The existence of an energy minimizing configuration for multiple solid objects floating in a bath of three liquids

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Abstract

We consider the physical configuration of a container which holds a finite number of movable solid objects and three immiscible fluids. Each fluid volume is prescribed, and the container is completely filled. The configuration is modeled using sets of finite perimeter, and the energy of the configuration is given using the theory of functions of bounded variation. We show that there exists an admissible configuration that attains the minimum energy.

Keywords Capillarity · Flotation · Direct methods of the calculus of variation

Mathematics Subject Classification Primary 35Q35; Secondary 76A02

1 Introduction

We consider a bounded container Ω to be a simply connected domain that is bounded in $\mathbb{R}^n$. We will present results for a Lipschitz domain Ω ; however, some of our hypotheses will benefit from showing that they are physically attainable from given criterion. For that reason, we will sometimes discuss the case when the container is bounded and cylindrical: $\Omega := G \times (0, p)$ where G is a bounded, simply connected domain in $\mathbb{R}^{n-1}$.

Into our container, we place a finite number of rigid but movable bodies $\mathcal{B}_k$, with $k = 1, 2, \dots, K$. We take each $\mathcal{B}_k$ to be the closure of an open set, with $\mathcal{B}_k^\circ \subset \Omega$ for $k = 1, 2, \dots, K$. We refer to the collection of $\mathcal{B}_k$ as $\mathfrak{B}$. These bodies have densities $\rho_{\mathcal{B}_k}$.

The remainder of Ω is filled with three fluids, denoted by the sets E_i where $i = 1, 2, 3$. See Fig. 1 for an example. We assume some structure on these sets, in that we require each E_i to be a Caccioppoli set. We recall that if $E \subset \Omega$ is a Caccioppoli set, otherwise known as a set of finite perimeter, then the characteristic function of E , denoted by ϕ_E , is in $BV(\Omega)$, which is the class of functions of bounded variation on Ω . This allows us to measure the perimeter

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Fig. 1 A container with three fluids and one floating object

in terms of the total variation of the characteristic function of each set. To be precise, given a domain U , and a Caccioppoli set E , we have

$$P(E, U) := \int_U |D\phi_E| := \sup \left\{ \int_U \phi_E \operatorname{div} g \, dx : g \in C_c^1(U; \mathbb{R}^n), \|g\|_{C^0} \leq 1 \right\}.$$

Naturally, the density of each fluid is prescribed and given by the quantity ρ_{E_i} . We also use the notation $\mathfrak{E} = \{E_1, E_2, E_3\}$. The interface between fluids has an associated surface tension, and we denote these quantities by σ_{ij} , which are symmetric in i and j , corresponding to the interface between fluids E_i and E_j . We define

$$\alpha_i := \frac{1}{2}(\sigma_{ij} + \sigma_{ik} - \sigma_{jk}), \quad (1)$$

where $i, j, k = 1, 2, 3$ taken to be mutually distinct, so that we can use the perimeters of the sets E_i to describe the surface tensions on the interfaces $S_{ij} := \partial^* E_i \cap \partial^* E_j$. Here, we use $\partial^* E$ to denote the reduced boundary of a set of finite perimeter. (For background on the reduced boundary of a set and functions of bounded variation in general, see Massari and Miranda [6]. We select this particular treatment of BV theory from the growing list of good books on the topic as it uniquely contains some particular material specifically related to our project.) We observe that $\alpha_i + \alpha_j = \sigma_{ij}$, and for a given domain U , we have

$$\sum \alpha_i \int_U |D\phi_{E_i}| = \sum \sigma_{ij} |S_{ij} \cap U|.$$

We will also require that each body $\mathcal{B}_k$ is represented by a Caccioppoli set. Each $\mathcal{B}_k$ may be subjected to rigid motions and can be described as $\mathcal{B}_k = \mathcal{B}_k(c_k, R_k) := \{y = c_k + R_k x : x \in \mathcal{B}_{k,0}\}$, where an initial description of each shape is given by $\mathcal{B}_{k,0}$. Here, $c_k \in \mathbb{R}^n$ denotes a translation and $R_k = R_k(d, \theta) \in SO(n)$ describes a rotation with respect to some axis generated by a unit vector d and about some angle $\theta \in [0, \pi]$. The quantities c_k and R_k are restricted by two assumptions. First, we assume that $\mathcal{B}_k^\circ \subset \Omega$, though we allow the possibility that $\partial\Omega \cap \mathcal{B}_k \neq \emptyset$. Further, we assume that there are a finite number of contact points: $\partial\Omega \cap \mathfrak{B}$ and $\cap \mathfrak{B}$ are both finite collections of points. At these contact points between the bodies, we allow $\mathcal{B}_k \cap \mathcal{B}_\ell = \{p_m\}$ for a finite number of points $p_m \in \mathbb{R}^n$ when $k \neq \ell$. We do, however, make the assumption that the interiors of the bodies do not intersect. It follows that each $E_i \subset \Omega \setminus \mathfrak{B}$ is measurable, and we denote $V_i = |E_i|$ to be the volume of that fluid. We will use the notation (x', x_n) for $x \in \mathbb{R}^n$.

