



Heat conservation for generalized Dirac Laplacians on manifolds with boundary

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Abstract

We consider a notion of conservation for the heat semigroup associated with a generalized Dirac Laplacian acting on sections of a vector bundle over a noncompact manifold with a (possibly noncompact) boundary under mixed boundary conditions. Assuming that the geometry of the underlying manifold is controlled in a suitable way and imposing uniform lower bounds on the zero-order piece (Weitzenböck potential) of the Dirac Laplacian, and on the endomorphism defining the mixed boundary condition, we show that the corresponding conservation principle holds. A key ingredient in the proof is a domination property for the heat semigroup which follows from an extension to this setting of a Feynman–Kac formula recently proved by the author de Lima (Pac J Math 292(1):177–201, 2018) in the context of differential forms. When applied to the Hodge Laplacian acting on differential forms satisfying absolute boundary conditions, this extends previous results by Vesentini (Ann Math Pura Appl 182:1–19, 2003) and Masamune (Atti Accad Naz Lincei Rend Lincei Mat Appl 18(4):351–358, 2007) in the boundaryless case. Along the way, we also prove a vanishing result for L^2 harmonic sections in the broader context of generalized (not necessarily Dirac) Laplacians. These results are further illustrated with applications to the Dirac Laplacian acting on spinors and to the Jacobi operator acting on sections of the normal bundle of a free boundary minimal immersion.

Keywords Heat conservation principle · Generalized Dirac Laplacians · Mixed boundary conditions · Feynman–Kac formula · Reflected Brownian motion

Mathematics Subject Classification 58J35 · 58J32 · 58J65

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1 Introduction and statement of the main result

Throughout this note, we consider a noncompact, oriented Riemannian manifold (X, g) of dimension $n \geq 2$. We assume that X carries a (possibly noncompact) boundary Σ , on which an inwardly oriented unit normal vector ν is globally defined. Also, we assume that X is geodesically complete in the sense that any geodesic avoiding Σ is defined for all time. We denote by d_X the intrinsic distance on X , by ∇ the Levi-Civita connection on tensors on X and by $B = -\nabla\nu$ the shape operator of Σ .

Let $\mathcal{E} \rightarrow X$ be a Riemannian (or Hermitean) vector bundle endowed with a fiber metric $\langle \cdot, \cdot \rangle$ and a compatible connection, still denoted by ∇ . Recall that a *generalized Laplacian* acting on sections of \mathcal{E} is a second-order elliptic operator given by

$$\Delta = \nabla^*\nabla + W,$$

where $\nabla^*\nabla$ is the Bochner Laplacian associated with ∇ and $W \in \Gamma(X, \text{End}(\mathcal{E}))$ is a pointwise selfadjoint bundle endomorphism. We will refer to W as the *Weitzenböck operator* or *potential*. Also, we denote the standard functional spaces of sections of \mathcal{E} by $L^p(X, \mathcal{E})$, etc.

In the presence of Σ , we need to attach to Δ suitable boundary conditions of elliptic type. Here, we adopt a certain class of mixed boundary conditions which are determined by an orthogonal decomposition

$$\mathcal{E}|_\Sigma = \mathcal{F}_+ \oplus \mathcal{F}_-$$

corresponding to the eigenbundles of a selfadjoint involution $\mathcal{I} \in \Gamma(X, \text{End}(\mathcal{E}|_\Sigma))$ and a pointwise selfadjoint endomorphism $S \in \Gamma(\Sigma, \text{End}(\mathcal{F}_+))$; see Sect. 3. Also, we assume throughout the text that both

$$W \in L^2_{\text{loc}}(X, \text{End}(\mathcal{E})) \quad \text{and} \quad S \in L^2_{\text{loc}}(\Sigma, \text{End}(\mathcal{F}_+))$$

are uniformly bounded from below. These requirements are better stated in terms of the functions

$$w : X \rightarrow \mathbb{R}, \quad w(x) = \inf_{|\phi|=1} \langle W(x)\phi, \phi \rangle, \tag{1}$$

and

$$\sigma : \Sigma \rightarrow \mathbb{R}, \quad \sigma(x) = \inf_{|\phi|=1} \langle S(x)\phi, \phi \rangle. \tag{2}$$

Assumption 1 There exist constants $c_1, c_2 > -\infty$ such that $w \geq c_1$ and $\sigma \geq c_2$.

Under this assumption and imposing mixed boundary conditions as above, Δ admits a natural selfadjoint extension which we denote by $\Delta_{W,S}$. Hence, we may apply the spectral theorem to define the corresponding heat semigroup

$$e^{-\frac{1}{2}t\Delta_{W,S}} : L^2(M, \mathcal{E}) \rightarrow L^2(M, \mathcal{E}), \quad t > 0.$$

In this setting, we denote by $\mathcal{D}_S(\mathcal{E})$ the space of compactly supported, smooth sections of \mathcal{E} meeting the given mixed boundary conditions. Also, (\cdot, \cdot) will denote the standard L^2 pairing between sections of \mathcal{E} .

We need a further specialization on the structure of Δ . Recall that a *Dirac operator* on \mathcal{E} is a first-order differential operator such that D^2 is a generalized Laplacian. We then say that $\Delta = D^2$ is a *generalized Dirac Laplacian*. We note that the existence of D is equivalent to requiring that \mathcal{E} is a Dirac bundle with respect to which D is the corresponding Dirac operator [25, Proposition 11.1.7]. In particular, we have the Leibniz rule

$$D(\xi \cdot \phi) = D_c\xi \cdot \phi + \xi \cdot D\phi, \tag{3}$$

for $\phi \in \Gamma(X, \mathcal{E})$, $\xi \in \Gamma(X, \text{Cl}(TX))$, where $\text{Cl}(TX)$ is the Clifford bundle of (X, g) , the dot is Clifford multiplication and D_c is the Dirac operator on $\text{Cl}(TX)$, viewed as a Dirac bundle over itself under left Clifford multiplication [21, Chapter II, Example 5.8]. In this setting, we define $\mathcal{H}_D(\mathcal{E})$ to be the space of D -harmonic sections (i.e., sections η satisfying $D\eta = 0$) and meeting the given mixed boundary conditions.

The definition below is motivated by [23,32], where it is discussed in the context of differential forms on boundaryless manifolds.

Definition 1 We say that the heat conservation principle holds for $\Delta_{W,S}$, the natural selfadjoint extension of a generalized Dirac Laplacian $\Delta = D^2$ as above, if the equality

$$\left(e^{-\frac{1}{2}t\Delta_{W,S}} \phi, \eta \right) = (\phi, \eta), \quad t > 0, \quad (4)$$

holds for any $\phi \in \mathcal{D}_S(\mathcal{E})$ and any $\eta \in \mathcal{H}_D(\mathcal{E}) \cap L^\infty(X, \mathcal{E})$.

This means that bounded D -harmonic sections are preserved by the heat semigroup. When \mathcal{E} is the trivial line bundle, $\Delta = \Delta_0$, the (nonnegative) Laplacian acting on functions, and we impose Neumann boundary conditions, this boils down to requiring that X is stochastically complete (with respect to normally reflected Brownian motion); see Sect. 2 for a discussion of this point. Thus, Definition 1 is a straightforward generalization of a much studied property of a natural diffusion process on manifolds with boundary.

Our main result provides a simple criterium for the validity of this principle. For technical reasons, we need to control the geometry of the underlying manifold (X, g) both at infinity and around the boundary. Thus, throughout the text, we assume that the following holds.

Assumption 2 The Ricci tensor Ric is bounded from below and

- Either Σ is convex (i.e., $B \geq 0$);
- Or

1. B is bounded;
2. there exists $r_0 > 0$ such that the geodesic collar map

$$\Lambda_{r_0} : [0, r_0) \times \Sigma \rightarrow X, \quad \Lambda_{r_0}(r, x) = \exp_x(r\nu),$$

is a diffeomorphism onto its image;

3. the sectional curvature is uniformly bounded from above on the image of Λ_{r_0} .

This kind of assumption appears in [33, Section 3.2.3]. As proved in [33, Theorem 3.2.9], it leads to an integrability result for the exponentiated boundary local time associated with reflected Brownian motion; see Theorem 3. Another useful consequence of Assumption 2 is that X is stochastically complete in the sense that the sample paths of the reflected Brownian motion remain in X for any positive time; see Theorem 2.

With this terminology at hand, we can state our main result.

Theorem 1 *If (X, g) satisfies Assumption 2 and a generalized Dirac Laplacian $\Delta = D^2$ acting on sections of $\mathcal{E} \rightarrow X$ satisfies Assumptions 1, then the heat conservation principle holds for $\Delta_{W,S}$.*

This paper is organized as follows. In Sect. 2, we review the properties of Brownian motion and Brownian bridge in the reflected case and in Sect. 3, we discuss mixed boundary conditions. The proof of Theorem 1 is included in Sect. 5 and makes use of a Feynman–Kac formula (Theorem 5), which allows us to obtain a path integral representation for the heat kernel associated with $e^{-\frac{1}{2}t\Delta_{W,S}}$ (Theorem 6). This is a key step in establishing the

corresponding semigroup domination property (Theorem 7 and Corollary 2). We stress that the proof of this fundamental property applies to *any* generalized Laplacian Δ satisfying Assumption 1, with no need to require it to be a Dirac Laplacian; see Remark 2. In particular, we are able to obtain a vanishing result in this rather general setting (Corollary 1). Finally, in Sect. 6, we discuss applications of our results to certain generalized Laplacians appearing in geometry, namely, the Hodge Laplacian acting on differential forms, the Dirac Laplacian acting on spinors and the Jacobi operator acting on sections of the normal bundle of a free boundary minimal submanifold.

Finally, we mention that a preliminary version of this article, with a sketch of the proof of our main result in the context of the Hodge Laplacian, has been published in [7].

2 Preliminary results on reflected Brownian motion

In this section, we collect a few technical results on the reflected Brownian motion on the underlying Riemannian manifold (X, g) . Besides reviewing the stochastic notions needed in the sequel, this is intended to justify the claim in the introduction that Definition 1 can be viewed as a natural generalization of X being stochastically complete with respect to this diffusion process. We then discuss the associated reflected Brownian bridge, which happens to be a key ingredient in establishing a path integral representation for the heat semigroup $e^{-\frac{1}{2}t\Delta_{W,S}}$.

