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A class of weighted Hardy inequalities and applications to evolution problems

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Abstract
We state the following weighted Hardy inequality:

$$c_{o,\mu} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}\, d\mu \le \int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu + K \int_{\mathbb{R}^N} \varphi^2\, d\mu \quad \forall\, \varphi \in H^1_\mu,$$

in the context of the study of the Kolmogorov operators:

$$Lu = \Delta u + \frac{\nabla\mu}{\mu} \cdot \nabla u,$$

perturbed by inverse square potentials and of the related evolution problems. The function μ in the drift term is a probability density on $\mathbb{R}^N$. We prove the optimality of the constant $c_{o,\mu}$ and state existence and nonexistence results following the Cabré–Martel's approach (Cabré and Martel in C R Acad Sci Paris 329 (11): 973–978, 1999) extended to Kolmogorov operators.

Keywords Weighted Hardy inequality · Optimal constant · Kolmogorov operators · Singular potentials

Mathematics Subject Classification 35K15 · 35K65 · 35B25 · 34G10 · 47D03

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1 Introduction

This paper on weighted Hardy inequalities fits in the framework of the study of Kolmogorov operators on smooth functions:

$$Lu = \Delta u + \frac{\nabla \mu}{\mu} \cdot \nabla u,$$

where μ is a probability density on $\mathbb{R}^N$, and of the related evolution problems:

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & x \in \mathbb{R}^N, \quad t > 0, \\ u(\cdot, 0) = u_0 \geq 0 \in L^2_\mu. \end{cases}$$

The operator L in (P) is perturbed by the singular potential $V(x) = \frac{c}{|x|^2}$, $c > 0$, and $L^2_\mu := L(\mathbb{R}^N, d\mu)$, with $d\mu(x) = \mu(x)dx$.

The interest in inverse square potentials of type $V \sim \frac{c}{|x|^2}$ relies in their criticality: the strong maximum principle and Gaussian bounds fail (see [2]). Furthermore, interest in singular potentials is due to the applications to many fields, for example in many physical contexts as molecular physics [23], quantum cosmology (see, e.g., [5]), quantum mechanics [4] and combustion models [19].

The operator $\Delta + V$, $V(x) = \frac{c}{|x|^2}$, has the same homogeneity as the Laplacian and does not belong to the Kato class, then, V cannot be regarded as a lower order perturbation term.

A remarkable result stated in 1984 by P. Baras and J. A. Goldstein in [3] shows that the evolution problem (P) with $L = \Delta$ admits a unique positive solution if $c \leq c_o = \left(\frac{N-2}{2}\right)^2$ and no positive solutions exist if $c > c_o$. When it exists, the solution is exponentially bounded, on the contrary, if $c > c_o$, there is the so-called instantaneous blow-up phenomenon.

In order to extend these results to Kolmogorov operators, the technique must be different.

A result analogous to that stated in [3] has been obtained in 1999 by X. Cabré and Y. Martel [8] for more general potentials $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$ with a different approach.

To state the existence and nonexistence results, we follow the Cabré–Martel's approach. We use the relation between the weak solution of (P) and the *bottom of the spectrum* of the operator $-(L + V)$:

$$\lambda_1(L+V) := \inf_{\varphi \in H^1_\mu \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu - \int_{\mathbb{R}^N} V\varphi^2 \, d\mu}{\int_{\mathbb{R}^N} \varphi^2 \, d\mu} \right)$$

where H^1_μ is the suitable weighted Sobolev space.

When $\mu = 1$, Cabré and Martel showed that the boundedness of $\lambda_1(\Delta + V)$, $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$, is a necessary and sufficient condition for the existence of positive exponentially bounded in time solutions to the associated initial value problem. Later in [9,20], similar results have been extended to Kolmogorov operators. The proof uses some properties of the operator L and of its corresponding semigroup in $L^2_\mu(\mathbb{R}^N)$.

For Ornstein–Uhlenbeck-type operators, $Lu = \Delta u - \sum_{i=1}^n A(x - a_i) \cdot \nabla u$, $a_i \in \mathbb{R}^N$, $i = 1, \ldots, n$, perturbed by multipolar inverse square potentials, weighted multipolar Hardy inequalities and related existence and nonexistence results were stated in [11]. In such a case, the invariant measure for these operators is $d\mu = \mu_A(x)dx = Ke^{-\frac{1}{2}\sum_{i=1}^n \langle A(x-a_i), x-a_i \rangle}dx$.

