Relative Čech–Dolbeault homology and applications

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Abstract

We define the relative Dolbeault homology of a complex manifold with currents via a Čech approach, and we prove its equivalence with the relative Čech–Dolbeault cohomology as defined by Suwa (in: Singularities–Niigata–Toyama 2007. Advanced studies in pure mathematics, vol 56. Mathematical Society of Japan, Tokyo, pp 321–340, 2009). This definition is then used to compare the relative Dolbeault cohomology groups of two complex manifolds of the same dimension related by a suitable proper surjective holomorphic map. Finally, an application to blow-ups is considered and a blow-up formula for the Dolbeault cohomology in terms of relative cohomology is presented.

Keywords Dolbeault cohomology · Relative cohomology · Currents

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Introduction

On a complex manifold an important global invariant is represented by the Dolbeault cohomology which can be described equivalently using complex differential forms or currents. In particular, this double interpretation was used fruitfully by Wells in [17] in order to compare the Dolbeault cohomology of two complex manifolds of the same dimensions related by a proper holomorphic surjective map. More precisely, he proved that if $\pi : \tilde{X} \rightarrow X$ is a proper holomorphic surjective map between two complex manifolds of the same dimension, then the induced map in cohomology

$$\pi^* : H_{\bar{\partial}}^{\bullet, \bullet}(X) \rightarrow H_{\bar{\partial}}^{\bullet, \bullet}(\tilde{X})$$

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is injective. In particular, if X and $\tilde{X}$ are compact, we have the dimensional inequalities $h_{\partial}^{\bullet,\bullet}(X) \leq h_{\partial}^{\bullet,\bullet}(\tilde{X})$ for their Hodge numbers. In fact, this result can be weakened to almost-complex manifolds as done in [14] for pseudo-holomorphic maps, where the Dolbeault cohomology is replaced by the cohomology groups introduced by Li and Zhang [6].

The Dolbeault cohomology can also be interpreted via a Čech approach, as done in [12], and it turns out that this approach is also useful in order to define its relative version. The relative Čech–Dolbeault cohomology, which is equivalent to the relative Dolbeault cohomology as defined, for instance, in [5] (cf. [13]), can be used to describe the localization theory of characteristic classes ([1, 12]) and recently has found applications to the Sato hyperfunction theory [4].

In the present paper, we define the relative Čech–Dolbeault homology of a complex manifold in terms of currents, and we prove in Theorem 2.5 that this description is equivalent to Suwa's one using complex differential forms. This different interpretation is used in Theorem 3.1 in order to prove a Wells-type result for relative cohomology; in particular, we prove the following

Theorem *Let $\pi : \tilde{X} \rightarrow X$ be a proper, surjective, holomorphic map between two complex manifolds of the same dimension. Suppose that X is connected and let S and $\tilde{S}$ be closed complex submanifolds of X and $\tilde{X}$, respectively, such that $\pi(\tilde{S}) \subset S$ and $\pi(\tilde{X} \setminus \tilde{S}) \subset X \setminus S$. Then,*

$$\pi^* : H_D^{p,q}(X, X \setminus S) \longrightarrow H_D^{p,q}(\tilde{X}, \tilde{X} \setminus \tilde{S})$$

is injective for any p, q .

Differently from the classical Dolbeault theory, a dimensional inequality in this setting cannot be expected even if X and $\tilde{X}$ are compact; indeed, in general the relative Dolbeault cohomology groups of a compact complex manifold are infinite-dimensional. A similar result can be proved for the relative de Rham cohomology (cf. Theorem 3.2).

Notice that the hypothesis in the previous theorem is satisfied by modifications. In particular, if $\tau : \tilde{X} \rightarrow X$ is the blow-up of a complex manifold X along a closed submanifold Z , then the previous assumptions are satisfied with $S = Z$ and $\tilde{S} = E := \pi^{-1}(Z)$ the exceptional divisor. Then, as a consequence, the Dolbeault cohomology of $\tilde{X}$ can be expressed in terms of the Dolbeault cohomology of X and the relative cohomology, more precisely (cf. Corollary 4.1)

Corollary *Let $\tau : \tilde{X} \rightarrow X$ be the blow-up of a complex manifold X along a closed submanifold Z . Then, there are isomorphisms*

$$H_{\partial}^{p,q}(\tilde{X}) \simeq \tau^* H_{\partial}^{p,q}(X) \oplus \frac{H_D^{p,q}(\tilde{X}, \tilde{X} \setminus E)}{\tau^* H_D^{p,q}(X, X \setminus Z)},$$

for any p, q , where E is the exceptional divisor of the blow-up.

This blow-up formula for the Dolbeault cohomology will find further applications in [2]. For other formulas of this kind, we refer the reader to [7–10, 16].

1 Preliminaries and notations

We start by fixing some notations and recalling some results about relative Čech–Dolbeault cohomology as presented in [12].