Let X_t^x be reflected Brownian motion starting at $x \in X$ [1,5,19,20,33]. This is a continuous stochastic process driven by $-\frac{1}{2}\Delta_0$, where Δ_0 is the (nonnegative) Laplacian acting on bounded functions satisfying Neumann boundary condition along Σ .¹ Recall that $X_t = \pi \tilde{X}_t$, where $\pi : P_{SO}(X) \rightarrow X$ is the principal bundle of oriented orthonormal frames and \tilde{X}_t is the *horizontal* reflected Brownian motion starting at some $\tilde{x} \in \pi^{-1}(x)$, whose antidevelopment is the standard Brownian motion b_t in \mathbb{R}^n . Formally, \tilde{X}_t satisfies the stochastic differential equation

$$d\tilde{X}_t = \sum_{i=1}^n H_i(\tilde{X}_t) \circ db_t^i + \nu^\dagger(\tilde{X}_t)d\lambda_t, \tag{5}$$

where $\{H_i\}_{i=1}^n$ are the fundamental horizontal vector fields on $P_{SO}(X)$, the dagger means the standard equivariant lift (scalarization) of tensor fields on X to $P_{SO}(X)$ and λ_t is the boundary local time associated with X_t . We recall that λ_t is a nondecreasing process which only increases when the Brownian path hits the boundary.

In general, X_t might fail to be a Markov process. More precisely, let $\hat{X} = X \cup \{\infty\}$ be the one-point compactification of the pair (X, Σ) and define

$$e(x) = \inf\{t \geq 0; X_t^x = \infty\}, \quad x \in X.$$

For obvious reasons, e is called the extinction time of X_t . Now, the Markov property for X_t might not hold precisely because the process might be explosive in the sense that $e \neq +\infty$.

This somewhat annoying explosiveness property can be reformulated in analytical terms as follows. A version of the Feynman–Kac formula in this setting says that the (local) semigroup generated by $-\frac{1}{2}\Delta_0$ is given by

$$\left(e^{-\frac{1}{2}t\Delta_0} f\right)(x) = \mathbb{E}_x[f(X_t^x)\chi_{\{t < e(x)\}}], \tag{6}$$

¹ Thus, our sign convention is so that $\Delta_0 = -d^2/dx^2$ on \mathbb{R} .

where \mathbb{E}_x is the expectation associated with the law \mathbb{P}_x of X_t^x , $f \in L^2(X) \cap L^\infty(X)$ satisfies Neumann boundary condition and χ is the indicator function. It follows that $t \mapsto e^{-\frac{1}{2}t\Delta_0}$ is a positive preserving, contraction semigroup on the space of all such functions, so by interpolation, it can be extended as a contraction semigroup to $L^p(X)$, $1 \leq p \leq \infty$. Thus, we may apply (6) with $f = \mathbf{1}$, the function identically equal to 1, in order to get

$$\left(e^{-\frac{1}{2}t\Delta_0}\mathbf{1}\right)(x) = \mathbb{P}[t < \mathbf{e}(x)]. \quad (7)$$

So in general, we have $e^{-\frac{1}{2}t\Delta_0}\mathbf{1} \leq \mathbf{1}$ and being explosive means precisely that $e^{-\frac{1}{2}t\Delta_0}\mathbf{1} \not\equiv \mathbf{1}$ for some (and hence any) $t > 0$. This means that constant functions are *not* preserved by the semigroup.

Another way of expressing this sub-Markov property of X_t relies on the well-known fact that the semigroup action can be represented by convolution against a smooth kernel. More precisely,

$$\left(e^{-\frac{1}{2}t\Delta_0}f\right)(x) = \int_X K_0(t; x, y)f(y)dX_y,$$

where K_0 is the Neumann heat kernel, that is, the fundamental solution of the initial value problem associated with the heat operator

$$L = \frac{\partial}{\partial t} + \frac{1}{2}\Delta_0$$

with Neumann boundary condition along Σ . Thus, by (7) in general we have

$$\int_X K_0(t; x, y)dX_y \leq 1,$$

and we see once again that in the explosive case the strict equality holds for some $t > 0$. Thus, in general, we are not allowed to interpret K_0 as a transition probability density function for X_t .

The following well-known proposition summarizes the discussion above. Here, $(\cdot, \cdot)_0$ is the standard L^2 pairing on functions.

Proposition 1 *The following are equivalent:*

1. X_t is nonexplosive in the sense that $\mathbf{e} \equiv +\infty$;
2. For some/any $t > 0$ and any $x \in X$, $K_0(t; x, \cdot)$ is a probability density function on X .
3. For some/any $t > 0$, $e^{-\frac{1}{2}t\Delta_0}\mathbf{1} = \mathbf{1}$;
4. For some/any $t > 0$, $(e^{-\frac{1}{2}t\Delta_0}f, \mathbf{1})_0 = (f, \mathbf{1})_0$, for any compactly supported function f on X satisfying Neumann boundary condition.

We now recall a standard terminology.

Definition 2 If any of the conditions in Proposition 1 happens, then we say that X is stochastically complete.

The validity of this property means that the desired probabilistic interpretation for K_0 has been restored so that X_t is turned into a genuine Markov process. Equivalently, constant functions are preserved by the associated semigroup. Also, in view of item (4), we see that X being stochastically complete is equivalent to the heat conservation principle holding for Δ_0 . This provides the link between this classical notion and our Definition 1.

It is not hard to exhibit examples of noncompact, geodesically complete manifolds which fail to be stochastically complete; see [13] for a rather complete survey in the boundaryless

case. On the other hand, a celebrated criterium due to Gregor'yan [13, Theorem 9.1], which certainly can be adapted to our setting, provides a sufficient condition for stochastic completeness in terms of volume growth. However, from our viewpoint, it is natural to consider instead the following test which involves imposing curvature bounds both in the interior and along the boundary. In the boundaryless case, where only the lower bound on the Ricci tensor is required, this is due to Yau [34].

Theorem 2 *If Assumption 2 is satisfied, then X is stochastically complete.*

Proof See Remark 4 for a simple proof based on the semigroup domination property proved in Sect. 4. \square

As already mentioned, Assumption 2 also yields an integrability result for the boundary local time λ_t . Clearly, we may assume that the lower bound for B , say $\underline{\kappa}$, is negative.

Theorem 3 [33, Theorem 3.2.9] *If Assumption 2 holds then for any $p \in [1, +\infty)$, there exist $K_1^{(p)}, K_2^{(p)} > 0$ such that*

$$\mathbb{E}_x[e^{-p\kappa\lambda_t}] \leq K_1^{(p)} e^{K_2^{(p)}t},$$

for all $t \geq 0$ and $x \in X$.

We now turn to the so-called *reflected Brownian bridge* associated with X_t ; see [6, Appendix A] for details. For each $t > 0$ and $x, y \in X$, this is the process $\mathfrak{X}_{s;x,y}$, $0 \leq s \leq t$, which starts at x , follows the reflected Brownian motion X_s^x and is further conditioned to hit y in time t . At least for $0 \leq s < t$, it is immediate to check that its law $\mathbb{P}_{t;x,y}$ satisfies

$$\frac{d\mathbb{P}_{t;x,y}}{d\mathbb{P}_x} \Big|_{\mathcal{G}_s} = \frac{K_0(t-s; X_s^x, y)}{K_0(t; x, y)}, \quad (8)$$

where \mathcal{G}_s is the standard filtration associated with X_t . It then follows that the reflected Brownian bridge is just reflected Brownian motion with an added drift involving the logarithmic derivative of K_0 . In particular, $\mathfrak{X}_{s;x,y}$ is a $\mathbb{P}_{t;x,y}$ -semimartingale in the range $0 \leq s < t$. It is crucial in applications to be able to extend this property to $s = t$.

Proposition 2 *If Assumption 2 holds, then reflected Brownian bridge $\mathfrak{X}_{s;x,y}$ is a $\mathbb{P}_{t;x,y}$ -semimartingale in the whole interval $[0, t]$.*

Proof We only sketch the proof, as it follows by adapting standard results in the available literature for the boundaryless case. First, as explained in [33, Section 3.2.3], Assumption 2 implies that, by eventually passing to a conformally deformed metric, we may assume Σ is convex. This guarantees that any two points in X can be joined by at least one minimizing geodesic. By using standard comparison theory, this implies that, at least locally, we have at our disposal the usual package of geometric bounds, which includes the Bishop–Gromov inequality, the doubling volume property and Gaussian bounds for K_0 , where the controlling constants entering in these estimates depend only on the local geometry; see [15, Appendix A]. We then argue as in [15, Appendix B] to obtain a localized gradient estimate for $\log K_0$, which adapts an argument in [2]. With this information at hand, we can easily establish local estimates of the types

$$D_1 t^{-n/2} e^{-D_2 \frac{d_X(x,y)^2}{t}} \leq K_0(t; x, y) \leq D_3 t^{-n/2} e^{-D_4 \frac{d_X(x,y)^2}{t}},$$

and

$$|\nabla \log K_0(t; x, y)| \leq D_5 (t^{-1/2} + t^{-1} d_X(x, y)),$$

where the constants D_j only depend on the local geometry of X ; cf. [15, Proposition 2.8]. From this point, we may proceed as in the proof of [15, Theorem 2.7] to check that the following localized inequality holds:

$$\mathbb{E}_{t;x,y} \left[\int_0^t |\nabla \log K_0(t-s; \mathfrak{X}_{s;x,y}, y)| |\nabla f(\mathfrak{X}_{s;x,y})| ds \right] < +\infty,$$

where $\mathbb{E}_{t;x,y}$ is the expectation associated with $\mathbb{P}_{t;x,y}$ and the compactly supported function f is supposed to satisfy Neumann boundary conditions in case $\text{supp } f \cap \Sigma \neq \emptyset$. As explained in [15], this suffices to complete the proof. \square

3 Mixed boundary conditions for generalized Laplacians

Rather complete studies of elliptic boundary conditions for generalized Laplacians, including the delicate issue of the existence and explicit computation of the corresponding heat kernel asymptotics, can be found in the available literature; see [3, 12, 14], for instance. Here, we single out a class of such boundary conditions which suffices for the applications we have in mind.

We start with a pointwise selfadjoint involution $\mathcal{I} \in \Gamma(X, \text{End}(\mathcal{E}|_\Sigma))$, which we extend to a collared neighborhood of Σ so that $\nabla_\nu \mathcal{I} = 0$. Let

$$\Pi_\pm = \frac{1}{2} (I \pm \mathcal{I})$$

be the corresponding projections onto the eigenbundles $\mathcal{F}_\pm = \Pi_\pm \mathcal{E}|_\Sigma$ of \mathcal{I} . Clearly,

$$\nabla_\nu \Pi_\pm = \Pi_\pm \nabla_\nu.$$

Now, take a pointwise selfadjoint endomorphism $S \in \Gamma(\Sigma, \text{End}(\mathcal{F}_+))$ and extend it to $\mathcal{E}|_\Sigma$ by declaring that $S = 0$ on \mathcal{F}_- . We may assume that the extension of S to the collared neighborhood, still denoted S , satisfies $\nabla_\nu S = 0$. It then follows that

$$S \Pi_\pm = \Pi_\pm S.$$

Definition 3 A section $\phi \in \Gamma(\mathcal{E})$ satisfies mixed boundary conditions if its restriction to Σ , still denoted ϕ , satisfies

$$\Pi_+(\nabla_\nu - S)\phi = 0, \quad \Pi_-\phi = 0. \quad (9)$$

The qualification ‘‘mixed’’ of course is due to the fact that this kind of boundary condition is Dirichlet in the \mathcal{F}_- -direction and Robin in the \mathcal{F}_+ -direction. This seems to be the largest class of local elliptic boundary conditions to which the stochastic methods in Sect. 4 apply; see Remark 1 below. The relevance of mixed boundary conditions in quantum field theory is explained in [3, 31].