There is a close relation between the estimate of the bottom of the spectrum $\lambda_1(L + V)$ and the weighted Hardy inequality with $V(x) = \frac{c}{|x|^2}$, $c \leq c_{o,\mu}$,

$$\int_{\mathbb{R}^N} V\varphi^2 \, d\mu \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2 d\mu + K \int_{\mathbb{R}^N} \varphi^2 d\mu \quad \forall \varphi \in H^1_\mu, \qquad K > 0 \tag{1}$$

with the best possible constant $c_{o,\mu}$.

In particular, the existence of positive solutions to (P) is related to the Hardy inequality (1) and the nonexistence is due to the optimality of the constant $c_{o,\mu}$.

The main results in the paper are, in Sect. 2, the weighted Hardy inequality (1) with measures which satisfy fairly general conditions and the optimality of the constant $c_{o,\mu}$ in Sect. 3.

The proof of the weighted Hardy inequality is different from the others in the literature. It is based on the introduction of a suitable C^∞ function, and it can be used to prove inequality (1) with $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ of a more general type, in other words Hardy type inequalities.

In [9], the authors state a weighted Hardy inequality using a different approach and improved Hardy inequalities. This requires suitable conditions on μ. Our technique, with different assumptions on μ, allows us to achieve the best constant (cf. [9, Theorem 3.3]) for a wide class of functions μ. To state the optimality of the constant in the estimate, we need further assumptions on μ as usually it is done. We find a suitable function φ for which the inequality (1) does not hold if $c > c_{o,\mu}$, and this is a crucial point in the proof. The way to estimate the bottom of the spectrum is close to the one used in [9]. We remark that the inequality obtained under our hypotheses applies in the context of weighted multipolar Hardy inequalities stated in the forthcoming paper [12].

Finally, we state an existence and nonexistence result in Sect. 4 following the Cabré–Martel's approach and using some results stated in [9,20] when the function μ belongs to $C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$ or belongs to $C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$, for some $\lambda \in (0,1)$.

Some classes of functions μ satisfying the hypotheses of the main Theorems are given in Sect. 2.

2 Weighted Hardy inequalities

Let μ be a weight function in $\mathbb{R}^N$. We define the weighted Sobolev space $H^1_\mu = H^1(\mathbb{R}^N, \mu(x)\mathrm{d}x)$ as the space of functions in $L^2_\mu := L^2(\mathbb{R}^N, \mu(x)\mathrm{d}x)$ whose weak derivatives belong to $(L^2_\mu)^N$.

As first step, we consider the following conditions on μ which we need to state a preliminary weighted Hardy inequality:

(H_1) $\mu \ge 0$, $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$;

(H_2) $\nabla\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$;

(H_3) there exist constants $k_1, k_2 \in \mathbb{R}$, $k_2 > 2-N$, such that if

$$f_\varepsilon = (\varepsilon + |x|^2)^{\frac{\alpha}{2}}, \quad \alpha < 0, \quad \varepsilon > 0,$$

it holds

$$\frac{\nabla f_\varepsilon}{f_\varepsilon} \cdot \nabla\mu = \frac{\alpha x}{\varepsilon + |x|^2} \cdot \nabla\mu \le \left(k_1 + \frac{k_2\alpha}{\varepsilon + |x|^2}\right)\mu$$

for any $\varepsilon > 0$.

The condition (H_3) contains the requirement that the scalar product $\alpha x \cdot \frac{\nabla\mu}{\mu}$ is bounded in B_R, $R > 0$, while $\frac{\alpha x}{\varepsilon+|x|^2} \cdot \frac{\nabla\mu}{\mu}$ is bounded in $\mathbb{R}^N\setminus B_R$, where B_R is a ball of radius R centered in zero.

The reason we use the function f_ε, introduced in [17], will be clear in the proof of the weighted Hardy inequality which we will state below. Finally, we observe that we need the condition $k_2 > 2-N$ to apply Fatou's lemma in the proof of Theorem 1.

Theorem 1 *Under conditions (H_1–H_3), there exists a positive constant c such that*

$$c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,\mathrm{d}\mu \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,\mathrm{d}\mu + k_1\int_{\mathbb{R}^N}\varphi^2\,\mathrm{d}\mu, \tag{2}$$

for any function $\varphi \in C_c^\infty(\mathbb{R}^N)$, where $c \in (0, c_o(N+k_2)]$ with $c_o(N+k_2) = \left(\frac{N+k_2-2}{2}\right)^2$.

