Volume growth for geodesic balls of static vacuum space on 3-manifolds

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Abstract

The purpose of this article is to study the geometry of static space–times. We show that the energy density and pressure vanish on the boundary of a perfect fluid space–time, provided that it satisfies a suitable condition. Moreover, we provide an upper bound volume growth for geodesic balls of the base of static vacuum space similar to a classical result due to Bishop. In addition, we derive a weak version of the maximum principle of Omori–Yau at infinity for such spaces.

Keywords Static space–time · Vacuum space · Volume growth · Maximum principle

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1 Introduction

In 1915, Einstein introduced a geometric theory for gravitation, which is now popularly known as *general relativity*. The general relativity shed light in the interplay between physics and

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geometry of the space–time. This theory became one of the main approaches to investigate the past and the future of the universe.

Before proceeding, we fix some notation. Let $(\widehat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$ be a static space–time endowed with metric

$$\hat{g} = g - f^2 dt^2, \quad (1.1)$$

where (M^n, g) , $n \geq 3$, is a noncompact, connected, oriented Riemannian manifold and $f : M^n \rightarrow (0, +\infty)$ is a positive smooth warped function. In this context, the Einstein equation with perfect fluid as a matter field is given by

$$\hat{Ric} - \frac{\hat{R}}{2} \hat{g} = \mu f^2 dt^2 + \rho g, \quad (1.2)$$

where $\hat{Ric}$ and $\hat{R}$ stand for the Ricci tensor and the scalar curvature with respect to $\hat{g}$, respectively. Moreover, μ and ρ are nonnegative smooth functions, namely the energy density and pressure, respectively; for more background, we refer the reader to [10, 11]. Here, we are assuming that M^n satisfies the *dominant energy condition* when $\mu \geq \rho$ (see [11, p. 91]). As it was pointed out by Hawking and Ellis [11], this holds for all known forms of matter and there is in fact good reason for believing that this should be the case in all situations; see also [17, p. 347]. Indeed, searching for different solutions to Einstein’s field equations became a problem of great interest in this theory. Among all solutions, static space–times are special and very important global solutions.

At the same time, we remark that (1.1) and (1.2) imply

$$f \hat{Ric} = \hat{\nabla}^2 f, \quad (1.3)$$

$$\mu = \frac{R}{2} \quad \text{and} \quad \rho f = \frac{n-1}{n} \left(\Delta f - \frac{n-2}{2(n-1)} R f \right), \quad (1.4)$$

where $\nabla^2 f$ is the Hessian of f and $\hat{T}$ stands for the traceless of T . Besides, Δ denotes the Laplacian and R is the scalar curvature with respect to g (see [5, 13, 17]). We further remember that when μ and ρ vanish in (1.2), we obtain the well-known *static vacuum Einstein equations*. Following the trend used in [2, 3, 10], we deduce from (1.3) and (1.4) that a semi-Riemannian manifold $(\widehat{M}, \hat{g})$ is Ricci-flat (i.e., $\hat{Ric} = 0$) if and only if the (positive) warped function f and the metric g satisfy the *static vacuum equations*

$$f Ric = \nabla^2 f \quad \text{and} \quad \Delta f = 0. \quad (1.5)$$

In this case, it is straightforward to check that the scalar curvature R is identically zero. These equations have been extensively studied in the classical general relativity. Some explicit examples can be found, for instance, in [2, 3, 11, 13].

It is not hard to check that there are no nonflat solutions to (1.5) on closed manifolds, and so it will be assumed that M is an open, connected oriented manifold. Therefore, let $\overline{M}$ be the metric (or Cauchy) completion of M and ∂M the metric boundary of M , so that $\overline{M} = M \cup \partial M$ is complete as a metric space. Moreover, it should be emphasized that if a three-dimensional manifold (M^3, g) satisfying (1.5) and $f > 0$ everywhere is complete, then the warped function f must be constant and the space must be the flat Minkowsky space–time (cf. Theorem 3 in Sect. 2). Since we are interested in nontrivial solutions of (1.5), then ∂M must be nonempty. Furthermore, the set formally given by

$$\Sigma = \{f = 0\} \subset \overline{M}$$

is called *horizon*. Although the set Σ may be an empty subset, most solutions of physical interest do have $\Sigma \neq \emptyset$; for more details, we refer the reader to [3].