For the next proposition, recall that $\mathcal{D}_S(\mathcal{E})$ is the space of smooth, compactly supported sections satisfying (9).

Proposition 3 *If a generalized Laplacian Δ satisfies Assumption 1 with S as in (9), then the bilinear form*

$$Q : \mathcal{D}_S(\mathcal{E}) \times \mathcal{D}_S(\mathcal{E}) \rightarrow \mathbb{R}, \quad Q(\phi, \eta) = \int_X \langle \Delta \phi, \eta \rangle dX,$$

is symmetric and bounded from below.

Proof By adding a sufficiently large positive multiple of the identity to S , we may assume that $c_2 \geq 0$. Recall that the Bochner Laplacian is locally given by

$$\nabla^* \nabla = - \sum_{i=1}^n \left(\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i} \right).$$

By choosing the orthonormal frame $\{e_i\}$ so that $\nabla_{e_i} e_j = 0$ at the given point and defining a vector field Z on M by $\langle Z, Y \rangle = \langle \nabla_Y \phi, \eta \rangle$, we have

$$\operatorname{div} Z = \sum_i e_i \langle \nabla_{e_i} \phi, \eta \rangle = - \langle \nabla^* \nabla \phi, \eta \rangle + \langle \nabla \phi, \nabla \eta \rangle,$$

so that

$$\int_M \langle \nabla^* \nabla \phi, \eta \rangle \, dM = \int_M \langle \nabla \phi, \nabla \eta \rangle \, dM + \int_\Sigma \langle \nabla_\nu \phi, \eta \rangle \, d\Sigma, \tag{10}$$

But

$$\begin{aligned} \int_\Sigma \langle \nabla_\nu \phi, \eta \rangle \, d\Sigma &= \int_\Sigma \langle \nabla_\nu (\Pi_+ \phi + \Pi_- \phi), \Pi_+ \eta + \Pi_- \eta \rangle \, d\Sigma \\ &= \int_\Sigma \langle \Pi_+ \nabla_\nu \phi, \Pi_+ \eta \rangle \, d\Sigma \\ &= \int_\Sigma \langle \Pi_+ (\nabla_\nu - S) \phi, \Pi_+ \eta \rangle \, d\Sigma + \int_\Sigma \langle \Pi_+ S \phi, \Pi_+ \eta \rangle \, d\Sigma, \end{aligned}$$

so that

$$Q(\phi, \eta) = \int_X \langle \nabla \phi, \nabla \eta \rangle \, dX + \int_X \langle W \phi, \eta \rangle \, dX + \int_\Sigma \langle S \phi, \eta \rangle \, d\Sigma. \tag{11}$$

From this, it is immediate that Q is both symmetric and bounded from below. □

Let us take W and S as in Theorem 1. Thus, Proposition 3 applies and the quadratic form Q , which is associated with the densely defined unbounded operator $\Delta : \mathcal{D}_S(\mathcal{E}) \subset L^2(X, \mathcal{E}) \rightarrow L^2(X, \mathcal{E})$, is closable and its closure, whose domain is contained in $H^1(X, \mathcal{E})$, is still given by (11). It is in this sense that the Friedrichs extension of Δ , denoted $\Delta_{W,S}$, satisfies the given mixed boundary conditions. In particular, we may appeal to the spectral theorem to canonically construct the associated heat semigroup

$$e^{-\frac{1}{2}t\Delta_{W,S}} : L^2(X, \mathcal{E}) \rightarrow L^2(X, \mathcal{E}), \quad t > 0.$$

In particular, if $\phi \in \mathcal{D}_S(\mathcal{E})$ then $\phi_t = e^{-\frac{1}{2}t\Delta_{W,S}} \phi$ solves the corresponding heat equation:

$$\frac{\partial \phi_t}{\partial t} + \frac{1}{2} \Delta_{W,S} \phi_t = 0, \quad \lim_{t \rightarrow 0} \phi_t = \phi, \quad \Pi_+ (\nabla_\nu - S) \phi_t = 0, \quad \Pi_- \phi_t = 0. \tag{12}$$

Of course, it is precisely this semigroup that appears in Definition 1.

Remark 1 The most general kind of (differential) boundary conditions for generalized Laplacians takes the form

$$A\phi = 0, \quad C\phi + \sum_{j=1}^{n-1} E_j \nabla_{e_j} \phi + E \nabla_\nu \phi = 0, \tag{13}$$

where $\{e_j\}$ is a local orthonormal frame along Σ and the coefficients (capital letters) are locally defined matrices acting on the components of ϕ and $\nabla_{e_j}\phi$. In this setting, the so-called Lopatinskiĭ–Shapiro ellipticity condition reduces to verifying that the \mathbb{C} -linear map

$$b_{(\xi,z)}\phi = \left(\begin{array}{c} A\phi \\ (i \sum_j E_j \xi_j - \sqrt{|\xi|^2 - z} E) \phi \end{array} \right)$$

is an isomorphism onto its image for any $(0, 0) \neq (\xi, z) \in T^*\Sigma \times \mathcal{K}$, where $\mathcal{K} = \mathbb{C} - (0, +\infty)$ and we use here an appropriate branch for the square root; see [12, Lemma 1.4.8]. As expected, the matrix C plays no role here, since only the symbol of the differential term in the second condition in (13) really matters. The usage of stochastic methods in Sect. 4, which relies on certain curvature-driven multiplicative functionals, forces us to choose the coefficients so as to eliminate the tangential derivatives in (13) while still keeping ellipticity. Given these constraints, we are basically led to set $A = \Pi_-$, $C = -\Pi_+ S$, $E = \Pi_+$ and $E_j = 0$ as in Definition 3, so that

$$b_{(\xi,z)}\phi = \left(\begin{array}{c} \Pi_- \phi \\ -\sqrt{|\xi|^2 - z} \Pi_+ \phi \end{array} \right),$$

is an isomorphism indeed. Notice that this latter assertion only depends on the existence of the involution \mathcal{I} , which determines the complementary projections Π_{\pm} . In particular, we see that the selfadjoint endomorphism S only plays a role in assuring that the quadratic form Q in Proposition 3 is symmetric.

4 The semigroup domination property

In this section, we prove the main technical result in the paper, namely, the domination property for the heat semigroup $e^{-\frac{1}{2}t\Delta_{W,S}}$ introduced in the previous section. The crucial point here is to make sure that $e^{-\frac{1}{2}t\Delta_{W,S}}\phi \in L^1(X, \mathcal{E})$ whenever $\phi \in \mathcal{D}_S(\mathcal{E})$, with an exponential bound on the norm of the corresponding linear map depending on the lower bounds imposed on W and S ; see Corollary 4.2. A key ingredient in the proof is a Feynman–Kac formula generalizing a previous result in [5] for differential forms, from which a path integral representation for the associated heat kernel follows.

Remark 2 As already observed, all the results in this section hold true for any generalized (not necessarily Dirac) Laplacian $\Delta_{W,S}$ satisfying Assumption 1 and defined over a manifold whose geometry is controlled as in Assumption 2. This latter hypothesis is used to make sure that Theorem 3, which provides an exponential bound for the expectation of the boundary local time, is valid. It is tempting to ask whether Theorem 3 still holds if the requirements in Assumption 2 are weakened to uniform lower bounds for the Ricci tensor of g and the shape operator of Σ .

Let $X_t = X_t^x$, $t \geq 0$, be reflected Brownian motion on X starting at some x . Since Assumption 2 is taken for granted, by Theorem 2, we know that X is stochastically complete (with respect to X_t). In view of Proposition 1, this means that X_t is nonexplosive, so the sample paths X_t^x remain in X for all time.

Although this is not strictly required in the following, for simplicity, we assume that \mathcal{E} is tensorial in the sense that it is associated with some orthogonal representation ρ of SO_n , the rotation group in dimension n . As a consequence, any section $\phi \in \Gamma(X, \mathcal{E})$ can be identified

to its ρ -equivariant lift $\phi^\dagger : P_{SO}(X) \rightarrow V$, where V is the representation space of ρ . Also, the heat operator L in (12) lifts to

$$L^\dagger = \frac{\partial}{\partial t} + \frac{1}{2} \Delta_{W,S}^\dagger,$$

where

$$\Delta_{W,S}^\dagger = \nabla^* \nabla^\dagger + W^\dagger,$$

and

$$\nabla^* \nabla^\dagger = - \sum_{i=1}^n \mathcal{L}_{H_i}^2$$

is the *horizontal* Bochner Laplacian. Here, \mathcal{L} is Lie derivative. Also, the boundary conditions in (9) lift to

$$\Pi_+^\dagger (\mathcal{L}_{\nu^\dagger} - S^\dagger) \phi^\dagger = 0, \quad \Pi_-^\dagger \phi^\dagger = 0.$$

The advantage of lifting everything in sight to $P_{SO}(X)$ is that, when doing computations in the framework of Itô’s stochastic calculus, we may work on the trivial vector bundle $\mathbb{R}^N \rightarrow \mathbb{R}^n$, $N = \text{rank } \mathcal{E}$, where the antidevelopment of \tilde{X}_t lives (as already mentioned, this happens to be the standard Brownian motion b_t in \mathbb{R}^n); see [9,18,20] for details on this so-called Eells–Elworthy–Malliavin approach to diffusions on manifolds.

We use this formalism to obtain a stochastic representation for the action of the heat semigroup $e^{-\frac{1}{2}t\Delta_{W,S}}$ on $\mathcal{D}_S(\mathcal{E})$; see Theorem 5 below. We start by observing that for each $W \in L_{\text{loc}}^2(M, \text{End}(\mathcal{E}))$ and $S \in L_{\text{loc}}^2(\Sigma, \text{End}(\mathcal{E}|_\Sigma))$, we may consider the pathwise solution $M_{W,S,t} \in \text{End}(\mathbb{R}^N)$ of

$$dM_{W,S,t} + M_{W,S,t} \left(\frac{1}{2} W^\dagger(\tilde{X}_t) dt + S^\dagger(\tilde{X}_t) d\lambda_t \right) = 0, \quad M_{W,S,0} = I; \tag{14}$$

see [8]. Note that the inverse process $M_{W,S,t}^{-1}$ satisfies

$$dM_{W,S,t}^{-1} - \left(\frac{1}{2} W^\dagger(\tilde{X}_t) dt + S^\dagger(\tilde{X}_t) d\lambda_t \right) M_{W,S,t}^{-1} = 0, \quad M_{W,S,0}^{-1} = I. \tag{15}$$

For each $\epsilon > 0$ and S as above defining mixed boundary conditions, let us set

$$S^\epsilon = S + \epsilon^{-1} \Pi_-.$$

Notice that

$$(S^\epsilon)^\dagger \phi^\dagger = S^\dagger \phi^\dagger, \quad \phi \in \mathcal{D}_S(\mathcal{E}). \tag{16}$$

Also, in the following, $\| \cdot \|$ is the operator norm in $\text{End}(\mathbb{R}^N)$.