The energy of our configuration can then be described by the functional

$$\begin{aligned} \mathcal{F}(\mathfrak{B}, \mathfrak{E}) := & \sum_{i=1}^3 \left(\alpha_i \int_{\Omega \setminus \mathfrak{B}} |D\phi_{E_i}| + g \rho_{E_i} \int_{\Omega \setminus \mathfrak{B}} x_n \phi_{E_i} \, dx \right. \\ & \left. + \beta_i \int_{\partial\Omega} \phi_{E_i} \, d\mathcal{H}^{n-1} + \sum_{k=1}^K \tau_{ik} \int_{\partial\mathcal{B}_k} \phi_{E_i} \, d\mathcal{H}^{n-1} \right) + \sum_{k=1}^K g \rho_{\mathcal{B}_k} \int_{\mathcal{B}_k} x_n \, dx. \end{aligned} \quad (2)$$

The integrals over $\partial\Omega$ and $\partial\mathfrak{B}$ measure the wetting energy along the container and the solids, respectively, and the remaining integrals measure the gravitational potential energy. Here, we use the gravitational constant g , and the adhesion energy between the fluid E_i and $\partial\Omega$ is given by β_i , while the adhesion energy between the fluid E_i and each $\partial\mathcal{B}_k$ is given by τ_{ik} . We seek a configuration $(\mathfrak{B}, \mathfrak{E})$ that minimizes $\mathcal{F}$ over the class

$$\begin{aligned} \mathcal{C} := & \{(\mathfrak{B}, \mathfrak{E}) : c_k \in \mathbb{R}^n, R_k \in SO(n) \text{ such that } \mathcal{B}_k^\circ(c_k, R_k) \subset \Omega, \text{ for } k = 1, 2, \dots, K, \\ & \mathcal{B}_k^\circ(c_k, R_k) \cap \mathcal{B}_\ell^\circ(c_\ell, R_\ell) = \emptyset, \text{ where } \partial\mathcal{B}_k \cap \partial\Omega = \{p_j\} \text{ and } \partial\mathcal{B}_k \cap \partial\mathcal{B}_\ell = \{p_m\} \\ & \text{are both finite sets, and each } E_i \subset \Omega \setminus \mathfrak{B} \text{ is a set of finite perimeter with } |E_i| = V_i\}. \end{aligned} \quad (3)$$

The results and approach in this paper are generalizations of two previous works. The first is a proof given by Massari [4] of the existence of a minimizer treating the case where there are no floating objects, and the second is an approach by Bemelmans, Galdi, and Kyed that spans (parts of) two papers, [1, 2], treating the case where there are two fluids and one floating object. These results are in turn generally based on the proof of existence of a minimizer for the two-fluid, no-solid problem given by Emmer [3], which can also be read about in [6].

2 Main result

We begin by establishing a lower bound on $\mathcal{F}$ and then forming a minimizing sequence of $(\mathfrak{B}^h, \mathfrak{E}^h)_{h=1}^\infty$. We then prove that $\mathcal{F}$ is lower semicontinuous with respect to this sequence. Finally, we will prove the existence of an energy minimizer of $\mathcal{F}$ in the class $\mathcal{C}$.

We have required that the bodies $\mathcal{B}_k$ and $\mathcal{B}_\ell$ contact at most at a finite number of points. As in [2], we can use conditions to enforce a stricter geometric requirement if we consider the bounded cylindrical container $\Omega = G \times (0, p)$ with ∂G and each $\partial\mathcal{B}_k$ to be of class C^2 and $\mathcal{B}_k(c_k, R_k)$ to have a projection $P_k(c_k, R_k)$ into G such that

$$\min_{R \in SO(3), x' \in \partial P_k(c_k, R_k)} K(\partial P_k(c_k, R_k), x') > \max_{x' \in \partial\Omega} K(\partial\Omega, x'). \quad (4)$$

This restriction ensures that all possible translations and rotations of each $\mathcal{B}_k$ give projections $P_k(c_k, R_k)$ with curvature strictly larger than that of G , so if a solid and container touch, then they only touch at a single point p_k . This also implies that the bodies $\mathcal{B}_k$ and $\mathcal{B}_\ell$ contact at most one point. That there are a finite number of contact points is a crucial condition, and we proceed with this assumed to be the case, no matter what restrictions led to this condition.

