = (f_1, \ldots, f_{N-1})$. Multiplying both sides of the above equation by $g(z)^d \overline{g(z)}^d$, we obtain that

$$\begin{aligned} & f_N(z) g(z)^{d-1} \overline{g(z)}^d - \overline{f_N(z)} g(z)^d \overline{g(z)}^{d-1} - 2i \sum_{j=1}^{N-1} |f_j(z)|^2 g(z)^{d-1} \overline{g(z)}^{d-1} \\ & - 2i g(z)^d \overline{g(z)}^d \phi\left(\frac{\tilde{f}(z)}{g(z)}, \frac{\overline{\tilde{f}(z)}}{\overline{g(z)}} \right) \equiv 0 \end{aligned} \tag{2.2}$$

For $z = (z_1, \tilde{z})$, we set $z^* = (\rho(\overline{z_1}, \tilde{z}, \overline{\tilde{z}}), \tilde{z})$, $s^*(z) = \overline{s(z^*)}$ for any function s. Then,

$$\begin{aligned} & f_N(z) g(z)^{d-1} \overline{g(z^*)}^d - \overline{f_N(z^*)} g(z)^d \overline{g(z^*)}^{d-1} - 2i \langle \tilde{f}(z), \tilde{f}(z^*) \rangle g(z)^{d-1} \overline{g(z^*)}^{d-1} \\ & - 2i g(z)^d \overline{g(z^*)}^d \phi\left(\frac{\tilde{f}(z)}{g(z)}, \frac{\overline{\tilde{f}(z^*)}}{\overline{g(z^*)}} \right) \end{aligned} \tag{2.3}$$

is a holomorphic function of z_1, and by (2.2), it is equal to 0 whenever $z_1 = \rho(\overline{z_1}, \tilde{z}, \overline{\tilde{z}})$, that is, when $z = z^*$. Here, $\langle , \rangle$ denotes the standard inner product, that is, $\langle a, b \rangle = \sum_{i=1}^N a_i \overline{b_i}$ for $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ in $\mathbb{C}^N$. For a fixed $\tilde{z}_0$, the real codimension of the set $\{z_1 = \rho(\overline{z_1}, \tilde{z}_0, \overline{\tilde{z}_0})\}$ in $\mathbb{C}^n$ is at most the sum of real codimensions of M and $\{\tilde{z} = \tilde{z}_0\}$. Hence, the real dimension of the set $\{z_1 = \rho(\overline{z_1}, \tilde{z}_0, \overline{\tilde{z}_0})\}$ is at least 1. It follows that the function above is identically 0 as a function of z_1.

Using the identities $f = uD + R$, $g = uh$ and $\tilde{f}(z^*) = \overline{\tilde{f}^*(z)}$, it follows from (2.3) that

$$\begin{aligned} &u^d u^{*d}(D_N h^{d-1} h^{*d} - D_N^* h^d h^{*(d-1)} - 2i\langle \tilde{D}, \overline{\tilde{D}^*}\rangle h^{d-1} h^{*(d-1)}) \\ &\quad + u^{d-1} u^{*d}(R_N h^{d-1} h^{*d} - 2i h^{d-1} h^{*(d-1)}\langle \tilde{R}, \overline{\tilde{D}^*}\rangle) \\ &\quad + u^d u^{*(d-1)}(R_N^* h^d h^{*(d-1)} - 2i h^{d-1} h^{*(d-1)}\langle \tilde{D}, \overline{\tilde{R}^*}\rangle) \\ &\quad - 2i u^{d-1} u^{*(d-1)} h^{d-1} h^{*(d-1)}\langle \tilde{R}, \overline{\tilde{R}^*}\rangle - 2i g(z)^d \overline{g(z^*)}^d \phi\left(\frac{\tilde{f}(z)}{g(z)}, \overline{\frac{\tilde{f}(z^*)}{g(z^*)}}\right) \equiv 0 \end{aligned} \tag{2.4}$$

where $\tilde{D} = (D_1, \ldots, D_{N-1})$ and $\tilde{R} = (R_1, \ldots, R_{N-1})$.