Čech–Dolbeault cohomology. Let X be a complex manifold of complex dimension n . We denote by $A^{p,q}$ the space of smooth (p, q) -forms on X . Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of X and consider

$$A^{p,q}(\mathcal{U}) := A^{p,q}(U_0) \oplus A^{p,q}(U_1) \oplus A^{p,q-1}(U_{01})$$

where by definition $U_{01} := U_0 \cap U_1$, with the differential operator $\bar{D} : A^{p,q}(\mathcal{U}) \rightarrow A^{p,q+1}(\mathcal{U})$ defined on every element $(\xi_0, \xi_1, \xi_{01}) \in A^{p,q}(\mathcal{U})$ by

$$\bar{D}(\xi_0, \xi_1, \xi_{01}) = (\bar{\partial}\xi_0, \bar{\partial}\xi_1, \xi_1 - \xi_0 - \bar{\partial}\xi_{01}).$$

The Čech–Dolbeault cohomology associated with the covering $\mathcal{U}$ is defined by $H_D^{\bullet,\bullet}(\mathcal{U}) = \text{Ker } \bar{D} / \text{Im } \bar{D}$ (cf. [12] where this definition is given for an arbitrary open covering of X). In [12] it is proven that the morphism $A^{p,q}(X) \rightarrow A^{p,q}(\mathcal{U})$ given by $\xi \mapsto (\xi|_{U_0}, \xi|_{U_1}, 0)$ induces an isomorphism

$$H_D^{\bullet,\bullet}(X) \xrightarrow{\sim} H_D^{\bullet,\bullet}(\mathcal{U}),$$

where $H_D^{p,q}(X)$ denotes the Dolbeault cohomology of X . In particular, the definition does not depend on the choice of the covering of X . The inverse map is given by assigning to the class of $\xi = (\xi_0, \xi_1, \xi_{01})$ the class of global $\bar{\partial}$ -closed form $\rho_0\xi_0 + \rho_1\xi_1 - \bar{\partial}\rho_0 \wedge \xi_{01}$, where (ρ_0, ρ_1) is a partition of unity subordinate to the covering $\mathcal{U}$.

One can define naturally the cup product, integration on top-degree cohomology and the Kodaira–Serre duality, and they turn out to be compatible with the above isomorphism (see [12] for more details).

Relative Dolbeault cohomology. Let S be a closed set in X . We let $U_0 = X \setminus S$ and U_1 be an open neighborhood of S in X and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of X . We set, for any p, q ,

$$A^{p,q}(\mathcal{U}, U_0) := \{\xi \in A^{p,q}(\mathcal{U}) \mid \xi_0 = 0\} = A^{p,q}(U_1) \oplus A^{p,q-1}(U_{01}).$$

Then, $(A^{p,\bullet}(\mathcal{U}, U_0), \bar{D})$ is a subcomplex of $(A^{p,\bullet}(\mathcal{U}), \bar{D})$. Let $H_D^{p,q}(\mathcal{U}, U_0)$ be the cohomology of $(A^{p,\bullet}(\mathcal{U}, U_0), \bar{D})$. From the exact sequence

$$0 \rightarrow A^{p,\bullet}(\mathcal{U}, U_0) \rightarrow A^{p,\bullet}(\mathcal{U}) \rightarrow A^{p,\bullet}(U_0) \rightarrow 0$$

where the first map is the inclusion and the second map is the projection on the first element, we obtain the long exact sequence in cohomology

$$\dots \rightarrow H_D^{p,q-1}(U_0) \xrightarrow{\delta} H_D^{p,q}(\mathcal{U}, U_0) \xrightarrow{j^*} H_D^{p,q}(\mathcal{U}) \xrightarrow{i^*} H_D^{p,q}(U_0) \rightarrow \dots$$

Therefore, $H_D^{\bullet,\bullet}(\mathcal{U}, U_0)$ is determined uniquely modulo canonical isomorphism. Thus, we denote it by $H_D^{\bullet,\bullet}(X, X \setminus S)$ and we call it the *relative Dolbeault cohomology* of X .

Together with integration theory, the relative Dolbeault cohomology has been used to study the localization of characteristic classes (cf. [1, 12]) and has found more recent applications to hyperfunction theory (cf. [4]).

However notice that the explicit computation of these relative Dolbeault cohomology groups can be very difficult even in some easy situations.