Proposition 4 *If Assumption 1 holds and if $\epsilon > 0$ satisfies $\epsilon^{-1} \geq c_2$, then*

$$\|M_{W,S^\epsilon,t}\| \leq \exp \left(-\frac{1}{2} \int_0^t w(\tilde{X}_s) ds - \int_0^t \sigma(\tilde{X}_s) d\lambda_s \right),$$

where w and σ are given by (1) and (2), respectively.

Proof The key point here is to make sure that the righthand side does not depend on ϵ , so it can be further estimated solely in terms of the lower bounds on W and S . Following [19], we observe that it suffices to prove the result for $M_{W,S^\epsilon,t}^\bullet$, where the bullet means transposition. Take $v \in \mathbb{R}^N$ and set $f(t) = |M_{W,S^\epsilon,t}^\bullet v|^2$. Then

$$\begin{aligned} df(t) &= -2v^\bullet M_{W,S^\epsilon,t} \left(\frac{1}{2} W^\dagger(\tilde{X}_t) dt + (S^\epsilon)^\dagger(\tilde{X}_t) d\lambda_t \right) M_{W,S^\epsilon,t}^\bullet v \\ &\leq -f(t) (w(\tilde{X}_t) dt + 2\sigma(\tilde{X}_t) d\lambda_t), \end{aligned}$$

and the result follows after integration. \square

The following proposition is a key technical ingredient in our argument. It allows us to establish a Feynman–Kac formula for $e^{-\frac{1}{2}t\Delta_{W,S}}$ under the more restrictive assumption that W and S are *uniformly bounded*, i.e., bounded from above and below; see Theorem 4 below.

Proposition 5 *Take W and S as above, with both being uniformly bounded and with S defining mixed boundary conditions. Then, as $\epsilon \rightarrow 0$, $M_{W,S^\epsilon,t}$ converges in L^2 to an adapted, right-continuous process M_t with left limits. Furthermore,*

$$M_t \Pi_-^\dagger(\tilde{X}_t) = 0, \quad (17)$$

whenever $\tilde{X}_t \in \pi^{-1}\Sigma$.

Proof This has been first proved in [19] for 1-forms, i.e., $\mathcal{E} = \wedge^1 T^*X$, $W = \text{Ric}$ and $S = B$, under the assumption that X is compact. It has been observed in [5] that the same proof works for p -forms on a noncompact manifold with bounded geometry in the sense of [29]; hence, using the notation in Subsection 6.1, in this case, we take $\mathcal{E} = \wedge^p T^*X$, $W = R_p$ and $S = \mathcal{B}_p$. As a careful analysis of the original proof confirms, the same argument still works fine if more generally X has controlled geometry in the sense of Assumption 2, so that the integrability result in Theorem 3 holds, and both W and S are uniformly bounded. We leave the details to the interested reader. \square

Now, let $\phi \in \mathcal{D}_S(\mathcal{E})$, so that $\phi_t^\dagger = e^{-\frac{1}{2}t\Delta_{W,S}^\dagger} \phi^\dagger$ is the solution to

$$L^\dagger \phi_t^\dagger = 0, \quad \lim_{t \rightarrow +\infty} \phi_t^\dagger = \phi^\dagger, \quad \Pi_+^\dagger(\mathcal{L}_{v^\dagger} - S^\dagger)\phi_t^\dagger = 0, \quad \Pi_-^\dagger \phi_t^\dagger = 0. \quad (18)$$

Then a simple application of Itô's formula to the process $M_{W,S^\epsilon,t} \phi_{T-t}^\dagger(\tilde{X}_t)$, $0 \leq t \leq T$, yields in the limit $\epsilon \rightarrow 0$ the following fundamental Feynman–Kac formula, which generalizes [5, Theorem 5.2].

Theorem 4 *Assume that W and S are as above, with both being uniformly bounded and with S defining mixed boundary conditions. Then*

$$\phi_t^\dagger(\tilde{x}) = \mathbb{E}_{\tilde{x}}(M_t \phi^\dagger(\tilde{X}_t)). \quad (19)$$

Equivalently,

$$(e^{-\frac{1}{2}t\Delta_{W,S}} \phi)(x) = \mathbb{E}_x(M_t J_t \phi(X_t^x)), \quad (20)$$

where J_t is the (reversed) stochastic parallel transport acting on sections of \mathcal{E} and we use the standard identification $\phi^\dagger = J_t \phi$.

Proof With the help of (5), Itô’s formula gives

$$dM_{W,S^\epsilon,t} \phi_{T-t}^\dagger(\tilde{X}_t^x) = \left\langle M_{W,S^\epsilon,t} \mathcal{L}_H \phi_{T-t}^\dagger(\tilde{X}_t^x), db_t \right\rangle - M_{W,S^\epsilon,t} L^\dagger \phi_{T-t}^\dagger(\tilde{X}_t^x) dt + M_{W,S^\epsilon,t} \left(\mathcal{L}_{v^\dagger} - S^\dagger - \epsilon^{-1} \Pi_-^\dagger \right) \phi_{T-t}^\dagger(\tilde{X}_t^x) d\lambda_t,$$

where $\mathcal{L}_H = (\mathcal{L}_{H_1}, \dots, \mathcal{L}_{H_n})$. Due to (18), both the second term and the term involving ϵ^{-1} on the righthand side vanish. Sending $\epsilon \rightarrow 0$ and using Proposition 5, we end up with

$$dM_t \phi_{T-t}^\dagger(\tilde{X}_t^x) = \left\langle M_t \mathcal{L}_H \phi_{T-t}^\dagger(\tilde{X}_t^x), db_t \right\rangle + M_t \Pi_+^\dagger (\mathcal{L}_{v^\dagger} - S^\dagger) \phi_{T-t}^\dagger(\tilde{X}_t^x) d\lambda_t,$$

where the insertion of Π_+^\dagger in the last term is justified by (17). Again by (18), this reduces to

$$dM_t \phi_{T-t}^\dagger(\tilde{X}_t^x) = \left\langle M_t \mathcal{L}_H \phi_{T-t}^\dagger(\tilde{X}_t^x), db_t \right\rangle,$$

thus showing that $M_t \phi_{T-t}^\dagger(\tilde{X}_t^x)$ is a (local) martingale. The result now follows by equating expectations of this process at $t = 0$ and $t = T$. \square

Our aim now is to extend the Feynman–Kac formula (20) to the case in which R and S are merely assumed to be bounded from below; see Theorem 5 below. For this, we rely on the results above to implement an approximation scheme adapted from [16]; see also [8] for similar arguments.

We start with a comparison estimate holding in the general context of solutions of (14). In order to simplify the notation in the following, we sometimes write $w(t) = w(\tilde{X}_t)$, $W_1(t) = W_1(\tilde{X}_t)$, etc. Also, recall that $N = \text{rank } \mathcal{E}$.

Proposition 6 For each $t > 0$, we have the pathwise estimate

$$\|M_{W_1,S_1,t} - M_{W_2,S_2,t}\| \leq e^{\int_0^t \left(\frac{1}{2} \|W_1(s)\| ds + \|S_1(s)\| d\lambda_s \right) + 2 \int_0^t \left(\frac{1}{2} \|W_2(s)\| ds + \|S_2(s)\| d\lambda_s \right)} \times \int_0^t \left(\frac{1}{2} \|W_1(s) - W_2(s)\| ds + \|S_1(s) - S_2(s)\| d\lambda_s \right).$$

Proof From (14) and (15),

$$d \left(M_{W_2,S_2,t}^{-1} M_{W_1,S_1,t} \right) = M_{W_2,S_2,t}^{-1} \times \left(\frac{1}{2} (W_2(t) - W_1(t)) dt + (S_2(t) - S_1(t)) d\lambda_t \right) M_{W_1,S_1,t},$$

so that

$$M_{W_1,S_1,t} = M_{W_2,S_2,t} + M_{W_2,S_2,t} \int_0^t M_{W_2,S_2,s}^{-1} \left(\frac{1}{2} (W_1(s) - W_2(s)) dt + (S_1(s) - S_2(s)) d\lambda_s \right) M_{W_1,S_1,s}.$$

Thus,

$$\|M_{W_1,S_1,t} - M_{W_2,S_2,t}\| \leq \|M_{W_2,S_2,t}\| \times \int_0^t \|M_{W_2,S_2,s}^{-1}\| \|M_{W_1,S_1,s}\| \left(\frac{1}{2} \|W_1(s) - W_2(s)\| ds + \|S_1(s) - S_2(s)\| d\lambda_s \right).$$

The result now follows since we can easily estimate the norms $\|M_{W_i,S_i,t}\|$ and $\|M_{W_i,S_i,t}^{-1}\|$ in the indicated way via Gronwall’s inequality. \square

Now, we will be able to implement the approximation scheme. So, we consider W and S , both bounded from below. Define a sequence $\{W_i\}$ by setting $W_i = \min\{W, i \text{ Id}\}$ fiberwise and similarly for $\{S_i\}$. It follows that W_i and S_i are uniformly bounded and $\|W_i(x) - W(x)\| \rightarrow 0$ and $\|S_i(x) - S(x)\| \rightarrow 0$ as $i \rightarrow +\infty, x \in X$. Also, the convergences are monotone nondecreasing in the obvious sense. Moreover, as a result of this procedure, we see that any $\phi \in \mathcal{D}_S(\mathcal{E})$ can be written as $\phi = \lim_{i \rightarrow +\infty} \phi_i, \phi_i \in \mathcal{D}_{S_i}(\mathcal{E})$.