Let $\tilde{z} = 0$. We note that

$$u^*(z_1, 0) = \overline{u(\rho(\overline{z_1}, 0, 0), 0)} = \overline{\rho(\overline{z_1}, 0, 0)^l} = z_1^l$$

and $u(z_1, 0) = z_1^l$. Let us assume that $R(z_1, 0) \not\equiv 0$ and the multiplicity of z_1 in $R(z_1, 0)$ is a for some $0 \le a < l$. That is, $R(z_1, 0) = z_1^a Q(z_1)$ for some holomorphic function Q such that $Q(0) \ne 0$. The multiplicity of z_1 in the first summand of the function in (2.4) is greater than or equal to $2dl$. In the second and the third summands, the multiplicity of z_1 is greater than or equal to $(2d-1)l + a$. In the fourth summand, the multiplicity of z_1 is greater than or equal to $2(d-1)l + 2a$. Equation (2.4) implies that

$$\min\{2dl, (2d-1)l + a, 2(d-1)l + 2a\} = 2(d-1)l + 2a \tag{2.5}$$

must be smaller than or equal to the multiplicity of z_1 in the last term $g(z)^d \overline{g(z^*)}^d \phi\left(\frac{\tilde{f}(z)}{g(z)}, \overline{\frac{\tilde{f}(z^*)}{g(z^*)}}\right)$.

Note that the multiplicity of z_1 in $f = uD + R$ and in $g = uh$ are a and l, respectively. By writing

$$g(z)^d \overline{g(z^*)}^d \phi\left(\frac{\tilde{f}(z)}{g(z)}, \overline{\frac{\tilde{f}(z^*)}{g(z^*)}}\right) = \sum_{|I|,|J| \le d} \alpha_{IJ} \tilde{f}(z)^I g(z)^{d-|I|} \overline{\tilde{f}(z^*)}^J \overline{g(z^*)}^{d-|J|}$$

we see that the multiplicity of z_1 in the last summand in (2.4) is equal to

$$\min_{|I|,|J|,\alpha_{IJ} \ne 0}\{a|I| + l(d-|I|) + a|J| + l(d-|J|)\} \le \min_{|J|}\{ad + ld + |J|(a-l)\}. \tag{2.6}$$

The inequality above is obtained by taking $|I| = d$. We have the following cases for d.

Case 1: $d = 2$. Since the total degree of ϕ is bigger than or equal to 3, when $|I| = 2$, $|J|$ must be at least one. Then, it follows from (2.5) and (2.6) that

$$2l + 2a \le \min_{|J|}\{2a + 2l + |J|(a-l)\} \le 3a + l.$$

The second inequality above follows from the fact that $|J| \ge 1$ and $a - l < 0$. But this implies that $l \le a$, which contradicts to the choice of a, and hence $R(z_1, 0) \equiv 0$.

Case 2: $d > 2$. It follows from (2.5) and (2.6) that

$$2(d-1)l + 2a \le \min_{|J|}\{ad + ld + |J|(a-l)\} \le ad + ld.$$

But this implies that $d \le 2$. Hence, again we have that $R(z_1, 0) \equiv 0$.

Now, we suppose that $R \not\equiv 0$. We may assume that $R(a) \neq 0$ for some $a = (a_1, \dots, a_n)$ such that $a_2 \neq 0$. We change the coordinates from z to ζ defined by

$$z_1 = a_1\zeta_2 + \zeta_1,\ z_i = a_i\zeta_2 + \sum_{j=3}^{n} b_{ij}\zeta_j, \quad i = 2, \dots, n.$$

Since $(a_2, \dots, a_n) \neq 0$, b_{ij} can be chosen so that ζ gives a non-singular linear change of coordinates. In these new coordinates, $R(0, 1, 0, \dots, 0) = R(a) \neq 0$, $R(\zeta_1, 0) \equiv 0$, $f_j(\zeta_1, 0) \not\equiv 0$, $g(\zeta_1, 0) \not\equiv 0$ and $\frac{\partial\phi}{\partial\zeta_1}(0) = \frac{\partial\phi}{\partial z_1}(0) \neq 0$. We denote these new coordinates by z again. Then

$$R(z_1, 0) \equiv 0,\ R(z_1, z_2, 0 \dots, 0) \not\equiv 0,$$

f_j and g do not vanish on z_1-axis, and M can be written as $z_1 = \rho(\overline{z_1}, \tilde{z}, \overline{\tilde{z}})$ near the origin.