If X and $\tilde{X}$ are complex manifolds, S and $\tilde{S}$ are closed subsets in X and $\tilde{X}$, respectively, and $f : \tilde{X} \rightarrow X$ is a holomorphic map such that $f(\tilde{S}) \subset S$ and $f(\tilde{X} \setminus \tilde{S}) \subset f(X \setminus S)$, then f induces a natural map in relative cohomology. Indeed, setting $U_0 := X \setminus S$, $\tilde{U}_0 := \tilde{X} \setminus \tilde{S}$ and let $U_1, \tilde{U}_1$ be open neighborhoods of S and $\tilde{S}$ in X and $\tilde{X}$, respectively, chosen in such

a way that $f(\tilde{U}_1) \subset U_1$. Take the open coverings $\mathcal{U} := \{U_0, U_1\}$ and $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$ of X and $\tilde{X}$, respectively, then we have a homomorphism

$$f^* : A^{\bullet, \bullet}(\mathcal{U}, U_0) \rightarrow A^{\bullet, \bullet}(\tilde{\mathcal{U}}, \tilde{U}_0)$$

defined on every element $(\xi_1, \xi_{01}) \in A^{\bullet, \bullet}(\mathcal{U}, U_0)$ as

$$f^*(\xi_1, \xi_{01}) := (f^*\xi_1, f^*\xi_{01})$$

which induces a homomorphism in relative cohomology

$$f^* : H_D^{\bullet, \bullet}(X, X \setminus S) \rightarrow H_D^{\bullet, \bullet}(\tilde{X}, \tilde{X} \setminus \tilde{S}).$$

2 Relative Čech–Dolbeault homology

In this section we describe a Čech interpretation of the Dolbeault homology and we give a definition for its relative counterpart. Finally, in Theorem 2.5 we prove that the relative Dolbeault cohomology can be computed equivalently using forms and currents.

Čech–Dolbeault homology. Before defining the relative Dolbeault homology, we discuss a Čech interpretation of the Dolbeault homology. Let X be a complex manifold of complex dimension n and denote by $\mathcal{K}^{p,q}(X) = \mathcal{K}_{n-p, n-q}(X)$ the space of currents of bidegree (p, q) , or equivalently of bidimension $(n-p, n-q)$, on X , namely the topological dual of the space of $(n-p, n-q)$ -forms with compact support in X .

The differential operator $\bar{\partial} : \mathcal{K}^{p,q}(X) \rightarrow \mathcal{K}^{p,q+1}(X)$ is defined as usual, for any $T \in \mathcal{K}^{p,q}(X)$, $\varphi \in A^{n-p, n-q-1}(X)$ with compact support, as

$$\langle \bar{\partial}T, \varphi \rangle := (-1)^{p+q+1} \langle T, \bar{\partial}\varphi \rangle,$$

where $\langle -, - \rangle$ stands for the duality pairing. Then, for any p , we denote by $H_{\bar{\partial}\mathcal{K}}^{p, \bullet}(X)$ the cohomology of the complex $(\mathcal{K}^{p, \bullet}(X), \bar{\partial})$ which is called the *Dolbeault homology of X* .

Now let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of X where I is an ordered set and let $I^{(r)} := \{(\alpha_0, \dots, \alpha_r) \mid \alpha_0 < \dots < \alpha_r, \alpha_v \in I\}$. We set, for any r, p, q ,

$$B^r(\mathcal{U}, \mathcal{K}^{p,q}) := \prod_{(\alpha_0, \dots, \alpha_r) \in I^{(r)}} \mathcal{K}^{p,q}(U_{\alpha_0 \dots \alpha_r})$$

where, by definition, $U_{\alpha_0 \dots \alpha_r} := U_{\alpha_0} \cap \dots \cap U_{\alpha_r}$, and we define the boundary operator as

$$\begin{aligned} \delta : B^r(\mathcal{U}, \mathcal{K}^{p,q}) &\rightarrow B^{r+1}(\mathcal{U}, \mathcal{K}^{p,q}) \\ (\delta T)_{\alpha_0 \dots \alpha_{r+1}} &:= \sum_{v=0}^{r+1} (-1)^v T_{\alpha_0 \dots \hat{\alpha}_v \dots \alpha_{r+1}} \end{aligned}$$

where all the currents $T_{\alpha_0 \dots \hat{\alpha}_v \dots \alpha_{r+1}}$ have to be restricted to $U_{\alpha_0 \dots \alpha_{r+1}}$. Moreover, setting $\bar{\partial} : B^r(\mathcal{U}, \mathcal{K}^{p,q}) \rightarrow B^r(\mathcal{U}, \mathcal{K}^{p,q+1})$ the extension of the operator $\bar{\partial}$ to every components, one gets that $B^\bullet(\mathcal{U}, \mathcal{K}^{p, \bullet})$ endowed with the operators δ and $\bar{\partial}$ is a double complex. The associated total complex will be denoted by $(\mathcal{K}^{p, \bullet}(\mathcal{U}), \bar{D}_{\mathcal{K}})$, namely

$$\mathcal{K}^{p,q}(\mathcal{U}) := \bigoplus_{s+r=q} B^s(\mathcal{U}, \mathcal{K}^{p,s})$$

and

$$\bar{D}_{\mathcal{K}}(T_{\alpha_0 \dots \alpha_r}) = \sum_{v=0}^r (-1)^v T_{\alpha_0 \dots \hat{\alpha}_v \dots \alpha_r} + (-1)^r \bar{\partial} T_{\alpha_0 \dots \alpha_r}.$$

In particular, for $r = 1$ we have

$$\bar{D}_{\mathcal{K}}(T_{\alpha_0\alpha_1}) := T_{\alpha_1} - T_{\alpha_0} - \bar{\partial} T_{\alpha_0\alpha_1}.$$

Definition 2.1 The cohomology of the complex $(\mathcal{K}^{p,\bullet}(\mathcal{U}), \bar{D}_{\mathcal{K}})$ will be denoted by $H_{\bar{D}_{\mathcal{K}}}^{\bullet,\bullet}(\mathcal{U})$ and will be called *Čech Dolbeault homology* associated with the covering $\mathcal{U}$.