We next construct open (almost) neighborhoods of $\partial\Omega$ and $\partial\mathcal{B}_k$ in strips that “dimple” near each contact point. See Fig. 2. We first consider contact points $p_m \in \partial\Omega$, where the index m ranges over all of the contact points with the container boundary. Then, for a neighborhood U_{ϵ_m} of p_m , we can define a tangent plane Π and a neighborhood $A_{\epsilon_m} \subset \Pi$ at p_m . We can then

Fig. 2 Detail near a contact point

describe the boundaries $\partial\Omega$ and $\partial\mathcal{B}_k$ by the graphs of the respective functions $y_n = \omega_m(y')$ and $y_n = \beta_m(y')$, where $y' \in A_{\epsilon_m} \subset \Pi$ are coordinates in the tangent plane at p_m , centered at $y'_0 = 0$. We restrict our neighborhood so that A_{ϵ_m} is the domain of both ω_m and β_m . We then have $\beta_m(y') > \omega_m(y')$ for all nonzero $y' \in A_{\epsilon_m}$. (We are locally using β_m and τ_k below as functions, not to be confused with the wetting energies β_i and τ_{ik} .)

Next, we consider contact points $p_\ell \notin \partial\Omega$, where the index ℓ ranges over all contact points between bodies. We will use the convention that the contact point is on $\mathcal{B}_\ell$ and occurs with some other bodies $\mathcal{B}_k$. Then, for a neighborhood $U_{\epsilon_k}^\ell$ of p_ℓ , we can define a tangent plane Π and a neighborhood $A_{\epsilon_k}^\ell \subset \Pi$ at p_ℓ . We can then describe the boundaries $\partial\mathcal{B}_\ell$ and $\partial\mathcal{B}_k$ by the graphs of the respective functions $y_n = \beta_\ell(y')$ and $y_n = \beta_k(y')$, where $y' \in A_{\epsilon_k}^\ell \subset \Pi$ are coordinates in the tangent plane to p_ℓ , centered at $y'_0 = 0$. As before, we restrict our neighborhood so that $A_{\epsilon_k}^\ell$ is the domain of both β_ℓ and β_k . We then have $\beta_k(y') > \beta_\ell(y')$ for all nonzero $y' \in A_{\epsilon_k}^\ell$.

If Ω or $\mathfrak{B}$ are not smooth enough to uniquely define tangent planes, in place of Π , we choose one of the supporting planes for Ω or $\mathcal{B}_\ell$ that is also a supporting plane for $\mathcal{B}_k$. Since all of these objects are Lipschitz, we can choose such a plane so that the above discussion still holds.

Fix $\epsilon > 0$. First, define the following for each p_m

$$\tau_m(p') := \begin{cases} \frac{\beta_m(p') - \omega_m(p')}{3\epsilon}, & \text{if } \beta_m(p') - \omega_m(p') < 3\epsilon, \\ 1, & \text{if } \beta_m(p') - \omega_m(p') \geq 3\epsilon, \end{cases} \quad (5)$$

$$\beta_m^*(\delta) := \{p = (p', p_n) : p_n = \beta_m(p') - \delta\tau_m(y)\}, \quad (6)$$

$$\omega_m^*(\delta) := \{p = (p', p_n) : p_n = \omega_m(p') - \delta\tau_m(y)\}, \quad (7)$$

$$(\mathcal{B}_m^0)_\epsilon^* := \cup_{\delta \in (0, \epsilon)} \beta_m^*(\delta), \quad (8)$$

$$(\Omega_m)_\epsilon^* := \cup_{\delta \in (0, \epsilon)} \omega_m^*(\delta). \quad (9)$$