Proof As first step, we start from the integral of the square of the gradient of the function φ. Then, we introduce $\psi = \frac{\varphi}{f_\varepsilon}$, with f_ε defined in (H_3), and integrate by parts taking in mind (H_1) and (H_2).

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla\varphi|^2\,\mathrm{d}\mu &= \int_{\mathbb{R}^N}|\nabla(\psi f_\varepsilon)|^2\,\mathrm{d}\mu\\ &= \int_{\mathbb{R}^N}|\nabla\psi f_\varepsilon + \nabla f_\varepsilon\psi|^2\,\mathrm{d}\mu\\ &= \int_{\mathbb{R}^N}|\nabla\psi|^2 f_\varepsilon^2\,\mathrm{d}\mu + \int_{\mathbb{R}^N}\psi^2|\nabla f_\varepsilon|^2\,\mathrm{d}\mu + 2\int_{\mathbb{R}^N} f_\varepsilon\psi\nabla\psi\cdot\nabla f_\varepsilon\,\mathrm{d}\mu\\ &= \int_{\mathbb{R}^N}|\nabla\psi|^2 f_\varepsilon^2\,\mathrm{d}\mu + \int_{\mathbb{R}^N}\psi^2|\nabla f_\varepsilon|^2\,\mathrm{d}\mu\\ &\quad - \int_{\mathbb{R}^N}\psi^2|\nabla f_\varepsilon|^2\,\mathrm{d}\mu - \int_{\mathbb{R}^N} f_\varepsilon^2\psi^2\frac{\Delta f_\varepsilon}{f_\varepsilon}\,\mathrm{d}\mu - \int_{\mathbb{R}^N} f_\varepsilon^2\psi^2\frac{\nabla f_\varepsilon}{f_\varepsilon}\cdot\nabla\mu\,\mathrm{d}x.\end{aligned} \tag{3}$$

Observing that

$$\Delta f_\varepsilon = \frac{\alpha(N-2+\alpha)|x|^2 + \alpha\varepsilon N}{(\varepsilon+|x|^2)^{2-\frac{\alpha}{2}}}$$

and using hypothesis (H_3), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla\varphi|^2\,\mathrm{d}\mu &\ge -\int_{\mathbb{R}^N}\frac{\Delta f_\varepsilon}{f_\varepsilon}\varphi^2\,\mathrm{d}\mu - \int_{\mathbb{R}^N}\frac{\nabla f_\varepsilon}{f_\varepsilon}\cdot\nabla\mu\,\varphi^2\,\mathrm{d}x\\ &\ge -\left[\alpha(N-2)+\alpha^2\right]\int_{\mathbb{R}^N}\frac{|x|^2}{(\varepsilon+|x|^2)^2}\varphi^2\,\mathrm{d}\mu - \varepsilon\alpha N\int_{\mathbb{R}^N}\frac{\varphi^2}{(\varepsilon+|x|^2)^2}\,\mathrm{d}\mu\\ &\quad - k_1\int_{\mathbb{R}^N}\varphi^2\,\mathrm{d}\mu - k_2\alpha\int_{\mathbb{R}^N}\frac{\varphi^2}{\varepsilon+|x|^2}\,\mathrm{d}\mu\\ &= [-\alpha(N-2+k_2)-\alpha^2]\int_{\mathbb{R}^N}\frac{|x|^2}{(\varepsilon+|x|^2)^2}\varphi^2\,\mathrm{d}\mu\\ &\quad - \varepsilon\alpha(N+k_2)\int_{\mathbb{R}^N}\frac{\varphi^2}{(\varepsilon+|x|^2)^2}\,\mathrm{d}\mu - k_1\int_{\mathbb{R}^N}\varphi^2\,\mathrm{d}\mu.\end{aligned} \tag{4}$$

The constant $-\alpha(N-2+k_2)-\alpha^2$ is greater than zero for $-(N-2+k_2) < \alpha < 0$ and $k_2 > 2-N$, so by Fatou's lemma, we state the following estimate letting $\varepsilon \to 0$:

$$\int_{\mathbb{R}^N}|\nabla\varphi|^2\,\mathrm{d}\mu + k_1\int_{\mathbb{R}^N}\varphi^2\,\mathrm{d}\mu \ge c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2}\,\mathrm{d}\mu,$$

where $c = -\alpha(N-2+k_2)-\alpha^2$. Finally, we observe that

$$\max_\alpha[-\alpha(N+k_2-2)-\alpha^2] = \left(\frac{N+k_2-2}{2}\right)^2 =: c_o(N+k_2),$$

attained for $\alpha_o = -\frac{N+k_2-2}{2}$. □

Remark 1 In an alternative way, we can define f_ε in (H_3) setting $\alpha = \alpha_o$ and get the estimate (2) with $c = c_o(N + k_2)$. Although the result goes in the same direction, in the proof we point out that $c_o(N + k_2)$ is the maximum value of the constant c.