This definition does not depend on the open covering; indeed, one has the following

Theorem 2.2 *The restriction map $\mathcal{K}^{p,q}(X) \rightarrow B^0(\mathcal{U}, \mathcal{K}^{p,q})$ induces a natural isomorphism*

$$H_{\bar{\partial}_{\mathcal{K}}}^{p,q} \rightarrow H_{\bar{D}_{\mathcal{K}}}^{p,q}(\mathcal{U}).$$

Proof The argument works exactly as in [12], since the complex of (p, q) -currents on $\mathcal{U}$ is acyclic, being a $\mathcal{C}^\infty(X)$ -module. For completeness we recall here the proof. We consider the first spectral sequence associated with the double complex $B^\bullet(\mathcal{U}, \mathcal{K}^{p,\bullet})$. In particular, at the second page one has

$$'E_2^{q,r} := H_{\bar{\partial}}^q H_{\delta}^r(B^\bullet(\mathcal{U}, \mathcal{K}^{p,\bullet}))$$

which converges to $H_{\bar{D}_{\mathcal{K}}}^{p,q+r}(\mathcal{U})$. By the acyclic property, for $r > 0$, $H_{\delta}^r(B^\bullet(\mathcal{U}, \mathcal{K}^{p,\bullet})) = 0$ and for $r = 0$, $H_{\delta}^0(B^\bullet(\mathcal{U}, \mathcal{K}^{p,\bullet})) = \mathcal{K}^{p,\bullet}(X)$. Therefore, $H_{\bar{D}_{\mathcal{K}}}^{p,q}(\mathcal{U}) \simeq 'E_2^{q,0} \simeq H_{\bar{\partial}_{\mathcal{K}}}^{p,q}(X)$. $\square$

Remark 2.3 Since the Dolbeault cohomology can be computed using currents, it follows by the previous theorem that there is also an isomorphism with the Dolbeault cohomology of X , namely

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) \simeq H_{\bar{\partial}_{\mathcal{K}}}^{\bullet,\bullet}(X) \simeq H_{\bar{D}_{\mathcal{K}}}^{\bullet,\bullet}(\mathcal{U}).$$

Since to every (p, q) -form φ on X we can associate a (p, q) -current $i(\varphi) := \int_X \varphi \wedge \cdot$, then we have a natural injective map

$$i : A^{p,q}(\mathcal{U}) \rightarrow \mathcal{K}^{p,q}(\mathcal{U})$$

where by definition (see [12] for more details)

$$A^{p,q}(\mathcal{U}) := \oplus_{s+r=q} \left(\prod_{(\alpha_0, \dots, \alpha_r) \in I(r)} A^{p,s}(U_{\alpha_0 \dots \alpha_r}) \right).$$

Relative Čech–Dolbeault homology. Now we define the relative Čech Dolbeault homology. Let S be a closed set in X . We let $U_0 = X \setminus S$ and U_1 be an open neighborhood of S in X and consider the open covering $\mathcal{U} = \{U_0, U_1\}$ of X . In particular, in this situation we have

$$\mathcal{K}^{p,q}(\mathcal{U}) = B^0(\mathcal{U}, \mathcal{K}^{p,q}) \oplus B^1(\mathcal{U}, \mathcal{K}^{p,q-1})$$

and an element of $\mathcal{K}^{p,q}(\mathcal{U})$ can be written as a triple (T_0, T_1, T_{01}) where T_0 is a (p, q) -current on U_0 , T_1 is a (p, q) -current on U_1 and T_{01} is a $(p, q-1)$ -current on U_{01} .

We set

$$\mathcal{K}^{p,q}(\mathcal{U}, U_0) := \{T \in \mathcal{K}^{p,q}(\mathcal{U}) \mid T_0 = 0\} = \mathcal{K}^{p,q}(U_1) \oplus \mathcal{K}^{p,q-1}(U_{01}).$$

Then, $\mathcal{K}^{p,\bullet}(\mathcal{U}, U_0)$ is a subcomplex of $\mathcal{K}^{p,\bullet}(\mathcal{U})$. Therefore, we have the following short exact sequence

$$0 \rightarrow \mathcal{K}^{p,\bullet}(\mathcal{U}, U_0) \rightarrow \mathcal{K}^{p,\bullet}(\mathcal{U}) \rightarrow \mathcal{K}^{p,\bullet}(U_0) \rightarrow 0$$

where the first map is the inclusion and the second map is the projection on the first factor and clearly by definition

$$\bar{D}_{\mathcal{K}} : \mathcal{K}^{p,q}(\mathcal{U}, U_0) \rightarrow \mathcal{K}^{p,q+1}(\mathcal{U}, U_0).$$