Then, define the following for each p_ℓ

$$\tau_k^\ell(p') := \begin{cases} \frac{\beta_k(p') - \beta_\ell(p')}{3\epsilon}, & \text{if } \beta_k(p') - \beta_\ell(p') < 3\epsilon, \\ 1, & \text{if } \beta_k(p') - \beta_\ell(p') \geq 3\epsilon, \end{cases} \quad (10)$$

$$\beta_{k,\ell}^*(\delta) := \{p = (p', p_n) : p_n = \beta_k(p') - \delta\tau_k^\ell(y)\}, \quad (11)$$

$$\beta_{\ell,k}^*(\delta) := \{p = (p', p_n) : p_n = \beta_\ell(p') - \delta \tau_k^\ell(y)\}, \quad (12)$$

$$(\mathcal{B}_{k,\ell})_\epsilon^* := \cup_{\delta \in (0,\epsilon)} \beta_{k,\ell}^*(\delta), \quad (13)$$

$$(\mathcal{B}_{\ell,k})_\epsilon^* := \cup_{\delta \in (0,\epsilon)} \beta_{\ell,k}^*(\delta). \quad (14)$$

Finally, we take intersections of these sets to obtain dimpled open neighborhoods of the boundaries of the container and each body. We define

$$(\mathcal{B}_k)_\epsilon^* := (\cap((\mathcal{B}_k)_m^0)_\epsilon^*) \cap (\cap(\mathcal{B}_{k,\ell})_\epsilon^*), \quad (15)$$

$$\mathfrak{B}_\epsilon^* := \cup(\mathcal{B}_k)_\epsilon^* \quad (16)$$

$$\Omega_\epsilon^* := \cap(\Omega_m)_\epsilon^*, \quad (17)$$

where we have made use of a convention that if a contact point is not nearby, the neighborhood has width ϵ in the sense that given a domain U , $U_\epsilon := \{x \in U : \text{dist}(x, \partial U) < \epsilon\}$.

Bemelmans et al. [2] proved a version of the following lemma, and their proof holds on our additionally restricted sets.

Lemma 2.1 *Let $\partial\Omega$ and $\mathfrak{B}$ touch in at most a finite number of points. Then, for $u \in BV(\Omega \setminus \mathfrak{B})$, we have*

$$\int_{\partial\Omega} u \, d\mathcal{H}^{n-1} \leq \sqrt{1+L^2} \int_{\Omega_\epsilon^*} |Du| + c \int_{\Omega_\epsilon^*} u \, dx, \quad (18)$$

$$\int_{\partial\mathcal{B}_k} u \, d\mathcal{H}^{n-1} \leq \sqrt{1+L^2} \int_{(\mathcal{B}_k)_\epsilon^*} |Du| + c \int_{(\mathcal{B}_k)_\epsilon^*} u \, dx, \quad (19)$$

where c is a constant, L is an upper bound for $|D\omega_m|$ and $|D\beta_m|$ in A_{ϵ_m} or $|D\beta_\ell|$ and $|D\beta_k|$ in A_{ϵ_k} when the objects are smooth, and L is the Lipschitz constant otherwise.

They also make the following applicable remark as to the usage of their lemma:

Under the assumption that $\partial\mathcal{B}_k$ and $\partial\Omega$ are regular surfaces, L can be made small by covering $\partial\mathcal{B}_k$ and $\partial\Omega$ by small open sets near each p_k or p_ℓ . Hence, the value of L is no extra restriction on the other physical parameters under that consideration.

We should point out that these types of inequalities also hold (for $L = 0$) when the container has an interior sphere condition, which is certainly satisfied when the boundary of the domain is regular. See Tamanini [5].

Next, we establish a lower bound on $\mathcal{F}$.

Lemma 2.2 *Let $\alpha_i, \rho_i, \rho_{\mathcal{B}_k} \in \mathbb{R}$ be such that $\rho_i \geq 0, \rho_{\mathcal{B}_k} \geq 0, \alpha_i \geq 0, \alpha_i + \alpha_j > 0$. Then, $\mathcal{F}$ is bounded below.*

Proof Since Ω is a bounded set in $\mathbb{R}^n$, we can define $m := \min\{x_n : (x', x_n) \in \Omega\}$. Set $d := \max\{\rho_{E_i}, \rho_{\mathcal{B}_k}\}$. Then, we can bound all of the terms related to gravity below by the quantity $gdm|\Omega|$. Similarly, we can completely wet the boundary $\partial\Omega$ with the set E_i that corresponds to the smallest β_i , and we can completely wet each body $\mathcal{B}_k$ with the set E_i corresponding to the smallest τ_{ik} . Then, it only remains to bound the terms corresponding to interfaces between fluids. It is clear that these terms are bounded below by 0 if and only if each pair $\alpha_i + \alpha_j > 0$. Therefore, $\mathcal{F}$ is bounded below. $\square$

We will break the energy functional into parts. The gravity terms are more well behaved than the free surface and wetting terms, so we postpone their consideration and define

$$\mathcal{A}(\mathfrak{B}, \mathfrak{E}) := \sum_{i=1}^3 \left(\alpha_i \int_{\Omega \setminus \mathfrak{B}} |D\phi_{E_i}| + \beta_i \int_{\partial\Omega} \phi_{E_i} d\mathcal{H}^{n-1} + \sum_{k=1}^K \tau_{ik} \int_{\partial\mathcal{B}_k} \phi_{E_i} d\mathcal{H}^{n-1} \right). \quad (20)$$

The next lemma is the key technical result of this paper.

