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On Hopf hypersurfaces of the homogeneous nearly Kähler $S^3 \times S^3$

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Abstract
In this paper, extending our previous joint work (Hu et al., Math Nachr 291:343–373, 2018), we initiate the study of Hopf hypersurfaces in the homogeneous NK (nearly Kähler) manifold $\mathbf{S}^3 \times \mathbf{S}^3$. First, we show that any Hopf hypersurface of the homogeneous NK $\mathbf{S}^3 \times \mathbf{S}^3$ does not admit two distinct principal curvatures. Then, for the important class of Hopf hypersurfaces with three distinct principal curvatures, we establish a complete classification under the additional condition that their holomorphic distributions $\{U\}^\perp$ are preserved by the almost product structure P of the homogeneous NK $\mathbf{S}^3 \times \mathbf{S}^3$.

Keywords Nearly Kähler manifold $\mathbf{S}^3 \times \mathbf{S}^3$ · Hopf hypersurface · Principal curvature · Holomorphic distribution · Almost product structure

Mathematics Subject Classification 53B25 · 53B35 · 53C30 · 53C42

1 Introduction

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J. Given a connected orientable real hypersurface M of $\bar{M}$, there appears an important notion the *structure vector field* defined by $U := -J\xi$, where ξ is the unit normal vector field. If the integral curves of U are geodesics, then it is well known that M is called a *Hopf hypersurface*. During the last four decades, Hopf hypersurfaces of the complex space forms and several other almost Hermitian manifolds have been extensively and deeply investigated, for details we refer to [4,10,20,21,24] and [5,6,15] and the references therein. Recall that a nearly Kähler (NK) manifold is an almost Hermitian manifold such that the covariant derivative of the almost

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complex structure J is skew-symmetric. It is well known from Nagy's classification of nearly Kähler manifolds [23] that the six-dimensional ones are important construction factor, and from Butruille [8,9] that the only homogeneous 6-dimensional NK manifolds are the 6-sphere $\mathbf{S}^6$, the $\mathbf{S}^3\times\mathbf{S}^3$, the complex projective space $\mathbf{CP}^3$ and the flag manifold $SU(3)/U(1)\times U(1)$, and moreover from Foscolo and Haskins [13] that both $\mathbf{S}^6$ and $\mathbf{S}^3\times\mathbf{S}^3$ admit inhomogeneous NK structures.

Notice that the Riemannian geometric invariants of the homogeneous NK $\mathbf{S}^3\times\mathbf{S}^3$ were systematically presented by Bolton et al. [7]. Since then the study of the canonical submanifolds of the homogeneous NK $\mathbf{S}^3\times\mathbf{S}^3$ becomes quite active and many interesting results have been obtained. This includes the results about almost complex surfaces in [7,11,18], about Lagrangian and CR submanifolds in [1–3,12,19,26]. Nevertheless, about hypersurfaces the results are few that appear only in [16,17].

The goal of this paper is to study Hopf hypersurfaces in the homogeneous NK $\mathbf{S}^3\times\mathbf{S}^3$. In this situation, according to Proposition 1 of [5], the Hopf condition is equivalent to that the structure vector field is a principal curvature vector field of the hypersurface.

Our first concern is Hopf hypersurfaces with two distinct principal curvatures. The result we obtain is the following:

Theorem 1.1 *No Hopf hypersurface in the homogeneous NK $\mathbf{S}^3\times\mathbf{S}^3$ admits exactly two distinct principal curvatures.*

Our next concern is Hopf hypersurfaces with three distinct principal curvatures. It turns out that hypersurfaces of this class are quite complicated and examples of at least three families appear. As the second main result of this paper, we obtain a classification of them under the additional/natural condition that their holomorphic distributions $\{U\}^\perp$ are preserved by the almost product structure P of the homogeneous NK $\mathbf{S}^3\times\mathbf{S}^3$. Before stating the result, we would recall that, according to Moruz and Vrancken [22] and Podestà and Spiro [25], the following three maps

(1) $\mathcal{F}_1:\ \mathbf{S}^3\times\mathbf{S}^3\to\mathbf{S}^3\times\mathbf{S}^3$ with $\mathcal{F}_1(p,q)=(q,p)$,
(2) $\mathcal{F}_2:\ \mathbf{S}^3\times\mathbf{S}^3\to\mathbf{S}^3\times\mathbf{S}^3$ with $\mathcal{F}_2(p,q)=(\bar p,q\bar p)$,
(3) $\mathcal{F}_{abc}:\ \mathbf{S}^3\times\mathbf{S}^3\to\mathbf{S}^3\times\mathbf{S}^3$ with $\mathcal{F}_{abc}(p,q)=(ap\bar c,bq\bar c)$ for any unitary quaternions a,b,c

are isometries of the NK $\mathbf{S}^3\times\mathbf{S}^3$. Then, the result can be stated as follows:

Theorem 1.2 *Let M be a Hopf hypersurface of the homogeneous NK $\mathbf{S}^3\times\mathbf{S}^3$ with three distinct principal curvatures. If $P\{U\}^\perp=\{U\}^\perp$, then, up to isometries of type $\mathcal{F}_{abc}$, M is locally given by one of the following embeddings f_r, f_r' and f_r'' : $\mathbf{S}^3\times\mathbf{S}^2\to\mathbf{S}^3\times\mathbf{S}^3$ defined by:*

$$f_r(x,y)=(x,\sqrt{1-r^2}+ry),\quad f_r'=\mathcal{F}_1\circ f_r,\quad f_r''=\mathcal{F}_2\circ f_r,$$

where $0<r\le 1$, $x\in\mathbf{S}^3$, $y\in\mathbf{S}^2\subset\mathbb{R}^3$, and as usual $\mathbf{S}^3$ (resp. $\mathbf{S}^2$) is regarded as the set of the unitary (resp. imaginary) quaternions in the quaternion space $\mathbb{H}$.

Remark 1.1 Let $M_1^{(r)},M_2^{(r)},M_3^{(r)}$ denote the images of the three embeddings f_r,f_r',f_r'', respectively. Then, for $0<r\le 1$, $M_1^{(r)}$, $M_2^{(r)}$ and $M_3^{(r)}$ correspond to the three possibilities of the action P on the unit normal vector field ξ, which we shall establish in Proposition 5.1.

Remark 1.2 Theorem 1.2 is an extension of the previous result in [16], where the hypersurfaces $M_1^{(r)},M_2^{(r)},M_3^{(r)}$ corresponding to $r=1$ were characterized by the property of satisfying $A\phi = \phi A$, where A is the shape operator of the hypersurfaces and ϕ is the almost contact structure induced from J. Moreover, it is worthy to mention that each of the hypersurfaces $M_1^{(r)}$, $M_2^{(r)}$ and $M_3^{(r)}$ is minimal if and only if $r = 1$.

Remark 1.3 Theorem 1.2 shows that Niebergall and Ryan's observation (cf. p.234 of [24]), which states that certain interesting classes of hypersurfaces in the complex space forms can be characterized by conditions on the holomorphic distribution $\{U\}^\perp$, is similarly valid for the homogeneous NK $\mathbf{S}^3 \times \mathbf{S}^3$. On the other hand, at the moment we do not know if there exist Hopf hypersurfaces of the homogeneous NK $\mathbf{S}^3 \times \mathbf{S}^3$ that have three distinct principal curvatures and satisfy $P\{U\}^\perp \neq \{U\}^\perp$.

2 Preliminaries

2.1 The homogeneous NK structure on $\mathbf{S}^3 \times \mathbf{S}^3$

One can look the classical and comprehensive study of the NK manifolds from [14]. In this section, we first collect some necessary materials from [7]. Let us denote by $\mathbf{S}^3$ the 3-sphere in $\mathbb{R}^4$ as the set of all unitary quaternions. By the natural identification $T_{(p,q)}(\mathbf{S}^3 \times \mathbf{S}^3) \cong T_p\mathbf{S}^3 \oplus T_q\mathbf{S}^3$, we write a tangent vector at $(p, q) \in \mathbf{S}^3 \times \mathbf{S}^3$ as $Z(p, q) = (U_{(p,q)}, V_{(p,q)})$ or simply $Z = (U, V)$. The well-known almost complex structure J on $\mathbf{S}^3 \times \mathbf{S}^3$ is defined by

$$JZ(p, q) = \frac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V). \tag{2.1}$$

On $\mathbf{S}^3 \times \mathbf{S}^3$, we can define a Hermitian metric g compatible with J by

$$\begin{aligned} g(Z, Z') &= \tfrac{1}{2}(\langle Z, Z'\rangle + \langle JZ, JZ'\rangle) \\ &= \tfrac{4}{3}(\langle U, U'\rangle + \langle V, V'\rangle) - \tfrac{2}{3}(\langle p^{-1}U, q^{-1}V'\rangle + \langle p^{-1}U', q^{-1}V\rangle), \end{aligned} \tag{2.2}$$

where $Z = (U, V)$ and $Z' = (U', V')$ are tangent vectors, and $\langle\cdot,\cdot\rangle$ is the standard product metric on $\mathbf{S}^3 \times \mathbf{S}^3$. Then, $\{g, J\}$ gives the homogeneous NK structure on $\mathbf{S}^3 \times \mathbf{S}^3$.

Let $\tilde{\nabla}$ be the Levi-Civita connection with respect to g, and as usual we define a $(1, 2)$-tensor field G by $G(X, Y) := (\tilde{\nabla}_X J)Y$ for $X, Y \in T(\mathbf{S}^3 \times \mathbf{S}^3)$. Then, we have the following formulas for G:

$$G(X, Y) + G(Y, X) = 0, \tag{2.3}$$

$$G(X, JY) + JG(X, Y) = 0, \tag{2.4}$$

$$g(G(X, Y), Z) + g(G(X, Z), Y) = 0, \tag{2.5}$$

$$\begin{aligned} g(G(X, Y), G(Z, W)) = \tfrac{1}{3}\big[&g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \\ &+ g(JX, Z)g(JW, Y) - g(JX, W)g(JZ, Y)\big]. \end{aligned} \tag{2.6}$$

An almost product structure P on $\mathbf{S}^3 \times \mathbf{S}^3$ is introduced by

$$PZ = (pq^{-1}V, qp^{-1}U), \ \ \forall Z = (U, V) \in T_{(p,q)}(\mathbf{S}^3 \times \mathbf{S}^3). \tag{2.7}$$

It is easily seen that P is compatible with the metric g, i.e., P is symmetric with respect to g. Also P is anti-commutative with J. Moreover, with respect to G and P, we further have

$$2(\tilde{\nabla}_X P)Y = JG(X, PY) + JPG(X, Y), \tag{2.8}$$

$$PG(X, Y) + G(PX, PY) = 0. \tag{2.9}$$

Note also that in terms of P the usual product structure Q, defined by $Q(Z) = (-U, V)$ for $Z = (U, V)$, can be expressed by

$$QZ = \tfrac{1}{\sqrt{3}}(2PJZ - JZ). \tag{2.10}$$

For the NK $\mathbf{S}^3 \times \mathbf{S}^3$, we also need the useful relation between the NK connection $\tilde{\nabla}$ and the usual Euclidean connection ∇^E (cf. Lemma 2.2 of [11] and Remark 2.5 of [12]):

$$\nabla^E_X Y = \tilde{\nabla}_X Y + \tfrac{1}{2}[JG(X, PY) + JG(Y, PX)]. \tag{2.11}$$

The Riemannian curvature tensor $\tilde{R}$ of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z =& \tfrac{5}{12}\big[g(Y, Z)X - g(X, Z)Y\big]\\ &+ \tfrac{1}{12}\big[g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\big]\\ &+ \tfrac{1}{3}\big[g(PY, Z)PX - g(PX, Z)PY\\ &+ g(JPY, Z)JPX - g(JPX, Z)JPY\big].\end{aligned} \tag{2.12}$$

2.2 Hypersurfaces of the NK $\mathbf{S}^3 \times \mathbf{S}^3$

Let M be a hypersurface of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ with unit normal vector field ξ. For any vector field X tangent to M, we have the decomposition

$$JX = \phi X + \eta(X)\xi, \tag{2.13}$$

where ϕX and $\eta(X)\xi$ are the tangent and normal parts of JX, respectively. Then, ϕ is a tensor field of type (1, 1), η is a 1-form on M. By definition, the following relations hold:

$$\begin{cases}\eta(X) = g(X, U), \;\; \eta(\phi X) = 0, \;\; \phi^2 X = -X + \eta(X)U, \;\; \phi U = 0,\\ g(\phi X, Y) = -g(X, \phi Y), \;\; g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\end{cases} \tag{2.14}$$

where $U := -J\xi$ is called the *structure vector field* of M. Equation (2.14) shows that (ϕ, U, η, g) determines an *almost contact metric structure* over M.

Let ∇ be the induced connection on M and R its Riemannian curvature tensor. The formulas of Gauss and Weingarten state that

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -AX, \quad \forall X, Y \in TM, \tag{2.15}$$

where h is the second fundamental form and A is the shape operator. They are related by $h(X, Y) = g(AX, Y)\xi$. Using the formulas of Gauss and Weingarten, we can easily show that

$$\nabla_X U = \phi AX - G(X, \xi). \tag{2.16}$$

The Gauss and Codazzi equations of M are given by

$$\begin{aligned}R(X, Y)Z =& \tfrac{5}{12}\big[g(Y, Z)X - g(X, Z)Y\big]\\ &+ \tfrac{1}{12}\big[g(JY, Z)\phi X - g(JX, Z)\phi Y - 2g(JX, Y)\phi Z\big]\\ &+ \tfrac{1}{3}\Big[g(PY, Z)(PX)^\top - g(PX, Z)(PY)^\top\\ &+ g(JPY, Z)(JPX)^\top - g(JPX, Z)(JPY)^\top\Big]\\ &+ g(AZ, Y)AX - g(AZ, X)AY,\end{aligned} \tag{2.17}$$

and

$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X =& \tfrac{1}{12}\big[g(X,U)\phi Y - g(Y,U)\phi X - 2g(JX,Y)U\big] \\
&+ \tfrac{1}{3}\Big[g(PX,\xi)(PY)^{\top} - g(PY,\xi)(PX)^{\top} \\
&+ g(PX,U)(JPY)^{\top} - g(PY,U)(JPX)^{\top}\Big],
\end{aligned}
\tag{2.18}
$$

where $\cdot^{\top}$ means the tangential part.