Proposition 7 *For each $t > 0$ and $\epsilon > 0$, we have the pathwise convergence*

$$\lim_{i \rightarrow +\infty} \|M_{W_i, S_i^\epsilon, t} - M_{W, S^\epsilon, t}\| = 0$$

Proof From Proposition 6 and the nondecreasing monotone convergence,

$$\begin{aligned} \|M_{W_i, S_i^\epsilon, t} - M_{W, S^\epsilon, t}\| &\leq e^{3 \int_0^t \left(\frac{1}{2} \|W(s)\| ds + \|S^\epsilon(s)\| d\lambda_s\right)} \\ &\quad \times \int_0^t \left(\frac{1}{2} \|W_i(s) - W(s)\| ds + \|S_i(s) - S(s)\| d\lambda_s\right). \end{aligned}$$

Consider $w = \|W\| \in L^2_{\text{loc}}(X)$ and $s^\epsilon = \|S^\epsilon\| \in L^2_{\text{loc}}(\Sigma)$. It is well known that for any $t > 0$ and almost every path X_s^x , we have

$$\int_0^t |w(X_s^x)| ds < +\infty \quad \text{and} \quad \int_0^t |s^\epsilon(X_s^x)| d\lambda_s < +\infty.$$

Thus,

$$\|M_{W_i, S_i^\epsilon, t} - M_{W, S^\epsilon, t}\| \leq C_{t, \epsilon} \int_0^t \left(\frac{1}{2} \|W_i(s) - W(s)\| ds + \|S_i(s) - S(s)\| d\lambda_s\right),$$

and the result follows by dominated convergence. □

Proposition 8 *For each $\epsilon > 0$,*

$$\lim_{i \rightarrow +\infty} \mathbb{E}_x \|M_{W_i, S_i^\epsilon, t} - M_{W, S^\epsilon, t}\|^2 = 0.$$

Proof Let $\{f_1, \dots, f_N\}$ be an orthonormal frame locally trivializing \mathcal{E} and set $Z_{i,t}^\epsilon = M_{W_i, S_i^\epsilon, t} - M_{W, S^\epsilon, t}$. We have

$$\begin{aligned} d\|Z_{i,t}^\epsilon f_\alpha\|^2 &= -2 \left\langle Z_{i,t}^\epsilon \left(\frac{1}{2} (W(t) - W_i(t)) dt + (S(t) - S_i(t)) d\lambda_t\right) f_\alpha, Z_{i,t}^\epsilon f_\alpha \right\rangle \\ &\leq -2 \left(\frac{1}{2} w^{(i)}(t) dt + \sigma^{(i)}(t) d\lambda_t\right) \|Z_{i,t}^\epsilon f_\alpha\|^2, \end{aligned}$$

where $W - W_i \geq w^{(i)} \text{Id}$ and $S - S_i \geq \sigma^{(i)} \text{Id}$. Recalling that the convergences $W_i \rightarrow W$ and $S_i \rightarrow S$ are monotone nondecreasing, we may assume that both $w^{(i)}$ and $\sigma^{(i)}$ are nonnegative, so $d\|Z_{i,t}^\epsilon f_\alpha\|^2 \leq 0$ and hence $\|Z_{i,t}^\epsilon\|^2 \leq 1$. The result then follows from Proposition 7 and dominated convergence. □

We know from Proposition 5 that for each i , $M_{W_i, S_i^\epsilon, t}$ converges in L^2 to a process, say $M_{i,t}$, as $\epsilon \rightarrow 0$. Moreover, by Theorem 4, this leads to a Feynman–Kac formula, namely,

$$\left(e^{-\frac{1}{2} t \Delta_{W_i, S_i}} \phi\right)(x) = \mathbb{E}_x [M_{i,t} J_t \phi(X_t^x)], \quad \phi \in \mathcal{D}_{S_i}(\mathcal{E}). \tag{21}$$

Now set $\mathbb{E}_x^{(2)} \|M_{i,t} - M_{j,t}\| = (\mathbb{E}_x \|M_{i,t} - M_{j,t}\|^2)^{1/2}$, etc. Then, Proposition 8 and the triangle inequality

$$\begin{aligned} \mathbb{E}_x^{(2)} \|M_{i,t} - M_{j,t}\| &\leq \mathbb{E}_x^{(2)} \|M_{i,t} - M_{W_i, S_i^c, t}\| \\ &\quad + \mathbb{E}_x^{(2)} \|M_{W_i, S_i^c, t} - M_{W_j, S_j^c, t}\| + \mathbb{E}_x^{(2)} \|M_{W_j, S_j^c, t} - M_{j,t}\| \end{aligned}$$

imply that $\{M_{i,t}\}_i$ is Cauchy in L^2 , so it converges as $i \rightarrow +\infty$ to a process, say \mathcal{M}_t . Passing the limit in (21) and making use of a standard result on the monotone convergence of quadratic forms [22, Theorem 3.18], we obtain a Feynman–Kac formula for the heat semigroup $e^{-\frac{1}{2}t\Delta_{W,S}}$.

Theorem 5 *If Assumption 1 holds, then*

$$\left(e^{-\frac{1}{2}t\Delta_{W,S}} \phi \right) (x) = \mathbb{E}_x \left[\mathcal{M}_t J_t \phi (X_t^x) \right], \quad \phi \in \mathcal{D}_S(\mathcal{E}). \tag{22}$$

This immediately yields a path integral representation for the heat kernel $K_{W,S}$ of $e^{-\frac{1}{2}t\Delta_{W,S}}$.

Theorem 6 *We have*

$$K_{W,S}(t; x, y) = K_0(t; x, y) \mathbb{E}_{t;x,y} [\mathcal{M}_t J_t], \tag{23}$$

where here J_t is the stochastic parallel transport along the (reversed) reflected Brownian bridge path joining y to x .

Proof If $\phi_i \in \mathcal{D}_{S_i}(\mathcal{E})$, then

$$M_{i,t} J_t \phi_i (X_t^x) = \int_X K_{W_i, S_i}(0; X_t^x, y) M_{i,t} J_t \phi_i (y) dX_y.$$

By taking expectation and using (8) and Proposition 2,

$$\begin{aligned} \mathbb{E}_x [M_{i,t} J_t \phi_i (X_t^x)] &= \int_X K_0(t; x, y) \left(\mathbb{E}_x \left[\frac{K_0^{\otimes N^2}(0; X_t^x, y)}{K_0(t; x, y)} M_{i,t} J_t \phi_i (y) \right] \right) dX_y \\ &= \int_X K_0(t; x, y) \mathbb{E}_{t;x,y} [M_{i,t} J_t \phi_i (\mathfrak{X}_{t;x,y})] dX_y, \end{aligned}$$

so after passing the limit, we get

$$\mathbb{E}_x [\mathcal{M}_t J_t \phi (X_t^x)] = \int_X K_0(t; x, y) \mathbb{E}_{t;x,y} [\mathcal{M}_t J_t \phi (\mathfrak{X}_{t;x,y})] dX_y, \quad \phi \in \mathcal{D}_S(\mathcal{E}).$$

On the other hand, from (22), we have

$$\mathbb{E}_x [\mathcal{M}_t J_t \phi (X_t^x)] = \int_X K_{W,S}(t; x, y) \phi (y) dX_y.$$

Since $\mathfrak{X}_{t;x,y} = y$ and ϕ is arbitrary, the result follows. □

Finally, we can establish the semigroup domination property for $K_{W,S}$.

Theorem 7 *If Assumption 1 holds, then there exist $C_1, C_2 > 0$ such that*

$$\|K_{W,S}(t; x, y)\| \leq C_1 e^{C_2 t} K_0(t; x, y), \tag{24}$$

for any $t > 0, x, y \in X$ and $\phi \in \mathcal{D}_S(\mathcal{E})$.

Proof It suffices to prove that

$$\left| \int_X \int_X \langle K_{W,S}(t; x, y), \phi(x) \otimes \psi(y) \rangle dX_x dX_y \right| \leq C_1 e^{C_2 t} \times \int_X K_0(t; x, y) |\phi(x)| |\psi(y)| dX_x dX_y,$$

where $\phi, \psi \in \mathcal{D}_S(\mathcal{E})$, and then send $\phi \otimes \psi \rightarrow \delta_x^{\otimes N} \otimes \delta_y^{\otimes N}$. For this, first note that

$$\begin{aligned} & \left| \int_X \int_X \langle K_{W,S}(t; x, y), \phi(x) \otimes \psi(y) \rangle dX_x dX_y \right| \\ & \leq \int_X \left| \int_X \langle K_{W,S}(t; x, y) \phi(x) dX_x, \psi(y) \rangle \right| dX_y \\ & = \int_X \left| \int_X K_0(t; x, y) \langle \mathbb{E}_{t;x,y} [\mathcal{M}_t J_t \phi(x)] dX_x, \phi(y) \rangle \right| dX_y, \end{aligned}$$

where we used (23) in the last step. On the other hand, since J_t is an isometry, Proposition 4 implies

$$\left| \mathbb{E}_{t;x,y} [M_{W_i, S_i^\epsilon} J_t] \right| \leq C_N e^{-\frac{1}{2} c_{1,i} t} |\mathbb{E}_{t;x,y} [e^{-c_{2,i} \lambda_t}]|,$$

where $c_{1,i} \text{Id}$ and $c_{2,i} \text{Id}$ are lower bounds for W_i and S_i , respectively. By sending $\epsilon \rightarrow 0$, we get

$$\left| \mathbb{E}_{t;x,y} [M_{i,t} J_t] \right| \leq C_N e^{-\frac{1}{2} c_{1,i} t} |\mathbb{E}_{t;x,y} [e^{-c_{2,i} \lambda_t}]|.$$

Clearly, we may assume that $c_{1,i} \rightarrow c_1$ and $c_{2,i} \rightarrow c_2$ as $i \rightarrow +\infty$ and that $c_2 < 0$, so after taking the limit in i , we may apply Theorem 3 and the ensuing discussion with $c_2 \geq p\kappa$ for some $p \in [1, +\infty)$ to get

$$\left| \mathbb{E}_{t;x,y} [\mathcal{M}_t J_t \phi(x)] \right| \leq C_1 e^{C_2 t} |\phi(x)|,$$

for $C_1 = C_N K_1^{(p)}$ and $C_2 = -c_1/2 + K_2^{(p)}$. This clearly proves the integral inequality above and completes the proof. \square

Corollary 1 *If we may take $c_2 = 0$, then*

$$\|K_{W,S}(t; x, y)\| \leq C_1 e^{-\frac{1}{2} c_1 t} K_0(t; x, y).$$

In particular, if $c_1 > -\lambda_0$, where λ_0 is the bottom of the spectrum of Δ_0 , then $\mathcal{H}(\mathcal{E}) \cap L^2(X, \mathcal{E}) = \{0\}$, where $\mathcal{H}(\mathcal{E}) = \ker \Delta_{W,S}$ is the space of harmonic sections.

Proof If $c_2 = 0$, then $\mathbb{E}_{t;x,y} [e^{-c_2 \lambda_t}] \leq 1$ and it is clear from the proof above that we may take $C_2 = -c_1/2$, so the estimate on $\|K_{W,S}\|$ follows. From this, the vanishing result can be easily obtained by means of a well-known argument [10,28]. \square

Corollary 2 *There exist $C_1, C_2 > 0$ such that*

$$\left\| e^{-\frac{1}{2} t \Delta_{W,S}} \phi \right\|_{L^1(X, \mathcal{E})} \leq C_1 e^{C_2 t} \|\phi\|_{L^1(X, \mathcal{E})}, \tag{25}$$

for any $t > 0$ and $\phi \in \mathcal{D}_S(\mathcal{E})$.