To make our approach more understandable, we need to recall some terminology.

Condition 1 [3] *We say that (M, g) is complete away from the horizon if for any sequence of points $\{p_i\}$ such that $p_i \rightarrow p \in \partial M$ on the metric of $\overline{M}$ one has $f(p_i) \rightarrow 0$. Conversely, if $\{p_i\}$ is a bounded sequence of M such that $f(p_i) \rightarrow 0$, then the definition of (M, g, f) implies that a subsequence of $\{p_i\}$ converges to a point $p \in \partial M$.*

Based on the above condition and according to [3], it is natural to consider $\partial M = \Sigma$.

Condition 2 [3] *We assume that the boundary Σ is compact, nonempty and such that g and f extends smoothly to Σ , namely,*

- (1) Σ is smooth and $M \cup \Sigma$ is a smooth Riemannian manifold with boundary;
- (2) the metric g extends smoothly to a smooth Riemannian metric in $M \cup \Sigma$;
- (3) the lapse function f extends smoothly to $\Sigma = \{f = 0\}$.

In the spirit of [2,3,10], throughout this article we consider nontrivial solutions to (1.3) or (1.5) which are connected and *complete up to the boundary* or, equivalently, *complete away from the horizon* (see Theorem 3.2 in [2], or Theorem 3 in Sect. 2, see also [3] p. 996). Moreover, we assume Condition 2. As an immediate consequence of Condition 2 we conclude that Σ is totally geodesic in $\overline{M}$, and hence, it follows from (1.5) that

$$\nabla^2 f = 0 \quad \text{in } \Sigma,$$

and Σ is therefore minimal. We further remark that Σ may be defined as the set of limit points of Cauchy sequences on (M^n, g) on which f converges to 0 (see [3]). In general relativity, the horizon is closely related to the event horizon, i.e., the boundary of a black hole. The apparent horizon always lies inside of (or coincides with) the event horizon. For more details on such a subject, we indicate [2,3,5,10–12].

In the first part of this article, we shall present a simple proof that the energy density μ and pressure ρ vanish on the boundary $\Sigma = \{f = 0\}$ of manifolds M^n satisfying (1.3), (1.4) and Condition 2. More precisely, we have established the following result.

Theorem 1 *Let (M^n, g) be a Riemannian manifold with boundary Σ and function $f : M \rightarrow \mathbb{R}$ satisfying (1.3) and (1.4). Then the energy density and pressure vanish on Σ .*

A classical theorem by Calabi [7] and Yau [23] asserts that the geodesic balls of complete noncompact manifolds with nonnegative Ricci tensor have at least linear volume growth, that is,

$$\text{Vol}(B_p(r)) \geq cr,$$

for any $r > r_0$, where r_0 is a positive constant, $B_p(r)$ is the geodesic ball of radius r centered at $p \in M^n$ and c is a constant that does not depend on r . In [15], Munteanu and Sesum obtained the same type of growth for steady gradient Ricci soliton. Afterward, Barros et al. [4] were able to prove this volume growth for m -quasi-Einstein manifolds with $\lambda = 0$ (and $m > 1$). Regarding the upper bound estimate, the classical Bishop volume comparison theorem guarantees that the geodesic balls of complete noncompact manifolds with nonnegative Ricci tensor must have the following volume growth rate

$$\text{Vol}(B_p(r)) \leq c_1 r^n,$$

for some positive constant c_1 and $r > 0$ sufficiently large. In [6], H.-D. Cao and D. Zhou proved an analog of Bishop's theorem for shrinking gradient Ricci solitons. While Munteanu and Sesum [15] showed that the geodesic balls of steady gradient Ricci soliton have at most exponential volume growth, namely, there exist uniform constants c , a and r_0 so that for any $r > r_0$ we have

$$\text{Vol}(B_p(r)) \leq ce^{a\sqrt{r}}.$$

Indeed, volume growth rate is an important piece of geometric information.

From now on, we focus on three-dimensional static vacuum spaces. In this scenario, motivated by the above discussion, we going to provide an upper bound on the growth of volume of geodesic balls (away from the horizon) for spatial factor of a three-dimensional static vacuum space which is similar to Bishop's estimate. To be precise, we have the following result.