Similar to that of the complex space forms, a hypersurface M of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ is a Hopf hypersurface if and only if the integral curves of its structure vector field U are geodesics, i.e., $\nabla_U U = 0$. We denote by α the principal curvature function corresponding to the structure vector field U, i.e., $AU = \alpha U$. First of all, we shall present two elementary lemmas for Hopf hypersurfaces of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ as follows:

Lemma 2.1 (cf. [17]) *Let M be a Hopf hypersurface in the NK $\mathbf{S}^3 \times \mathbf{S}^3$. Then, we have*

$$
\begin{aligned}
&\tfrac{1}{6}g(\phi X,Y) - \tfrac{2}{3}\big[g(PX,\xi)g(PY,U) - g(PX,U)g(PY,\xi)\big] \\
&\quad = g((\alpha I - A)G(X,\xi),Y) + g(G((\alpha I - A)X,\xi),Y) \\
&\quad\ - \alpha g((A\phi + \phi A)X,Y) + 2g(A\phi AX,Y), \quad X,Y \in \{U\}^{\perp},
\end{aligned}
\tag{2.19}
$$

where $\{U\}^{\perp}$ denotes the subdistribution of TM that is orthogonal to U, and I denotes the identity transformation.

Lemma 2.2 *Let M be a Hopf hypersurface in the NK $\mathbf{S}^3 \times \mathbf{S}^3$ satisfying $P\{U\}^{\perp} = \{U\}^{\perp}$. Then, the function α is constant.*

Proof By using the Codazzi equation and the symmetry of A, we have the calculation

$$
0 = g((\nabla_U A)Y - (\nabla_Y A)U, U) = g((\nabla_U A)U, Y) - g((\nabla_Y A)U, U) = -Y\alpha, \ \ Y \in \{U\}^{\perp}.
$$

It follows that $\nabla\alpha = (U\alpha)\,U$. Then, for $X, Y \in \{U\}^{\perp}$, we have

$$
0 = X(Y\alpha) - Y(X\alpha) = [X,Y]\alpha = g([X,Y],U)\,U\alpha. \tag{2.20}
$$

If $U\alpha \neq 0$ holds on some open set, then (2.20) implies that $[X,Y] \in \{U\}^{\perp}$. Thus, $\{U\}^{\perp}$ is integrable which gives four-dimensional almost complex submanifolds of the NK $\mathbf{S}^3 \times \mathbf{S}^3$. This is impossible because, according to Lemma 2.2 of [25], any six-dimensional compact non-Kähler NK manifold admits no almost complex four-dimensional submanifold. Hence, $U\alpha = 0$ and α is constant. □

2.3 A canonical distribution related to hypersurfaces of the NK $\mathbf{S}^3 \times \mathbf{S}^3$

In order for choosing an appropriate local orthonormal frame of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ along its hypersurface M, following that in [17] we consider

$$
\mathfrak{D}(p) := \mathrm{Span}\,\{\xi(p), U(p), P\xi(p), PU(p)\}, \quad p \in M.
$$

It is easily seen that, since P is anti-commutative with J, $\mathfrak{D}$ defines a distribution on M with dimension exact 2 or 4, and that it is invariant under both J and P. Along M, let $\mathfrak{D}^{\perp}$ denote the distribution in $T(\mathbf{S}^3 \times \mathbf{S}^3)$ that is orthogonal to $\mathfrak{D}$ at each $p \in M$. For later's purpose, we shall make some remarks about $\dim \mathfrak{D}$:

(1) If $\dim \mathfrak{D} = 4$ holds in an open set, then there exists a unit tangent vector field $e_1 \in \{U\}^\perp$ and functions a, b, c with $c > 0$ such that

$$P\xi = a\xi + bU + ce_1, \quad a^2 + b^2 + c^2 = 1. \tag{2.21}$$

Put $e_2 = Je_1$. Moreover, from the fact $\dim \mathfrak{D}^\perp = 2$ and that $\mathfrak{D}^\perp$ is invariant under the action of both J and P, we can choose a local unit vector field $e_3 \in \mathfrak{D}^\perp$ such that $Pe_3 = e_3$. Now, putting $e_4 = Je_3$ and $e_5 = U$, then $\{e_i\}_{i=1}^5$ is a well-defined orthonormal basis of TM and, acting by P, it has the following properties:

$$\begin{cases} P\xi = a\xi + ce_1 + be_5, & Pe_1 = c\xi - ae_1 - be_2, \\ Pe_2 = ce_5 - be_1 + ae_2, & Pe_3 = e_3, \\ Pe_4 = -e_4, & Pe_5 = b\xi + ce_2 - ae_5. \end{cases} \tag{2.22}$$

(2) If $\dim \mathfrak{D} = 2$ holds in an open set, then $P\{U\}^\perp = \{U\}^\perp$ and we can write

$$P\xi = a\xi + bU, \quad a^2 + b^2 = 1. \tag{2.23}$$

Now, $\mathfrak{D}^\perp$ is a 4-dimensional distribution that is invariant under the action of both J and P. Hence, we can choose unit vector fields $e_1, e_3 \in \mathfrak{D}^\perp$ such that $Pe_1 = e_1, \ Pe_3 = e_3$. Put $e_2 = Je_1$, $e_4 = Je_3$ and $e_5 = U$. In this way, we obtain an orthonormal basis $\{e_i\}_{i=1}^5$ of TM. However, we would remark that such choice of $\{e_1, e_3\}$ (resp. $\{e_2, e_4\}$) is unique up to an orthogonal transformation.

3 The proof of Theorem 1.1

Suppose on the contrary that M is a Hopf hypersurface in the NK $\mathbf{S}^3 \times \mathbf{S}^3$ which has two distinct principal curvatures, say α and λ, with $AU = \alpha U$. We denote by V_α and V_λ the corresponding eigen-distributions. By the continuity of the principal curvature functions, we know that the dimensions $(\dim V_\alpha, \dim V_\lambda)$ of the two eigen-distributions have to be one of the four possibilities: (1, 4), (2, 3), (3, 2) and (4, 1).

Next, we separate the proof of Theorem 1.1 into the proofs of two lemmas, depending on the dimension of $\mathfrak{D}$.

Lemma 3.1 *The case* $\dim \mathfrak{D} = 4$ *does not occur.*

Proof To argue by contradiction, we assume that $\dim \mathfrak{D} = 4$ does hold on an open set. Now we check each possibility of $(\dim V_\alpha, \dim V_\lambda)$.

(i) $(\dim V_\alpha, \dim V_\lambda) = (1, 4)$ **on** M.

In this case, it is easy to see that $A\phi = \phi A$ holds. This is impossible because, according to Theorem 4.1 of [16], hypersurfaces satisfying $A\phi = \phi A$ must have three distinct principal curvatures.

(ii) $(\dim V_\alpha, \dim V_\lambda) = (2, 3)$ **on** M.

In this case, we can take a local orthonormal frame field $\{X_i\}_{i=1}^5$ such that

$$AX_i = \alpha X_i, \ i = 1, 5; \quad AX_j = \lambda X_j, \ j = 2, 3, 4,$$

where $X_2 = JX_1$, $X_4 = JX_3$, $X_5 = U$. Then by using (2.3)–(2.6), we get

$$\begin{aligned} G(X_1, X_4) = G(X_2, X_3) = -JG(X_1, X_3), \\ g(G(X_1, X_3), X_i) = 0 \ \text{ for } \ 1 \le i \le 4, \end{aligned} \tag{3.1}$$

$$g(G(X_1, X_3), G(X_1, X_3)) = \tfrac{1}{3}. \tag{3.2}$$

Let $\{e_i\}_{i=1}^5$ be the orthonormal basis as described in (2.22). Then,

$$X_1 = me_1 + ne_2 + ue_3 + ve_4, \quad X_3 = -ue_1 + ve_2 + me_3 - ne_4,$$

for some functions m, n, u, v; and

$$X_2 = -ne_1 + me_2 - ve_3 + ue_4, \quad X_4 = -ve_1 - ue_2 + ne_3 + me_4.$$

Now, taking in (2.19), respectively, $(X, Y) = (X_1, X_3), (X_1, X_4), (X_2, X_3), (X_2, X_4)$, we can obtain

$$\tfrac{2}{3}c^2mv + \tfrac{2}{3}c^2nu = (\lambda - \alpha)g(G(X_1, \xi), X_3), \tag{3.3}$$
$$-\tfrac{2}{3}c^2mu + \tfrac{2}{3}c^2nv = (\lambda - \alpha)g(G(X_1, \xi), X_4), \tag{3.4}$$
$$-\tfrac{2}{3}c^2nv + \tfrac{2}{3}c^2mu = 2(\lambda - \alpha)g(G(X_2, \xi), X_3), \tag{3.5}$$
$$\tfrac{2}{3}c^2nu + \tfrac{2}{3}c^2mv = 2(\lambda - \alpha)g(G(X_2, \xi), X_4). \tag{3.6}$$

From (3.4) and (3.5), and, respectively, (3.3) and (3.6), we deduce that

$$g(G(X_1, X_3), U) = 0, \quad g(G(X_1, X_3), \xi) = 0.$$

This combining with (3.1) implies that $G(X_1, X_3) = 0$, a contradiction to (3.2).

(iii) $(\dim V_\alpha, \dim V_\lambda) = (3, 2)$ **on** M.

In this case, as $U \in V_\alpha$, we have $\dim(V_\alpha \cap \{U\}^\perp) = \dim V_\lambda = 2$. For an orthonormal basis $\{X_1, X_2\}$ of $V_\alpha \cap \{U\}^\perp$, we consider $|g(JX_1, X_2)|$, which is obviously independent of the choice of $\{X_1, X_2\}$, thus gives a well-defined function $\theta := |g(JX_1, X_2)|$ on M, with $0 \le \theta \le 1$. Since our concern is only local, in order to prove that Case (iii) does not occur, we are sufficient to show that the following three subcases do not occur on M.

(iii)-(1) $0 < \theta < 1$.

In this subcase, we can take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1, \; AX_2 = \alpha X_2, \; AX_3 = \lambda X_3, \; AX_4 = \lambda X_4, \; X_5 = U,$$

where $X_3 = (JX_1 - \theta X_2)/\sqrt{1 - \theta^2}$, $X_4 = (JX_2 + \theta X_1)/\sqrt{1 - \theta^2}$ and $\theta = g(JX_1, X_2)$.

Moreover, direct calculations give the following relations:

$$\begin{cases} JX_1 = \sqrt{1-\theta^2}X_3 + \theta X_2, \quad JX_2 = \sqrt{1-\theta^2}X_4 - \theta X_1, \\ JX_3 = -\sqrt{1-\theta^2}X_1 - \theta X_4, \quad JX_4 = -\sqrt{1-\theta^2}X_2 + \theta X_3, \\ g(JX_1, X_2) = -g(JX_3, X_4) = \theta, \quad g(JX_1, X_3) = g(JX_2, X_4) = \sqrt{1-\theta^2}, \\ g(JX_1, X_4) = g(JX_2, X_3) = 0, \quad G(X_3, X_4) = -G(X_1, X_2), \\ G(X_1, X_3) = \frac{-\theta}{\sqrt{1-\theta^2}}G(X_1, X_2), \quad G(X_1, X_4) = \frac{-1}{\sqrt{1-\theta^2}}JG(X_1, X_2), \\ G(X_2, X_3) = \frac{1}{\sqrt{1-\theta^2}}JG(X_1, X_2), \quad G(X_2, X_4) = \frac{-\theta}{\sqrt{1-\theta^2}}G(X_1, X_2). \end{cases} \tag{3.7}$$

Let $\{e_i\}_{i=1}^5$ be the orthonormal basis as described in (2.22) and assume that

$$X_i = \sum_{j=1}^{4} a_{ij}e_j, \quad 1 \le i \le 4.$$

Then, by the definition of X_3 and X_4, we can derive

$$\begin{cases} a_{31}=\frac{-a_{12}-a_{21}\theta}{\sqrt{1-\theta^2}},\ a_{32}=\frac{a_{11}-a_{22}\theta}{\sqrt{1-\theta^2}},\ a_{33}=\frac{-a_{14}-a_{23}\theta}{\sqrt{1-\theta^2}},\ a_{34}=\frac{a_{13}-a_{24}\theta}{\sqrt{1-\theta^2}}; \\ a_{41}=\frac{-a_{22}+a_{11}\theta}{\sqrt{1-\theta^2}},\ a_{42}=\frac{a_{21}+a_{12}\theta}{\sqrt{1-\theta^2}},\ a_{43}=\frac{-a_{24}+a_{13}\theta}{\sqrt{1-\theta^2}},\ a_{44}=\frac{a_{23}+a_{14}\theta}{\sqrt{1-\theta^2}}. \end{cases} \tag{3.8}$$