Proof From (24), we have

$$\begin{aligned} \left| \left(e^{-\frac{1}{2}t\Delta_{W,S}}\phi \right) (x) \right| &= \left| \int_X K_{W,S}(t; x, y)\phi(y)dX_y \right| \\ &\leq C_1 e^{C_2t} \int_X K_0(t; x, y)|\phi(y)|dX_y \\ &= C_1 e^{C_2t} (e^{-\frac{1}{2}t\Delta_0}|\phi|)(x), \end{aligned}$$

and after integration, we obtain

$$\left\| e^{-\frac{1}{2}t\Delta_{W,S}}\phi \right\|_{L^1(X,\mathcal{E})} \leq C_1 e^{C_2t} \left\| e^{-\frac{1}{2}t\Delta_0}|\phi| \right\|_{L^1(X)} \leq C_1 e^{C_2t} \|\phi\|_{L^1(X,\mathcal{E})},$$

where in the last step, we used that $e^{-\frac{1}{2}t\Delta_0}$ defines a contraction on $L^1(X)$. □

As we shall see below, this semigroup domination property is going to play a key role in the proof of our main result.

5 The proof of Theorem 1

In this section, we present the proof of Theorem 1 following the lines of the argument in [23]. We start with a useful integral identity.

Proposition 9 *Let $\phi \in \mathcal{D}_S(\mathcal{E})$ and $\xi \in \text{Dom}(\Delta_{W,S})$. Then, for any $t > 0$,*

$$\left(e^{-\frac{1}{2}t\Delta_{W,S}}\phi - \phi, \xi \right) = -\frac{1}{2} \int_0^t \int_X \left\langle e^{-\frac{1}{2}\tau\Delta_{W,S}}\phi, \Delta_{W,S}\xi \right\rangle dXd\tau. \tag{26}$$

Proof We compute:

$$\begin{aligned} \left(e^{-\frac{1}{2}t\Delta_{W,S}}\phi - \phi, \xi \right) &= \int_X \left\langle e^{-\frac{1}{2}t\Delta_{W,S}}\phi - e^{-\frac{1}{2}0\Delta_{W,S}}\phi, \xi \right\rangle dX \\ &= \int_0^t \int_X \left\langle \partial_\tau e^{-\frac{1}{2}\tau\Delta_{W,S}}\phi, \xi \right\rangle dXd\tau \\ &\stackrel{(12)}{=} -\frac{1}{2} \int_0^t \int_X \left\langle \Delta_{W,S}e^{-\frac{1}{2}\tau\Delta_{W,S}}\phi, \xi \right\rangle dXd\tau \\ &= -\frac{1}{2} \int_0^t \int_X \left\langle e^{-\frac{1}{2}\tau\Delta_{W,S}}\phi, \Delta_{W,S}\xi \right\rangle dXd\tau, \end{aligned}$$

where we used Proposition 3 in the last step. □

We now take a sequence of smooth, compactly supported functions h_i on X such that $0 \leq h_i \leq h_{i+1} \leq 1$, $h_i \rightarrow \mathbf{1}$ as $i \rightarrow +\infty$ and $\partial h_i / \partial \nu = 0$ along Σ .

Proposition 10 *If Assumption 2 is satisfied, then*

$$\zeta_i(x) = \int_0^{+\infty} e^{-t} \int_X K_0(t; x, y)h_i(y)dX_y dt, \quad x \in X,$$

is smooth and satisfies: a) $\zeta_i \rightarrow 1$; b) $\frac{1}{2}\Delta_0\zeta_i = h_i - \zeta_i \rightarrow 0$; and c) $\partial\zeta_i/\partial\nu = 0$ along Σ .

Proof In fact, we only use that X is stochastically complete by Theorem 2. By Proposition 1, (2), we have

$$\zeta_i(x) - 1 = \int_0^{+\infty} e^{-t} \int_X K_0(t; x, y) (h_i(y) - 1) dX_y dt,$$

from which a) follows easily. Also,

$$\begin{aligned} \frac{1}{2} \Delta_0 \zeta_i(x) &= \int_0^{+\infty} e^{-t} \int_X \frac{1}{2} \Delta_0 K_0(t; x, y) h_i(y) dX_y dt \\ &= - \int_X \left(\int_0^{+\infty} e^{-t} \frac{\partial}{\partial t} K_0(t; x, y) dt \right) h_i(y) dX_y \\ &= - \int_X \left(-K_0(0; x, y) + \int_0^{+\infty} e^{-t} K_0(t; x, y) dt \right) h_i(y) dX_y, \end{aligned}$$

which yields b). The proof of c) is obvious. \square

We now have all the ingredients needed in the proof of Theorem 1. Indeed, take ϕ as in Definition 1 and $\xi = \zeta_i \eta$, where η is as in Definition 1. Since $\partial \zeta_i / \partial \nu = 0$, $\zeta_i \eta \in \text{Dom}(\Delta_{W,S})$. Also, since $\eta \in \mathcal{H}_D(\mathcal{E})$, we may use (3) to check that $D(\zeta_i \eta) = D_C \zeta_i \cdot \eta$, so that

$$\Delta_{W,S}(\zeta_i \eta) = D_C^2 \zeta_i \cdot \eta = (\Delta_0 \zeta_i) \eta.$$

Hence, from Proposition 9 and Corollary 2, we get for each $t > 0$,

$$\begin{aligned} \left| \left(e^{-\frac{1}{2}t \Delta_{W,S}} \phi - \phi, \zeta_i \eta \right) \right| &\leq \frac{1}{2} \|\Delta_0 \zeta_i\|_{L^\infty(X)} \|\eta\|_{L^\infty(X, \mathcal{E})} \int_0^t \left\| e^{-\frac{1}{2}\tau \Delta_{W,S}} \phi \right\|_{L^1(X, \mathcal{E})} d\tau \\ &\leq \frac{C_1}{2} \|\Delta_0 \zeta_i\|_{L^\infty(X)} \|\eta\|_{L^\infty(X, \mathcal{E})} \|\phi\|_{L^1(X, \mathcal{E})} \int_0^t e^{C_2 \tau} d\tau. \end{aligned}$$

By sending $i \rightarrow +\infty$, Proposition 10 guarantees that the righthand side goes to 0. Since $\zeta_i \eta \rightarrow \eta$, we obtain (4), which completes the proof of Theorem 1.

6 Some examples

In this section, we indicate a few applications of our results to some generalized Laplacians appearing in geometry. As always, we assume that Assumption 2 is satisfied by the base manifold (X, g) .

6.1 The Hodge Laplacian

For $0 \leq p \leq n$, we denote by $\mathcal{A}^p(X) = \Gamma(X, \wedge^p T^*X)$ the space of differential p -forms on X . Let d be the exterior differential acting on forms and $d^* = \pm \star d \star$ be the co-differential, where \star is the Hodge star operator.

Recall that the Hodge Laplacian acting on p -forms is given by

$$\Delta_p = (d + d^*)^2 = dd^* + d^*d. \quad (27)$$

This is a generalized Laplacian due to the so-called Weitzenböck decomposition, namely,

$$\Delta_p = \nabla^* \nabla_p + R_p,$$

where $\nabla^* \nabla_p$ is the Bochner Laplacian associated with the standard Levi–Civita connection on $\wedge^p T^*M$ and R_p is the Weitzenböck operator, a (pointwise) selfadjoint element in $\Gamma(X, \text{End}(\wedge^p T^*X))$ whose local expression depends on the curvature tensor of (X, g) [28]. We note that $R_1 = \text{Ric}$. Also, recall that the Clifford bundle $\text{Cl}(TX)$ may be viewed as a Dirac bundle over itself under left Clifford multiplication. Moreover, under the standard vector bundle identification $\wedge T^*X = \text{Cl}(TX)$, one has $D_C = d + d^*$ [21, Chapter II, Theorem 5.12], so Δ_p is a generalized Dirac Laplacian by (27).

To implement boundary conditions in this setting, we note that, given $\alpha \in \mathcal{A}^p(X)$, its restriction to Σ decomposes into its tangential and normal components, namely,

$$\alpha = \alpha_t + \alpha_n. \tag{28}$$

Definition 4 We say that a p -form α is absolute if $\alpha_n = 0$ and $(d\alpha)_n = 0$.

It turns out that the differential condition in Definition 4 can be expressed in terms of the shape operator $B = -\nabla_\nu$ of Σ . To see this, extend B to $TM|_\Sigma$ by declaring that $B\nu = 0$ and then extend this further to $\wedge^p T^*X|_\Sigma$ as the selfadjoint operator \mathcal{B}_p given by

$$(\mathcal{B}_p \alpha)(e_1, \dots, e_p) = \sum_i \alpha(e_1, \dots, Be_i, \dots, e_p),$$

where $\{e_i\}$ is a local orthonormal frame. Notice that \mathcal{B}_p preserves the decomposition given by (28). More precisely, if Π_t and Π_n denote the orthogonal projections onto the tangential and normal factors, respectively, with the corresponding orthonormal bundle decomposition $\wedge^p T^*X|_\Sigma = \mathcal{F}_t \oplus \mathcal{F}_n$, then \mathcal{B}_p commutes with both projections. In particular, if α is absolute then $\mathcal{B}_p \alpha \in \Gamma(\Sigma, \mathcal{F}_t)$.

If we choose e_i so that $Be_j = \kappa_j e_j, j = 1, \dots, n-1$, where κ_j are the principal curvatures of Σ , it is immediate to check that

$$(\mathcal{B}_p \alpha)(e_{j_1}, \dots, e_{j_p}) = \left(\sum_k \kappa_{j_k} \right) \alpha(e_{j_1}, \dots, e_{j_p}), \quad \alpha \in \Gamma(\mathcal{F}_t),$$

which shows that the sums in the brackets are the eigenvalues of $\mathcal{B}_p|_{\mathcal{F}_t}$. The remarks above allow us to redefine \mathcal{B}_p so that $\mathcal{B}_p|_{\mathcal{F}_n} = 0$.

The next result shows that absolute boundary conditions are of mixed type.

Proposition 11 [5, Proposition 5.1] *A differential p -form α is absolute if and only if*

$$\Pi_t(\nabla_\nu - \mathcal{B}_p)\alpha = 0, \quad \Pi_n \alpha = 0. \tag{29}$$

This discussion shows that if we take $\mathcal{F}_+ = \mathcal{F}_t, \mathcal{F}_- = \mathcal{F}_n$ and $S = \mathcal{B}_p$, and of course if we assume that both R_p and \mathcal{B}_p are bounded from below, then the general setting in Sects. 3 and 4 applies here. In particular, we have the corresponding heat semigroup $e^{-\frac{1}{2}t \Delta_{R_p, \mathcal{B}_p}}$ at our disposal.