Theorem 2 *Let (M^3, g, f) be a three-dimensional Riemannian manifold complete away from the horizon satisfying (1.5). Then there exist positive constants a , c and r_0 so that for any $r > r_0$*

$$\text{Vol}(B_p(r)) \leq cr^{3+a}.$$

It is worthwhile to remark that the Schwarzschild space indicates that our upper bound estimate obtained in Theorem 2 is possibly optimal. Moreover, we emphasize that there are examples of nonasymptotically flat static vacuum spaces in higher dimension [cf. Example 3.11 in [3]; see also [5], pg. 271]. We also remark that the essential tool in the proof of Theorem 2 is an approach by using the Morse Index Theorem which was already used in [15, 16].

For what follows, we remember that the Omori–Yau maximum principle (at infinity) is a very powerful tool in geometric analysis and it is related to a number of properties of the underlying Riemannian manifold, ranging from the realm of stochastic analysis to that of geometry and PDEs. In [20], Pigola, Rigoli and Setti extended the Omori–Yau maximum principle to a larger class of manifolds and operators; see [20, Remark 1.2 & Examples 1.13, 1.14]. In particular, Pigola, Rigoli and Setti [19, 20] introduced the concept of *weak Omori–Yau maximum principle* as follows.

Definition 1 Let (M^n, g) be a complete Riemannian manifold. We say that the weak Omori–Yau maximum principle holds if for every function $u \in C^2(M)$ with $u^* := \sup_M u < +\infty$, there exists a sequence $\{x_k\} \subset M$, $k = 1, 2, \dots$, such that, for every k ,

$$u(x_k) > u^* - 1/k \quad \text{and} \quad \Delta u(x_k) < 1/k.$$

The validity of the weak maximum principle (at infinity) implies, for instance, stochastic completeness (see [20, 21]). Remember that the $\mathcal{L}_\varphi$ -Laplacian operator, or simply φ -Laplacian, is given by

$$\mathcal{L}_\varphi u = \text{div}(|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u), \quad (1.6)$$

for some function $u \in C^1(M)$ and $\varphi \in C^1((0, +\infty)) \cap C^0([0, +\infty))$. Notice that if the vector field in brackets is not C^1 , then the divergence in (1.6) must be considered in distributional sense. The φ -Laplacian arises from the Euler–Lagrange equation associated with the energy functional

$$\Lambda(u) = \int \phi(|\nabla u|),$$

where $\phi(t) = \int_0^t \varphi(s) ds$. It is important to mention that when $\varphi(t) = t$ in (1.6) the φ -Laplacian reduces to Laplace–Beltrami operator Δu . Moreover, when $\varphi(t) = t^{p-1}$ in (1.6) the φ -Laplacian becomes the p -Laplacian $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$. Further, when $\varphi(t) = 1/(1+t^2)^\alpha$, it becomes the generalized mean curvature operator, $\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^\alpha}\right)$, for $\alpha > 0$.

Inspired in the weak maximum principle to the Laplacian Δ , Rigoli and Setti [22] studied its validity for the φ -Laplacian operator. They were able to prove under suitable geometric assumptions a weak version of the Omori–Yau maximum principle for the φ -Laplacian. For more details on this subject, we refer the reader to [22].

Next, as an application of Theorem 2, we have established a weak maximum principle at infinity for the φ -Laplacian on three-dimensional Riemannian manifolds satisfying (1.5).

Corollary 1 *Let (M^3, g, f) be a three-dimensional Riemannian manifold complete away from the horizon satisfying (1.5). Let u be a smooth function on M^3 with $u^* = \sup_M u < +\infty$ and such that the vector field $|\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least C^1 . Then the weak maximum principle at infinity holds for φ -Laplacian on M^3 .*

It is interesting to observe that the regularity condition in the statement of Corollary 1 is certainly satisfied in the case of the Laplacian, p -Laplacian or the generalized mean curvature operator once u is assumed to be at least C^2 .

2 Background

In this section, we review some information that will be useful in the proof of our main results. To start with, we recall a result obtained previously in [2,10].

