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On a class of generalized solutions to equations describing incompressible viscous fluids

Anna Abbatiello[2] · **Eduard Feireisl**[1]

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Abstract
We consider a class of viscous fluids with a general monotone dependence of the viscous stress on the symmetric velocity gradient. We introduce the concept of *dissipative solution* to the associated initial boundary value problem inspired by the measure-valued solutions for the inviscid (Euler) system. We show the existence as well as the weak–strong uniqueness property in the class of dissipative solutions. Finally, the dissipative solution enjoying certain extra regularity coincides with a strong solution of the same problem.

Keywords Generalized viscous fluid · Weak solution · Weak–strong uniqueness

Mathematics Subject Classification 35 Q 35 · 35 A 01 · 35 A 02

1 Introduction

The main goal of the present paper is to develop a mathematical theory of viscous fluids in the case of very low regularity of solutions of the underlying evolutionary equations, similar to the Euler system describing the inviscid fluids. In particular, we allow the viscous stress tensor to be merely bounded function of the velocity gradient, where the corresponding energy estimate provides only bounds of the total variation (in fact total "deformation") of the

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✉ Anna Abbatiello
anna.abbatiello@tu-berlin.de

Eduard Feireisl
feireisl@math.cas.cz

1 Faculty of Mathematics and Physics, Charles University, Sokolovská 83, CZ-186 75 Prague 8, Czech Republic

2 Institute of Mathematics, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

velocity. We refer to the recent survey by Málek, Blechta, and Rajagopal [1] for the relevant physical background of the equations and systems studied below.

The motion of a general viscous incompressible fluid is described in terms of its *velocity* $\mathbf{v} = \mathbf{v}(t, x)$ satisfying the following system of equations:

$$\mathrm{div}_x \mathbf{v} = 0, \tag{1.1}$$

$$\partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}. \tag{1.2}$$

Here Π is the associated pressure and $\mathbb{S}$ denotes the viscous stress tensor related to the symmetric velocity gradient

$$\mathbb{D}\mathbf{v} \equiv \frac{1}{2} \left(\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} \right)$$

through a general "implicit" rheological law

$$\mathbb{S} : \mathbb{D}\mathbf{v} = F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}) \tag{1.3}$$

where

$$F : R^{d\times d}_{\mathrm{sym}} \to [0, \infty), \ F(0) = 0, \ \mathrm{Dom}(F) = R^{d\times d}_{\mathrm{sym}}, \tag{1.4}$$

is a convex function, F^* denotes its conjugate, while $R^{d\times d}_{\mathrm{sym}}$ is the space of real symmetric tensors and $d = 2, 3$ is the dimension. Note that (1.3) means

$$\mathbb{S} \in \partial F(\mathbb{D}\mathbf{v}), \text{ or, equivalently, } \mathbb{D}\mathbf{v} \in \partial F^*(\mathbb{S}),$$

in particular, the mapping

$$\mathbb{D} \in R^{d\times d}_{\mathrm{sym}} \to \mathbb{S} \in \partial F(\mathbb{D}) \in R^{d\times d}_{\mathrm{sym}}$$

is monotone. Here and hereafter, the symbol ∂ denotes the subdifferential of a convex function. To avoid technicalities connected with the kinematic boundary, we restrict ourselves to the spatially periodic solutions defined on the flat torus

$$\mathbb{T}^d = \left([-1, 1]|_{\{-1,1\}} \right)^d . \tag{1.5}$$

The problem is formally closed by imposing the initial data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{1.6}$$

1.1 Weak and strong solutions

Problem (1.1)–(1.6) is essentially well posed locally in time in the class of strong (classical) solutions for non-degenerate F, see, e.g., Bothe and Prüss [2]. They consider a generalized Newtonian fluid

$$\mathbb{S} = \mu(|\mathbb{D}\mathbf{v}|^2)\mathbb{D}\mathbf{v} \tag{1.7}$$

which possesses the additional restriction to be non-degenerate compared to (1.3)–(1.4). By non-degenerate, we mean that μ is twice continuously differentiable function, such that

$$\mu(Z) > 0, \ \mu(Z) + 2Z\mu'(Z) > 0 \text{ for any } Z \in [0, \infty). \tag{1.8}$$

An iconic example is the power law fluid, for which

$$\mathbb{S} = \mu(|\mathbb{D}\mathbf{v}|)\mathbb{D}\mathbf{v}, \ \mu(|\mathbb{D}|) = (\mu_1 + \mu_2|\mathbb{D}|^2)^{\frac{p-2}{2}}, \ p \geq 1, \ \mu_1 \geq 0, \ \mu_2 > 0.$$

Note that the above example becomes non-degenerate when $\mu_1 > 0$. It is known that if $p \geq \frac{11}{5}$, and $d = 3$, the system (1.1)–(1.6) possesses global in time strong solutions, see, e.g., Málek, Nečas, Růžička [12]. The existence of weak solutions has been established in several cases, see, e.g., the monograph by Málek et al. [11]. The best result dealing with the limit case $d = 3$, $p > \frac{6}{5}$ was obtained by Diening, Růžička, and Wolf [8]. More general rheological laws have been studied by Bulíček et al. [5], see also [1] for the analysis of incompressible non-Newtonian fluids. The hypothesis $p > \frac{6}{5}$ represents a threshold in the 3D case. Indeed, the energy balance (inequality) associated with (1.1), (1.2) reads

$$\frac{1}{2}\int_{\mathbb{T}^d} |\mathbf{v}|^2(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\mathbb{D}\mathbf{v}) : \mathbb{D}\mathbf{v}\ \mathrm{d}x\,\mathrm{d}t \leq \frac{1}{2}\int_{\mathbb{T}^d} |\mathbf{v}_0|^2\ \mathrm{d}x. \tag{1.9}$$

In view of the standard Sobolev embedding $W^{1,p} \hookrightarrow L^q$, $1 \leq q \leq \frac{3p}{3-p}$, the solutions are *a priori* bounded in the space L^q, $q > 2$ only if $p > \frac{6}{5}$. In the opposite case, the convective term $\mathbf{v} \otimes \mathbf{v}$ is only bounded in $L^\infty(0,T;L^1(\mathbb{T}^d;R^{d\times d}_{\rm sym}))$, which makes the analysis rather delicate. The theory of dissipative solutions proposed in this paper goes beyond this threshold considering the class of velocities for which $\mathbb{D}\mathbf{v}$ is merely a measure. Possible concentrations that may be produced by the convective term are captured by a Reynolds viscous stress $\mathfrak{R}_v$ introduced below.