To rephrase Theorem 1 in this setting, we attach to the curvature invariants above the functions

$$r_{(p)} : X \rightarrow \mathbb{R}, \quad r_{(p)}(x) = \inf_{|\alpha|=1} \langle R_p(x)\alpha, \alpha \rangle,$$

and

$$\kappa_{(p)} : \Sigma \rightarrow \mathbb{R}, \quad \kappa_{(p)}(x) = \inf_{1 \leq j_1 < \dots < j_p \leq n-1} \kappa_{j_1}(x) + \dots + \kappa_{j_p}(x).$$

With this notation at hand, we can state the main result of this subsection, which is a straightforward application of Theorem 1.

Theorem 8 *If Assumption 2 is satisfied and for some $1 \leq p \leq n - 1$ we have $r_{(p)} \geq c_1$ for some $c_1 > -\infty$, then the heat conservation principle holds for Δ_{R_p, B_p} .*

Proof Use that $\kappa_{(p)} \geq c_2 > -\infty$ because B is bounded from below in view of Assumption 2. \square

Corollary 3 *If Assumption 2 is satisfied, then the heat conservation principle holds for $\Delta_{R_1, B}$.*

Proof Combine Theorem 8 with Theorem 2 and observe that here both lower bounds $r_{(1)} \geq c_1$ and $\sigma_{(1)} \geq c_2$ already follow from Assumption 2. \square

Remark 3 From Corollary 1, we obtain a vanishing result for absolute L^2 harmonic p -forms under the assumptions that $c_1 > -\lambda_0$ and Σ is (weakly) p -convex in the sense that

$$\inf_{x \in \Sigma} \kappa_{(p)}(x) \geq 0.$$

This strengthens [5, Theorem 5.3], where the result was obtained under the assumption that X has bounded geometry.

Remark 4 A simpler variant of the argument leading to Theorem 1, which dispenses with Proposition 10, yields a proof of Theorem 2. We first note that by geodesic completeness, we may assume that $\|dh_i\|_{L^\infty(X, \wedge^1 T^*X)} \rightarrow 0$. Thus, using (26) with $\phi = f$ a function as in item (4) of Proposition 1 and $\xi = h_i$, we have

$$\begin{aligned} \left(e^{-\frac{1}{2}t\Delta_0} f - f, h_i \right)_0 &= -\frac{1}{2} \int_0^t \int_X \left\langle e^{-\frac{1}{2}\tau\Delta_0} f, \Delta_0 h_i \right\rangle dX d\tau \\ &= -\frac{1}{2} \int_0^t \int_X \left\langle e^{-\frac{1}{2}\tau\Delta_0} f, d^* dh_i \right\rangle dX d\tau \\ &= -\frac{1}{2} \int_0^t \int_X \left\langle d e^{-\frac{1}{2}\tau\Delta_0} f, dh_i \right\rangle dX d\tau \\ &= -\frac{1}{2} \int_0^t \int_X \left\langle e^{-\frac{1}{2}\tau\Delta_{R_1, B}} df, dh_i \right\rangle dX d\tau, \end{aligned}$$

here we assume that $t < \mathbf{e}$, the extinction time of X_t . It follows from Corollary 2 applied to 1-forms that

$$\begin{aligned} \left| \left(e^{-\frac{1}{2}t\Delta_{R_1, B}} f - f, h_i \right)_0 \right| &\leq \frac{1}{2} \|dh_i\|_{L^\infty(X, \wedge^1 T^*X)} \int_0^t \left\| e^{-\frac{1}{2}\tau\Delta_{R_1, B}} df \right\|_{L^1(X, \wedge^1 T^*X)} d\tau \\ &\leq \frac{C_1}{2} \|dh_i\|_{L^\infty(X, \wedge^1 T^*X)} \|df\|_{L^1(X, \mathcal{E})} \int_0^t e^{C_2\tau} d\tau. \end{aligned}$$

By sending $i \rightarrow +\infty$, we then recover item (4) in Proposition 1 for some $t > 0$, which proves Theorem 2. Note that in order to avoid circularity in the argument, it is crucial here not using the functions ζ_i in Proposition 10. Finally, we observe that the argument above is a concrete manifestation of an abstract reasoning in [4, Theorem 3.2.6].

6.2 The Dirac Laplacian

Let X be a spin^c manifold and fix a spin^c structure. In [5, Section 5], it is proved a Feynman–Kac formula for the semigroup $e^{-\frac{1}{2}t\Delta}$ associated with the Dirac Laplacian $\Delta = D^2$, where D is the Dirac operator acting on spinors associated with a metric g on X and a unitary connection on the auxiliary complex line bundle \mathcal{U} . This formula was established under the assumption that the pair (X, Σ) has bounded geometry and by imposing suitable boundary conditions on spinors along Σ . As a consequence, a semigroup domination result for $e^{-\frac{1}{2}t\Delta}$ was derived in this setting. We now show that more generally, i.e., under Assumption 2, we may also derive a semigroup domination inequality for $e^{-\frac{1}{2}t\Delta}$ under suitable mixed boundary conditions. As a consequence, we will show that the corresponding heat conservation principle for Δ holds.

Let $\mathbb{S}X = P_{\text{Spin}^c}(X) \times_{\zeta} V$ be the spin bundle of X , where ζ is the complex spin representation [11, 21]. Thus, $P_{\text{Spin}^c}(X)$ is a Spin_n^c -principal bundle double covering $P_{SO}(X) \times P_{U_1}(X)$, where $P_{U_1}(X)$ is the U_1 -principal bundle associated with $\mathcal{U} \rightarrow X$, so the Levi–Civita connection on TX induces a metric connection on $\mathbb{S}X$, still denoted ∇ . The corresponding Dirac operator $D : \Gamma(X, \mathbb{S}X) \rightarrow \Gamma(X, \mathbb{S}X)$ is locally given by

$$D\psi = \sum_{i=1}^n \gamma(e_i)\nabla_{e_i}\psi, \quad \psi \in \Gamma(X, \mathbb{S}X),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame and $\gamma : TX \rightarrow \text{End}(\mathbb{S}X)$ is the Clifford product by tangent vectors. In this setting, the Dirac Laplacian operator $\Delta = D^2$ satisfies the Lichnerowicz decomposition

$$\Delta = \nabla^*\nabla + \mathfrak{R}, \quad \mathfrak{R} = \frac{\varrho}{4} + \frac{1}{2}\gamma(i\Theta), \tag{30}$$

where ϱ is the scalar curvature of X and $i\Theta$ is the curvature 2-form of the given unitary connection on \mathcal{F} . Clearly, this is a generalized Dirac Laplacian.

In the presence of the boundary, we must also consider the restricted spin bundle $\mathbb{S}X|_{\Sigma}$. By defining the restricted Clifford product and the restricted connection by

$$\gamma^{\top}(X)\psi = \gamma(X)\gamma(\nu)\psi, \quad X \in \Gamma(\Sigma, TX), \quad \psi \in \Gamma(\Sigma, \mathbb{S}X|_{\Sigma}),$$

and

$$\nabla_X^{\top}\psi = \nabla_X\psi - \frac{1}{2}\gamma^{\top}(BX)\psi, \tag{31}$$

respectively, where as usual $B = -\nabla\nu$ is the shape operator of Σ , then $\mathbb{S}X|_{\Sigma}$ becomes a Dirac bundle over $\text{Cl}(TX|_{\Sigma})$ [17, 24]. The associated Dirac operator $D^{\top} : \Gamma(\Sigma, \mathbb{S}X|_{\Sigma}) \rightarrow \Gamma(\Sigma, \mathbb{S}X|_{\Sigma})$ is

$$D^{\top} = \sum_{j=1}^{n-1} \gamma^{\top}(e_j)\nabla_{e_j}^{\top},$$

where the frame has been adapted so that $e_n = \nu$.

To see the relevance of this tangential Dirac operator, assume $Be_j = \kappa_j e_j$, where κ_j are the principal curvatures of Σ . It follows that

$$D^{\top} = \frac{H}{2} + \sum_{j=1}^{n-1} \gamma(e_j)\nabla_{e_j},$$

where $H = \text{tr } B$ is the mean curvature. Hence, $D = -\gamma(\nu)D$ is given by

$$D = D^\top + \nabla_\nu - \frac{H}{2}. \quad (32)$$

We now specify mixed boundary conditions in this setting. We start with an involutive endomorphism $\mathcal{I} \in \Gamma(X|_\Sigma, \mathbb{S}X)$, which we extend to a collared neighborhood of Σ such that $\nabla_\nu \mathcal{I} = 0$. Let Π_\pm be the corresponding projections and set $\mathcal{F}_\pm = \Pi_\pm \mathbb{S}X|_\Sigma$. In particular, $\nabla_\nu \Pi_\pm = \Pi_\pm \nabla_\nu$. We now recall a notion introduced in [5].

Definition 5 We say that the tangential Dirac operator D^\top intertwines the projections if $\Pi_\pm D^\top = D^\top \Pi_\mp$.

If this compatibility condition between D^\top and Π_\pm holds and $\psi, \eta \in \Gamma(\Sigma, \mathcal{F}_+)$, then $\langle D^\top \psi, \eta \rangle = 0$ and hence, by (32),

$$\begin{aligned} \langle D\psi, \eta \rangle &= \left\langle \left(\nabla_\nu - \frac{H}{2} \right) \psi, \eta \right\rangle \\ &= \left\langle \Pi_+ \left(\nabla_\nu - \frac{H}{2} \right) \psi, \eta \right\rangle \end{aligned} \quad (33)$$

Thus, we may proceed as in the proof of Proposition 3 to get

$$\begin{aligned} \int_\Sigma \langle \nabla_\nu \psi, \eta \rangle d\Sigma &= \int_\Sigma \langle D\psi, \eta \rangle d\Sigma + \int_\Sigma \frac{H}{2} \langle \psi, \eta \rangle d\Sigma \\ &= \int_\Sigma \left\langle \Pi_+ \left(\nabla_\nu - \frac{H}{2} \right) \psi, \eta \right\rangle d\Sigma + \int_\Sigma \frac{H}{2} \langle \psi, \eta \rangle d\Sigma. \end{aligned}$$

If we think of H as an endomorphism \widehat{H} of $\mathbb{S}X|_\Sigma$ such that $\widehat{H} = H \text{Id}$ on \mathcal{F}_+ and $\widehat{H} = 0$ on \mathcal{F}_- and impose the mixed boundary conditions

$$\Pi_+ \left(\nabla_\nu - \frac{\widehat{H}}{2} \right) \psi = 0, \quad \Pi_- \psi = 0, \quad (34)$$

then for compactly supported spinors ψ and η satisfying these conditions, we see that the bilinear form associated with Δ satisfies

$$\mathcal{Q}(\psi, \eta) = \int_X \langle \nabla \psi, \nabla \eta \rangle dX + \int_X \langle \mathfrak{R} \psi, \eta \rangle dX + \frac{1}{2} \int_\Sigma \langle \widehat{H} \psi, \eta \rangle d\Sigma.$$

Clearly, this is symmetric and bounded from below if \mathfrak{R} and H are uniformly bounded from below. It follows from (30) and Assumption 2 that \mathfrak{R} is bounded from below if and only if so does $i\Theta$. Moreover, H is always bounded from below. Thus, as an immediate application of Theorem 1, we obtain the main result of this subsection.