Lemma 1 *Let (M^n, g, f) be a Riemannian manifold satisfying (1.3) and (1.4). Then there is no critical point of f in Σ .*

Proof Since the proof is short, we include its proof here for the sake of completeness. Indeed, from (1.4) we obtain that $\Delta f = 0$ in Σ , where f is the potential function satisfying (1.3) and (1.4). Now, consider $\operatorname{Crit}(f) = \{x \in M; (\nabla f)(x) = 0\}$, and since $\Delta f = 0$ in Σ , we may use Hopf's lemma (cf. [9]) to conclude that $|\nabla f| > 0$ for any $p \in \Sigma$. Whence, $\operatorname{Crit}(f) \cap \Sigma = \emptyset$, which finishes the proof of the lemma. $\square$

Proceeding, we recall a well-known result by Anderson [2,3] which plays a crucial role in the proof of Theorem 2.

Theorem 3 (Theorem 3.2 in [2], or Theorem 1.1 in [3]) *Let (M^3, g, f) be a solution to the static vacuum equations (1.5).*

- (1) *Suppose that (M^3, g) is a complete Riemannian manifold and $f > 0$ on M^3 , then M^3 must be flat and f is constant.*
- (2) *Let $U \subset M$ be any domain with smooth boundary on which $f > 0$. Assume that $r(x) = \operatorname{dist}_M(x, \partial U)$, for $x \in U$. Then there is an absolute constant $K < +\infty$ such that*

$$\frac{|\nabla f|}{f}(x) \leq \frac{K}{r(x)}, \quad (2.1)$$

where the constant K does not depend on the domain U (since $f > 0$ on U), or on the static vacuum solution (M^3, g) .

To conclude this section, let us recall an useful result obtained previously by Rigoli and Setti [22]. It will be fundamental in the proof of Corollary 1.

Theorem 4 [22] *Let M^n be a complete Riemannian manifold. Suppose that*

$$\liminf_{r \rightarrow +\infty} \frac{\log \text{Vol}(B_p(r))}{r^{1+\delta}} < +\infty \quad (2.2)$$

and let u be a smooth function on M^n such that $u^ = \sup u < +\infty$. In addition, assume that the vector field $X = |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least C^1 . Then there exists a sequence $\{x_k\} \subset M$, $k = 1, 2, \dots$, such that*

$$u(x_k) \rightarrow u^* \quad \text{and} \quad \mathcal{L}_\varphi u(x_k) < 1/k.$$

Let us point out that if $X = |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u$ is of class at least C^1 , the conclusion of Theorem A in [18] is equivalent to the following version of the weak maximum principle, namely, that there exists a sequence $\{x_k\}$ such that

$$u(x_k) \rightarrow u^* \quad \text{and} \quad Q(r(x_k)) \mathcal{L}_\varphi u(x_k) < 1/k.$$

When $Q(r)$ is constant, this is precisely Theorem 4.

In [1], Albanese and Rigoli generalized the above weak maximum principle to noncompact manifolds with boundary by assuming some Neumann boundary-type conditions on the functions; for more details see [1, Theorem 3.5].

We are now in the position to present the proof of the main results.

3 Proof of the main results

3.1 Proof of Theorem 1

Proof In what follows, we assume that g and f satisfy (1.3), (1.4) and Condition 2. Next, we rewrite (1.3) in tensorial language as

$$f \left(R_{ij} - \frac{R}{n} g_{ij} \right) = \nabla_i \nabla_j f - \frac{\Delta f}{n} g_{ij}. \quad (3.1)$$

In particular, it is straightforward to check from (3.1) that

$$f \mathring{R}_{ij} \nabla_j f = \frac{1}{2} \nabla_i |\nabla f|^2 - \frac{\Delta f}{n} \nabla_i f, \quad (3.2)$$

where $\mathring{R}_{ij} = R_{ij} - \frac{R}{n} g_{ij}$.

On the other hand, taking the covariant derivative of (3.1) and then using the classical Ricci identity

$$\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = R_{ijkl} \nabla_l f,$$

we obtain

$$\begin{aligned} f (\nabla_i \mathring{R}_{jk} - \nabla_j \mathring{R}_{ik}) + (\mathring{R}_{jk} \nabla_i f - \mathring{R}_{ik} \nabla_j f) &= R_{ijkl} \nabla_l f \\ &+ \frac{1}{n} \left(\nabla_j (\Delta f) g_{ik} - \nabla_i (\Delta f) g_{jk} \right). \end{aligned} \quad (3.3)$$