1.2 Dissipative solutions

Inspired by [3], we introduce the concept of *dissipative solution* to problem (1.1)–(1.6). The essential features of the approach are (i) augmenting the family of unknowns by quantities that account for possible oscillations/concentrations, (ii) considering the energy balance as an integral part of the definition of generalized solutions.

Accordingly, we shall deal with the following quantities:

- The velocity $\mathbf{v} \in C_{\rm weak}([0,T];L^2(\mathbb{T}^d;R^d))$;
- The viscous stress tensor $\mathbb{S} \in L^1(0,T;L^1(\mathbb{T}^d;R^{d\times d}_{\rm sym}))$;
- The Reynolds viscous stress $\mathfrak{R}_v$ that accounts for possible concentrations in the convective term,

$$\mathfrak{R}_v \in L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^d;R^{d\times d}_{\rm sym})).$$

Remark 1.1 The symbol $\mathcal{M}^+(\mathbb{T}^d;R^{d\times d}_{\rm sym})$ denotes the space of finite vector valued signed measures on $\mathbb{T}^d$ ranging in the cone of positively definite symmetric matrices. Specifically,

$$\mathbb{M} \in \mathcal{M}^+(\mathbb{T}^d;R^{d\times d}_{\rm sym}) \ \Leftrightarrow\ \int_{\mathbb{T}^d} \varphi(y)\ \xi\otimes\xi : \mathrm{d}\mathbb{M}(y) \geq 0 \text{ for any } \xi \in R^d,\ \varphi \in C_c^\infty(\mathbb{T}^d),\ \varphi \geq 0.$$

The dissipative solutions, introduced in detail in Sect. 2, solve the following system of equations

$$\begin{gathered}
\mathrm{div}_x \mathbf{v} = 0,\\
\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S} - \mathrm{div}_x \mathfrak{R}_v,\ \ \mathbf{v}(0,\cdot) = \mathbf{v}_0,\\
\frac{1}{2}\int_{\mathbb{T}^d} |\mathbf{v}|^2(\tau,\cdot)\ \mathrm{d}x + \int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v](\tau) + \int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S})\Big]\ \mathrm{d}x\,\mathrm{d}t \leq \frac{1}{2}\int_{\mathbb{T}^d} |\mathbf{v}_0|^2\ \mathrm{d}x
\end{gathered} \tag{1.10}$$

in the sense of distributions.

Remark 1.2 Strictly speaking, the symmetric gradient $\mathbb{D}\mathbf{v}$ will be merely a measure on $(0, T) \times \mathbb{T}^d$. Accordingly, its composition $F(\mathbb{D}\mathbf{v})$ with a convex function F must be interpreted in a generalized sense proposed by Demengel and Temam [6], [7].

Although formally underdetermined, the problem (1.10) enjoys the *weak–strong uniqueness* property. The velocity $\mathbf{v}$ associated with the dissipative solutions coincides with the velocity satisfying (1.1), (1.2) in the classical sense as long as the latter exists. In such a case, the measure $\mathfrak{R}_v$ vanishes, while

$$\mathbb{S} \in \partial F(\mathbb{D}\mathbf{v}) \text{ for a.a. } (t, x) \in (0, T) \times \mathbb{T}^d.$$

The rest of the paper is organized as follows:

- In Sect. 2, we introduce the dissipative solutions and state our main results.
- In Sect. 3, we show existence of a dissipative solution for a fairly general class of initial data.
- In Sect. 4, we establish the relative energy inequality associated with system (1.10), which represents a basic tool for proving stability results. In particular, we establish the weak–strong uniqueness property in Sect. 5.
- In Sect. 6, we show that dissipative solutions that are sufficiently regular coincide with the standard strong solutions of the same problem.
- Finally, we discuss possible extensions of the method in Sect. 7.

2 Preliminaries, main results

We start by introducing the basic hypotheses imposed on the rheological relation (1.3) between the viscous stress $\mathbb{S}$ and the symmetric part of the velocity gradient $\mathbb{D}\mathbf{v}$. We recall Fenchel–Young inequality

$$\mathbb{S} : \mathbb{D} \leq F(\mathbb{D}) + F^*(\mathbb{S}) \text{ for any } \mathbb{D}, \mathbb{S} \in R^{d\times d}_{\text{sym}} \tag{2.1}$$

yielding

$$F^*(\mathbb{S}) = \sup_{\mathbb{D}\in R^{d\times d}_{\text{sym}}} \left[\mathbb{S} : \mathbb{D} - F(\mathbb{D})\right], \text{ in particular } F(0) = 0 \Rightarrow F^*(\mathbb{S}) \geq 0. \tag{2.2}$$

Moreover, in view of hypothesis (1.4), the domain of F is the whole space $R^{d\times d}_{\text{sym}}$ which implies that $\mathbb{F}^*$ is superlinear, specifically,

$$\liminf_{|\mathbb{S}|\to\infty} \frac{F^*(\mathbb{S})}{|\mathbb{S}|} = \infty. \tag{2.3}$$

For technical reasons specified below, we will assume that the domain of F^* contains a ball in $R^{d\times d}_{\text{sym}}$: there is $r > 0$ such that

$$0 \leq F^*(\mathbb{S}) < \infty \text{ for any } \mathbb{S} \in R^{d\times d}_{\text{sym}},\ |\mathbb{S}| < r. \tag{2.4}$$