Taking, in (2.19), $(X,Y)=(X_i,X_j)$ for $1\le i<j\le 4$, and using (3.7) and (2.22), we get

$$-\tfrac{1}{6}\theta+\tfrac{2}{3}c^2(a_{11}a_{22}-a_{12}a_{21})=0, \tag{3.9}$$

$$\tfrac{2}{3\sqrt{1-\theta^2}}c^2(a_{11}a_{21}+a_{12}a_{22})+\tfrac{1}{\sqrt{1-\theta^2}}(\alpha-\lambda)g(G(X_1,X_2),U)=0, \tag{3.10}$$

$$\tfrac{2}{3\sqrt{1-\theta^2}}c^2(a_{11}a_{21}+a_{12}a_{22})-\tfrac{1}{\sqrt{1-\theta^2}}(\alpha-\lambda)g(G(X_1,X_2),U)=0, \tag{3.11}$$

$$\begin{aligned}&\tfrac{2}{3\sqrt{1-\theta^2}}c^2(-a_{11}^2-a_{12}^2+(a_{22}a_{11}-a_{21}a_{12})\theta)+\tfrac{\sqrt{1-\theta^2}}{6}\\ &\quad-\tfrac{\theta}{\sqrt{1-\theta^2}}(\alpha-\lambda)g(G(X_1,X_2),\xi)+\alpha(\alpha-\lambda)\sqrt{1-\theta^2}=0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\tfrac{2}{3\sqrt{1-\theta^2}}c^2(-a_{21}^2-a_{22}^2+(a_{22}a_{11}-a_{21}a_{12})\theta)+\tfrac{\sqrt{1-\theta^2}}{6}\\ &\quad-\tfrac{\theta}{\sqrt{1-\theta^2}}(\alpha-\lambda)g(G(X_1,X_2),\xi)+\alpha(\alpha-\lambda)\sqrt{1-\theta^2}=0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\tfrac{2}{3(1-\theta^2)}c^2\big[a_{21}a_{12}-a_{11}a_{22}+(a_{22}^2+a_{11}^2+a_{21}^2+a_{12}^2)\theta+(a_{12}a_{21}-a_{11}a_{22})\theta^2\big]\\ &\quad-\tfrac{\theta}{6}-2(\alpha-\lambda)g(G(X_1,X_2),\xi)-2\lambda(\alpha-\lambda)\theta=0.\end{aligned} \tag{3.14}$$

From (3.10), (3.11) and $g(X_1,X_2)=0$, we have

$$g(G(X_1,X_2),U)=0,\ \ a_{11}a_{21}+a_{12}a_{22}=0,\ \ a_{13}a_{23}+a_{14}a_{24}=0.$$

From (3.9), (3.12), (3.13) and $g(X_1,X_1)=g(X_2,X_2)=1$, we have

$$a_{11}^2+a_{12}^2=a_{21}^2+a_{22}^2\neq 0,\ \ a_{13}^2+a_{14}^2=a_{23}^2+a_{24}^2.$$

Thus, we can write

$$\begin{cases} a_{11}=\sqrt{a_{11}^2+a_{12}^2}\cos\omega_1,\ \ a_{12}=\sqrt{a_{11}^2+a_{12}^2}\sin\omega_1; \\ a_{21}=\sqrt{a_{11}^2+a_{12}^2}\cos\omega_2,\ \ a_{22}=\sqrt{a_{11}^2+a_{12}^2}\sin\omega_2. \end{cases}$$

Then, the fact $0=a_{11}a_{21}+a_{12}a_{22}=(a_{11}^2+a_{12}^2)\cos(\omega_1-\omega_2)$ implies that $\omega_1-\omega_2=\frac{\pi}{2}(2k+1)$ for $k\in\mathbb{Z}$. Hence, $(a_{21},a_{22})=\pm(a_{12},-a_{11})$. On the other hand, (3.9) implies that $a_{11}a_{22}-a_{12}a_{21}=\frac{\theta}{4c^2}>0$, so it should be that $(a_{21},a_{22})=-(a_{12},-a_{11})$.

Similarly, we can prove that $(a_{23},a_{24})=(a_{14},-a_{13})$. It follows that $a_{11}^2+a_{12}^2=\frac{\theta}{4c^2}$ and $a_{13}^2+a_{14}^2=1-\frac{\theta}{4c^2}$. On the other hand, by definition, we can finally get

$$\theta=\sum a_{1i}a_{2j}g(Je_i,e_j)=a_{11}a_{22}-a_{12}a_{21}+a_{13}a_{24}-a_{14}a_{23}=\tfrac{\theta}{2c^2}-1,$$

and thus $\theta=\frac{2c^2}{1-2c^2}$.

Next, from the fact $g(G(X_1,X_2),X_i)=0$ for $1\le i\le 5$ and that, by (2.6),

$$g(G(X_1,X_2),G(X_1,X_2))=\tfrac{1}{3}(1-\theta^2),$$

we have $G(X_1, X_2) = \pm\sqrt{(1-\theta^2)/3}\,\xi$. Since the discussion is totally similar, we just consider the case $G(X_1, X_2) = \sqrt{(1-\theta^2)/3}\,\xi$. We calculate the connections $\{\nabla_{X_i} X_j\}$ so that we can apply for the Codazzi equations.

Put $\nabla_{X_i} X_j = \sum \Gamma_{ij}^k X_k$ with $\Gamma_{ij}^k = -\Gamma_{ik}^j$, $1 \le i, j, k \le 5$.

Then, on the one hand, by definition and the Gauss–Weingarten formulas, we have

$$G(X_1, \xi) = -\sum_{i=1}^{5} \Gamma_{15}^i X_i + \alpha J X_1.$$

On the other hand, using $G(X_1, \xi) = \sum_i g(G(X_1, \xi), X_i) X_i$, we easily get

$$G(X_1, \xi) = -\sqrt{\tfrac{1-\theta^2}{3}} X_2 + \tfrac{\sqrt{3}}{3} \theta X_3.$$

From the above calculations and (3.7), it follows that

$$\Gamma_{15}^1 = 0,\ \ \Gamma_{15}^2 = \alpha\theta + \sqrt{\tfrac{1-\theta^2}{3}},\ \ \Gamma_{15}^3 = \alpha\sqrt{1-\theta^2} - \tfrac{\sqrt{3}}{3}\theta,\ \ \Gamma_{15}^4 = 0. \tag{3.15}$$

Analogously, calculating $G(X_i, \xi) = (\tilde{\nabla}_{X_i} J)\xi$ for $2 \le i \le 4$, we can further obtain

$$\begin{cases} \Gamma_{25}^1 = -\alpha\theta - \sqrt{\tfrac{1-\theta^2}{3}},\ \ \Gamma_{25}^2 = 0,\ \ \Gamma_{25}^3 = 0,\ \ \Gamma_{25}^4 = \alpha\sqrt{1-\theta^2} - \tfrac{\sqrt{3}}{3}\theta, \\ \Gamma_{35}^1 = -\lambda\sqrt{1-\theta^2} + \tfrac{\sqrt{3}}{3}\theta,\ \ \Gamma_{35}^2 = 0,\ \ \Gamma_{35}^3 = 0,\ \ \Gamma_{35}^4 = -\lambda\theta - \sqrt{\tfrac{1-\theta^2}{3}}, \\ \Gamma_{45}^1 = 0,\ \ \Gamma_{45}^2 = -\lambda\sqrt{1-\theta^2} + \tfrac{\sqrt{3}}{3}\theta,\ \ \Gamma_{45}^3 = \lambda\theta + \sqrt{\tfrac{1-\theta^2}{3}},\ \ \Gamma_{45}^4 = 0. \end{cases} \tag{3.16}$$

Now, we are ready to calculate $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ for $1 \le i \le 4$.

On the one hand, using $e_i = \sum_{j=1}^4 a_{ji} X_j$ and the preceding results (3.15) and (3.16), direct calculations give the $\{U\}^\perp$-components of $(\nabla_U A)e_i - (\nabla_{e_i} A)U$:

$$\begin{pmatrix} (\nabla_U A)e_1 - (\nabla_{e_1} A)U \\ (\nabla_U A)e_2 - (\nabla_{e_2} A)U \\ (\nabla_U A)e_3 - (\nabla_{e_3} A)U \\ (\nabla_U A)e_4 - (\nabla_{e_4} A)U \end{pmatrix}_{\{U\}^\perp} = BC \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix},$$

where

$$B = (a_{ij})^T = \begin{pmatrix} a_{11} & -a_{12} & -a_{12}\sqrt{(1-\theta)/(1+\theta)} & -a_{11}\sqrt{(1-\theta)/(1+\theta)} \\ a_{12} & a_{11} & a_{11}\sqrt{(1-\theta)/(1+\theta)} & -a_{12}\sqrt{(1-\theta)/(1+\theta)} \\ a_{13} & a_{14} & -a_{14}\sqrt{(1+\theta)/(1-\theta)} & a_{13}\sqrt{(1+\theta)/(1-\theta)} \\ a_{14} & -a_{13} & a_{13}\sqrt{(1+\theta)/(1-\theta)} & a_{14}\sqrt{(1+\theta)/(1-\theta)} \end{pmatrix},$$

$$C = (C_{ij}) := \begin{pmatrix} U(\alpha) & 0 & (\alpha-\lambda)(\Gamma_{51}^3 - \Gamma_{15}^3) & (\alpha-\lambda)\Gamma_{51}^4 \\ 0 & U(\alpha) & (\alpha-\lambda)\Gamma_{52}^3 & (\alpha-\lambda)(\Gamma_{52}^4 - \Gamma_{25}^4) \\ (\lambda-\alpha)\Gamma_{53}^1 & (\lambda-\alpha)\Gamma_{53}^2 & U(\lambda) & (\lambda-\alpha)\Gamma_{35}^4 \\ (\lambda-\alpha)\Gamma_{54}^1 & (\lambda-\alpha)\Gamma_{54}^2 & (\lambda-\alpha)\Gamma_{45}^3 & U(\lambda) \end{pmatrix}.$$

On the other hand, using the Codazzi equation (2.18), $e_i = \sum_{j=1}^4 a_{ji} X_j$ and (2.22), another calculation for the $\{U\}^\perp$-components of $(\nabla_U A)e_i - (\nabla_{e_i} A)U$ can be carried out to obtain:

$$\begin{pmatrix} (\nabla_U A)e_1 - (\nabla_{e_1} A)U \\ (\nabla_U A)e_2 - (\nabla_{e_2} A)U \\ (\nabla_U A)e_3 - (\nabla_{e_3} A)U \\ (\nabla_U A)e_4 - (\nabla_{e_4} A)U \end{pmatrix}_{\{U\}^\perp} = (D+E)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix},$$

where

$$D = (D_{ij}) := \begin{pmatrix} -\frac{2ab}{3}a_{11} & \frac{2ab}{3}a_{12} & \frac{2aba_{12}}{3}\sqrt{\frac{1-\theta}{1+\theta}} & \frac{2aba_{11}}{3}\sqrt{\frac{1-\theta}{1+\theta}} \\ \frac{2ab}{3}a_{12} & \frac{2ab}{3}a_{11} & \frac{2aba_{11}}{3}\sqrt{\frac{1-\theta}{1+\theta}} & -\frac{2aba_{12}}{3}\sqrt{\frac{1-\theta}{1+\theta}} \\ \frac{b}{3}a_{13} & \frac{b}{3}a_{14} & -\frac{ba_{14}}{3}\sqrt{\frac{1+\theta}{1-\theta}} & \frac{ba_{13}}{3}\sqrt{\frac{1+\theta}{1-\theta}} \\ -\frac{b}{3}a_{14} & \frac{b}{3}a_{13} & -\frac{ba_{13}}{3}\sqrt{\frac{1+\theta}{1-\theta}} & -\frac{ba_{14}}{3}\sqrt{\frac{1+\theta}{1-\theta}} \end{pmatrix},$$

$$E = (E_{ij}) := \begin{pmatrix} \frac{8a^2-3}{12}a_{12} & \frac{8a^2-3}{12}a_{11} & \frac{(8a^2-3)a_{11}}{12}\sqrt{\frac{1-\theta}{1+\theta}} & -\frac{(8a^2-3)a_{12}}{12}\sqrt{\frac{1-\theta}{1+\theta}} \\ \frac{3-8b^2}{12}a_{11} & \frac{8b^2-3}{12}a_{12} & \frac{(8b^2-3)a_{12}}{12}\sqrt{\frac{1-\theta}{1+\theta}} & \frac{(8b^2-3)a_{11}}{12}\sqrt{\frac{1-\theta}{1+\theta}} \\ \frac{1-4a}{12}a_{14} & -\frac{1-4a}{12}a_{13} & \frac{(1-4a)a_{13}}{12}\sqrt{\frac{1+\theta}{1-\theta}} & \frac{(1-4a)a_{14}}{12}\sqrt{\frac{1+\theta}{1-\theta}} \\ -\frac{1+4a}{12}a_{13} & -\frac{1+4a}{12}a_{14} & \frac{(1+4a)a_{14}}{12}\sqrt{\frac{1+\theta}{1-\theta}} & -\frac{(1+4a)a_{13}}{12}\sqrt{\frac{1+\theta}{1-\theta}} \end{pmatrix}.$$

In this way, we obtain the equation $BC = D + E$. This can be written in equivalent form: $C_{ij} = \sum_k a_{ik}(D_{kj} + E_{kj})$ for $1 \le i, j \le 4$. Then, since by (3.16) we have

$$C_{11} - C_{22} = 0,\ C_{12} + C_{21} = 0,\ C_{33} - C_{44} = 0,\ C_{34} + C_{43} = 0,$$

it follows that $LF = 0$, where $L = (a_{11}^2 - a_{12}^2,\ a_{13}^2 - a_{14}^2,\ a_{11}a_{12},\ a_{13}a_{14})$, and

$$F = \begin{pmatrix} -2ab & a^2-b^2 & (b^2-a^2)(1-\theta)^2 & 2ab(1-\theta)^2 \\ b & a & -a(1+\theta)^2 & -b(1+\theta)^2 \\ 2(a^2-b^2) & 4ab & -4ab(1-\theta)^2 & 2(b^2-a^2)(1-\theta)^2 \\ -2a & 2b & -2b(1+\theta)^2 & 2a(1+\theta)^2 \end{pmatrix}.$$

Now, direct calculation gives that $\det F = -64\theta^2(a^2+b^2)^3$.

If $\det F = 0$, then $c = 1$ and this contradicts to $\theta = \frac{2c^2}{1-2c^2} \in (0, 1)$. If $\det F \neq 0$, then $L = 0$ and thus $a_{11} = a_{12} = a_{13} = a_{14} = 0$, which is also a contradiction.

In summary, we have shown that (iii)-(1) does not occur.

(iii)-(2) $\theta = 0$.