Besides, the twice contracted second Bianchi identity leads to

$$g^{ik} \nabla_i \mathring{R}_{jk} = \frac{n-2}{2n} \nabla_j R.$$

These data combined with the trace over i and k of (3.3) gives

$$\frac{1}{2(n-1)} \left[(n-2) f \nabla_j R - 2R \nabla_j f \right] = \nabla_j (\Delta f). \quad (3.4)$$

Proceeding, easily one verifies from (3.2) that

$$\frac{1}{2} \nabla_j |\nabla f|^2 - \frac{\Delta f}{n} \nabla_j f = 0 \quad \text{on } \Sigma. \quad (3.5)$$

But, since $\Sigma = \{f = 0\}$, we may use (1.4) to obtain $\Delta f = 0$ on Σ and then, by Lemma 1, we deduce that $|\nabla f|^2$ is a nonnull constant on Σ . Moreover, from (3.4), on Σ , we also have

$$-\frac{R}{n-1} \nabla_j f = \nabla_j (\Delta f). \quad (3.6)$$

Next, we already know from (1.4) that

$$\Delta f = \frac{n}{(n-1)} \rho f + \frac{(n-2)}{2(n-1)} R f. \quad (3.7)$$

Whence, it follows that

$$\nabla_j \Delta f = \frac{n}{n-1} (f \nabla_j \rho + \rho \nabla_j f) + \frac{(n-2)}{2(n-1)} (f \nabla_j R + R \nabla_j f).$$

Thus, using again that the horizon correspond to the set $\Sigma = \{f = 0\}$, we have

$$\nabla_j \Delta f = \frac{n}{n-1} \rho \nabla_j f + \frac{n-2}{2(n-1)} R \nabla_j f \quad \text{on } \Sigma. \quad (3.8)$$

This jointly with (3.6) gives

$$-\frac{R}{n-1} \nabla_j f = \frac{n}{n-1} \rho \nabla_j f + \frac{n}{2(n-1)} R \nabla_j f - \frac{R}{n-1} \nabla_j f,$$

and therefore, we achieve

$$0 = \left(\rho + \frac{R}{2} \right) \nabla f \quad \text{on } \Sigma. \quad (3.9)$$

Now, it suffices to use again Lemma 1 jointly with (1.4) to conclude that $\rho + \mu = 0$. Hence, since μ and ρ are nonnegative, we conclude that $\rho = \mu = \frac{R}{2} = 0$ on Σ . This finishes the proof of the theorem. $\square$

In the sequel, we shall present the proof of Theorem 2, which was mainly inspired by the trend developed by Munteanu and Wang [16] as well as Munteanu and Sesum [15].

3.2 Proof of Theorem 2

Proof Firstly, we denote by

$$dV|_{\exp_x(r\theta)} = J(x, r, \theta) dr d\theta$$

the volume form in the geodesic polar coordinates centered at x for $r > 0$, θ a tangent vector field on $x \in M$ sufficiently away from the horizon and the distance function

$$r = \text{dist}_M(x_0, x), \quad \text{for } x_0 \in M$$

also sufficiently away from the horizon, i.e., $\text{dist}_M(x_0, x) \leq \text{dist}_M(x, \Sigma)$, where $\Sigma = \partial M$. In particular, by Theorem 3, we have

$$\frac{|\nabla f|}{f}(x) \leq \frac{K}{\text{dist}_M(x_0, x)}.$$

From now on, we omit the dependence on θ . Thus, it is known that

$$\Delta r = \frac{J'}{J}(r).$$

Next, let $\gamma(s)$ be a minimizing geodesic starting from x_0 , such that $\gamma(0) = x_0$. In particular, given any orthonormal basis $\{E_1, E_2\}$ such that $\gamma' \perp E_i$ at $\gamma(s)$, Y_i is the Jacobi field along γ with $Y_i(0) = 0$ and $Y_i(s) = E_i$, for $i = 1, 2$. Of which, we have

$$\Delta r = \sum_{i=1}^2 I_t(Y_i, Y_i).$$

By using the Morse Index Theorem (see [8, 14]), for any piecewise differentiable vector field W_i along γ , we get

$$I_t(Y_i, Y_i) \leq \int_0^t \{g(W'_i, W'_i)(s) - g(R(\gamma', W_i)\gamma', W_i)(s)\} ds.$$