This implies that F grows at least linearly for large $\mathbb{D}$,

$$\liminf_{|\mathbb{D}|\to\infty} \frac{F(\mathbb{D})}{|\mathbb{D}|} > 0. \tag{2.5}$$

In view of the constitutive relation (1.3), we may anticipate that generalized solutions of (1.1), (1.2), satisfying some form of the energy balance (1.9), will be fields $\mathbf{v} \in C_{\rm weak}([0,T]; L^2(\mathbb{T}^d; R^d))$ with bounded deformation

$$\mathbb{D}\mathbf{v} \in \mathcal{M}((0,T) \times \mathbb{T}^d; R^{d\times d}_{\rm sym}) \cap L^\infty(0,T; W^{-1,2}(\mathbb{T}^d; R^{d\times d}_{\rm sym})),$$

where $\mathcal{M}((0,T) \times \mathbb{T}^d; R^{d\times d}_{\rm sym})$ denotes the set of finite tensor-valued Radon measures.

Now, we may use the machinery developed by Demengel and Temam [7] to define

$$F(\mathbb{D}\mathbf{v}) \in \mathcal{M}((0,T) \times \mathbb{T}^d).$$

To this end, a few technical hypotheses are needed. Let

$$F_\infty(\mathbb{D}) \equiv \lim_{s\to\infty} \frac{F(s\mathbb{D})}{s} \in [0,\infty]$$

be the asymptotic function of F. Following Demengel and Temam [7], we further suppose that there is $r > 0$ such that

$$\mathrm{Dom}(F^*) \subseteq \Lambda^0 + B(r, \mathbb{O}), \ \text{ where } \Lambda \equiv \mathrm{Dom}(F_\infty). \tag{2.6}$$

Here $B(r, \mathbb{O})$ is the ball of radius r centered at zero in $R^{d\times d}_{\rm sym}$, while Λ^0 is the polar set of Λ and $\mathrm{Dom}(\cdot)$ denotes the domain of a function. In view of [7, Proposition 1.2], hypothesis (2.6) is equivalent to

$$F_\infty(\mathbb{D}) \le r|\mathbb{D}| \text{ for all } \mathbb{D} \in \mathrm{Dom}(F_\infty).$$

Under these circumstances, one can define

$$F(\mathbb{D}\mathbf{v}) \in \mathcal{M}((0,T) \times \mathbb{T}^d),$$

see [7, Section 2].

Remark 2.1 The hypotheses (2.4), (2.6) may seem rather awkward at first glance. We claim they are automatically satisfied if $\mathrm{Dom}(F^*) = R^{d\times d}_{\rm sym}$, in which case F is superlinear in $\mathbb{D}$. We refer to [6], [7] for other interesting examples.

2.1 Dissipative solutions

As already pointed out, the dissipative solutions are defined in terms of the velocity field $\mathbf{v}$, the viscous stress $\mathbb{S}$, and the Reynolds viscous stress tensor $\mathfrak{R}_v$.

Definition 2.2 We say that $\mathbf{v} \in C_{\rm weak}([0,T]; L^2(\mathbb{T}^d; R^d))$ is a *dissipative solution* of the problem (1.1)–(1.6) if

$$\mathbb{D}\mathbf{v} \in \mathcal{M}((0,T) \times \mathbb{T}^d; R^{d\times d}_{\rm sym}), \tag{2.7}$$

and there exist

$$\mathbb{S} \in L^1((0,T) \times \mathbb{T}^d; R^{d\times d}_{\rm sym}),\ \mathfrak{R}_v \in L^\infty(0,T; \mathcal{M}^+(\mathbb{T}^d; R^{d\times d}_{\rm sym}))$$

such that the following holds:

- Incompressibility

$$\int_0^T \int_{\mathbb{T}^d} \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x\, \mathrm{d}t = 0 \tag{2.8}$$

for any $\varphi \in C^1([0,T] \times \mathbb{T}^d)$.

- Momentum equation

$$\int_{\mathbb{T}^d} \mathbf{v}\cdot\boldsymbol{\varphi}(\tau,\cdot)\ \mathrm{d}x - \int_{\mathbb{T}^d} \mathbf{v}_0\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x$$
$$= \int_0^\tau\int_{\mathbb{T}^d} [\mathbf{v}\cdot\partial_t\boldsymbol{\varphi} + (\mathbf{v}\otimes\mathbf{v}):\nabla_x\boldsymbol{\varphi}]\ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_{\mathbb{T}^d} \mathbb{S}:\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\mathbb{T}^d} \nabla_x\boldsymbol{\varphi}:\mathrm{d}\mathfrak{R}_v\,\mathrm{d}t \tag{2.9}$$

for any $0\le\tau\le T$, and $\boldsymbol{\varphi}\in C^1([0,T]\times\mathbb{T}^d;R^d)$, $\mathrm{div}_x\boldsymbol{\varphi}=0$.
- Energy inequality

$$\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}|^2(\tau,\cdot)\ \mathrm{d}x + \int_{\mathbb{T}^d}\mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)] + \int_0^\tau\int_{\mathbb{T}^d}\Big[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S})\Big]\mathrm{d}x\,\mathrm{d}t$$
$$\le \int_{\mathbb{T}^d}\frac{1}{2}|\mathbf{v}_0|^2\ \mathrm{d}x \tag{2.10}$$

for a.a. $\tau\in(0,T)$.