In this case, we have $J\{V_\alpha \cap \{U\}^\perp\} = V_\lambda$. Take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1,\ AX_2 = \alpha X_2,\ AX_3 = \lambda X_3,\ AX_4 = \lambda X_4,\ AX_5 = \alpha X_5,$$

where $X_3 = JX_1,\ X_4 = JX_2,\ X_5 = U$. It follows that

$$g(G(X_1, X_2), X_i) = 0,\ 1 \le i \le 4; \quad g(G(X_1, X_2), G(X_1, X_2)) = \tfrac{1}{3}.$$

Assume that $X_i = \sum_{j=1}^4 a_{ij}e_j$ for $1 \le i \le 4$. Then taking in (2.19) that $(X, Y) = (X_i, X_j)$ for each $1 \le i, j \le 4$, we can still get the equations from (3.9) up to (3.14) but with $\theta = 0$. From (3.9) and (3.14) corresponding to $\theta = 0$, we get $g(G(X_1, X_2), \xi) = 0$. Then, by (3.10) and (3.11), we obtain $g(G(X_1, X_2), U) = 0$.

It follows that $G(X_1, X_2) = 0$, a contradiction to $g(G(X_1, X_2), G(X_1, X_2)) = \frac{1}{3}$.

(iii)-(3) $\theta = 1$.

In this case, both $V_\alpha \cap \{U\}^\perp$ and V_λ are J-invariant. Then, it is easily seen that M satisfies $A\phi = \phi A$, and according to Theorem 4.1 of [16] once more we get as desired a contradiction.

(iv) $(\dim V_\alpha, \dim V_\lambda) = (4, 1)$ **on** M.

In this case, we can take a local orthonormal basis $\{X_i\}_{i=1}^5$ such that

$$AX_1 = \lambda X_1,\ AX_2 = \alpha X_2,\ AX_3 = \alpha X_3,\ AX_4 = \alpha X_4,\ AX_5 = \alpha X_5,$$

where $X_2 = JX_1,\ X_4 = JX_3,\ X_5 = U$. Then, as preceding we have

$$g(G(X_1, X_3), X_i) = 0,\ 1 \le i \le 4;\ \ |G(X_1, X_3)|^2 = \tfrac{1}{3}. \tag{3.17}$$

Let $\{e_i\}_{i=1}^5$ be the orthonormal basis as described in (2.22) and assume, for some functions m, n, u, v that $X_1 = me_1 + ne_2 + ue_3 + ve_4,\quad X_3 = -ue_1 + ve_2 + me_3 - ne_4$. Then, by definition, we have

$$X_2 = -ne_1 + me_2 - ve_3 + ue_4,\quad X_4 = -ve_1 - ue_2 + ne_3 + me_4.$$

Taking in (2.19), respectively, $(X, Y) = (X_1, X_3), (X_1, X_4), (X_3, X_2), (X_4, X_2)$, we get

$$\tfrac{2}{3}c^2mv + \tfrac{2}{3}c^2nu = (\lambda - \alpha)g(G(X_1, \xi), X_3), \tag{3.18}$$

$$-\tfrac{2}{3}c^2mu + \tfrac{2}{3}c^2nv = (\lambda - \alpha)g(G(X_1, \xi), X_4), \tag{3.19}$$

$$-\tfrac{2}{3}c^2mu + \tfrac{2}{3}c^2nv = 0, \tag{3.20}$$

$$\tfrac{2}{3}c^2nu + \tfrac{2}{3}c^2mv = 0. \tag{3.21}$$

From these equations, we immediately obtain

$$g(G(X_1, X_3), U) = 0,\quad g(G(X_1, X_3), \xi) = 0.$$

This together with (3.17) gives $G(X_1, X_3) = 0$, a contradiction to $|G(X_1, X_3)|^2 = \frac{1}{3}$.

This finally completes the proof of Lemma 3.1. □

Lemma 3.2 *The case* $\dim \mathfrak{D} = 2$ *does not occur.*

Proof Suppose on the contrary that $\dim \mathfrak{D} = 2$ does hold on M.

Then, we consider each possibility of the dimensions $(\dim V_\alpha, \dim V_\lambda)$.

(i) $(\dim V_\alpha, \dim V_\lambda) = (1, 4)$ **on** M.

In this case, we can easily show that M satisfies $A\phi = \phi A$. As before by Theorem 4.1 in [16], this is impossible.

(ii) $(\dim V_\alpha, \dim V_\lambda) = (2, 3)$ **on** M.

In this case, we take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1,\ AX_2 = \lambda X_2,\ AX_3 = \lambda X_3,\ AX_4 = \lambda X_4,\ AX_5 = \alpha X_5,$$

where $X_2 = JX_1,\ X_4 = JX_3,\ X_5 = U$. By (2.3)–(2.5), $G(X_1, \xi)$ is orthogonal to $\mathrm{Span}\{\xi, U, X_1, X_2\}$, so $AG(X_1, \xi) = \lambda G(X_1, \xi)$. Then, taking $X = X_1$ in (2.19), we can get

$$(\alpha - \lambda)g(G(X_1, \xi), Y) = (\alpha^2 - \alpha\lambda + \tfrac{1}{6})g(X_2, Y),\ \forall Y \in \{U\}^\perp. \tag{3.22}$$

Notice that $g(X_2, X_3) = g(X_2, X_4) = 0$ and $\alpha \neq \lambda$, so (3.22) implies that $G(X_1, \xi) = 0$. However, by (2.6) we have $|G(X_1, \xi)|^2 = \frac{1}{3}$. This is a contradiction.

(iii) $(\dim V_\alpha, \dim V_\lambda) = (3, 2)$ **on** M.

In this case, we take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1,\ AX_2 = \alpha X_2,\ AX_3 = \lambda X_3,\ AX_4 = \lambda X_4,\ AX_5 = \alpha X_5,$$

where $X_5 = U$. Taking in (2.19) $(X, Y) = (X_1, X_2)$ gives $g(\phi X_1, X_2) = 0$. It follows that $J\{V_\alpha \cap \{U\}^\perp\} = V_\lambda$. Then, we can choose a local orthonormal frame field $\{\tilde{X}_i\}_{i=1}^5$ such that $\tilde{X}_1 = X_1,\ \tilde{X}_2 = J\tilde{X}_1,\ \tilde{X}_3 = X_2,\ \tilde{X}_4 = J\tilde{X}_3,\ \tilde{X}_5 = U$, and moreover, $\tilde{X}_1, \tilde{X}_3, \tilde{X}_5 \in V_\alpha$ and $\tilde{X}_2, \tilde{X}_4 \in V_\lambda$. By identity (2.19) with (X, Y) equal to $(\tilde{X}_2, \tilde{X}_3),\ (\tilde{X}_2, \tilde{X}_4)$, respectively, we have $g(G(\tilde{X}_2, \xi), \tilde{X}_3) = g(G(\tilde{X}_2, \xi), \tilde{X}_4) = 0$. This implies that $G(\tilde{X}_2, \xi) = 0$ due to the obvious fact $G(\tilde{X}_2, \xi) \perp \mathrm{Span}\,\{\xi, U, \tilde{X}_1, \tilde{X}_2\}$.

However, by (2.6) we have $|G(\tilde{X}_2, \xi)|^2 = \frac{1}{3}$. This is also a contradiction.

(iv) $(\dim V_\alpha, \dim V_\lambda) = (4, 1)$ **on** M.

In this case, we take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \lambda X_1,\ AX_2 = \alpha X_2,\ AX_3 = \alpha X_3,\ AX_4 = \alpha X_4,\ AX_5 = \alpha X_5,$$

where $X_2 = JX_1,\ X_4 = JX_3,\ X_5 = U$. By (2.3)–(2.5), $G(X_1, \xi)$ is orthogonal to $\mathrm{Span}\{\xi, U, X_1, X_2\}$, so $AG(X_1, \xi) = \alpha G(X_1, \xi)$. Taking in (2.19) $X = X_1$, we get

$$(\alpha - \lambda)g(G(X_1, \xi), Y) = (\alpha^2 - \alpha\lambda + \tfrac{1}{6})g(X_2, Y),\ \forall\, Y \in \{U\}^\perp. \tag{3.23}$$

Then, similar as in case (ii), from (3.23), the fact $g(X_2, X_3) = g(X_2, X_4) = 0$ and $\alpha \neq \lambda$, we obtain $G(X_1, \xi) = 0$.

However, by (2.6), $|G(X_1, \xi)|^2 = \frac{1}{3}$. This is a contradiction. □

4 Examples of Hopf hypersurfaces in $\mathbf{S}^3 \times \mathbf{S}^3$

As usual we denote $\mathbf{S}^3$ (resp. $\mathbf{S}^2$) the set of the unitary (resp. imaginary) quaternions in the quaternion space $\mathbb{H}$. Then, in this short section, we can describe several of the simplest examples of Hopf hypersurfaces in the NK $\mathbf{S}^3 \times \mathbf{S}^3$.

Examples 4.1 For each $0 < r \le 1$, we define three families of hypersurfaces $M_1^{(r)}$, $M_2^{(r)}$ and $M_3^{(r)}$ in the NK $\mathbf{S}^3 \times \mathbf{S}^3$ as below:

$$\begin{aligned} M_1^{(r)} &:= \left\{(x, \sqrt{1 - r^2} + ry) \in \mathbf{S}^3 \times \mathbf{S}^3 \mid x \in \mathbf{S}^3,\ y \in \mathbf{S}^2\right\},\\ M_2^{(r)} &:= \mathcal{F}_1(M_1^{(r)}),\\ M_3^{(r)} &:= \mathcal{F}_2(M_1^{(r)}). \end{aligned}$$

Remark 4.1 Among the preceding hypersurfaces $M_1^{(r)}$, $M_2^{(r)}$ and $M_3^{(r)}$ of the NK $\mathbf{S}^3 \times \mathbf{S}^3$, $M_1^{(r)}$, $M_2^{(r)}$ and $M_3^{(1)}$ have been carefully discussed, respectively, in Examples 5.1, 5.2 and 5.3 of [16]. As a matter of fact, all of them are Hopf hypersurfaces with three distinct constant principal curvatures: $\alpha = 0$ (i.e., $AU = 0$) of multiplicity 1, $\lambda = \frac{\sqrt{1-r^2}}{2r} - \frac{\sqrt{3-2r^2}}{2\sqrt{3}r}$ of multiplicity 2, and $\beta = \frac{\sqrt{1-r^2}}{2r} + \frac{\sqrt{3-2r^2}}{2\sqrt{3}r}$ of multiplicity 2. The holomorphic distributions $\{U\}^\perp$ of these hypersurfaces are all preserved by the almost product structure P of the NK $\mathbf{S}^3 \times \mathbf{S}^3$, but P acts differently on their unit normal vector fields.

Examples 4.2 For each $0 < k, l < 1$, $k^2 + l^2 = 1$, we can define three families of hypersurfaces $M_4^{(k,l)}$, $M_5^{(k,l)}$ and $M_6^{(k,l)}$ in the NK $\mathbf{S}^3 \times \mathbf{S}^3$ as below:

$$\begin{aligned} M_4^{(k,l)} &:= \left\{(x, (y_1, y_2, y_3, y_4)) \in \mathbf{S}^3 \times \mathbf{S}^3 \mid x \in \mathbf{S}^3,\ y_1^2 + y_2^2 = k^2,\ y_3^2 + y_4^2 = l^2\right\},\\ M_5^{(k,l)} &:= \mathcal{F}_1(M_4^{(k,l)}),\\ M_6^{(k,l)} &:= \mathcal{F}_2(M_4^{(k,l)}). \end{aligned}$$

Remark 4.2 Direct calculations show that all of these three families of hypersurfaces are Hopf ones, and they have five distinct constant principal curvatures: $\alpha = 0$ (i.e., $AU = 0$), $\lambda_1 = \frac{3k-\sqrt{9k^2+3l^2}}{6l}$, $\lambda_2 = \frac{3k+\sqrt{9k^2+3l^2}}{6l}$, $\lambda_3 = \frac{-3l-\sqrt{3k^2+9l^2}}{6k}$, $\lambda_4 = \frac{-3l+\sqrt{3k^2+9l^2}}{6k}$. Similarly, the holomorphic distributions $\{U\}^\perp$ of these hypersurfaces are all preserved by the almost product structure P of the NK $\mathbf{S}^3 \times \mathbf{S}^3$, but P acts differently on their unit normal vector fields.

Remark 4.3 Theorem 1.2 gives a characterization of the Hopf hypersurfaces $M_1^{(r)}$, $M_2^{(r)}$ and $M_3^{(r)}$ in the NK $\mathbf{S}^3 \times \mathbf{S}^3$. We expect that a similar interesting characterization of the Hopf hypersurfaces $M_4^{(k,l)}$, $M_5^{(k,l)}$ and $M_6^{(k,l)}$ in the NK $\mathbf{S}^3 \times \mathbf{S}^3$ is possible, but at the moment it is still not achieved.

5 The proof of Theorem 1.2

This last section is devoted to the proof of Theorem 1.2, which is given in two steps. In the sequel, we assume that M is a Hopf hypersurface of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ with three distinct principal curvatures α, λ and β such that $AU = \alpha U$, and that $P\{U\}^\perp = \{U\}^\perp$. In particular, (2.23) holds.

5.1 The principal curvatures and their multiplicities

Let V_α, V_λ and V_β denote the eigenspaces corresponding to the principal curvatures α, λ and β, respectively. By the assumption of having three distinct principal curvatures and the continuity of the principal curvature functions, we know that the dimensions $(\dim V_\alpha, \dim V_\lambda, \dim V_\beta)$ remain unchanged on M, which, without loss of generality, have four possibilities: $(3, 1, 1)$, $(2, 2, 1)$, $(1, 3, 1)$ and $(1, 2, 2)$.

First of all, we shall determine the multiplicities of the principal curvatures.