Lemma 2.3 *Let $\alpha_i, \rho_i, \rho_{\mathcal{B}_k} \in \mathbb{R}$ be such that $\rho_i \geq 0, \rho_{\mathcal{B}_k} \geq 0, \alpha_i \geq 0, \alpha_i + \alpha_j > 0$. We also require two conditions which include a Lipschitz constant L :*

$$\alpha_i + \alpha_j \geq \sqrt{1 + L^2} |\beta_i - \beta_j|, \quad (21)$$

and

$$\alpha_i + \alpha_j \geq \sqrt{1 + L^2} |\tau_{ik} - \tau_{jk}|, \quad (22)$$

for each k . In addition suppose that we have a sequence $\{(\mathfrak{B}^h, \mathfrak{E}^h)\} \subset \mathcal{C}$ and a configuration $(\mathfrak{B}, \mathfrak{E}) \in \mathcal{C}$ with $(\mathfrak{B}^h, \mathfrak{E}^h) \rightarrow (\mathfrak{B}, \mathfrak{E})$ in the sense of characteristic functions converging in $L^1(\Omega)$. Then, $\mathcal{A}(\mathfrak{B}, \mathfrak{E}) \leq \liminf_h \mathcal{A}(\mathfrak{B}^h, \mathfrak{E}^h)$.

Proof In order to show our desired result, we prefer to work with the equivalent form of the conclusion given by

$$\limsup_h \left(\mathcal{A}(\mathfrak{B}, \mathfrak{E}) - \mathcal{A}(\mathfrak{B}^h, \mathfrak{E}^h) \right) \leq 0.$$

We are free to choose one ordering of the sets, and we fix the convention that the β_i terms are labeled in increasing order, i.e., $\beta_1 \leq \beta_2 \leq \beta_3$. We will also need an ordering on each of the τ_{ik} terms, so we introduce permutation functions $p_k(i) = j$ so that $\tau_{p_k(1)k} \leq \tau_{p_k(2)k} \leq \tau_{p_k(3)k}$ for each k .

The first technicality is in comparing the traces of sets E_i^h defined on $\partial\Omega$ using $\Omega_\epsilon^* = \cap(\Omega_m^h)_\epsilon^*$ with the traces of sets E_i defined on $\partial\Omega$ using $\Omega_\epsilon^* = \cap(\Omega_m)_\epsilon^*$. The main difficulty is that the contact points do not appear in the same places. To address this, for each fixed h , we use $(\cap_m(\Omega_m^h)_\epsilon^*) \cap \Omega_\epsilon^*$ when comparing the traces of ϕ_{E_i} and $\phi_{E_i^h}$.

For the traces of sets E_i^h defined on $\mathcal{B}_k^h$ to be compared to the traces of sets E_i defined on $\mathcal{B}_k$, we consider the linear transformation T_k^h that maps $\mathcal{B}_k^h$ onto $\mathcal{B}_k$. This can also be used to map $(\mathcal{B}_k^h)_\epsilon^*$ onto $(\mathcal{B}_k)_\epsilon^*$, or at least onto a portion of $(\mathcal{B}_k)_\epsilon^*$ that is sufficient for defining the trace of both $T_k^h E_i^h$ and E_i . This set is $T_k^h(\mathcal{B}_k^h)_\epsilon^* \cap (\mathcal{B}_k)_\epsilon^*$. One might wish to remove the dependence on a specific h ; however, if there is an infinite spin in our maps T_k^h , then it may be possible to have a countable dense set of contact points on $\partial\mathcal{B}_k$ and this would collapse our dimpled open neighborhood of the boundary on a compact set of positive measure. We do note that $T_k^h(\mathcal{B}_k^h)_\epsilon^* \cap (\mathcal{B}_k)_\epsilon^* \rightarrow (\mathcal{B}_k)_\epsilon^*$ as $h \rightarrow \infty$ in the sense of characteristic functions converging in $L^1(\Omega)$ and that there are only a finite number of contact points on any given $T_k^h(\mathcal{B}_k^h)_\epsilon^* \cap (\mathcal{B}_k)_\epsilon^*$.