We denote by $H_{\bar{D}_{\mathcal{K}}}^{p,q}(\mathcal{U}, U_0)$ the associated cohomology. We get the following long exact sequence in homology

$$\dots \rightarrow H_{\bar{\partial}_{\mathcal{K}}}^{p,q-1}(U_0) \xrightarrow{\delta} H_{\bar{D}_{\mathcal{K}}}^{p,q}(\mathcal{U}, U_0) \xrightarrow{j^*} H_{\bar{D}_{\mathcal{K}}}^{p,q}(\mathcal{U}) \xrightarrow{i^*} H_{\bar{\partial}_{\mathcal{K}}}^{p,q}(U_0) \rightarrow \dots$$

By Theorem 2.2, we see that $H_{\bar{\partial}_{\mathcal{K}}}^{p,q}(\mathcal{U}, U_0)$ is determined uniquely modulo canonical isomorphisms.

Definition 2.4 In the above situation, we set $H_{\bar{D}_{\mathcal{K}}}^{p,q}(X, X \setminus S) := H_{\bar{D}_{\mathcal{K}}}^{p,q}(\mathcal{U}, X \setminus S)$ and we call it the *relative Čech–Dolbeault homology* of X with respect to $X \setminus S$.

Notice that we have an injective map

$$i : A^{p,q}(\mathcal{U}, U_0) \rightarrow \mathcal{K}^{p,q}(\mathcal{U}, U_0)$$

which induces naturally a map

$$H_{\bar{D}}^{p,q}(X, X \setminus S) \rightarrow H_{\bar{D}_{\mathcal{K}}}^{p,q}(X, X \setminus S)$$

from the relative Dolbeault cohomology to the relative Dolbeault homology.

Theorem 2.5 *The map*

$$H_{\bar{D}}^{p,q}(X, X \setminus S) \rightarrow H_{\bar{D}_{\mathcal{K}}}^{p,q}(X, X \setminus S)$$

is an isomorphism.

Proof We have the following commutative diagram between forms and currents

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{p,q}(\mathcal{U}, U_0) & \longrightarrow & A^{p,q}(\mathcal{U}) & \longrightarrow & A^{p,q}(U_0) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}^{p,q}(\mathcal{U}, U_0) & \longrightarrow & \mathcal{K}^{p,q}(\mathcal{U}) & \longrightarrow & \mathcal{K}^{p,q}(U_0) \longrightarrow 0 \end{array}$$

which induces the commutative diagram with exact rows

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H_{\bar{\partial}}^{p,q-1}(X) & \longrightarrow & H_{\bar{\partial}}^{p,q-1}(X \setminus S) & \longrightarrow & H_D^{p,q}(X, X \setminus S) & \longrightarrow & H_{\bar{\partial}}^{p,q}(X) & \longrightarrow & H_{\bar{\partial}}^{p,q}(X \setminus S) & \longrightarrow \dots \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq & \\ \dots & \longrightarrow & H_{\bar{\partial}_{\mathcal{K}}}^{p,q-1}(X) & \longrightarrow & H_{\bar{\partial}_{\mathcal{K}}}^{p,q-1}(X \setminus S) & \longrightarrow & H_{\bar{D}_{\mathcal{K}}}^{p,q}(X, X \setminus S) & \longrightarrow & H_{\bar{\partial}_{\mathcal{K}}}^{p,q}(X) & \longrightarrow & H_{\bar{\partial}_{\mathcal{K}}}^{p,q}(X \setminus S) & \longrightarrow \dots \end{array}$$

Since the Dolbeault cohomology of X and $X \setminus S$ can be computed equivalently using differential forms or currents, we have that the first two and the last two vertical maps are isomorphisms. By the five lemma, we obtain the same conclusion for the central vertical morphism. $\square$

Now let $\pi : \tilde{X} \rightarrow X$ be a proper holomorphic map between two complex manifolds and let S be a closed subset of X and $\tilde{S}$ a closed subset of $\tilde{X}$ and take $U_0 = X \setminus S$ and U_1 a neighborhood of S in X , similarly $\tilde{U}_0 = \tilde{X} \setminus \tilde{S}$ and $\tilde{U}_1$ a neighborhood of $\tilde{S}$ in $\tilde{X}$. Suppose that $\pi(\tilde{S}) \subset S$, $\pi(\tilde{U}_0) \subset U_0$ and $\pi(\tilde{U}_1) \subset U_1$. Then, $\mathcal{U} := \{U_0, U_1\}$ and $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$ are compatible open coverings of X and $\tilde{X}$, respectively. We define the *push-forward* $\pi_* : \mathcal{K}^{p,q}(\tilde{\mathcal{U}}, \tilde{U}_0) \rightarrow \mathcal{K}^{p,q}(\mathcal{U}, U_0)$, namely