Since $R(z_1, z_2, 0, \dots, 0)$ is a nonzero analytic function of z_2 vanishing at $z_2 = 0$, there exists the largest integer $k \geq 1$ such that z_2^k divides $R(z_1, z_2, 0, \dots, 0)$. We define $G_i = \frac{R_i}{z_2^k}$, $G = (G_1, \dots, G_N)$ and $\tilde{G} = (G_1, \dots, G_{N-1})$. Note that

$$G(z_1, 0, \dots, 0) \neq 0. \tag{2.7}$$

Then, dividing the terms in (2.4) by $|z_2|^{2k}$, we obtain that

$$\begin{aligned}
&\frac{1}{|z_2|^{2k}} u^d u^{*d}(D_N h^{d-1} h^{*d} - D_N^* h^d h^{*(d-1)} - 2i\langle \tilde{D}, \overline{\tilde{D}^*}\rangle h^{d-1} h^{*(d-1)})\\
&\quad + \frac{1}{\overline{z_2}^k} u^{d-1} u^{*d}(G_N h^{d-1} h^{*d} - 2i h^{d-1} h^{*(d-1)}\langle \tilde{G}, \overline{\tilde{D}^*}\rangle)\\
&\quad + \frac{1}{z_2{}^k} u^d u^{*(d-1)}(G_N^* h^d h^{*(d-1)} - 2i h^{d-1} h^{*(d-1)}\langle \tilde{D}, \overline{\tilde{G}^*}\rangle)\\
&\quad - 2i u^{d-1} u^{*(d-1)} h^{d-1} h^{*(d-1)}\langle \tilde{G}, \overline{\tilde{G}^*}\rangle - \frac{2i}{|z_2|^{2k}} g(z)^d \overline{g(z^*)}^d \phi\left(\frac{\tilde{f}(z)}{g(z)}, \frac{\overline{\tilde{f}(z^*)}}{\overline{g(z^*)}}\right)\\
&\quad \equiv 0.
\end{aligned} \tag{2.8}$$

We take $(z_3, \dots, z_n) = 0$ and let $z_2 \to 0$ in above equation. Considering the order of z_1 in all terms in (2.8), as in above argument for $R(z_1, 0, \dots, 0)$, we obtain that $G(z_1, 0 \dots, 0) = 0$ when $z_1 = \rho(\overline{z_1}, 0, 0)$. But this contradicts to (2.7). Consequently, $R \equiv 0$ and $F = \frac{f}{g} = \frac{D}{h}$ defines a holomorphic mapping near $0 \in \mathbb{C}^n$. □

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References

1. Baouendi, M.S., Ebenfelt, P., Rothschild, L.P.: Real Submanifolds in Complex Space and Their Mappings. Princeton Mathematical Series, vol. 47. Princeton University Press, Princeton (1999)
2. Baouendi, M.S., Huang, X., Rothschild, L.P.: Regularity of CR mappings between algebraic hypersurfaces. Invent. Math. **125**, 13–36 (1996)
3. Chiappari, S.: Holomorphic extension of proper meromorphic mappings. Mich. Math. J. **38**, 167–174 (1991)

4. Cima, J.A., Suffridge, T.J.: Boundary behavior of rational proper maps. Duke Math. J. **60**(1), 135–138 (1990)
5. Forstnerič, F.: Extending proper holomorphic mappings of positive codimension. Invent. Math. **95**(1), 31–61 (1989)
6. Ivashkovich, S., Meylan, F.: An example concerning holomorphicity of meromorphic mappings along real hypersurfaces. Mich. Math. J. **64**(3), 487–491 (2015)
7. Mir, N.: Analytic regularity of CR maps into spheres. Math. Res. Lett. **10**(4), 447–457 (2003)