We begin with a pair of preliminary inequalities:

Claim 2.4 *For traces defined on the sets as above, we have*

$$\sum_{i=1}^3 \beta_i \int_{\partial\Omega} (\phi_{E_i} - \phi_{E_i^h}) d\mathcal{H}^{n-1} \leq \sum_{j=1,3} |\beta_j - \beta_2| \int_{\partial\Omega} (\phi_{E_j} - \phi_{E_j^h}) d\mathcal{H}^{n-1}, \quad (23)$$

and

$$\begin{aligned} & \sum_{i=1}^3 \tau_{ik} \int_{\partial \mathcal{B}_k} (\phi_{E_i} - \phi_{E_i^h}) \, d\mathcal{H}^{n-1} \\ & \leq \sum_{j=1,3} |\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\partial \mathcal{B}_k} (\phi_{E_{p_k(j)}} - \phi_{E_{p_k(j)}^h}) \, d\mathcal{H}^{n-1}. \end{aligned} \quad (24)$$

Proof Using $\sum \phi_{E_i} = 1$ and $\sum \phi_{E_i^h} = 1$, we have

$$2 \int_{\partial \Omega} (\phi_{E_1} - \phi_{E_1^h}) \, d\mathcal{H}^{n-1} = \int_{\partial \Omega} ((\phi_{E_1} - \phi_{E_2} - \phi_{E_3}) - (\phi_{E_1^h} - \phi_{E_2^h} - \phi_{E_3^h})) \, d\mathcal{H}^{n-1}.$$

Then, we multiply by β_1 and use the fact that $\beta_1 \leq \beta_2$ to find

$$2\beta_1 \int_{\partial \Omega} (\phi_{E_1} - \phi_{E_1^h}) \, d\mathcal{H}^{n-1} \leq \beta_2 \int_{\partial \Omega} [(\phi_{E_1} - \phi_{E_1^h}) - (\phi_{E_2} - \phi_{E_2^h}) - (\phi_{E_3} - \phi_{E_3^h})] \, d\mathcal{H}^{n-1}.$$

Then, we add $\beta_3 \int_{\partial \Omega} (\phi_{E_3} - \phi_{E_3^h}) \, d\mathcal{H}^{n-1}$ to both sides and rearrange to find

$$\begin{aligned} & \sum_{i=1}^3 \beta_i \int_{\partial \Omega} (\phi_{E_i} - \phi_{E_i^h}) \, d\mathcal{H}^{n-1} \leq \\ & (\beta_2 - \beta_1) \int_{\partial \Omega} (\phi_{E_1} - \phi_{E_1^h}) \, d\mathcal{H}^{n-1} + (\beta_3 - \beta_2) \int_{\partial \Omega} (\phi_{E_3} - \phi_{E_3^h}) \, d\mathcal{H}^{n-1}, \end{aligned}$$

establishing the first inequality. The second inequality is proved in the same manner. $\square$

Next, using our discussion on traces, Lemma 2.1, and our claim, we have

$$\begin{aligned} & \mathcal{A}(\mathfrak{Z}, \mathfrak{E}) - \mathcal{A}(\mathfrak{Z}^h, \mathfrak{E}^h) \\ & = \sum_{i=1}^3 \left(\alpha_i \int_{\Omega \setminus \mathfrak{Z}} |D\phi_{E_i}| - \alpha_i \int_{\Omega \setminus \mathfrak{Z}^h} |D\phi_{E_i^h}| \right. \\ & \quad \left. + \beta_i \int_{\partial \Omega} (\phi_{E_i} - \phi_{E_i^h}) \, d\mathcal{H}^{n-1} + \sum_{k=1}^K \tau_{ik} \int_{\partial \mathcal{B}_k} (\phi_{E_i} - \phi_{T_k^h E_i^h}) \, d\mathcal{H}^{n-1} \right) \end{aligned} \quad (25)$$