Remark 2 In the case $\mu = 1$, we obtain the classical Hardy inequality. We remark that if in the proof we introduce a function $f \in C^\infty(\mathbb{R}^N)$ in place of f_ε, the inequality (4) can be used to get Hardy type inequalities:

$$\int_{\mathbb{R}^N} V\varphi^2 \, \mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, \mathrm{d}x \tag{5}$$

where the potential $V = V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $V(x) \ge 0$, is such that

$$-\frac{\Delta f}{f} \ge V \qquad \forall\, x \in \mathbb{R}^N.$$

Operators perturbed by potentials of a more general type, for which the generation of semigroups was stated, have been investigated, for example, in [13–15] when $\mu = 1$ and in [10] in weighted spaces. For functions $\mu \ne 1$ such that $k_2 \ne 0$, we have to modify the condition (H_3) to get the Hardy type inequality (5) with respect to the measure $\mathrm{d}\mu$.

Now, we suppose that

(H_4) $\quad \mu \ge 0$, $\sqrt{\mu} \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$;

(H_5) $\quad \mu^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Let us observe that in the hypotheses $(H_4$–$H_5)$, the space $C_c^\infty(\mathbb{R}^N)$ is dense in H^1_μ, and H^1_μ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the Sobolev norm:

$$\|\cdot\|^2_{H^1_\mu} := \|\cdot\|^2_{L^2_\mu} + \|\nabla\cdot\|^2_{L^2_\mu}$$

(see [25]). For some interesting papers on density of smooth functions in weighted Sobolev spaces and related questions, we refer, for example, to [6,7,16,18,21,22,26].

So, we can deduce the following result from Theorem 1 by density argument.

Theorem 2 *Under conditions $(H_2$–$H_5)$, there exists a positive constant c such that*

$$c\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \, \mathrm{d}\mu \le \int_{\mathbb{R}^N} |\nabla\varphi|^2 \, \mathrm{d}\mu + k_1 \int_{\mathbb{R}^N} \varphi^2 \, \mathrm{d}\mu, \tag{6}$$

for any function $\varphi \in H^1_\mu$, where $c \in (0, c_o(N + k_2)]$ with $c_o(N + k_2) = \left(\frac{N+k_2-2}{2}\right)^2$.

We give some examples of functions μ which satisfy the hypotheses of Theorem 2.

We remark that, in the hypotheses $\mu = \mu(|x|) \in C^1$ for $|x| \in [r_0, +\infty[$, $r_0 > 0$, a class of weight functions μ which satisfies (H_3) is the following:

$$\mu(x) \ge Ce^{-\frac{k_1}{2|\alpha|}|x|^2}|x|^{k_2-\frac{k_1}{|\alpha|}\varepsilon}, \quad \text{for} \quad |x| \ge r_0, \tag{7}$$

where C is a constant depending on $\mu(r_0)$ and r_0.

Indeed, in the case of radial functions, $\mu(x) = \mu(|x|)$, if we set $|x| = \rho$, the condition (H_3) states that μ satisfies the following inequality:

$$\frac{\alpha\rho}{\varepsilon + \rho^2}\mu'(\rho) \le \left(k_1 + \frac{k_2\alpha}{\varepsilon + \rho^2}\right)\mu(\rho),$$

which implies

$$\mu'(\rho) \geq a(\rho)\mu(\rho)$$

where

$$a(\rho) = \frac{k_1}{\alpha}\left(\frac{\varepsilon + \rho^2}{\rho}\right) + \frac{k_2}{\rho}.$$