Lemma 5.1 *The multiplicities of the three distinct principal curvature functions α, λ, β can only be* $1, 2$ *and* 2, *respectively.*

Proof Suppose on the contrary that, for the multiplicities of the principal curvatures α, λ and β, one of the three possibilities $(3, 1, 1)$, $(2, 2, 1)$, $(1, 3, 1)$ does occur. Then, for each possible case, we shall derive a contradiction by using Lemma 2.1.

(i) $(\dim V_\alpha, \dim V_\lambda, \dim V_\beta) = (3, 1, 1)$ **on** M.

We take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \lambda X_1,\ AX_2 = \beta X_2,\ AX_3 = \alpha X_3,\ AX_4 = \alpha X_4,\ X_5 = U.$$

Taking in (2.19) $(X, Y) = (X_3, X_4)$, we get $g(\phi X_3, X_4) = 0$, which implies that $J\{V_\lambda \oplus V_\beta\} = V_\alpha \cap \{U\}^\perp$. So we can further choose $X_3 = JX_1$ and $X_4 = JX_2$. Then, we easily show that $G(X_1, X_2) \in \mathrm{Span}\{\xi, U\}$, and by (2.6), we have $|G(X_1, X_2)|^2 = \frac{1}{3}$.

Now, taking in (2.19) $(X, Y) = (X_1, X_3),\ (X_2, X_4),\ (X_2, X_3),\ (X_1, X_2)$, respectively, we obtain

$$\alpha^2 - \alpha\lambda = -\tfrac{1}{6},\ \ \alpha^2 - \alpha\beta = -\tfrac{1}{6}, \tag{5.1}$$

$$(\alpha - \beta)g(G(X_1, X_2), U) = 0,\ \ (2\alpha - \lambda - \beta)g(G(X_1, X_2), \xi) = 0. \tag{5.2}$$

From (5.2), $\alpha - \beta \neq 0$ and the preceding results, we see that $g(G(X_1, X_2), \xi) \neq 0$ and $\lambda + \beta = 2\alpha$. On the other hand, from (5.1) we get $2\alpha^2 - \alpha(\lambda + \beta) = -\frac{1}{3}$. But this is a contradiction to $\lambda + \beta = 2\alpha$.

(ii) $(\dim V_\alpha, \dim V_\lambda, \dim V_\beta) = (2, 2, 1)$ **on** M.

In this case, we can define a function $\theta := |g(JX, Y)|$ on M for unit vectors $X \in V_\alpha \cap \{U\}^\perp$ and $Y \in V_\beta$. Since $0 \le \theta \le 1$ and that our concern is only local, in order to prove that Case (ii) does not occur, it is sufficient to show that the following three subcases do not occur on M.

(ii)-(a) $0 < \theta < 1$.

In this subcase, we have the decomposition $JX = W + g(JX, Y)Y$ and $0 \neq W \in V_\lambda$. Then, we can take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1,\ AX_2 = \beta X_2,\ AX_3 = \lambda X_3,\ AX_4 = \lambda X_4,\ X_5 = U,$$

where $X_3 = (JX_1 - \theta X_2)/\sqrt{1-\theta^2}$, $X_4 = (JX_2 + \theta X_1)/\sqrt{1-\theta^2}$ and $\theta = g(JX_1, X_2)$.

It follows that $G(X_1, X_2) \in \mathrm{Span}\{\xi, U\}$ and, by (2.6), $|G(X_1, X_2)|^2 = (1-\theta^2)/3$. Moreover, it is easily seen that with respect to the frame field $\{X_i\}_{i=1}^5$, all relations of (3.7) hold.

Then, taking in (2.19) that $(X, Y) = (X_1, X_4)$ and making use of (3.7), we get

$$0 = (\lambda - \alpha) g(G(X_1, X_2), U).$$

It follows that $g(G(X_1, X_2), U) = 0$ and $G(X_1, X_2) = \pm\sqrt{(1-\theta^2)/3}\,\xi$.

In case $G(X_1, X_2) = -\sqrt{(1-\theta^2)/3}\,\xi$, with respect to the normal vector $\tilde{\xi} = -\xi$, we have $G(X_1, X_2) = \sqrt{(1-\theta^2)/3}\,\tilde{\xi}$, and the principal curvatures become $\tilde{\alpha} = -\alpha$, $\tilde{\lambda} = -\lambda$, $\tilde{\beta} = -\beta$, and $X_1, X_5 \in V_{\tilde{\alpha}}$, $X_2 \in V_{\tilde{\beta}}$, $X_3, X_4 \in V_{\tilde{\lambda}}$. So it is sufficient to show that $G(X_1, X_2) = \sqrt{(1-\theta^2)/3}\,\xi$.

Taking in (2.19), respectively, $(X, Y) = (X_1, X_2), (X_1, X_3), (X_2, X_4), (X_3, X_4)$, and making use of (3.7), we have

$$-\tfrac{\theta}{6} = (\alpha - \beta)\sqrt{\tfrac{1-\theta^2}{3}} + (\alpha^2 - \alpha\beta)\theta, \tag{5.3}$$

$$\sqrt{3}\alpha + \tfrac{\sqrt{3}}{6(\alpha-\lambda)} = \tfrac{\theta}{\sqrt{1-\theta^2}}, \tag{5.4}$$

$$-\tfrac{\sqrt{1-\theta^2}}{6} = -\tfrac{\sqrt{3}}{3}(2\alpha - \lambda - \beta)\theta + (\alpha\lambda + \alpha\beta - 2\lambda\beta)\sqrt{1-\theta^2}, \tag{5.5}$$

$$-\sqrt{3}\lambda - \tfrac{\sqrt{3}}{12(\alpha-\lambda)} = \tfrac{\sqrt{1-\theta^2}}{\theta}. \tag{5.6}$$

From these equations, we can derive a contradiction. Indeed, from (5.4) and (5.6), we have

$$\sqrt{3}(\alpha - \lambda) + \tfrac{\sqrt{3}}{12(\alpha-\lambda)} = \tfrac{1}{\theta\sqrt{1-\theta^2}}. \tag{5.7}$$

It follows that $\alpha - \lambda = \frac{1 \pm \sqrt{1-\theta^2+\theta^4}}{2\theta\sqrt{3(1-\theta^2)}}$. Then, from (5.4), (5.6) and (5.3) we get

$$\alpha = \tfrac{-1+\theta^2 \pm \sqrt{1-\theta^2+\theta^4}}{\theta\sqrt{3(1-\theta^2)}},\quad \lambda = \tfrac{-3+2\theta^2 \pm \sqrt{1-\theta^2+\theta^4}}{2\theta\sqrt{3(1-\theta^2)}},\quad \beta = \tfrac{\pm(2-\theta^2+\theta^4) - 2(1-\theta^2)\sqrt{1-\theta^2+\theta^4}}{2\sqrt{3}\theta\sqrt{(1-\theta^2)(1-\theta^2+\theta^4)}}.$$

Now, substituting α, λ and β into (5.5), we get the contradiction $\frac{\sqrt{1-\theta^2}}{3\sqrt{1-\theta^2+\theta^4}} = 0$.

(ii)-(b) $\theta = 1$.

In this subcase, both $(V_\alpha \cap \{U\}^\perp) \oplus V_\beta$ and V_λ are J-invariant. We take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1,\ AX_2 = \beta X_2,\ AX_3 = \lambda X_3,\ AX_4 = \lambda X_4,\ X_5 = U,$$

where $X_2 = JX_1$ and $X_4 = JX_3$. Then, $G(X_1, X_3) \in \text{Span}\{\xi, U\}$, and by (2.6), we have $|G(X_1, X_3)|^2 = \frac{1}{3}$. Taking in (2.19) $(X, Y) = (X_1, X_3)$ and (X_1, X_4), respectively, we easily get $(\alpha - \lambda)g(G(X_1, X_3), \xi) = (\alpha - \lambda)g(G(X_1, X_4), \xi) = 0$. This together with $G(X_1, X_4) = -JG(X_1, X_3)$ implies that $G(X_1, X_3) = 0$, which is a contradiction.

(ii)-(c) $\theta = 0$.

In this subcase, $J\{(V_\alpha \cap \{U\}^\perp) \oplus V_\beta\} = V_\lambda$. Then, we can take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \alpha X_1,\ AX_2 = \lambda X_2,\ AX_3 = \lambda X_3,\ AX_4 = \beta X_4,\ X_5 = U,$$

where $X_2 = JX_1$ and $X_4 = JX_3$. Then, $G(X_1, X_3) \in \text{Span}\{\xi, U\}$ and $|G(X_1, X_3)|^2 = \frac{1}{3}$. Taking in (2.19) $(X, Y) = (X_1, X_3)$ and (X_1, X_4), respectively, we get

$$(\alpha - \lambda)g(G(X_1, X_3), \xi) = (\alpha - \beta)g(G(X_1, X_4), \xi) = 0.$$

Then similar as the last subcase, we get $G(X_1, X_3) = 0$, which is a contradiction.

(iii) $(\dim V_\alpha, \dim V_\lambda, \dim V_\beta) = (1, 3, 1)$ **on** M.

In this case, we can take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1 = \beta X_1,\ AX_2 = \lambda X_2,\ AX_3 = \lambda X_3,\ AX_4 = \lambda X_4,\ X_5 = U,$$

where $X_2 = JX_1$, $X_4 = JX_3$. Then $G(X_1, X_3) \in \text{Span}\{\xi, U\}$ and $|G(X_1, X_3)|^2 = \frac{1}{3}$.

Taking in (2.19) $(X, Y) = (X_1, X_2)$, (X_1, X_3) and (X_1, X_4), respectively, we have

$$-\tfrac{1}{6} = \alpha\lambda + \alpha\beta - 2\lambda\beta, \tag{5.8}$$

$$(2\alpha - \lambda - \beta)g(G(X_1, X_3), \xi) = (2\alpha - \lambda - \beta)g(G(X_1, X_4), \xi) = 0. \tag{5.9}$$

Then, by (5.9) and the fact $g(G(X_1, X_4), \xi) = g(-JG(X_1, X_3), \xi) = -g(G(X_1, X_3), U)$, we get $2\alpha - \lambda - \beta = 0$. This together with (5.8) gives the contradiction $(\lambda - \beta)^2 = -\frac{1}{3}$.

We have completed the proof of Lemma 5.1. □

Next, we shall determine the principal curvatures and show that they are constants. Since we have the fact $\dim V_\alpha = 1$ and $\dim V_\lambda = \dim V_\beta = 2$, without loss of generality, we shall assume that $\lambda > \beta$. Then, we can state our result as follows:

Lemma 5.2 *All the three distinct principal curvatures α, λ and β are constants. More specifically, we have $\alpha = 0$, $\lambda = \frac{\sqrt{1-\theta^2}+1}{2\sqrt{3}\theta}$ and $\beta = \frac{\sqrt{1-\theta^2}-1}{2\sqrt{3}\theta}$ for some $0 < \theta \le 1$.*

Proof It is easily seen that $|g(JX, Y)|$, for an orthonormal basis $\{X, Y\}$ of V_λ, defines a well-defined function θ on M satisfying $0 \le \theta \le 1$. Since our concern is only local, in order to prove Lemma 5.2, by using the continuity of the principal curvature functions and θ, we are sufficient to consider the following three cases:

(1) $0 < \theta < 1$ **on** M.

In this case, we see that $JV_\lambda \neq V_\beta$ and V_λ is not J-invariant. Then, we can take a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that $\theta = g(JX_1, X_2)$ and

$$AX_1 = \lambda X_1,\ AX_2 = \lambda X_2,\ AX_3 = \beta X_3,\ AX_4 = \beta X_4,\ AX_5 = \alpha X_5, \tag{5.10}$$

where $X_5 = U,\ X_3 = \frac{JX_1 - \theta X_2}{\sqrt{1-\theta^2}},\ X_4 = \frac{JX_2 + \theta X_1}{\sqrt{1-\theta^2}}$. Thus, $G(X_1, X_2) \in \text{Span}\{\xi, U\}$ and, by (2.6), $|G(X_1, X_2)|^2 = \frac{1}{3}(1 - \theta^2)$. Moreover, it is easily seen that with respect to the frame field $\{X_i\}_{i=1}^5$, all relations of (3.7) hold.

Taking, in (2.19), $(X, Y) = (X_3, X_4)$ and $(X, Y) = (X_1, X_i)$ for $2 \le i \le 4$, respectively, and making use of (3.7), we have

$$-\tfrac{\theta}{6} = 2(\alpha-\lambda)g(G(X_1, X_2), \xi) + 2\lambda(\alpha-\lambda)\theta, \tag{5.11}$$
$$-\tfrac{1}{6}\sqrt{1-\theta^2} = -\tfrac{\theta(2\alpha-\lambda-\beta)}{\sqrt{1-\theta^2}}g(G(X_1, X_2), \xi) + (\alpha\lambda+\alpha\beta-2\lambda\beta)\sqrt{1-\theta^2}, \tag{5.12}$$
$$0 = (2\alpha-\lambda-\beta)g(G(X_1, X_2), U), \tag{5.13}$$
$$\tfrac{\theta}{6} = -2(\alpha-\beta)g(G(X_1, X_2), \xi) + 2\beta(\beta-\alpha)\theta. \tag{5.14}$$

If $2\alpha-\lambda-\beta = 0$, then together with (5.12) we derive a contradiction $(\lambda-\beta)^2 = -\frac{1}{3}$.

Hence, $2\alpha-\lambda-\beta \ne 0$. Then from (5.13), we get $g(G(X_1, X_2), U) = 0$, and therefore, we obtain $G(X_1, X_2) = \pm\sqrt{(1-\theta^2)/3}\,\xi$. Without loss of generality, we shall assume that $G(X_1, X_2) = -\sqrt{(1-\theta^2)/3}\,\xi$.