Remark 2.3 The integral $\int_0^\tau\int_{\mathbb{T}^d} F(\mathbb{D}\mathbf{v})\ \mathrm{d}x$ should be understood as

$$\int_0^\tau\int_{\mathbb{T}^d} F(\mathbb{D}\mathbf{v})\ \mathrm{d}x = \int_{(0,\tau)\times\mathbb{T}^d}\mathrm{d}F(\mathbb{D}\mathbf{v})$$

in the sense of Demengel and Temam [7]. Note that this extended definition is only necessary if $\mathrm{Dom}(F^*)$ is not the whole space $R^{d\times d}_{\mathrm{sym}}$, because in such case one can prove that F is superlinear and $\mathbb{D}\mathbf{v}\in L^1((0,T);L^1(\mathbb{T}^d,R^{d\times d}))$.

2.2 Main results

As the first result, we state the existence of dissipative solutions proved in Sect. 3.

Theorem 2.4 *Let $\mathbf{v}_0\in L^2(\mathbb{T}^d;R^d)$, $\mathrm{div}_x\mathbf{v}_0=0$, $d=2,3$, and $T>0$ be given. Suppose that F satisfies* (1.4)*, together with* (2.4)*,* (2.6).

Then the problem (1.1)–(1.6) *admits a dissipative solution $\mathbf{v}\in C_{\mathrm{weak}}([0,T];L^2(\mathbb{T}^d;R^d))$ in the sense of Definition* 2.2*.*

Remark 2.5 To simplify presentation, we deliberately omitted the issue of boundary conditions. However, the proof can be easily adapted to include some of the standard boundary conditions, in particular the complete slip or the no-slip boundary conditions. Note that in the latter case, the viscous stress $\mathbb{S}$ must be non-degenerate so that the trace of $\mathbf{v}$ is well-defined. We refer to Blechta, Málek and Rajagopal [1] for a detailed discussion of various boundary conditions.

The second result concerns the weak–strong uniqueness property. In view of the theory developed by Bothe and Prüss [2], we state it in the L^p-framework.

Theorem 2.6 *Let F satisfy the hypotheses* (1.4)*,* (2.4)*,* (2.6)*. Let $p>d+2$, $d=2,3$. Suppose that the system* (1.1)–(1.6) *admits a strong solution $\widehat{\mathbf{v}}$ in the class*

$$\widehat{\mathbf{v}}\in C([0,T];W^{2-\frac{2}{p},p}(\mathbb{T}^d;R^d)),\ \partial_t\widehat{\mathbf{v}}\in L^p(0,T;L^p(\mathbb{T}^d;R^d)),$$
$$\widehat{\mathbf{v}}\in L^p(0,T;W^{2,p}(\mathbb{T}^d;R^d)) \tag{2.11}$$

defined on a time interval $[0, T]$*, with the viscous stress*

$$\widehat{\mathbb{S}} \in L^p(0, T; W^{1,p}(\mathbb{T}^d, R^{d\times d}_{\rm sym})) \cap C([0, T] \times \mathbb{T}^d, R^{d\times d}_{\rm sym}), \tag{2.12}$$

and with the initial datum

$$\widehat{\mathbf{v}}(0, \cdot) = \mathbf{v}_0 \in W^{2-\frac{2}{p},p}(\mathbb{T}^d; R^d).$$

Let $\mathbf{v}$*, together with* $\mathbb{S}$ *and* $\mathfrak{R}_v$*, be a dissipative solution in the sense of Definition 2.2 of the same problem starting from the same initial data*

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0.$$

Then

$$\mathfrak{R}_v = 0,\ \mathbb{S}(t, x) \in \partial F(\mathbb{D}\mathbf{v}(t, x)) \textit{ for a.a. } (t, x) \in (0, T) \times \mathbb{T}^d,$$

and

$$\mathbf{v}(t, x) = \widehat{\mathbf{v}}(t, x) \textit{ for a.a. } (t, x) \in (0, T) \times \mathbb{T}^d.$$

Remark 2.7 Note that for $p > d + 2$, we have the embedding relation $W^{2-\frac{2}{p},p}(\mathbb{T}^d) \hookrightarrow C^1(\mathbb{T}^d)$.

The proof of Theorem 2.6 will be given in Sects. 4, 5. Note that *existence* of local-in-time strong solutions in the class specified in Theorem 2.6 was proved by Bothe and Prüss [2] for smooth non-degenerate viscous stress tensor satisfying (1.7), (1.8). In this case, (2.12) follows from (2.11).

Finally, we claim that dissipative solutions enjoying certain regularity are in fact strong solutions.

Theorem 2.8 *Let* F *satisfy the hypotheses* (1.4)*,* (2.4)*,* (2.6)*. Let* $p > d + 2$*,* $d = 2, 3$*. Let*

$$\mathbf{v}_0 \in W^{2-\frac{2}{p},p}(\mathbb{T}^d; R^d),\ \mathrm{div}_x \mathbf{v}_0 = 0$$

be given. Suppose that $\mathbf{v}$ *is a dissipative solution of the system* (1.1)–(1.6) *on the time interval* $[0, T]$ *belonging to the regularity class*

$$\begin{aligned} &\mathbf{v} \in C([0, T]; W^{2-\frac{2}{p},p}(\mathbb{T}^d; R^d)),\ \partial_t \mathbf{v} \in L^p(0, T; L^p(\mathbb{T}^d; R^d)),\\ &\mathbf{v} \in L^p(0, T; W^{2,p}(\mathbb{T}^d; R^d)). \end{aligned} \tag{2.13}$$

Then $\mathfrak{R}_v = 0$ *and*

$$\mathbb{S} : \mathbb{D}\mathbf{v} = F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}),$$

meaning $\mathbf{v}$ *is a strong solution of the system* (1.1)–(1.6).

The proof of Theorem 2.8 is given in Sect. 6.