Integrating in $[r_0, r]$, we get

$$\mu(r) \geq \mu(r_0)e^{\int_{r_0}^{r} a(s)\mathrm{d}s} = \mu(r_0)\left(\frac{r}{r_0}\right)^{k_2 - \frac{k_1}{|\alpha|}\varepsilon} e^{-\frac{k_1}{2|\alpha|}(r^2 - r_0^2)} \qquad \text{for} \quad r \geq r_0,$$

from which we deduce that

$$\mu(r) \geq \frac{\mu(r_0)}{r_0^{k_2 - \frac{k_1}{|\alpha|}\varepsilon}} e^{\frac{k_1}{2|\alpha|}r_0^2} r^{k_2 - \frac{k_1}{|\alpha|}\varepsilon} e^{-\frac{k_1}{2|\alpha|}r^2} \qquad \text{for} \quad r \geq r_0.$$

Example 1 Another class of weight functions satisfying (H_3), when $k_1 = k_2 = 0$, consists of the bounded increasing functions, as, for example, $\cos e^{-|x|^2}$. Such a function verifies the requirements of Theorem 2.

In the following example, we consider a wide class of functions which contains the Gaussian measure and polynomial-type measures. A class of functions which behaves as $\frac{1}{|x|^\gamma}$ when $|x|$ goes to zero.

Example 2 We consider the following weight functions:

$$\mu(x) = \frac{1}{|x|^\gamma} e^{-\delta|x|^m}, \quad \delta \geq 0, \quad \gamma < N - 2. \tag{8}$$

We state the values of γ and m for which the functions in (8) are "good" functions to get the weighted Hardy inequality (6).

The weight μ satisfies (H_2), (H_4) and (H_5) if $\gamma > -N$. The condition (H_3):

$$\frac{\alpha(-\gamma - \delta m|x|^m)}{\varepsilon + |x|^2} \leq k_1 + \frac{\alpha k_2}{\varepsilon + |x|^2},$$

is fulfilled if

$$-(\alpha\gamma + \alpha k_2 + k_1\varepsilon) - \alpha\delta m|x|^m - k_1|x|^2 \leq 0. \tag{9}$$

In the case $\delta = 0$, we only need to require that $\gamma \leq -k_2 - \frac{k_1}{\alpha}\varepsilon$, and we are able to get the Caffarelli–Nirenberg inequality:

$$\left(\frac{N - 2 - \gamma}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}|x|^{-\gamma}\,\mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2|x|^{-\gamma}\,\mathrm{d}x \qquad \forall\varphi \in H^1_\mu.$$

While if $\gamma = 0$, the inequality (6) holds, for k_1 large enough, with $k_2 = 0$ if $m = 2$ and with $k_2 < 0$ if $m < 2$.

In general to get (9), we need the following conditions on parameters and on the constant k_1:

(i) $\gamma \in (-N, -k_2]$, $\delta = 0$, $k_1 = 0$,
(ii) $\gamma \in (-N, -k_2]$, $k_1 \geq -2\alpha\delta$, $m = 2$,
(iii) $\gamma \in (-N, -k_2)$, $k_1 \geq \tilde{k}_1$, $m < 2$,

where $\tilde{k}_1 = \frac{\frac{m}{2}(1-\frac{m}{2})^{\frac{2}{m}-1}(-\alpha\delta m)^{\frac{2}{m}}}{[\alpha(\gamma+k_2)]^{\frac{2}{m}-1}}$, to get the inequality (6).

Example 3 The function $\mu(x) = [\log(1+|x|)]^{-\gamma}$, for $\gamma < N-2$, behaves as $\frac{1}{|x|^\gamma}$ when $|x|$ goes to 0. So, we can state the weighted Hardy inequality (6) with $k_1 = 0$ and $\gamma \in (-N, -k_2]$ as in the previous example.

3 Optimality of the constant

To state the optimality of the constant $c_o(N + K_2)$ in the estimate (6), we need further assumptions on μ as usually it is done. We remark that in the proof of optimality, the choice of the function φ plays a fundamental role.

We suppose

(H_6) $\quad \frac{\mu(x)}{|x|^\delta} \in L^1_{\text{loc}}(\mathbb{R}^N)$ iff $\delta \le N + k_2$.

We observe that the condition (H_6) is necessary for the technique used to estimate the bottom of the spectrum of the operator $-L - V$ in the proof of the optimality. For example, the functions μ such that

$$\lim_{|x|\to 0} \frac{\mu(|x|)}{|x|^{k_2}} = l, \qquad l > 0,$$

verify (H_6).

The result below states the optimality of the constant $c_o(N + k_2)$ in the Hardy inequality.