$$\begin{aligned} & \leq \sum_{i=1}^3 \alpha_i \left(\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{Z} \cup \mathfrak{Z}_\epsilon^*)} |D\phi_{E_i}| - \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{Z}^h \cup \mathfrak{Z}_\epsilon^{h*})} |D\phi_{E_i^h}| \right. \\ & \quad \left. + \int_{\Omega_\epsilon^* \cup \mathfrak{Z}_\epsilon^*} |D\phi_{E_i}| - \int_{\Omega_\epsilon^* \cup \mathfrak{Z}_\epsilon^{h*}} |D\phi_{E_i^h}| \right) \\ & \quad + \sum_{j=1,3} (|\beta_j - \beta_2| \int_{\partial \Omega} (\phi_{E_j} - \phi_{E_j^h}) \, d\mathcal{H}^{n-1} \\ & \quad + \sum_{k=1}^K |\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\partial \mathcal{B}_k} (\phi_{E_{p_k(j)}} - \phi_{T_k^h E_{p_k(j)}^h}) \, d\mathcal{H}^{n-1}) \end{aligned} \quad (26)$$

$$\leq \sum_{i=1}^3 \alpha_i \left(\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{Z} \cup \mathfrak{Z}_\epsilon^*)} |D\phi_{E_i}| - \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{Z}^h \cup \mathfrak{Z}_\epsilon^{h*})} |D\phi_{E_i^h}| \right)$$

$$\begin{aligned}
& + \int_{\Omega_\epsilon^* \cup \mathfrak{B}_\epsilon^*} |D\phi_{E_i}| - \int_{\Omega_\epsilon^* \cup \mathfrak{B}_\epsilon^{h*}} |D\phi_{E_i^h}| \Big) \\
& + \sum_{j=1,3} \left\{ \sqrt{1+L^2} |\beta_j - \beta_2| \left(\int_{\Omega_\epsilon^*} |D\phi_{E_i}| + \int_{\Omega_\epsilon^*} |D\phi_{E_i^h}| \right) \right. \\
& + c |\beta_j - \beta_2| \int_{\Omega_\epsilon^*} |\phi_{E_i} - \phi_{E_i^h}| dx \\
& + \sum_{k=1}^K \left[\sqrt{1+L^2} |\tau_{p_k(j)k} - \tau_{p_k(2)k}| \left(\int_{\mathfrak{B}_{k\epsilon}^*} |D\phi_{E_{p_k(j)}}| + \int_{\mathfrak{B}_{k\epsilon}^{h*}} |D\phi_{E_{p_k(j)}^h}| \right) \right. \\
& \left. \left. + c |\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\mathfrak{B}_{k\epsilon}^*} |\phi_{E_i} - \phi_{T_k^h E_{p_k(j)}^h}| dx \right] \right\} \quad (27)
\end{aligned}$$

$$\begin{aligned}
& \leq \sum_{i=1}^3 \alpha_i \left(\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*)} |D\phi_{E_i}| - \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} |D\phi_{E_i^h}| + \int_{\Omega_\epsilon^* \cup \mathfrak{B}_\epsilon^*} |D\phi_{E_i}| \right) \\
& - \alpha_2 \int_{\Omega_\epsilon^* \cup \mathfrak{B}_\epsilon^{h*}} |D\phi_{E_2^h}| + \sum_{j=1,3} \left\{ \sqrt{1+L^2} |\beta_j - \beta_2| \int_{\Omega_\epsilon^*} |D\phi_{E_j}| \right. \\
& + \left(\sqrt{1+L^2} |\beta_j - \beta_2| - \alpha_j \right) \int_{\Omega_\epsilon^*} |D\phi_{E_j^h}| + c |\beta_j - \beta_2| \int_{\Omega_\epsilon^*} |\phi_{E_j} - \phi_{E_j^h}| dx \\
& + \sum_{k=1}^K \left[\sqrt{1+L^2} |\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\mathfrak{B}_{k\epsilon}^*} |D\phi_{E_{p_k(j)}}| \right. \\
& + \left(\sqrt{1+L^2} |\tau_{p_k(j)k} - \tau_{p_k(2)k}| - \alpha_j \right) \int_{\mathfrak{B}_{k\epsilon}^{h*}} |D\phi_{E_{p_k(j)}^h}| \\
& \left. \left. + c |\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\mathfrak{B}_{k\epsilon}^*} |\phi_{E_{p_k(j)}} - \phi_{T_k^h E_{p_k(j)}^h}| dx \right] \right\}. \quad (28)
\end{aligned}$$

We treat the first terms of (28) in the following claim.