Actually, if it occurs $G(X_1, X_2) = \sqrt{(1-\theta^2)/3}\,\xi$, then $G(X_3, X_4) = -\sqrt{(1-\theta^2)/3}\,\xi$ and $g(JX_3, X_4) = -\theta < 0$. Now, with respect to the normal vector field $\tilde{\xi} = -\xi$, the principal curvatures become $\tilde{\alpha} = -\alpha$, $\tilde{\lambda} = -\beta$ and $\tilde{\beta} = -\lambda$, $\tilde{\lambda} > \tilde{\beta}$. Putting $\tilde{X}_1 = X_3$, $\tilde{X}_2 = -X_4$, $\tilde{X}_3 = \frac{J\tilde{X}_1-\theta\tilde{X}_2}{\sqrt{1-\theta^2}}$, $\tilde{X}_4 = \frac{J\tilde{X}_2+\theta\tilde{X}_1}{\sqrt{1-\theta^2}}$ and $\tilde{X}_5 = U$, then, with respect to the orthonormal frame field $\{\tilde{X}_i\}_{i=1}^5$, as assumed we have $G(\tilde{X}_1, \tilde{X}_2) = -\sqrt{(1-\theta^2)/3}\,\tilde{\xi}$ and $g(J\tilde{X}_1, \tilde{X}_2) = \theta > 0$.

Having the assumption $G(X_1, X_2) = -\sqrt{(1-\theta^2)/3}\,\xi$, Eqs. (5.11), (5.12) and (5.14) become

$$\theta = 4\sqrt{3}(\alpha-\lambda)\sqrt{1-\theta^2} + 12\lambda(\lambda-\alpha)\theta, \tag{5.15}$$
$$-\sqrt{1-\theta^2} = 2\sqrt{3}\theta(2\alpha-\lambda-\beta) + 6(\alpha\lambda+\alpha\beta-2\lambda\beta)\sqrt{1-\theta^2}, \tag{5.16}$$
$$\theta = 4\sqrt{3}(\alpha-\beta)\sqrt{1-\theta^2} + 12\beta(\beta-\alpha)\theta. \tag{5.17}$$

Then, solving λ and β from (5.15) and (5.17), we obtain

$$\lambda+\beta = \tfrac{3\alpha\theta+\sqrt{3(1-\theta^2)}}{3\theta},\quad \lambda\beta = \tfrac{4\alpha\sqrt{3(1-\theta^2)}-\theta}{12\theta}.$$

This combining with (5.16) gives $\alpha(\alpha\sqrt{1-\theta^2}+\frac{2\theta^2-1}{\sqrt{3}\theta}) = 0$. Hence, $\alpha = 0$ or $\alpha = \frac{1-2\theta^2}{\theta\sqrt{3-3\theta^2}}$.

In conclusion, we can solve the above equations to obtain two possibilities:

Case (1)-(i): $\alpha = 0,\quad \lambda = \frac{\sqrt{1-\theta^2}+1}{2\sqrt{3}\theta},\quad \beta = \frac{\sqrt{1-\theta^2}-1}{2\sqrt{3}\theta}$;

Case (1)-(ii): $\alpha = \frac{1-2\theta^2}{\theta\sqrt{3(1-\theta^2)}},\quad \lambda = \frac{2-3\theta^2+\theta}{2\theta\sqrt{3(1-\theta^2)}},\quad \beta = \frac{2-3\theta^2-\theta}{2\theta\sqrt{3(1-\theta^2)}}$.

Before dealing with these two subcases in more details, we need some preparations.

Put $\nabla_{X_i}X_j = \sum\Gamma_{ij}^k X_k$ with $\Gamma_{ij}^k = -\Gamma_{ik}^j$, $1 \le i, j, k \le 5$. First of all, we have

$$G(X_1, \xi) = -\sum_{i=1}^{5}\Gamma_{15}^i X_i + \lambda JX_1.$$

On the other hand, the facts $g(G(X_1, X_2), \xi) = -\sqrt{(1-\theta^2)/3}$ and $g(G(X_1, X_2), U) = 0$ imply that $G(X_1, \xi) = \sqrt{\frac{1-\theta^2}{3}}X_2 - \frac{\sqrt{3}}{3}\theta X_3$. Hence, we obtain

$$\Gamma_{15}^1 = 0,\ \ \Gamma_{15}^2 = \lambda\theta - \sqrt{\tfrac{1-\theta^2}{3}},\ \ \Gamma_{15}^3 = \lambda\sqrt{1-\theta^2} + \tfrac{\sqrt{3}}{3}\theta,\ \ \Gamma_{15}^4 = 0. \tag{5.18}$$

Similarly, calculating $G(X_i, \xi)$ for $2 \le i \le 4$, we can further obtain

$$\begin{cases} \Gamma_{25}^1 = -\lambda\theta + \sqrt{\frac{1-\theta^2}{3}}, \ \ \Gamma_{25}^2 = 0, \ \ \Gamma_{25}^3 = 0, \ \ \Gamma_{25}^4 = \lambda\sqrt{1-\theta^2} + \frac{\sqrt{3}}{3}\theta, \\ \Gamma_{35}^1 = -\beta\sqrt{1-\theta^2} - \frac{\sqrt{3}}{3}\theta, \ \ \Gamma_{35}^2 = 0, \ \ \Gamma_{35}^3 = 0, \ \ \Gamma_{35}^4 = -\beta\theta + \sqrt{\frac{1-\theta^2}{3}}, \\ \Gamma_{45}^1 = 0, \ \ \Gamma_{45}^2 = -\beta\sqrt{1-\theta^2} - \frac{\sqrt{3}}{3}\theta, \ \ \Gamma_{45}^3 = \beta\theta - \sqrt{\frac{1-\theta^2}{3}}, \ \ \Gamma_{45}^4 = 0. \end{cases} \tag{5.19}$$

Now, we calculate $g((\nabla_{X_i} A)X_j - (\nabla_{X_j} A)X_i, X_k)$ for each $1 \le i, j, k \le 4$.

First, by using (2.18) we easily see that $g((\nabla_{X_i} A)X_j - (\nabla_{X_j} A)X_i, X_k) = 0$.

On the other hand, by using (5.10) we can calculate $0 = g((\nabla_{X_i} A)X_j - (\nabla_{X_j} A)X_i, X_k)$ to conclude that $X_1\lambda = X_2\lambda = X_3\beta = X_4\beta = 0$ that is $X_i\theta = 0$ for $1 \le i \le 4$, and $\Gamma_{ij}^k = \Gamma_{ik}^j = 0$ for $i \in \{1, 2, 3, 4\}$, $j \in \{1, 2\}$ and $k \in \{3, 4\}$.

Next, by definition, the above information of $\{\Gamma_{ij}^k\}$ and (3.7), we can get

$$0 = g(G(X_1, X_2), X_3) = g((\tilde{\nabla}_{X_1} J)X_2, X_3) = \sqrt{1-\theta^2}\,(\Gamma_{14}^3 - \Gamma_{12}^1).$$

It follows that $\Gamma_{14}^3 = \Gamma_{12}^1$. Similarly, by calculating $0 = g(G(X_i, X_1), X_4)$ for $2 \le i \le 4$, we further get $\Gamma_{23}^4 = \Gamma_{21}^2$, $\Gamma_{33}^4 = \Gamma_{31}^2$ and $\Gamma_{43}^4 = \Gamma_{41}^2$.

Moreover, by using (3.7) we have $g(G(U, X_1), X_4) = -\frac{\sqrt{3}}{3}$, then direct calculation of its left hand side gives

$$(\Gamma_{53}^4 - \Gamma_{51}^2)\sqrt{1-\theta^2} + (\Gamma_{52}^4 + \Gamma_{51}^3)\theta = -\frac{\sqrt{3}}{3}. \tag{5.20}$$

Finally, from now on we assume that $PX_i = \sum_{j=1}^4 p_{ij}X_j$ for $1 \le i \le 4$, where $p_{ij} = p_{ji}$ and, by the definition of X_3 and X_4, we have the following relations:

$$\begin{cases} p_{23} = p_{14} - \frac{(p_{11}+p_{22})\theta}{\sqrt{1-\theta^2}}, \ \ p_{33} = \frac{\theta^2 p_{22} - p_{11} + 2\theta^2 p_{11}}{1-\theta^2} - \frac{2\theta p_{14}}{\sqrt{1-\theta^2}}, \\ p_{34} = \frac{(p_{13}-p_{24})\theta}{\sqrt{1-\theta^2}} - p_{12}, \ \ p_{44} = \frac{2\theta p_{14}}{\sqrt{1-\theta^2}} - \frac{p_{22}+\theta^2 p_{11}}{1-\theta^2}. \end{cases} \tag{5.21}$$

Now, we come to discuss Case (1)-(i) and show that in this subcase θ is constant.

For that purpose, we apply for the Codazzi equation (2.18) with $(X, Y) = (U, X_i)$ for $1 \le i \le 4$, and then checking the results we obtain the following equations:

$$3U\lambda - p_{11}b - a(p_{12}\theta + p_{13}\sqrt{1-\theta^2}\,) = 0, \tag{5.22}$$

$$ap_{11}\theta - p_{12}b - ap_{14}\sqrt{1-\theta^2} = 0, \tag{5.23}$$

$$2ap_{14}\theta - 1 - 2p_{13}b + \frac{2\sqrt{3}}{\theta}\Gamma_{51}^3 + 2ap_{11}\sqrt{1-\theta^2} = 0, \tag{5.24}$$

$$\frac{\sqrt{3}}{\theta}\Gamma_{51}^4 - p_{14}b - ap_{13}\theta + ap_{12}\sqrt{1-\theta^2} = 0, \tag{5.25}$$

$$3U\lambda - p_{22}b + ap_{12}\theta - ap_{24}\sqrt{1-\theta^2} = 0, \tag{5.26}$$

$$\begin{aligned} &\Gamma_{52}^3\sqrt{3(1-\theta^2)} + b\theta\big[(p_{11} + p_{22})\theta - p_{14}\sqrt{1-\theta^2}\,\big] \\ &\quad + a\theta(p_{12} - p_{12}\theta^2 + p_{24}\theta\sqrt{1-\theta^2}\,) = 0, \end{aligned} \tag{5.27}$$

$$\begin{aligned} &2\sqrt{3}\Gamma_{52}^4(\theta^2 - 1) - \theta\big\{\theta^2 - 1 + 2p_{24}b(\theta^2 - 1) \\ &\quad + 2a\big[p_{14}\theta(\theta^2 - 1) + \sqrt{1-\theta^2}(p_{22} + p_{11}\theta^2)\big]\big\} = 0, \end{aligned} \tag{5.28}$$

$$\begin{aligned} &2p_{14}b\theta(\theta^2 - 1) + \sqrt{1-\theta^2}\big[(2p_{11}b + p_{22}b + 3U\beta)\theta^2 - p_{11}b - 3U\beta\big] \\ &\quad + a(\theta^2 - 1)\big[p_{13} - \theta(p_{24}\theta + p_{12}\sqrt{1-\theta^2}\,)\big] = 0, \end{aligned} \tag{5.29}$$

$$\begin{aligned}&b(\theta^2-1)\big[(p_{24}-p_{13})\theta+p_{12}\sqrt{1-\theta^2}\,\big]\\&\quad+a\big[\theta\sqrt{1-\theta^2}(p_{22}+p_{11}\theta^2)+p_{14}(\theta^4-1)\big]=0,\end{aligned}\tag{5.30}$$

$$\begin{aligned}&2p_{14}b\theta(\theta^2-1)+\sqrt{1-\theta^2}\big[p_{22}b+3U\beta+(p_{11}b-3U\beta)\theta^2\big]\\&\quad-a(\theta^2-1)\big[p_{24}+\theta(p_{12}\sqrt{1-\theta^2}-p_{13}\theta\,)\big]=0.\end{aligned}\tag{5.31}$$

Calculating (5.22)–(5.26) and (5.29)+(5.31), respectively, we obtain

$$0=(p_{22}-p_{11})b+a\big[-2p_{12}\theta+(p_{24}-p_{13})\sqrt{1-\theta^2}\,\big],\tag{5.32}$$

$$\begin{aligned}0=\;&a(1-\theta^2)\big[(p_{24}-p_{13})(1+\theta^2)+2p_{12}\theta\sqrt{1-\theta^2}\,\big]\\&+b\big\{4p_{14}\theta(\theta^2-1)+\sqrt{1-\theta^2}\big[p_{22}-p_{11}+(3p_{11}+p_{22})\theta^2\big]\big\}.\end{aligned}\tag{5.33}$$

Now, we claim that $a\neq 0$ holds on M.

Indeed, if otherwise, we assume $a(z)=0$ for some $z\in M$. Then, carrying calculations below at z, we have $b=\pm 1$ and, by (5.32), (5.33), (5.23) and (5.30), we have

$$p_{22}-p_{11}=p_{12}=p_{24}-p_{13}=0,\quad p_{14}=\tfrac{p_{11}\theta}{\sqrt{1-\theta^2}}.\tag{5.34}$$

From (5.22) and (5.31), we obtain $U\lambda=-U\beta=\frac{1}{3}p_{11}b$ and thus $U(\lambda+\beta)=0$. Then, as $\lambda+\beta=\frac{\sqrt{1-\theta^2}}{\sqrt{3}\theta}$ and $0<\theta<1$, we get $U\theta=0$ and thus $U\lambda=U\beta=p_{11}=0$. From (5.34), we have $p_{11}=p_{12}=p_{22}=p_{14}=0$.

Finally, we apply for $0=g(G(PX_1,PX_2)+PG(X_1,X_2),U)$. By direct calculation of the right hand side, making use of the fact $G(X_1,X_2)=-\sqrt{\frac{1-\theta^2}{3}}\,\xi$, (3.7) and (5.21), we get the contradiction $\sqrt{1-\theta^2}b=0$, which verifies the claim.