Theorem 3 *In the hypotheses* $(H_2$–$H_6)$*, the Hardy inequality* (6) *does not hold for any* $\varphi \in H^1_\mu$ *if* $c > c_o(N + k_2) = \left(\frac{N+k_2-2}{2}\right)^2$.

Proof Let $\theta \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function, $0 \le \theta \le 1$, $\theta = 1$ in B_1 and $\theta = 0$ in B_2^c. We introduce the function:

$$\varphi_\varepsilon(x) = \begin{cases} (\varepsilon + |x|)^\eta & \text{if } |x| \in [0, 1[, \\ (\varepsilon + |x|)^\eta \theta(x) & \text{if } |x| \in [1, 2[, \\ 0 & \text{if } |x| \in [2, +\infty[, \end{cases}$$

where $\varepsilon > 0$ and the exponent η is such that

$$\max\left\{-\sqrt{c}, -\frac{N+k_2}{2}\right\} < \eta < \min\left\{-\frac{N+k_2-2}{2}, 0\right\}.$$

The function φ_ε belongs to H^1_μ for any $\varepsilon > 0$.

For this choice of η, we obtain $\eta^2 < c$, $|x|^{2\eta} \in L^1_{\text{loc}}(\mathbb{R}^N, d\mu)$ and $|x|^{2\eta-2} \notin L^1_{\text{loc}}(\mathbb{R}^N, d\mu)$.

Let us assume that $c > c_o(N + k_2)$. In order to state the result, we prove that bottom of the spectrum of the operator $-(L + V)$:

$$\lambda_1 = \inf_{\varphi \in H^1_\mu \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2 \, d\mu - \int_{\mathbb{R}^N} \frac{c}{|x|^2}\varphi^2 \, d\mu}{\int_{\mathbb{R}^N} \varphi^2 \, d\mu} \right), \tag{10}$$

is $-\infty$. For this purpose, we estimate at first the numerator in (10) with $\varphi = \varphi_\varepsilon$.

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left(|\nabla\varphi_\varepsilon|^2-\frac{c}{|x|^2}\varphi_\varepsilon^2\right)\mathrm{d}\mu\\
&=\int_{B_1}\left[|\nabla(\varepsilon+|x|)^\eta|^2-\frac{c}{|x|^2}(\varepsilon+|x|)^{2\eta}\right]\mathrm{d}\mu\\
&\quad+\int_{B_1^c}\left[|\nabla(\varepsilon+|x|)^\eta\theta|^2-\frac{c}{|x|^2}(\varepsilon+|x|)^{2\eta}\theta^2\right]\mathrm{d}\mu\\
&\le\int_{B_1}\left[\eta^2(\varepsilon+|x|)^{2\eta-2}-\frac{c}{|x|^2}(\varepsilon+|x|)^{2\eta}\right]\mathrm{d}\mu\\
&\quad+\eta^2\int_{B_1^c}(\varepsilon+|x|)^{2\eta-2}\theta^2\,\mathrm{d}\mu+\int_{B_1^c}(\varepsilon+|x|)^{2\eta}|\nabla\theta|^2\,\mathrm{d}\mu\\
&\quad+2\eta\int_{B_1^c}\theta(\varepsilon+|x|)^{2\eta-1}\frac{x}{|x|}\cdot\nabla\theta\,\mathrm{d}\mu\\
&\le\int_{B_1}(\varepsilon+|x|)^{2\eta}\left[\frac{\eta^2}{(\varepsilon+|x|)^2}-\frac{c}{|x|^2}\right]\mathrm{d}\mu\\
&\quad+2\eta^2\int_{B_1^c}(\varepsilon+|x|)^{2\eta-2}\theta^2\,\mathrm{d}\mu+2\int_{B_1^c}(\varepsilon+|x|)^{2\eta}|\nabla\theta|^2\,\mathrm{d}\mu\\
&\le\int_{B_1}(\varepsilon+|x|)^{2\eta}\left[\frac{\eta^2}{(\varepsilon+|x|)^2}-\frac{c}{|x|^2}\right]\mathrm{d}\mu+C_1,
\end{aligned}
\tag{11}
$$

where $C_1=\left(2\eta^2+2\|\nabla\theta\|_\infty\right)\int_{B_1^c}\mathrm{d}\mu$.