$$\pi_* : \mathcal{K}^{p,q}(\tilde{U}_1) \oplus \mathcal{K}^{p,q-1}(\tilde{U}_{01}) \rightarrow \mathcal{K}^{p,q}(U_1) \oplus \mathcal{K}^{p,q-1}(U_{01})$$

as

$$(T_1, T_{01}) \mapsto (\pi_* T_1, \pi_* T_{01}).$$

In particular, the push-forward commutes with the differential $\bar{D}_{\mathcal{K}}$; indeed,

$$\pi_* \bar{D}_{\mathcal{K}}(T_1, T_{01}) = (\pi_* \bar{\partial} T_1, \pi_* T_1 - \pi_* \bar{\partial} T_{01}) = (\bar{\partial} \pi_* T_1, \pi_* T_1 - \bar{\partial} \pi_* T_{01}) = \bar{D}_{\mathcal{K}} \pi_*(T_1, T_{01}),$$

where in the second equality we use that π_* commutes with the operator $\bar{\partial}$. Hence, the push-forward induces a map in relative Dolbeault homology

$$\pi_* : H_{\bar{D}_{\mathcal{K}}}^{\bullet, \bullet}(\tilde{X}, \tilde{X} \setminus \tilde{S}) \rightarrow H_{\bar{D}_{\mathcal{K}}}^{\bullet, \bullet}(X, X \setminus S).$$

We will use this map in the next section in order to prove that under suitable assumptions the pull-back map in relative cohomology is injective.

Remark 2.6 Similar considerations can be done for the relative Čech–de Rham cohomology (cf. [11] for its definition) using k -currents with compact support and the exterior derivative d .

3 Comparisons via proper holomorphic surjective maps

In this section, we study a Wells-type result for relative Dolbeault cohomology; in particular, we prove the following

Theorem 3.1 *Let $\pi : \tilde{X} \rightarrow X$ be a proper, surjective, holomorphic map between two complex manifolds of the same dimension. Suppose that X is connected and let S and $\tilde{S}$ be closed complex submanifolds of X and $\tilde{X}$, respectively, such that $\pi(\tilde{S}) \subset S$ and $\pi(\tilde{X} \setminus \tilde{S}) \subset X \setminus S$.*

Then,

$$\pi^* : H_D^{p,q}(X, X \setminus S) \longrightarrow H_D^{p,q}(\tilde{X}, \tilde{X} \setminus \tilde{S})$$

is injective for any p, q .

Proof We denote by $U_0 = X \setminus S$ and $\tilde{U}_0 = \tilde{X} \setminus \tilde{S}$. Let U_1 be a neighborhood of S in X and $\tilde{U}_1$ be a neighborhood of $\tilde{S}$ in $\tilde{X}$ such that $\pi(\tilde{U}_1) = U_1$ and $\pi(\tilde{U}_{01}) = U_{01}$. Hence, we consider the open coverings $\mathcal{U} := \{U_0, U_1\}$ of X and $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$ of $\tilde{X}$.

We take the following diagram, for any p, q

$$\begin{array}{ccc} A^{p,q}(\tilde{\mathcal{U}}, \tilde{U}_0) & \xrightarrow{\tilde{i}} & \mathcal{K}^{p,q}(\tilde{\mathcal{U}}, \tilde{U}_0) \\ \pi^* \uparrow & & \downarrow \pi_* \\ A^{p,q}(\mathcal{U}, U_0) & \xrightarrow{i} & \mathcal{K}^{p,q}(\mathcal{U}, U_0), \end{array}$$

or equivalently, by definition

$$\begin{array}{ccc} A^{p,q}(\tilde{U}_1) \oplus A^{p,q-1}(\tilde{U}_{01}) & \xrightarrow{\tilde{i}} & \mathcal{K}^{p,q}(\tilde{U}_1) \oplus \mathcal{K}^{p,q-1}(\tilde{U}_{01}) \\ \pi^* \uparrow & & \downarrow \pi_* \\ A^{p,q}(U_1) \oplus A^{p,q-1}(U_{01}) & \xrightarrow{i} & \mathcal{K}^{p,q}(U_1) \oplus \mathcal{K}^{p,q-1}(U_{01}), \end{array}$$

where $\tilde{i}$ and i denote the natural injections of forms into currents. By [17, Lemma 2.1] we have that the diagram

$$\begin{array}{ccc} A^{p,q}(\tilde{U}_1) & \xrightarrow{\tilde{i}} & \mathcal{K}^{p,q}(\tilde{U}_1) \\ \pi^* \uparrow & & \downarrow \pi_* \\ A^{p,q}(U_1) & \xrightarrow{i} & \mathcal{K}^{p,q}(U_1). \end{array}$$

commutes up to a constant; more precisely, we have $\mu i = \pi_* \tilde{i} \pi^*$, where μ is the degree of π . Recall that the degree of π is defined as $\pi_*(1)$, where 1 is thought as a current on $\tilde{X}$; in particular, it is a d -closed function on X . By the connectedness of X , we have that the degree of π is constant on X . Similarly, we also have the commutativity up to μ , of

$$\begin{array}{ccc} A^{p,q-1}(\tilde{U}_{01}) & \xrightarrow{\tilde{i}} & \mathcal{K}^{p,q-1}(\tilde{U}_{01}) \\ \pi^* \uparrow & & \downarrow \pi_* \\ A^{p,q-1}(U_{01}) & \xrightarrow{i} & \mathcal{K}^{p,q-1}(U_{01}). \end{array}$$