3 Existence

For any $n \in \mathbb{N}$, we look for approximations $\mathbf{v}^n$ and $\mathbb{S}^n$ that are solutions of the following system

$$\int_{\mathbb{T}^d} \partial_t \mathbf{v}^n \cdot \boldsymbol{\varphi}_i \, \mathrm{d}x + \int_{\mathbb{T}^d} \mathbb{S}^n : \mathbb{D}\boldsymbol{\varphi}_i \, \mathrm{d}x = \int_{\mathbb{T}^d} \mathbf{v}^n \otimes \mathbf{v}^n : \nabla_x \boldsymbol{\varphi}_i \, \mathrm{d}x, \ i = 1, \dots, n, \ \text{a.e. in } (0, T), \tag{3.1a}$$

$$\mathbf{v}^n(0, \cdot) = P^n \mathbf{v}_0, \tag{3.1b}$$

such that they satisfy

$$\mathbb{S}^n \in \partial F(\mathbb{D}\mathbf{v}^n), \tag{3.2}$$

and the following energy balance holds

$$\frac{1}{2} \int_{\mathbb{T}^d} |\mathbf{v}^n(\tau)|^2 \, \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} F^*(\mathbb{S}^n) + F(\mathbb{D}\mathbf{v}^n) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \frac{1}{2} \int_{\mathbb{T}^d} |P^n \mathbf{v}_0|^2 \, \mathrm{d}x \ \text{ for any } \tau \in (0, T). \tag{3.3}$$

Here, the sequence $\{\boldsymbol{\varphi}_i\}_{i \in \mathbb{N}}$ denotes the eigenfunctions of the Stokes operator, while P^n is the projection onto the finite-dimensional space $[\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n]$ for any $n \in \mathbb{N}$. The existence of such $\mathbf{v}^n$ and $\mathbb{S}^n$ can be achieved regularizing $\mathbb{S}^n$ by the introduction of a convolution kernel depending on $\mathbb{D}\mathbf{v}^n$. With this approximation scheme, the Galerkin method can be applied and then $\mathbf{v}^n$ and $\mathbb{S}^n$ are obtained as limit in the finite-dimensional spaces. A detailed proof was carried over in [5, Section 3.1].

The energy inequality (3.3) together with (3.2) implies the following

$$\mathbf{v}^n \rightharpoonup^* \mathbf{v} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d; R^d)), \ \text{as } n \to +\infty, \tag{3.4}$$

$$\mathbb{S}^n \rightharpoonup \mathbb{S} \text{ in } L^1(0, T; L^1(\mathbb{T}^d; R^{d \times d})), \ \text{as } n \to +\infty, \tag{3.5}$$

$$\mathbb{D}\mathbf{v}^n \rightharpoonup^* \mathbb{D}\mathbf{v} \text{ in } \mathcal{M}((0, T) \times \mathbb{T}^d; R^{d \times d}_{\text{sym}}), \ \text{as } n \to +\infty, \tag{3.6}$$

at least for suitable subsequences, which we do not relabel. Note that (3.5) follows from the superlinearity of F^* and de la Vallé-Poussin criterion, while for (3.6), we use the assumption (2.5) and the embedding $L^1((0, T) \times \mathbb{T}^d; R^{d \times d}_{\text{sym}}) \hookrightarrow \mathcal{M}((0, T) \times \mathbb{T}^d; R^{d \times d}_{\text{sym}})$.

As a consequence of (3.4), we have

$$0 = \int_0^T \int_{\mathbb{T}^d} \mathbf{v}^n \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} \mathbf{v} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \text{ as } n \to +\infty,$$

for any $\varphi \in C^1([0, T] \times \mathbb{T}^d)$, and thus the incompressibility condition (2.8) is satisfied.

Now, consider $\mathbf{v}^n \otimes \mathbf{v}^n - \mathbf{v} \otimes \mathbf{v}$. In view of the energy inequality (3.3), the sequence is uniformly bounded in the space $L^\infty(0, T; L^1(\mathbb{T}^d; R^{d \times d}))$, which is embedded into $L^\infty(0, T; \mathcal{M}(\mathbb{T}^d; R^{d \times d}))$. Therefore, there exists $\mathfrak{R}_v \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d; R^{d \times d}_{\text{sym}}))$ such that

$$(\mathbf{v}^n \otimes \mathbf{v}^n - \mathbf{v} \otimes \mathbf{v}) \rightharpoonup^* \mathfrak{R}_v \text{ in } L^\infty(0, T; \mathcal{M}(\mathbb{T}^d; R^{d \times d}_{\text{sym}})) \text{ as } n \to +\infty. \tag{3.7}$$

Now, through the same computations as in [9, Section 3.2] it can be shown that $\mathfrak{R}_v$ is positive semidefinite, thus

$$\mathfrak{R}_v \in L^\infty(0, T; \mathcal{M}^+(\mathbb{T}^d; R^{d \times d}_{\text{sym}})).$$

For the reader's convenience, we repeat the computations here. For any $\xi \in R^d$, $\varphi \in C_c^\infty(\mathbb{T}^d)$, $\varphi \geq 0$ and $\psi \in L^1(0,T)$, $\psi \geq 0$, it holds

$$\int_0^T \int_{\mathbb{T}^d} \psi(t)\varphi(x)\, \xi \otimes \xi : \mathrm{d}\mathfrak{R}_v \mathrm{d}t = \lim_{n\to+\infty} \int_0^T \int_{\mathbb{T}^d} \psi(t)\varphi(x)\, \xi \otimes \xi : (\mathbf{v}^n \otimes \mathbf{v}^n - \mathbf{v} \otimes \mathbf{v})\mathrm{d}x\mathrm{d}t$$

and

$$\xi \otimes \xi : \mathfrak{R}_v = \lim_{n\to+\infty} \xi \otimes \xi : (\mathbf{v}^n \otimes \mathbf{v}^n - \mathbf{v} \otimes \mathbf{v}) = \lim_{n\to+\infty} |\xi \cdot \mathbf{v}^n|^2 - |\xi \cdot \mathbf{v}|^2 \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^d),$$

then the weak lower semicontinuity of the convex function $\mathbf{v} \to |\xi \cdot \mathbf{v}|^2$, with $\xi \in R^d$, implies the desired conclusion. Finally, from (3.1a) fixing $\boldsymbol{\varphi} \in [\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_k]$ with $k \leq n$, multiplying by $\eta \in C^1(0,T)$, adding and subtracting $\int_0^t \int_{\mathbb{T}^d} \mathbf{v} \otimes \mathbf{v} : \nabla_x \boldsymbol{\varphi} \mathrm{d}x\ \eta(\tau)\,\mathrm{d}t$, taking the limit as $n \to +\infty$ and collecting the established convergences, using also a standard density argument, one gets (2.9).