Furthermore,

$$
\int_{\mathbb{R}^N}\varphi_\varepsilon^2\,\mathrm{d}\mu\ge\int_{B_2\setminus B_1}(\varepsilon+|x|)^{2\eta}\theta^2\,\mathrm{d}\mu=C_{2,\varepsilon}. \tag{12}
$$

Putting together (11) and (12), we get from (10):

$$
\lambda_1\le\frac{\int_{B_1}(\varepsilon+|x|)^{2\eta}\left[\frac{\eta^2}{(\varepsilon+|x|)^2}-\frac{c}{|x|^2}\right]\mathrm{d}\mu+C_1}{C_{2,\varepsilon}}.
$$

Letting $\varepsilon\to 0$ in the numerator above, taking in mind that $|x|^{2\eta}\in L^1_{\mathrm{loc}}(\mathbb{R}^N,\mathrm{d}\mu)$ and Fatou's lemma, we obtain:

$$
\lim_{\varepsilon\to 0}\int_{B_1}(\varepsilon+|x|)^{2\eta}\left[\frac{\eta^2}{(\varepsilon+|x|)^2}-\frac{c}{|x|^2}\right]\mathrm{d}\mu\le-(c-\eta^2)\int_{B_1}|x|^{2\eta-2}\,\mathrm{d}\mu=-\infty,
$$

and then, $\lambda_1=-\infty$. □

4 Kolmogorov operators and existence and nonexistence results

In the standard setting, one considers $\mu\in C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\lambda\in(0,1)$ and $\mu>0$ for any $x\in\mathbb{R}^N$.

We consider Kolmogorov operators:

$$
Lu=\Delta u+\frac{\nabla\mu}{\mu}\cdot\nabla u, \tag{13}
$$

on smooth functions, where the probability density μ in the drift term is not necessarily $(1,\lambda)$-Hölderian in the whole space but belongs to $C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$.

These operators arise from the bilinear form integrating by parts:

$$a_\mu(u,v)=\int_{\mathbb{R}^N}\nabla u\cdot\nabla v\,\mathrm{d}\mu=-\int_{\mathbb{R}^N}(Lu)v\,\mathrm{d}\mu.$$

The purpose is to get existence and nonexistence results for weak solutions to the initial value problem on L^2_μ corresponding to the operator L perturbed by an inverse square potential:

$$(P)\quad\begin{cases}\partial_t u(x,t)=Lu(x,t)+V(x)u(x,t), & x\in\mathbb{R}^N, t>0,\\ u(\cdot,0)=u_0\geq 0\in L^2_\mu,\end{cases}$$

where $V(x)=\frac{c}{|x|^2}$, with $c>0$.

We say that u is a weak solution to (P) if, for each T, $R>0$, we have:

$$u\in C([0,T],L^2_\mu),\quad Vu\in L^1(B_R\times(0,T),\mathrm{d}\mu\mathrm{d}t)$$

and

$$\int_0^T\int_{\mathbb{R}^N}u(-\partial_t\phi-L\phi)\,\mathrm{d}\mu\mathrm{d}t-\int_{\mathbb{R}^N}u_0\phi(\cdot,0)\,\mathrm{d}\mu=\int_0^T\int_{\mathbb{R}^N}Vu\phi\,\mathrm{d}\mu\mathrm{d}t$$

for all $\phi\in W_2^{2,1}(\mathbb{R}^N\times[0,T])$ having compact support with $\phi(\cdot,T)=0$, where B_R denotes the open ball of $\mathbb{R}^N$ of radius R centered at 0. For any $\Omega\subset\mathbb{R}^N$, $W_2^{2,1}(\Omega\times(0,T))$ is the parabolic Sobolev space of the functions $u\in L^2(\Omega\times(0,T))$ having weak space derivatives $D^\alpha_x u\in L^2(\Omega\times(0,T))$ for $|\alpha|\leq 2$ and weak time derivative $\partial_t u\in L^2(\Omega\times(0,T))$ equipped with the norm:

$$\|u\|_{W_2^{2,1}(\Omega\times(0,T))}:=\left(\|u\|^2_{L^2(\Omega\times(0,T))}+\|\partial_t u\|^2_{L^2(\Omega\times(0,T))}\right.$$
$$\left.+\sum_{1\leq|\alpha|\leq 2}\|D^\alpha u\|^2_{L^2(\Omega\times(0,T))}\right)^{\frac{1}{2}}.$$

Let us assume that the function μ is a probability density on $\mathbb{R}^N$, $\mu>0$. In the hypothesis