Now, if E_i is the parallel unit field generated by $Y_i(t)$ and μ is a piecewise differentiable function, taking $W_i = \mu E_i$, we deduce

$$I_t(Y_i, Y_i) \leq \int_0^t ((\mu')^2 - \mu^2 g(R(\gamma', E_i)\gamma', E_i)) ds, \quad (3.10)$$

where the expression in the integral depends on s . Besides, taking the trace of (3.10) over i , with $X = \gamma'(s)$, we arrive at

$$\Delta r \leq \int_0^t (2(\mu')^2 - \mu^2 \text{Ric}(X, X)) ds.$$

Also, taking into account that $\Delta r = \frac{J'}{J}$, we may express $\mu(s) = \frac{s}{t}$ to infer

$$\frac{J'}{J}(t) \leq \frac{2}{t} - \frac{1}{t^2} \int_0^t s^2 \text{Ric}(X, X) ds. \quad (3.11)$$

On the other hand, since f is positive, we may use the static vacuum equation (1.5) to obtain

$$\begin{aligned} \text{Ric}(X, X) &= \frac{1}{f} \nabla^2 f(X, X) \\ &= \left(\frac{f'}{f} \right)' + \left(\frac{f'}{f} \right)^2 \geq \left(\frac{f'}{f} \right)'. \end{aligned}$$

This substitution in (3.11) yields

$$\frac{J'}{J}(t) \leq \frac{2}{t} - \frac{1}{t^2} \int_0^t s^2 \left(\frac{f'}{f} \right)'(s) ds,$$

where $f(s) = f(\gamma(s))$ and $0 \leq s \leq t$. Hence, upon integrating by parts we achieve

$$\frac{J'}{J}(t) \leq \frac{2}{t} + \frac{2}{t^2} \int_0^t s \frac{f'}{f}(s) ds - \frac{f'}{f}(t). \quad (3.12)$$

Moreover, we already know from Eq. (2.1) (see also Theorem 3.2 in [2]) that $\left(\frac{f'}{f}\right)$ is bounded from above. With the aid of (3.12), we obtain

$$\begin{aligned} \frac{J'}{J}(t) &\leq \frac{2}{t} + \frac{2}{t^2} \int_0^t s \frac{f'}{f}(s) ds - \frac{f'}{f}(t) \\ &\leq \frac{2}{t} + \frac{2}{t^2} \int_0^t s \left(\frac{K}{s} \right) ds + \frac{K}{t} \\ &\leq \frac{2}{t} + \frac{2K}{t} + \frac{K}{t} = \frac{2+3K}{t}, \end{aligned}$$

which reduces to

$$\left(\log J(t) \right)' \leq \frac{2+3K}{t}.$$

Thus, upon integrating from $t = 1$ to $t = T$, we get

$$\frac{J(T)}{J(1)} \leq T^{2+3K}.$$

Consequently,

$$\text{Area}(\partial B_{x_0}(r)) \leq \text{Area}(\partial B_{x_0}(1)) r^{2+3K}. \quad (3.13)$$

Finally, it suffices to integrate once more (3.13) in order to obtain

$$\text{Vol}(B_{x_0}(r)) \leq cr^{3+a},$$

where c depends on the volume of the unitary ball and a is positive constant. This is what we wanted to prove. $\square$

3.3 Proof of Corollary 1

Proof We start invoking Theorem 2 to deduce that there exist positive constants a , c and r_0 , so that for any $r > r_0$, it holds

$$\text{Vol}(B_p(r)) \leq cr^{3+a}.$$

Easily one verifies that

$$\begin{aligned} \frac{\log \text{Vol}(B_p(r))}{r^{1+\delta}} &\leq \frac{\log cr^{3+a}}{r^{1+\delta}} \\ &= \frac{\log c}{r^{1+\delta}} + (3+a) \frac{\log r}{r^{1+\delta}}. \end{aligned}$$

Letting $r \rightarrow +\infty$, we obtain

$$\liminf_{r \rightarrow +\infty} \frac{\log \text{Vol}(B_p(r))}{r^{1+\delta}} < +\infty.$$

Therefore, it suffices to apply Theorem 4 (see also [1], Theorem 3.5) to conclude that there exists a sequence $\{x_n\} \subset M$, $n = 1, 2, \dots$, such that

$$u(x_n) \rightarrow u^*$$

and

$$\mathcal{L}_\varphi u(x_n) < \frac{1}{n}.$$

So, the proof is completed. $\square$

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