Therefore, one can pass to (co)homology considering the following

$$\begin{array}{ccc} H_D^{p,q}(\tilde{X}, \tilde{X} \setminus \tilde{S}) & \xrightarrow{\tilde{i}_*} & H_{D\mathcal{K}}^{p,q}(\tilde{X}, \tilde{X} \setminus \tilde{S}) \\ \pi^* \uparrow & & \downarrow \pi_* \\ H_D^{p,q}(X, X \setminus S) & \xrightarrow{i_*} & H_{D\mathcal{K}}^{p,q}(X, X \setminus S). \end{array}$$

We have shown in Theorem 2.5 that the maps $\tilde{i}_*$ and i_* in this last diagram are isomorphisms. Using this fact we prove that π^* is injective; indeed, let $a \in H_D^{p,q}(X, X \setminus S)$ and suppose that $\pi^*a = 0$, then $\mu i_*a = \pi_* \tilde{i}_* \pi^*a = 0$. Then, $i_*a = 0$ and by injectivity we can conclude that $a = 0$, proving the assertion. $\square$

As already noticed, one can define the relative Čech–de Rham homology and with similar techniques one can prove an injectivity result also for the relative Čech–de Rham cohomology (cf. [17, Theorem 3.1]). For completeness, we write down the theorem explicitly without the proof since it is the same for the Dolbeault case.

Theorem 3.2 *Let $\pi : \tilde{X} \rightarrow X$ be a proper, surjective, holomorphic map between two complex manifolds of the same dimension. Suppose that X is connected and let S and $\tilde{S}$ be closed complex submanifolds of X and $\tilde{X}$, respectively, such that $\pi(\tilde{S}) \subset S$ and $\pi(\tilde{X} \setminus \tilde{S}) \subset X \setminus S$.*

Then,

$$\pi^* : H_{dR}^k(X, X \setminus S) \longrightarrow H_{dR}^k(\tilde{X}, \tilde{X} \setminus \tilde{S})$$

is injective for any k .

Remark 3.3 Notice that even if $\tilde{X}$ and X are compact, the relative cohomology groups can be infinite-dimensional; therefore, in general we do not have a comparison between the dimensions of these cohomology groups.

4 Application to blow-ups

In these last years an increased interest for blow-up formulas for cohomology groups of complex manifolds arose with the purpose of studying the behavior of the $\partial\bar{\partial}$ -lemma under modifications. Let $\tau : \tilde{X} \rightarrow X$ be the blow-up of a compact complex manifold X along a closed complex submanifold Z . Then, by a classical result if X is Kähler, the same holds for $\tilde{X}$ and one can express the de Rham cohomology of $\tilde{X}$ in terms of the de Rham cohomology groups of X and Z (cf. [15]). In fact, the Kähler hypothesis can be dropped and several formulas for the de Rham, Dolbeault and Bott–Chern cohomologies can be found in [2, 7–10, 16]. Here we discuss a blow-up formula for the Dolbeault cohomology in terms of the relative cohomology groups which follows directly by the previous results and which finds further applications in [2].

Let X be a compact complex manifold and let Z be a closed complex submanifold. Then, the blow-up $\tau : \tilde{X} \rightarrow X$ of X along Z satisfies the previous assumptions with $S = Z$ and $\tilde{S} = E := \pi^{-1}(Z)$ the exceptional divisor. Then, by Theorem 3.1 the induced maps in relative cohomology

$$\tau^* : H_D^{p,q}(X, X \setminus Z) \longrightarrow H_D^{p,q}(\tilde{X}, \tilde{X} \setminus E)$$

are injective for any p, q .

Therefore, one has the following commutative diagram with exact rows (cf. [2])

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_{\partial}^{p,q-1}(X \setminus Z) & \longrightarrow & H_D^{p,q}(X, X \setminus Z) & \longrightarrow & H_{\partial}^{p,q}(X) & \longrightarrow & H_{\partial}^{p,q}(X \setminus Z) & \longrightarrow & H_D^{p,q+1}(X, X \setminus Z) & \longrightarrow \cdots \\ & & \downarrow \tau^* & & \downarrow \tau^* & & \downarrow \tau^* & & \downarrow \tau^* & & \downarrow \tau^* & \\ \cdots & \longrightarrow & H_{\partial}^{p,q-1}(\tilde{X} \setminus E) & \longrightarrow & H_D^{p,q}(\tilde{X}, \tilde{X} \setminus E) & \longrightarrow & H_{\partial}^{p,q}(\tilde{X}) & \longrightarrow & H_{\partial}^{p,q}(\tilde{X} \setminus E) & \longrightarrow & H_D^{p,q+1}(\tilde{X}, \tilde{X} \setminus E) & \longrightarrow \cdots \end{array}$$