(H_7) $\quad\mu\in C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$, $\lambda\in(0,1)$,

it is known that the operator L with domain

$$D_{\max}(L)=\{u\in C_b(\mathbb{R}^N)\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)\text{ for all }1<p<\infty,\,Lu\in C_b(\mathbb{R}^N)\}$$

is the weak generator of a not necessarily C_0-semigroup in $C_b(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N}Lu\,\mathrm{d}\mu=0$ for any $u\in C_c^\infty(\mathbb{R}^N)$, $\mathrm{d}\mu=\mu(x)\mathrm{d}x$ is the invariant measure for this semigroup in $C_b(\mathbb{R}^N)$. So, we can extend it to a positivity preserving and analytic C_0-semigroup $\{T(t)\}_{t\geq 0}$ on L^2_μ, whose generator is still denoted by L (see [24]).

When the assumptions on μ allow degeneracy at one point, we require the following conditions to get that L generates a semigroup:

(H_8) $\quad\mu\in C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})$, $\lambda\in(0,1)$, $\mu\in H^1_{\mathrm{loc}}(\mathbb{R}^N)$, $\frac{\nabla\mu}{\mu}\in L^r_{\mathrm{loc}}(\mathbb{R}^N)$ for some $r>N$, and $\inf_{x\in K}\mu(x)>0$ for any compact set $K\subset\mathbb{R}^N$.

So by [1, Corollary 3.7], we have that the closure of $(L,C_c^\infty(\mathbb{R}^N))$ on L^2_μ generates a strongly continuous and analytic Markov semigroup $\{T(t)\}_{t\geq 0}$ on L^2_μ.

We observe that the function $e^{-\delta|x|^m}$ fully satisfies the condition (H_8) while $\cos e^{-|x|^2}$ is $(1,\lambda)$-Hölderian in $\mathbb{R}^N$ (see Examples in Sect. 2).

For weight functions μ satisfying assumption (H_7) or (H_8), there are some interesting properties regarding the semigroup $\{T(t)\}_{t\geq 0}$ generated by the operator L. These properties listed in the Proposition below are well known under hypothesis (H_7) (see [24]) and have been proved in [9] if μ satisfies (H_8).

Proposition 1 *Assume that μ satisfies (H_7) or (H_8). Then, the following assertions hold:*

(i) $D(L) \subset H^1_\mu$.
(ii) *For every $f \in D(L)$, $g \in H^1_\mu$ we have:*

$$\int Lfg\,\mathrm{d}\mu = -\int \nabla f \cdot \nabla g\,\mathrm{d}\mu.$$

(iii) $T(t)L^2_\mu \subset D(L)$ for all $t > 0$.

The following Theorem stated in [20] for functions μ satisfying condition (H_7) was proved in [9] for functions μ under condition (H_8).

Theorem 4 *Let $0 \leq V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Assume that the weight function μ satisfies H_4), H_5) and H_8). Then, the following assertions hold:*

(i) *If $\lambda_1(L+V) > -\infty$, then there exists a positive weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying*

$$\|u(t)\|_{L^2_\mu} \leq Me^{\omega t}\|u_0\|_{L^2_\mu}, \quad t \geq 0 \tag{14}$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$.
(ii) *If $\lambda_1(L+V) = -\infty$, then for any $0 \leq u_0 \in L^2_\mu \setminus \{0\}$, there is no positive weak solution of (P) satisfying (14).*

To get existence and nonexistence of solutions to (P), we put together the weighted Hardy inequality (2), Theorems 3 and 4. So, we can state the following result.

Theorem 5 *Assume that the weight function μ satisfies hypotheses $(H_2$–$H_6)$, (H_8) and $0 \leq V(x) \leq \frac{c}{|x|^2}$. The following assertions hold:*

(i) *If $0 \leq c \leq c_o(N+k_2) = \left(\frac{N+k_2-2}{2}\right)^2$, then there exists a positive weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying*

$$\|u(t)\|_{L^2_\mu} \leq Me^{\omega t}\|u_0\|_{L^2_\mu}, \quad t \geq 0 \tag{15}$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$, and any $u_0 \in L^2_\mu$.
(ii) *If $c > c_o(N+k_2)$, then for any $0 \leq u_0 \in L^2_\mu$, $u_0 \neq 0$, there is no positive weak solution of (P) with $V(x) = \frac{c}{|x|^2}$ satisfying (15).*

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