As $a\neq 0$, from (5.23) we solve $p_{14}=\frac{ap_{11}\theta-p_{12}b}{a\sqrt{1-\theta^2}}$. Then, from (5.32), (5.33) and (5.30), we obtain a matrix equation $AB=0$, where

$$A=(p_{22}-p_{11},\;p_{12},\;p_{24}-p_{13}),$$
$$B=\begin{pmatrix} b & b(1+\theta^2) & -a\\ -2a\theta & \frac{4b^2\theta+2a^2\theta(1-\theta^2)}{a} & -2b\theta\\ a\sqrt{1-\theta^2} & a\sqrt{1-\theta^2}(1+\theta^2) & b\sqrt{1-\theta^2}\end{pmatrix}.$$

The fact $\det B=\frac{4\theta\sqrt{1-\theta^2}}{a}\neq 0$ implies that $p_{22}-p_{11}=p_{12}=p_{24}-p_{13}=0$. By (5.22) and (5.31), we have $U\lambda=-U\beta=\frac{1}{3}(p_{11}b+ap_{13}\sqrt{1-\theta^2}\,)$. The fact $0<\theta<1$ and $\lambda+\beta=\frac{\sqrt{1-\theta^2}}{\sqrt{3}\theta}$ then implies that $U\theta=0$. This combining with $X_i\lambda=X_i\beta=0$ for $1\leq i\leq 4$ shows that θ and so that λ and β are constants on M.

Moreover, from (5.22) up to (5.31), we can finally obtain:

$$p_{13}=-\tfrac{p_{11}b}{a\sqrt{1-\theta^2}},\;\;p_{14}=\tfrac{p_{11}\theta}{\sqrt{1-\theta^2}},\;\;\Gamma_{51}^3=\Gamma_{52}^4=\tfrac{\theta(-2p_{11}+a\sqrt{1-\theta^2})}{2a\sqrt{3-3\theta^2}},\;\;\Gamma_{51}^4=\Gamma_{52}^3=0.\tag{5.35}$$

Then, by $\sum_{i=1}^4(p_{1i})^2=1$, we get $(p_{11})^2=a^2(1-\theta^2)$.

Now, calculating the curvature tensor, we obtain

$$g(R(X_1,X_3)X_3,X_1)=\Gamma_{31}^5\Gamma_{53}^1-\Gamma_{13}^5\Gamma_{35}^1-\Gamma_{13}^5\Gamma_{53}^1=\tfrac{4p_{11}(1+\theta^2)-a\sqrt{1-\theta^2}(5+3\theta^2)}{12a\sqrt{1-\theta^2}}.\tag{5.36}$$

On the other hand, by Gauss equation (2.17) and the fact $a^2+b^2=1$, we have

$$g(R(X_1,X_3)X_3,X_1)=\tfrac{a^2(10\theta^2-7-3\theta^4)-4(p_{11})^2(\theta^2-2)}{12a^2(\theta^2-1)}.\tag{5.37}$$

Comparing these two calculations, we get

$$(p_{11})^2(2-\theta^2)+3a^2(\theta^2-1)+ap_{11}\sqrt{1-\theta^2}(1+\theta^2)=0.$$

Then, by using $(p_{11})^2 = a^2(1-\theta^2)$, we finally get $p_{11} = a\sqrt{1-\theta^2}$. It follows that, by (5.20), (5.35) and the previous results about p_{ij}, we have

$$\begin{cases} p_{11}=p_{22}=-p_{33}=-p_{44}=a\sqrt{1-\theta^2}, \quad p_{12}=p_{34}=0, \\ p_{13}=p_{24}=-b, \ p_{14}=-p_{23}=a\theta, \\ \Gamma_{51}^3=\Gamma_{52}^4=-\frac{\theta}{2\sqrt{3}}, \quad \Gamma_{53}^4=\Gamma_{51}^2-\sqrt{\frac{1-\theta^2}{3}}. \end{cases} \tag{5.38}$$

Later, in Lemma 5.3, we will show that Case (1)-(ii) occurs only if $\theta = \frac{\sqrt{2}}{2}$. But this implies that Case (1)-(ii) is actually a special situation of Case (1)-(i) with $\theta = \frac{\sqrt{2}}{2}$.

(2) $\theta = 1$ **on** M.

In this case, it is easy to see that M satisfies $A\phi = \phi A$. According to Proposition 5.7 of [16], the principal curvatures of M are $\alpha = 0$, $\lambda = \frac{\sqrt{3}}{6}$ and $\beta = -\frac{\sqrt{3}}{6}$. This exactly shows that expressions of the principal curvatures stated in Case (1)-(i) are valid also for $\theta = 1$.

(3) $\theta = 0$ **on** M.

In this case, we choose a local orthonormal frame field $\{X_i\}_{i=1}^5$ of M such that

$$AX_1=\lambda X_1,\ AX_2=\beta X_2,\ AX_3=\lambda X_3,\ AX_4=\beta X_4,\ X_5=U,$$

where $X_2 = JX_1$ and $X_4 = JX_3$. Then $G(X_1, X_3) \in \mathrm{Span}\{\xi, U\}$ and $|G(X_1, X_3)|^2 = \frac{1}{3}$. Now, taking in (2.19) $(X, Y) = (X_1, X_2)$, (X_1, X_3) and (X_1, X_4), respectively, we obtain

$$\alpha\beta+\alpha\lambda-2\lambda\beta=-\tfrac{1}{6}, \tag{5.39}$$

$$(\alpha-\lambda)g(G(X_1,X_3),\xi)=0, \quad (2\alpha-\lambda-\beta)g(G(X_1,X_3),U)=0. \tag{5.40}$$

From (5.40), $\alpha \neq \lambda$ and $|G(X_1, X_3)|^2 = \frac{1}{3}$, we get $2\alpha-\lambda-\beta = 0$. This combining with (5.39) gives the contradiction $(\lambda-\beta)^2 = -\frac{1}{3}$.

We have completed the proof of Lemma 5.2. □

Lemma 5.3 *If Case (1)-(ii) in the proof of Lemma 5.2 does occur, then* $\theta = \frac{\sqrt{2}}{2}$.

Proof First of all, according to Lemma 2.2, α is constant. Hence, by the formulas for Case (1)-(ii) of the proof of Lemma 5.2, also θ, λ and β are constants. Now, since the local orthonormal frame field $\{X_i\}_{i=1}^5$ of M satisfy (5.10), we apply for the Codazzi equation (2.18) with $(X, Y) = (U, X_i)$ for $1 \le i \le 4$. Then, by checking the results, as in Case (1)-(i) we obtain Eqs. (5.22), (5.23), (5.26) and (5.29)–(5.31) with $U\lambda = U\beta = 0$. Moreover, we have the following additional four equations:

$$\theta\big\{2\sqrt{3}\Gamma_{51}^3+\theta-2p_{13}b\sqrt{1-\theta^2}+2a\big[p_{11}(1-\theta^2)+p_{14}\theta\sqrt{1-\theta^2}\,\big]\big\}-1=0, \tag{5.41}$$

$$\sqrt{3}\Gamma_{51}^4-p_{14}b\sqrt{1-\theta^2}+a\big[p_{12}(1-\theta^2)-p_{13}\theta\sqrt{1-\theta^2}\,\big]=0, \tag{5.42}$$

$$\sqrt{3}\Gamma_{52}^3+(p_{11}+p_{22})b\theta-p_{14}b\sqrt{1-\theta^2}+a\big[p_{12}(1-\theta^2)+p_{24}\theta\sqrt{1-\theta^2}\,\big]=0, \tag{5.43}$$

$$\theta\big\{2\sqrt{3}\Gamma_{52}^4+\theta-2p_{24}b\sqrt{1-\theta^2}+2a\big[p_{22}+\theta(p_{11}\theta-p_{14}\sqrt{1-\theta^2}\,)\big]\big\}-1=0. \tag{5.44}$$

It follows that (5.32) and (5.33) are still valid. Then, similar discussions as in dealing with Case (1)-(i), we have

$$a \neq 0,\ (p_{11})^2 = a^2(1-\theta^2),\ p_{22} = p_{11},\ p_{12} = 0,\ p_{13} = p_{24} = -\frac{p_{11}b}{a\sqrt{1-\theta^2}},\ p_{14} = \frac{p_{11}\theta}{\sqrt{1-\theta^2}}.$$

Moreover, by using Eqs. (5.41)–(5.44), we can get

$$\Gamma_{51}^3 = \Gamma_{52}^4 = \frac{a-2p_{11}\theta-a\theta^2}{2\sqrt{3}a\theta},\quad \Gamma_{51}^4 = \Gamma_{52}^3 = 0.$$

Now, calculating the curvature tensor, we obtain

$$g(R(X_1,X_3)X_4,X_2) = \Gamma_{34}^5\Gamma_{15}^2 - \Gamma_{13}^5\Gamma_{54}^2 + \Gamma_{31}^5\Gamma_{54}^2 = \frac{a(6\theta^2-4-3\theta^4)-4p_{11}\theta(\theta^2-2)}{12a\theta^2},$$
$$g(R(X_1,X_3)X_3,X_1) = \Gamma_{31}^5\Gamma_{53}^1 - \Gamma_{13}^5\Gamma_{35}^1 - \Gamma_{13}^5\Gamma_{53}^1 = \frac{a(11\theta^2-8-3\theta^4)-4p_{11}\theta(\theta^2-2)}{12a\theta^2}.$$

On the other hand, by the Gauss equation (2.17) and the fact $a^2+b^2=1$, we have

$$g(R(X_1,X_3)X_4,X_2) = \frac{4(p_{11})^2\theta^2+a^2(2-3\theta^2)(1-\theta^2)}{12a^2(1-\theta^2)},$$
$$g(R(X_1,X_3)X_3,X_1) = \frac{4(p_{11})^2\theta^2(2-\theta^2)-a^2(1-\theta^2)^2(4+3\theta^2)}{12a^2\theta^2(\theta^2-1)}.$$

Comparing these two calculations, respectively, we can obtain

$$(p_{11})^2\theta^4 - ap_{11}\theta(2-\theta^2)(1-\theta^2) + a^2(1-\theta^2)^2 = 0, \tag{5.45}$$
$$(p_{11})^2\theta^2(\theta^2-2) - ap_{11}\theta(2-\theta^2)(1-\theta^2) + 3a^2(1-\theta^2)^2 = 0. \tag{5.46}$$

Now calculation (5.45)–(5.46) gives that

$$(p_{11})^2\theta^2 = a^2(1-\theta^2)^2,$$

and, by using the fact $(p_{11})^2 = a^2(1-\theta^2)$, we obtain $\theta = \frac{\sqrt{2}}{2}$.

This completes the proof of Lemma 5.3. □

Based on Lemma 5.2, we can prove the following result for Hopf hypersurfaces which is an interesting counterpart of Proposition 5.8 in [16].

Proposition 5.1 *Let M be a Hopf hypersurface of the NK $\mathbf{S}^3\times\mathbf{S}^3$ with three distinct principal curvatures and assume that the almost product structure P of M preserves the holomorphic distribution, i.e., $P\{U\}^\perp = \{U\}^\perp$. Then either $P\xi = \frac{1}{2}\xi + \frac{\sqrt{3}}{2}J\xi$, or $P\xi = \frac{1}{2}\xi - \frac{\sqrt{3}}{2}J\xi$, or $P\xi = -\xi$.*

Proof We first assume that $0<\theta<1$. Let $\{X_i\}_{i=1}^5$ be as described by (5.10). Then, by using (3.7), (5.38) and the fact $G(X_1,X_2) = -\sqrt{(1-\theta^2)/3}\,\xi$, we can show that the equation $0 = g(G(PX_1,PX_2)+PG(X_1,X_2),\xi)$ becomes equivalently

$$(1-2a)(1+a) = 0.$$

This implies the assertion that we have three possibilities for $P\xi$, namely,

(1) $a=\frac{1}{2}$ and $b=-\frac{\sqrt{3}}{2}$, (2) $a=\frac{1}{2}$ and $b=\frac{\sqrt{3}}{2}$, (3) $a=-1$ and $b=0$.

Next, if $\theta=1$, then as stated before the hypersurface satisfies $A\phi=\phi A$ and the assertion follows from Proposition 5.8 of [16]. □

For the sake of later's purpose, we summarize the following conclusion that we have established.

Lemma 5.4 *For $0 < \theta < 1$ with $\alpha = 0$, $\lambda = \frac{\sqrt{1-\theta^2}+1}{2\sqrt{3}\theta}$ and $\beta = \frac{\sqrt{1-\theta^2}-1}{2\sqrt{3}\theta}$, the vector $P\xi$ has three possibilities: $\frac{1}{2}\xi + \frac{\sqrt{3}}{2}J\xi$, $\frac{1}{2}\xi - \frac{\sqrt{3}}{2}J\xi$, $-\xi$. For each of these cases, we have a local orthonormal frame $\{X_i\}_{i=1}^5$, which is described by (5.10), such that $PX_i = \sum_{j=1}^4 p_{ij}X_j$ for $1 \le i \le 4$, and $\{p_{ij}\}$ satisfy (5.38). Moreover, with respect to $\{X_i\}_{i=1}^5$, the connection coefficients $\{\Gamma_{ij}^k\}$ satisfy (5.18), (5.19), (5.38), as well as the following relations:*

$$\begin{cases} \Gamma_{ij}^k = 0, \ \text{if } i \in \{1,2,3,4\}, \ j \in \{1,2\}, \ k \in \{3,4\}; \\ \Gamma_{14}^3 = \Gamma_{12}^1, \ \Gamma_{23}^4 = \Gamma_{21}^2, \ \Gamma_{33}^4 = \Gamma_{31}^2, \ \Gamma_{43}^4 = \Gamma_{41}^2, \ \Gamma_{51}^4 = \Gamma_{52}^3 = 0. \end{cases} \tag{5.47}$$

5.2 Proof of Theorem 1.2

We get the proof of Theorem 1.2 as a direct consequence of three results concerning the three possibilities for $P\xi$ described in Proposition 5.1. First of all, we prove the following result:

Theorem 5.1 *Let M be a Hopf hypersurface of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ which possesses three distinct principal curvatures and satisfies $P\{U\}^\perp = \{U\}^\perp$ on M. If $P\xi = \frac{1}{2}\xi + \frac{\sqrt{3}}{2}J\xi$, then, up to isometries of type $\mathcal{F}_{abc}$, M is locally given by the embedding f_r $(0 < r \le 1)$ in Theorem 1.2.*

Proof We first assume that $0 < \theta < 1$ and let $\{X_i\}_{i=1}^5$ be as described by (5.10). Put

$$\begin{cases} \bar{e}_1 = \frac{\sqrt{2+\sqrt{1-\theta^2}}}{2}X_1 - \frac{\sqrt{3}}{2\sqrt{2+\sqrt{1-\theta^2}}}X_3 + \frac{\theta}{2\sqrt{2+\sqrt{1-\theta^2}}}X_4, \ \bar{e}_5 = X_5 = U, \\ \bar{e}_2 = \frac{\sqrt{2+\sqrt{1-\theta^2}}}{2}X_2 - \frac{\theta}{2\sqrt{2+\sqrt{1-\theta^2}}}X_3 - \frac{\sqrt{3}}{2\sqrt{2+\sqrt{1-\theta^2}}}X_4, \\ \bar{e}_3 = \frac{\theta}{\sqrt{2+2\sqrt{1-\theta^2}}}X_2 + \frac{\sqrt{1+\sqrt{1-\theta^2}}}{\sqrt{2}}X_3, \ \ \bar{e}_4 = \frac{\theta}{\sqrt{2+2\sqrt{1-\theta^2}}}X_1 - \frac{\sqrt{1+\sqrt{1-\theta^2}}}{\sqrt{2}}X_4. \end{cases} \tag{5.48}$$

Then, $\{\bar{e}_i\}_{i=1}^5$ is a local (non-orthonormal) frame field of M. We consider the following decomposition of the tangent bundle of M: $TM = \mathrm{Span}\{\bar{e}_1, \bar{e}_2\} \oplus \mathrm{Span}\{\bar{e}_3, \bar{e}_4, \bar{e}_5\}$.