In order to prove (2.10), we rewrite (3.3) in the weak "differential" form:

$$-\int_0^T \partial_t \psi \int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}^n|^2\ \mathrm{d}x\,\mathrm{d}t + \int_0^T \psi \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}^n) + F^*(\mathbb{S}^n)\Big]\ \mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^d} \frac{1}{2}|P^n\mathbf{v}_0|^2\ \mathrm{d}x \tag{3.8}$$

for any $\psi \in C_c^\infty[0,T)$, $\psi(0) = 1$, $\partial_t \psi \leq 0$. Next, we write

$$\frac{1}{2}|\mathbf{v}^n|^2 = \frac{1}{2}\mathrm{trace}\left[\mathbf{v}^n \otimes \mathbf{v}^n - \mathbf{v} \otimes \mathbf{v}\right] + \frac{1}{2}|\mathbf{v}|^2.$$

Employing [7, Lemma 3.1], we obtain

$$\int_0^T \int_{\mathbb{T}^d} \widetilde{\psi}\mathrm{d}F(\mathbb{D}\mathbf{v}) \equiv \int_0^T \int_{\mathbb{T}^d} \widetilde{\psi} F(\mathbb{D}\mathbf{v})\ \mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to+\infty} \int_0^T \int_{\mathbb{T}^d} \widetilde{\psi} F(\mathbb{D}\mathbf{v}^n)\ \mathrm{d}x\,\mathrm{d}t \tag{3.9}$$

for any $\widetilde{\psi} \in C_c^\infty(0,T)$, $0 \leq \widetilde{\psi} \leq \psi$, where the measure $F(\mathbb{D}\mathbf{v})$ is understood in the sense of Demengel and Temam [7]. Moreover, since F^* is a lower semicontinuous convex function, then

$$\int_0^T \psi \int_{\mathbb{T}^d} F^*(\mathbb{S})\ \mathrm{d}x\,\mathrm{d}t \leq \liminf_{n\to+\infty} \int_0^T \psi \int_{\mathbb{T}^d} F^*(\mathbb{S}^n)\ \mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

Consequently, using (3.7), (3.9), and (3.10) we may perform the limit $n \to +\infty$ in (3.8) to obtain (2.10). This completes the proof of Theorem 2.4.

4 Relative energy and weak–strong uniqueness

We show the relative energy inequality that is a crucial tool for proving the weak strong uniqueness property claimed in Theorem 2.6.

4.1 Relative energy inequality

Let $\mathbf{U}$ be a continuously differentiable vector field, $\mathrm{div}_x \mathbf{U} = 0$. It follows considering $\boldsymbol{\varphi} = -\mathbf{U}$ as test function in the momentum balance (2.9), adding to both sides $\int_0^\tau \int_{\mathbb{T}^d} \partial_t \mathbf{U} \cdot \mathbf{U}\ \mathrm{d}x\,\mathrm{d}t$ and summing up the resulting expression with the energy inequality (2.10) that

$$\left[\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}-\mathbf{U}|^2 \ \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)] + \int_0^\tau \int_{\mathbb{T}^d} \left[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S})\right] \ \mathrm{d}x \, \mathrm{d}t$$
$$\leq - \int_0^\tau \int_{\mathbb{T}^d} [\mathbf{v}\cdot\partial_t \mathbf{U} + (\mathbf{v}\otimes\mathbf{v}) : \nabla_x \mathbf{U}] \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S} : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^\tau \int_{\mathbb{T}^d} \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R}_v \, \mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \partial_t \mathbf{U}\cdot\mathbf{U} \ \mathrm{d}x \, \mathrm{d}t.$$

Thus, regrouping some terms, we get

$$\left[\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}-\mathbf{U}|^2 \ \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)] + \int_0^\tau \int_{\mathbb{T}^d} \left[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D}\mathbf{U}\right] \ \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_0^\tau \int_{\mathbb{T}^d} [(\mathbf{U}-\mathbf{v})\cdot\partial_t \mathbf{U} - (\mathbf{v}\otimes\mathbf{v}) : \nabla_x \mathbf{U}] \ \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^\tau \int_{\mathbb{T}^d} \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R}_v \, \mathrm{d}t. \tag{4.1}$$

Finally,

$$\int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v}\otimes\mathbf{v}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^\tau \int_{\mathbb{T}^d} \mathbf{v}\otimes(\mathbf{v}-\mathbf{U}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v}\otimes\mathbf{U}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v}-\mathbf{U})\otimes(\mathbf{v}-\mathbf{U}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} \mathbf{U}\otimes(\mathbf{v}-\mathbf{U}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t,$$

where we have used that

$$\int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v}\otimes\mathbf{U}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t = 0$$

thanks to the incompressibility constraint (2.8).

Accordingly, relation (4.1) reduces to

$$\left[\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}-\mathbf{U}|^2 \ \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)] + \int_0^\tau \int_{\mathbb{T}^d} \left[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D}\mathbf{U}\right] \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_0^\tau \int_{\mathbb{T}^d} [(\mathbf{U}-\mathbf{v})\cdot(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U})] \ \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^\tau \int_{\mathbb{T}^d} \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R}_v \, \mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v}-\mathbf{U})\otimes(\mathbf{v}-\mathbf{U}) : \nabla_x \mathbf{U} \ \mathrm{d}x \, \mathrm{d}t \tag{4.2}$$

Relation (4.2) is called *relative energy inequality*. Its validity can be easily extended by density argument to any vector field $\mathbf{U}$ belonging to the regularity class (2.11) from Theorem 2.6.