where the maps

$$\tau^* : H_{\partial}^{\bullet,\bullet}(X \setminus Z) \rightarrow H_{\partial}^{\bullet,\bullet}(\tilde{X} \setminus E)$$

are isomorphisms since X and $\tilde{X}$ are biholomorphic outside the exceptional divisor. Furthermore, the maps

$$\tau^* : H_{\partial}^{\bullet,\bullet}(X) \rightarrow H_{\partial}^{\bullet,\bullet}(\tilde{X})$$

are injective by Wells' Theorem [17, Theorem 3.1] and finally the maps

$$\tau^* : H_D^{\bullet,\bullet}(X, X \setminus Z) \rightarrow H_D^{\bullet,\bullet}(\tilde{X}, \tilde{X} \setminus E)$$

are injective by Theorem 3.1. As a consequence, for instance, by [3, Lemme II.6], one obtains the following isomorphisms

Corollary 4.1 *Let $\tau : \tilde{X} \rightarrow X$ be the blow-up of a compact complex manifold X along a closed complex submanifold Z and let E be the exceptional divisor. Then, there are isomorphisms*

$$H_{\partial}^{p,q}(\tilde{X}) \simeq \tau^* H_{\partial}^{p,q}(X) \oplus \frac{H_{\bar{D}}^{p,q}(\tilde{X}, \tilde{X} \setminus E)}{\tau^* H_{\bar{D}}^{p,q}(X, X \setminus Z)},$$

for any p, q .

Notice that in [8] and [9] the authors prove a Dolbeault blow-up formula using a sheaf-theoretic approach. However, their definition of relative Dolbeault cohomology is different from Suwa's one.

Remark 4.2 In fact the previous formula is a bit more general. Let $\pi : \tilde{X} \rightarrow X$ be a proper, surjective, holomorphic map between two complex manifolds of the same dimension. Suppose that X is connected and let S and $\tilde{S}$ be closed complex submanifolds of X and $\tilde{X}$, respectively, such that $\pi(\tilde{S}) \subset S$ and $\pi(\tilde{X} \setminus \tilde{S}) \subset X \setminus S$. Fix p and q and consider the following diagram with exact rows

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_{\partial}^{p,q-1}(X \setminus S) & \longrightarrow & H_{\bar{D}}^{p,q}(X, X \setminus S) & \longrightarrow & H_{\partial}^{p,q}(X) & \longrightarrow & H_{\partial}^{p,q}(X \setminus S) & \longrightarrow & H_{\bar{D}}^{p,q+1}(X, X \setminus S) & \longrightarrow \cdots \\ & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & \\ \cdots & \longrightarrow & H_{\partial}^{p,q-1}(\tilde{X} \setminus \tilde{S}) & \longrightarrow & H_{\bar{D}}^{p,q}(\tilde{X}, \tilde{X} \setminus \tilde{S}) & \longrightarrow & H_{\partial}^{p,q}(\tilde{X}) & \longrightarrow & H_{\partial}^{p,q}(\tilde{X} \setminus \tilde{S}) & \longrightarrow & H_{\bar{D}}^{p,q+1}(\tilde{X}, \tilde{X} \setminus \tilde{S}) & \longrightarrow \cdots \end{array}$$

Then, as above, the maps

$$\pi^* : H_{\partial}^{\bullet,\bullet}(X) \rightarrow H_{\partial}^{\bullet,\bullet}(\tilde{X}) \quad \text{and} \quad \pi^* : H_{\bar{D}}^{\bullet,\bullet}(X, X \setminus S) \rightarrow H_{\bar{D}}^{\bullet,\bullet}(\tilde{X}, \tilde{X} \setminus \tilde{S})$$

are injective, respectively, by [17, Theorem 3.1] and Theorem 3.1. If

$$\pi^* : H_{\partial}^{p,q-1}(X \setminus S) \rightarrow H_{\partial}^{p,q-1}(\tilde{X} \setminus \tilde{S})$$

is surjective and

$$\pi^* : H_{\partial}^{p,q}(X \setminus S) \rightarrow H_{\partial}^{p,q}(\tilde{X} \setminus \tilde{S})$$

is an isomorphism, then there is an isomorphism

$$H_{\partial}^{p,q}(\tilde{X}) \simeq \pi^* H_{\partial}^{p,q}(X) \oplus \frac{H_{\bar{D}}^{p,q}(\tilde{X}, \tilde{X} \setminus \tilde{S})}{\pi^* H_{\bar{D}}^{p,q}(X, X \setminus S)}.$$

Namely, the relative cohomology groups measure the gap between $H_{\partial}^{p,q}(\tilde{X})$ and $H_{\partial}^{p,q}(X)$.

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