Using Lemma 5.4, we have

$$\nabla_{\bar{e}_i}\bar{e}_j \in \mathrm{Span}\{\bar{e}_1, \bar{e}_2, \bar{e}_5\} \text{ for } i,j = 1,2; \quad \nabla_{\bar{e}_i}\bar{e}_j \in \mathrm{Span}\{\bar{e}_3, \bar{e}_4, \bar{e}_5\} \text{ for } i,j = 3,4,5.$$

Moreover, by direct calculation, we can show that

$$[\bar{e}_i, \bar{e}_j] \in \mathrm{Span}\{\bar{e}_1, \bar{e}_2\} \text{ for } i,j = 1,2; \quad [\bar{e}_i, \bar{e}_j] \in \mathrm{Span}\{\bar{e}_3, \bar{e}_4, \bar{e}_5\} \text{ for } i,j = 3,4,5.$$

It follows that both $\mathrm{Span}\{\bar{e}_1, \bar{e}_2\}$ and $\mathrm{Span}\{\bar{e}_3, \bar{e}_4, \bar{e}_5\}$ are integrable distributions. Let M_1 and M_2 be the integral manifolds of $\mathrm{Span}\{\bar{e}_3, \bar{e}_4, \bar{e}_5\}$ and $\mathrm{Span}\{\bar{e}_1, \bar{e}_2\}$, respectively. Note also that now we have

$$g(A\bar{e}_i, \bar{e}_j) = 0 \text{ for } i,j = 3,4,5; \quad g(A\bar{e}_i, \bar{e}_j) = \frac{\sqrt{3(1-\theta^2)}}{4\theta}\delta_{ij} \text{ for } i,j = 1,2.$$

So we have $\tilde{\nabla}_{\bar{e}_i}\bar{e}_j \in TM_1$ for $i,j = 3,4,5$; and $\tilde{\nabla}_{\bar{e}_i}\bar{e}_j = \hat{\nabla}_{\bar{e}_i}\bar{e}_j + \hat{h}(\bar{e}_i, \bar{e}_j)$ for $i,j = 1,2$, where $\hat{\nabla}$ is the Levi-Civita connection of M_2, and $\hat{h}$ is the second fundamental form of the submanifold $M_2 \hookrightarrow \mathbf{S}^3 \times \mathbf{S}^3$. Moreover, by direct calculations we can show that $\hat{h}(\bar{e}_i, \bar{e}_j) = (\frac{\sqrt{1-\theta^2}}{4\theta}U + \frac{\sqrt{3(1-\theta^2)}}{4\theta}\xi)\delta_{ij}, i,j = 1,2$. Hence, M_1 is a totally geodesic submanifold of $\mathbf{S}^3 \times \mathbf{S}^3$, whereas M_2 is a totally umbilical submanifold of $\mathbf{S}^3 \times \mathbf{S}^3$.

Applying for (2.12), we further see that M_1 and M_2 have constant sectional curvature $\frac{3}{4}$ and $\frac{1+2\theta^2}{4\theta^2}$, respectively. Thus, M_1 (resp. M_2) is locally isometric to $\mathbf{S}^3$ (resp. $\mathbf{S}^2$) equipped with metric $\frac{4}{3}g_0$ (resp. $\frac{4\theta^2}{1+2\theta^2}g_0$), where g_0 denotes the standard metric of constant sectional curvature 1 on $\mathbf{S}^3$ (resp. $\mathbf{S}^2$). In particular, M is locally diffeomorphic to the product manifold $\mathbf{S}^3 \times \mathbf{S}^2$.

By the identification of M with an open subset of $\mathbf{S}^3 \times \mathbf{S}^2$, we can express the hypersurface M by an immersion $f = (p, q)$ with the parametrization (x, y) of $\mathbf{S}^3 \times \mathbf{S}^2$ such that

$$f : \mathbf{S}^3 \times \mathbf{S}^2 \longrightarrow \mathbf{S}^3 \times \mathbf{S}^3, \quad (x, y) \mapsto (p(x, y), q(x, y)).$$

From (2.10), $P\xi = \frac{1}{2}\xi - \frac{\sqrt{3}}{2}U$, (3.7), (5.38) and (5.48), it can be verified that

$$Q\bar{e}_1 = \bar{e}_1, \quad Q\bar{e}_2 = \bar{e}_2, \quad Q\bar{e}_3 = -\bar{e}_3, \quad Qe_4 = -\bar{e}_4, \quad QU = -U.$$

Then, by the definition of Q, it follows that $dp, dq : T(\mathbf{S}^3 \times \mathbf{S}^2) \to T\mathbf{S}^3$ have the following properties:

$$\begin{cases} (dp(v), 0) = \frac{1}{2}(df(v) - Qdf(v)) = df(v), \\ (0, dq(v)) = \frac{1}{2}(df(v) + Qdf(v)) = 0, \end{cases} \quad \forall\, v \in T(\mathbf{S}^3 \times \{pt\}). \tag{5.49}$$

$$\begin{cases} (dp(w), 0) = \frac{1}{2}(df(w) - Qdf(w)) = 0, \\ (0, dq(w)) = \frac{1}{2}(df(w) + Qdf(w)) = df(w), \end{cases} \quad \forall\, w \in T(\{pt\} \times \mathbf{S}^2). \tag{5.50}$$

The first equation of (5.50) shows that p depends only on the first entry x, and hence, it can be regarded as a mapping from $\mathbf{S}^3$ to $\mathbf{S}^3$. From (5.49), we see that $p : \mathbf{S}^3 \to \mathbf{S}^3$ is a local diffeomorphism. Noting that the pull-back metric f^*g restricted on $\mathbf{S}^3 \times \{pt\}$ is exactly $\frac{4}{3}g_0$, p is actually an isometry. By a re-parametrization of the preimage $\mathbf{S}^3$, we can assume that $p(x) = x$.

Similarly, from the second equation in (5.49) we derive that q depends only on the second entry y; thus, q is actually a mapping from $\mathbf{S}^2$ to $\mathbf{S}^3$. As the second equation in (5.50) shows that dq is of rank 2, then $q(\mathbf{S}^2)$ is a 2-dimensional submanifold in $\mathbf{S}^3$. Noting that the pull-back metric f^*g restricted on $\{pt\} \times \mathbf{S}^2$ is $\frac{4\theta^2}{1+2\theta^2}g_0$. It follows that $\mathbf{S}^2$ is totally umbilical immersed in $\mathbf{S}^3$ and, up to an isometry of $\mathbf{S}^3$, we can assume that $q(y) = \sqrt{1-r^2} + ry$, where $r = \frac{\sqrt{3}\theta}{\sqrt{1+2\theta^2}}$ and $y \in \mathbf{S}^3 \cap \operatorname{Im}\mathbb{H}$.

Hence, up to isometries of type $\mathcal{F}_{abc}$, M is locally the image of the embedding f_r, corresponding to $0 < r < 1$, as described in Theorem 1.2.

Next, we consider the case $\theta = 1$. As we mentioned earlier, in this case M satisfies $A\phi = \phi A$. Then, according to Theorem 5.9 of [16], M is locally given by the embedding f_1 as described in Theorem 1.2.

This completes the proof of Theorem 5.1. □

Theorem 5.2 *Let M be a Hopf hypersurface of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ which possesses three distinct principal curvatures and satisfies $P\{U\}^\perp = \{U\}^\perp$ on M. If $P\xi = \frac{1}{2}\xi - \frac{\sqrt{3}}{2}J\xi$, then, up to isometries of type $\mathcal{F}_{abc}$, M is locally given by the embedding f'_r $(0 < r \leq 1)$ in Theorem 1.2.*

Proof Given M, by using the isometry $\mathcal{F}_1$, we obviously get another Hopf hypersurface $\mathcal{F}_1(M)$ of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ which also possesses three distinct principal curvatures. From Theorem 5.1 of [22], we know that the differential of the isometry $\mathcal{F}_1$ anticommutes with the almost complex structure J, and commutes with the almost product structure P, that is,

$$d\mathcal{F}_1 \circ J = -J \circ d\mathcal{F}_1, \quad d\mathcal{F}_1 \circ P = P \circ d\mathcal{F}_1.$$

Noticing that $\xi' := d\mathcal{F}_1(\xi)$ and $U' := -J\xi' = -d\mathcal{F}_1(U)$ are the unit normal vector field and the structure vector field of $\mathcal{F}_1(M)$. By using $P\xi = \frac{1}{2}\xi - \frac{\sqrt{3}}{2}J\xi$, we have

$$\begin{aligned} P\xi' &= Pd\mathcal{F}_1(\xi) = d\mathcal{F}_1 P(\xi) = d\mathcal{F}_1(\tfrac{1}{2}\xi - \tfrac{\sqrt{3}}{2}J\xi) \\ &= \tfrac{1}{2}d\mathcal{F}_1(\xi) + \tfrac{\sqrt{3}}{2}Jd\mathcal{F}_1(\xi) = \tfrac{1}{2}\xi' + \tfrac{\sqrt{3}}{2}J\xi'. \end{aligned}$$

It follows that $P\{U'\}^\perp = \{U'\}^\perp$ holds on $\mathcal{F}_1(M)$.

Noticing that, for any unitary quaternions a, b, c, the isometries $\mathcal{F}_{abc}$ and $\mathcal{F}_1$ satisfy $(\mathcal{F}_1)^2 = \mathrm{id}$ and $\mathcal{F}_{abc} \circ \mathcal{F}_1 = \mathcal{F}_1 \circ \mathcal{F}_{bac}$. Then, applying for Theorem 5.1 to the hypersurface $\mathcal{F}_1(M)$, we immediately conclude the proof of Theorem 5.2. □

Theorem 5.3 *Let M be a Hopf hypersurface of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ which possesses three distinct principal curvatures and satisfies $P\{U\}^\perp = \{U\}^\perp$ on M. If $P\xi = -\xi$, then, up to isometries of type $\mathcal{F}_{abc}$, M is locally given by the embedding f_r'' $(0 < r \le 1)$ in Theorem 1.2.*

Proof Given M, by using the isometry $\mathcal{F}_2$, we get another Hopf hypersurface $\mathcal{F}_2(M)$ of the NK $\mathbf{S}^3 \times \mathbf{S}^3$ which also possesses three distinct principal curvatures. From Theorem 5.2 of [22], the differential of the isometry $\mathcal{F}_2$ satisfies the following relationship with J and P:

$$d\mathcal{F}_2 \circ J = -J \circ d\mathcal{F}_2, \quad d\mathcal{F}_2 \circ P = (-\tfrac{1}{2}P + \tfrac{\sqrt{3}}{2}JP) \circ d\mathcal{F}_2.$$

Noticing that $\xi'' := d\mathcal{F}_2(\xi)$ and $U'' := -J\xi'' = -d\mathcal{F}_2(U)$ are the unit normal vector field and the structure vector field of $\mathcal{F}_2(M)$. By using $P\xi = -\xi$, we have

$$\begin{aligned} P\xi'' &= Pd\mathcal{F}_2(\xi) = -2d\mathcal{F}_2 P(\xi) + \sqrt{3}JPd\mathcal{F}_2(\xi) \\ &= 2d\mathcal{F}_2(\xi) + \sqrt{3}JP\xi'' = 2\xi'' + \sqrt{3}JP\xi''. \end{aligned}$$

It follows that $P\xi'' = \frac{1}{2}(\xi'' - \sqrt{3}PJP\xi'') = \frac{1}{2}\xi'' + \frac{\sqrt{3}}{2}J\xi''$, and $P\{U''\}^\perp = \{U''\}^\perp$ holds on $\mathcal{F}_2(M)$.

Noticing also that, for any unitary quaternions a, b, c, the isometries $\mathcal{F}_{abc}$ and $\mathcal{F}_2$ satisfy $(\mathcal{F}_2)^2 = \mathrm{id}$ and $\mathcal{F}_{abc} \circ \mathcal{F}_2 = \mathcal{F}_2 \circ \mathcal{F}_{cba}$. Then, applying for Theorem 5.1 to the hypersurface $\mathcal{F}_2(M)$, we immediately conclude the proof of Theorem 5.3. □

Finally, combining Proposition 5.1 and Theorems 5.1–5.3, we have completed the proof of Theorem 1.2. □

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