5 Weak strong uniqueness property

The weak strong uniqueness follows immediately from the relative energy inequality (4.2) applied to $\mathbf{U} = \widehat{\mathbf{v}}$ - the strong solution emanating from the initial data $\mathbf{v}_0$. Indeed, we have

$$\int_0^\tau \int_{\mathbb{T}^d} (\widehat{\mathbf{v}}-\mathbf{v})\cdot(\partial_t \widehat{\mathbf{v}} + \widehat{\mathbf{v}}\cdot\nabla_x \widehat{\mathbf{v}}) \ \mathrm{d}x \, \mathrm{d}t = \int_0^\tau \int_{\mathbb{T}^d} (\widehat{\mathbf{v}}-\mathbf{v})\cdot \mathrm{div}_x \widehat{\mathbb{S}} \ \mathrm{d}x \, \mathrm{d}t,$$

where, in accordance with (1.3),

$$\widehat{\mathbb{S}} : \mathbb{D}\widehat{\mathbf{v}} = F(\mathbb{D}\widehat{\mathbf{v}}) + F^*(\widehat{\mathbb{S}}).$$

Consequently, the relative energy inequality (4.2) yields

$$\begin{aligned}
&\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v} - \widehat{\mathbf{v}}|^2(\tau, \cdot)\ \mathrm{d}x + \int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)] + \int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D}\widehat{\mathbf{v}}\Big]\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v} - \widehat{\mathbf{v}}) \cdot \mathrm{div}_x \widehat{\mathbb{S}}\ \mathrm{d}x\,\mathrm{d}t \\
&\quad \leq - \int_0^\tau \int_{\mathbb{T}^d} \nabla_x \widehat{\mathbf{v}} : \mathrm{d}\mathfrak{R}_v\,\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v} - \widehat{\mathbf{v}}) \otimes (\mathbf{v} - \widehat{\mathbf{v}}) : \nabla_x \widehat{\mathbf{v}}\ \mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{5.1}$$

for a.a. $\tau \in (0, T)$.

Strictly speaking, $F(\mathbb{D}\mathbf{v})$ is a measure defined on an open set $(0, \tau) \times \mathbb{T}^d$. Accordingly, we should interpret

$$\int_0^\tau \int_{\mathbb{T}^d} F(\mathbb{D}\mathbf{v}) dx\,\mathrm{d}t \equiv \int_0^\tau \int_{\mathbb{T}^d} \mathrm{d}F(\mathbb{D}\mathbf{v})\,\mathrm{d}t = \lim_{n\to+\infty} \int_0^\tau \psi_n \int_{\mathbb{T}^d} \mathrm{d}F(\mathbb{D}\mathbf{v})\,\mathrm{d}t$$

where $\psi_n \in C_c^\infty(0, \tau]$, $\psi_n \nearrow 1_{(0,\tau)}$, $0 \leq \psi_n \leq 1$.
Similarly, we have

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v} - \widehat{\mathbf{v}}) \cdot \mathrm{div}_x \widehat{\mathbb{S}}\ \mathrm{d}x\,\mathrm{d}t = \lim_{n\to+\infty} \int_0^\tau \psi_n(t) \int_{\mathbb{T}^d} (\mathbf{v} - \widehat{\mathbf{v}}) \cdot \mathrm{div}_x \widehat{\mathbb{S}}\ \mathrm{d}x\,\mathrm{d}t \\
&= - \lim_{n\to+\infty} \int_0^\tau \psi_n(t) \int_{\mathbb{T}^d} (\mathbb{D}\mathbf{v} - \mathbb{D}\widehat{\mathbf{v}}) : \widehat{\mathbb{S}}\ \mathrm{d}x\,\mathrm{d}t = \int_0^\tau \int_{\mathbb{T}^d} \widehat{\mathbb{S}} : \mathbb{D}\widehat{\mathbf{v}}\ \mathrm{d}x - \int_0^\tau \int_{\mathbb{T}^d} \widehat{\mathbb{S}} : \mathrm{d}\mathbb{D}\mathbf{v}.
\end{aligned}$$

Thus, the integrals on the left-hand side of (5.1) can be handled by means of Fenchel–Young inequality:

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D}\widehat{\mathbf{v}}\Big]\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^d} (\mathbf{v} - \widehat{\mathbf{v}}) \cdot \mathrm{div}_x \widehat{\mathbb{S}}\ \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) - \widehat{\mathbb{S}} : (\mathbb{D}\mathbf{v} - \mathbb{D}\widehat{\mathbf{v}}) - F(\mathbb{D}\widehat{\mathbf{v}})\Big]\ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\widehat{\mathbf{v}}) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D}\widehat{\mathbf{v}}\Big]\ \mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) - \widehat{\mathbb{S}} : (\mathbb{D}\mathbf{v} - \mathbb{D}\widehat{\mathbf{v}}) - F(\mathbb{D}\widehat{\mathbf{v}})\Big]\ \mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Note that strictly speaking as $\mathbb{D}\mathbf{v}$ is only a measure we should interpret

$$\int_0^\tau \int_{\mathbb{T}^d} \widehat{\mathbb{S}} : \mathbb{D}\mathbf{v} = \int_0^\tau \int_{\mathbb{T}^d} \widehat{\mathbb{S}} : \mathrm{d}\mathbb{D}\mathbf{v}.$$

Moreover, as $\widehat{\mathbb{S}} \in \partial F(\mathbb{D}\widehat{\mathbf{v}})$, we may infer that

$$\int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) - \widehat{\mathbb{S}} : (\mathbb{D}\mathbf{v} - \mathbb{D}\widehat{\mathbf{v}}) - F(\mathbb{D}\widehat{\mathbf{v}})\Big]\ \mathrm{d}x\,\mathrm{d}t \geq 0.$$

Indeed, the inequality

$$\Big[F(\mathbb{D}\mathbf{v}) - \widehat{\mathbb{S}} : (\mathbb{D}\mathbf{v} - \mathbb{D}\widehat{\mathbf{v}}) - F(\mathbb{D}\widehat{\mathbf{v}})\Big] \geq 0$$

can be derived by regularizing $\mathbb{D}\mathbf{v}$ similarly to [7, Lemma 3.2].