Theorem 9 *Let X be a spin^c manifold satisfying Assumption 2 and assume that $i\Theta$ is bounded from below. Then, the heat conservation principle holds for Δ .*

We note that examples of boundary conditions satisfying (34) include both chirality and MIT bag boundary conditions; see Remarks 5.1 and 5.2 in [5].

Remark 5 From Corollary 1, we obtain a vanishing result for L^2 harmonic spinors satisfying the given boundary conditions if we further assume that $\mathfrak{R} \geq c > -\lambda_0$ and Σ is mean convex ($H \geq 0$). This strengthens [5, Theorem 5.5], where the result was obtained under the assumption that X has bounded geometry.

6.3 The Jacobi operator on free boundary minimal immersions

Let (\bar{X}, \bar{g}) be a noncompact Riemannian manifold of dimension $\bar{n} > n$ and with boundary $\bar{\Sigma}$. Let $\Psi : (X, g) \looparrowright (\bar{X}, \bar{g})$ be a noncompact isometric immersion with boundary $\Sigma = X \cap \bar{\Sigma}$. If TX^\perp is the normal bundle of X , $\mathfrak{B} \in \Gamma(X, \text{Hom}(TX \otimes TX, TX^\perp))$ is the second fundamental form of X . Also, we denote by \bar{R} the curvature tensor of (\bar{X}, \bar{g}) .

Any compactly supported vector field $U \in \Gamma(X, T\bar{X}|_X)$ which is *admissible* in the sense that it is tangent to $\bar{\Sigma}$ along Σ gives rise to a one-parameter family of isometric immersions $t \in (-\varepsilon, \varepsilon) \mapsto \Psi_t : (X, g_t) \looparrowright (\bar{X}, \bar{g})$, $\varepsilon > 0$, such that $\Psi_0 = \Psi$ and

$$\frac{\partial \Psi_t}{\partial t} \Big|_{t=0} = U.$$

We then say that U is the variational field associated with the variation Ψ_t . A direct computation gives the first variation of the area functional

$$(\delta_{(X,g)} \text{Area})(U) = \frac{d}{dt} \text{Area}(X_t, g_t) \Big|_{t=0}$$

along a variational field U . We have

$$(\delta_{(X,g)} \text{Area})(U) = - \int_X \langle \mathcal{H}, U \rangle dX - \int_\Sigma \langle U, \nu \rangle d\Sigma, \tag{35}$$

where $\mathcal{H} = \text{trace } \mathfrak{B}$ is the mean curvature vector and ν is the inward pointing unit co-normal vector along Σ .

Definition 6 We say that X is a free boundary minimal immersion if it is a critical point for the functional Area under compactly supported variations.

By (35), this means that $\mathcal{H} = 0$ along X (this is the minimality condition) and $\langle U, \nu \rangle = 0$ along Σ for any U . This latter condition means that Σ meets $\bar{\Sigma}$ orthogonally (this is the free boundary condition). Notice that this implies that ν is normal to $\bar{\Sigma}$. In particular, it makes sense to consider $B_{\bar{\Sigma}}^\nu$, the shape operator of $\bar{\Sigma}$ in the direction of ν .

If (X, g) is a free boundary minimal immersion, it is natural to compute the second variation of the area along admissible variational fields U and V as above. The result is

$$\left(\delta_{(X,g)}^2 \text{Area} \right) (U, V) = \int_X \langle \mathcal{J}U, V \rangle dX - \int_\Sigma \left\langle \left(\nabla_\nu^\perp + B_{\bar{\Sigma}}^\nu \right) U, V \right\rangle d\Sigma. \tag{36}$$

Here, ∇^\perp is the normal connection on TX^\perp and the *Jacobi operator* is given by

$$\mathcal{J} = \nabla^* \nabla^\perp - W,$$

where $\nabla^* \nabla^\perp$ is the Bochner Laplacian associated with ∇^\perp , $W = R + B$, $B = \mathfrak{B} \circ \mathfrak{B}^\bullet \in \Gamma(X, \text{End}(TX^\perp))$ and $R \in \Gamma(X, \text{End}(TX^\perp))$ is given by

$$\langle RU, V \rangle = \sum_{i=1}^n \langle \bar{R}_{U, e_i} e_i, V \rangle.$$

Since W is clearly selfadjoint, \mathcal{J} is a generalized Laplacian. But notice that it is *not* a generalized Dirac Laplacian, so a heat conservation principle corresponding to Theorem 1 does not necessarily hold here; however, see Remark 8.

As a consequence of (36), any Morse-theoretic notion involving this variational problem (like index, nullity, etc.) should be addressed by imposing to variational fields the Robin-type boundary condition

$$(\nabla_v^\perp + B_{\Sigma}^v)U = 0. \tag{37}$$

In particular, *Jacobi fields*, i.e., solutions of $\mathcal{J}U = 0$, should be studied under this boundary condition. We refer to [30] for details.

Remark 6 Note that, strictly speaking, (37) is of mixed type. Indeed, in the language of Sect. 3, it is obtained by taking $\mathcal{I} = \text{Id}$, so that $\Pi_+ = \text{Id}$ and $\Pi_- = 0$, and $S = -B_{\Sigma}^v$.

Now, by (10), we can rewrite (36) as

$$(\delta_{(X,g)}^2 \text{Area})(U, V) = \int_X (\langle \nabla^\perp U, \nabla^\perp V \rangle - \langle WU, V \rangle) dX - \int_{\Sigma} \langle B_{\Sigma}^v U, V \rangle d\Sigma.$$

Hence, the bilinear form

$$Q(U, V) = \int_X \langle \mathcal{J}U, V \rangle dX$$

is given by

$$\begin{aligned} Q(U, V) &= \int_X (\langle \nabla^\perp U, \nabla^\perp V \rangle - \langle WU, V \rangle) dX + \int_{\Sigma} \langle \nabla_v^\perp U, V \rangle d\Sigma \\ &= \int_X (\langle \nabla^\perp U, \nabla^\perp V \rangle - \langle WU, V \rangle) dX + \int_{\Sigma} \langle (\nabla_v^\perp + B_{\Sigma}^v)U, V \rangle d\Sigma \\ &\quad - \int_{\Sigma} \langle B_{\Sigma}^v U, V \rangle d\Sigma. \end{aligned}$$

Thus, Q is symmetric and bounded from below if we assume that the variational fields U and V satisfy (37) and impose lower bounds of the type

$$-W \geq c_1 \text{Id}, \quad -B_{\Sigma}^v \geq c_2 \text{Id}. \tag{38}$$

Under these assumptions, all the results in Sect. 4 hold for $\mathcal{J}_{-W, -B_{\Sigma}^v}$. In particular, the following vanishing result, corresponding to Corollary 1, holds true.

Theorem 10 *Under the conditions above, assume that $c_1 > -\lambda_0$ and $c_2 = 0$ in (38). Then, X carries no L^2 Jacobi field satisfying (37).*

Example 1 Let \bar{X} be the exterior of an open geodesic ball in hyperbolic space \mathbb{H}^n , so that $\bar{\Sigma}$ is the geodesic sphere bounding this ball. Now take any totally geodesic submanifold passing through the center of the ball and take X to be the portion of this submanifold outside the ball. Then, Theorem 10 clearly applies to the free boundary minimal submanifold X .

Remark 7 We note that the proof of the domination property in this setting is substantially simplified in the sense that we can get rid of the parameter $\epsilon > 0$ appearing in Sect. 4. In fact, this kind of simplification will take place whenever, in the notation of Section 3, we take $\mathcal{I} = \text{Id}$ as in Remark 6. To see this, take ϕ satisfying (18) with $\Pi_+ = \text{Id}$ and $\Pi_- = 0$ and directly apply Itô’s formula to $M_{W,S,t} \phi_{T-t}^\dagger(\tilde{X}_t)$ (no mention to ϵ) as in the proof of Theorem 4, where we assume that both W and S are uniformly bounded. We end up with

$$dM_{W,S,t} \phi_{T-t}^\dagger(\tilde{X}_t^x) = \left\langle M_{W,S,t} \mathcal{L}_H \phi_{T-t}^\dagger(\tilde{X}_t^x), db_t \right\rangle - M_{W,S,t} L^\dagger \phi_{T-t}^\dagger(\tilde{X}_t^x) dt + M_{W,S,t} (\mathcal{L}_{v^\dagger} - S^\dagger) \phi_{T-t}^\dagger(\tilde{X}_t^x) d\lambda_t,$$

and since the last two terms vanish, $M_{W,S,t} \phi_{T-t}^\dagger(\tilde{X}_t^x)$ is found to be a martingale. In this way, we obtain a proof of the Feynman–Kac formula in Theorem 4 without having to appeal to the rather technical ϵ^{-1} -perturbation in Propositions 4 and 5. From this point, we may use the approximation scheme to remove the upper bounds on W and S just as we did in Sect. 4.

Remark 8 Let (\bar{X}, g) as above be a Kähler manifold and assume that the free boundary minimal submanifold $X \subset \bar{X}$ of dimension $n/2$ is Lagrangian in the sense that $\Omega|_X = 0$, where Ω is the underlying symplectic form. The map that to each normal vector $u \in TX_x^\perp$ associates the 1-form $\alpha_u = i_u \Omega \in T^*X$ defines a bundle isomorphism between TX^\perp and T^*X , so that to each admissible variation vector field $U \in \Gamma(X, TX^\perp)$ there corresponds a 1-form $\alpha_U \in \mathcal{A}^1(X)$. If we assume further that \bar{X} is a Ricci flat, Kähler–Einstein manifold, then under this identification, we have $\mathcal{J} = \Delta_1$, the Hodge Laplacian acting on 1-forms [26, Proposition 4.1]. In particular, by Sect. 6.1, \mathcal{J} is a generalized Dirac Laplacian. Recalling that $\Delta_1 = \nabla^* \nabla + \text{Ric}$ and that Assumption 2 already implies that Ric is bounded from below, an application of Theorem 1 gives the following result: *if $-B_{\frac{v}{\Sigma}}$ is bounded from below, then the heat conservation principle holds for \mathcal{J} .*

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