Thus, by virtue of the regularity of $\widehat{\mathbf{v}}$, we derive from (5.1)

$$\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v} - \widehat{\mathbf{v}}|^2(\tau, \cdot)\ \mathrm{d}x + \int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)]$$
$$\leq \|\nabla_x \widehat{\mathbf{v}}\|_{L^\infty((0,\tau)\times\mathbb{T}^d)} \int_0^\tau \int_{\mathbb{T}^d} \mathrm{d}\mathfrak{R}_v\ \mathrm{d}t + \|\nabla_x \widehat{\mathbf{v}}\|_{L^\infty((0,\tau)\times\mathbb{T}^d)} \int_0^\tau \int_{\mathbb{T}^d} |\mathbf{v} - \widehat{\mathbf{v}}|^2\ \mathrm{d}x\,\mathrm{d}t \tag{5.2}$$

for a.a. $\tau \in (0, T)$. Finally, as $\mathfrak{R}_v$ is positively semidefinite, we have $|\mathfrak{R}_v| \stackrel{<}{\sim} \mathrm{trace}[\mathfrak{R}_v]$, and , applying Gronwall's lemma to (5.2), we obtain the desired conclusion $\mathbf{v} = \widehat{\mathbf{v}}$, $\mathfrak{R}_v = 0$, and, finally, $\mathbb{S} = \widehat{\mathbb{S}}$. We have proved Theorem 2.6.

6 Conditional regularity

Our ultimate goal is to show Theorem 2.8. To this end, observe that $\mathbf{v}$ belonging to the regularity class (2.13) can be used as a test function in the momentum balance (2.9). After a routine manipulation, we obtain the energy balance in the form

$$\int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}|^2(\tau, \cdot)\ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S} : \mathbb{D}\mathbf{v}\ \mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^d} \nabla_x \mathbf{v} : \mathrm{d}\mathfrak{R}_v\ \mathrm{d}t = \int_{\mathbb{T}^d} \frac{1}{2}|\mathbf{v}_0|^2\ \mathrm{d}x. \tag{6.1}$$

Relation (6.1) subtracted from the energy inequality (2.10) gives rise to

$$\int_{\mathbb{T}^d} \mathrm{d}\frac{1}{2}\mathrm{trace}[\mathfrak{R}_v(\tau)] + \int_0^\tau \int_{\mathbb{T}^d} \Big[F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D}\mathbf{v}\Big]\ \mathrm{d}x\,\mathrm{d}t \leq \int_0^\tau \int_{\mathbb{T}^d} \nabla_x \mathbf{v} : \mathrm{d}\mathfrak{R}_v\ \mathrm{d}t \tag{6.2}$$

for a.a. $\tau \in (0, T)$. Consequently, similarly to (5.2), by the standard Gronwall's argument, we first deduce $\mathfrak{R}_v = 0$ and then

$$\mathbb{S} : \mathbb{D}\mathbf{v} = F(\mathbb{D}\mathbf{v}) + F^*(\mathbb{S}).$$

We have proved Theorem 2.8.

7 Concluding remarks

The existence of local-in-time strong solutions has been shown by Bothe and Prüss [2] in the case of non-degenerate positive viscosity coefficient μ. Similar result can be shown for the degenerate case $\mathbb{S} \equiv 0$ corresponding to the Euler system. (Note also that solutions of the Euler system are regular for regular initial data if $d = 2$.) However, strictly speaking, the Euler system is excluded by hypothesis (2.4). Obviously, the same approach works in this case as well, cf. Brenier, De Lellis, and Székelyhidi [4], Gwiazda et al. [10], and Székelyhidi and Wiedemann [13]. Although we could not find any relevant existence result in the degenerate mixed case (the fluid with activated viscosity according to Blechta, Málek, and Rajagopal [1]), we believe the local-in-time existence of strong solution is in reach of the available analytic methods as soon as the viscosity coefficient μ is a sufficiently smooth function of $\mathbb{D}$.

For the sake of simplicity, we have also omitted the effect of external bulk force. It is easy to see that the latter can be accommodated in a straightforward manner.

Finally, we propose an extension of the method that accommodates both the inviscid (Euler) system and some viscous fluids with anisotropic viscous stress. Let

$$L : R^{d\times d}_{\rm sym} \to R^{d\times d}_{\rm sym} \text{ be a linear mapping.}$$

For a convex function F satisfying the hypotheses of Theorem 2.4, we consider

$$F_L(\mathbb{D}) = F(L \circ \mathbb{D}),$$

which is again a convex function with $\mathrm{Dom}(F_L) = R^{d\times d}_{\rm sym}$. The associated conjugate function reads

$$F_L^* = \text{(closure of) } \mathbb{S} \mapsto \inf_{\mathbb{M}\in R^{d\times d}_{\rm sym}} \left\{ F^*(\mathbb{M}),\ L^t\mathbb{M} = \mathbb{S} \right\}.$$

The dissipative solutions are now defined exactly in Definition 2.2, where (2.7) is replaced by

$$L[\mathbb{D}\mathbf{v}] \in \mathcal{M}((0,T)\times\mathbb{T}^d; R^{d\times d}_{\rm sym}),$$

and with F_L, F_L^* in (2.10). Note that the choice $L \equiv 0$ gives rise to the Euler system, while $L = P$, where P is a projection on a subspace of $R^{d\times d}_{\rm sym}$, corresponds to the anisotropic viscosity acting only in certain directions. The proofs of Theorems 2.4, 2.6, 2.8 can be adapted in a direct manner.

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