Claim 2.5 *Let $\epsilon > 0$ be a fixed number. As $h \rightarrow \infty$, we have*

$$\limsup_h \sum_{i=1}^3 \alpha_i \left(\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*)} |D\phi_{E_i}| - \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} |D\phi_{E_i^h}| \right) \leq 0.$$

Proof We first observe that for each $i = 1, 2, 3$, as $h \rightarrow \infty$, we have $\phi_{E_i^h} \rightarrow \phi_{E_i}$ in $L^1(\Omega)$. Combining this convergence with the fact that for each $i = 1, 2, 3$, we have $E_i^h \subset \Omega \setminus \mathfrak{B}^h$, and this implies the expected fact that

$$\int_{\mathfrak{B}} \phi_{E_i} dx = 0.$$

Then, we will argue using the definition of the total variation by fixing a function $g \in C_c^1(\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*), \mathbb{R}^n)$ where $\|g\|^0 \leq 1$. With this function, and the above convergence, we have, for each $i = 1, 2, 3$,

$$\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*)} \phi_{E_i} \operatorname{div} g dx = \lim_{h \rightarrow \infty} \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} \phi_{E_i^h} \operatorname{div} g dx.$$

Next, we observe that $|\mathfrak{B} \Delta \mathfrak{B}^h| \rightarrow 0$ in a uniform manner, as each body $\mathcal{B}_k^h$ is a rigid motion of $\mathcal{B}_k$. Then, using the fact that $\text{supp } g \subset \subset \Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*)$, there exists $h_0 \in \mathbb{N}$ large enough so that $\text{supp } g \subset \subset \Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})$ when $h \geq h_0$. Given such an $h \geq h_0$, it follows that we can use this g as an admissible function when defining $\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} |D\phi_{E_i}^h|$ for each $i = 1, 2, 3$. Taking a supremum over all $g \in C_c^1(\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*), \mathbb{R}^n)$ with $\|g\|^0 \leq 1$, we have

$$\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} \phi_{E_i}^h \operatorname{div} g \, dx \leq \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} |D\phi_{E_i}^h|$$

for each $i = 1, 2, 3$, and when $h > h_0$. Then, combining the two quantities displayed above, we have

$$\int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*)} \phi_{E_i} \operatorname{div} g \, dx \leq \liminf_{h \rightarrow \infty} \int_{\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B}^h \cup \mathfrak{B}_\epsilon^{h*})} |D\phi_{E_i}^h|$$

for each $i = 1, 2, 3$. Finally, we take the supremum over all $g \in C_c^1(\Omega \setminus (\Omega_\epsilon^* \cup \mathfrak{B} \cup \mathfrak{B}_\epsilon^*), \mathbb{R}^n)$ with $\|g\|^0 \leq 1$ to complete the proof of the claim. $\square$

As $\phi_{E_j}^h \rightarrow \phi_{E_j}$ in $L^1(\Omega)$, we have

$$c|\beta_j - \beta_2| \int_{\Omega_\epsilon^*} |\phi_{E_j} - \phi_{E_j}^h| \, dx \rightarrow 0.$$

Also, as $\phi_{T_k^h E_j}^h \rightarrow \phi_{E_j}$ in $L^1(\Omega)$, we have

$$c|\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\mathfrak{B}_{k\epsilon}^*} |\phi_{E_{p_k(j)}} - \phi_{T_k^h E_{p_k(j)}}^h| \, dx \rightarrow 0.$$

Then, as $\epsilon \rightarrow 0$, both $|\Omega_\epsilon^*| \rightarrow 0$ and $|\mathfrak{B}_\epsilon^*| \rightarrow 0$; thus, we have $\int_{\Omega_\epsilon^*} |D\phi_{E_i}| \rightarrow 0$, $\int_{\mathfrak{B}_\epsilon^*} |D\phi_{E_i}| \rightarrow 0$,

$$\sqrt{1+L^2}|\beta_j - \beta_2| \int_{\Omega_\epsilon^*} |D\phi_{E_j}| \rightarrow 0,$$

and

$$\sqrt{1+L^2}|\tau_{p_k(j)k} - \tau_{p_k(2)k}| \int_{\mathfrak{B}_{k\epsilon}^*} |D\phi_{E_{p_k(j)}}| \rightarrow 0.$$

Next, we have the terms

$$-\alpha_2 \int_{\Omega_\epsilon^*} |D\phi_{E_2}^h| + \sum_{j=1,3} \left(\sqrt{1+L^2}|\beta_j - \beta_2| - \alpha_j \right) \int_{\Omega_\epsilon^*} |D\phi_{E_j}^h|. \quad (29)$$

This quantity is negative when $\sqrt{1+L^2}|\beta_j - \beta_2| - \alpha_j \leq 0$, so we consider the case where $\alpha_1 < \sqrt{1+L^2}|\beta_1 - \beta_2|$. Using $\sum \phi_{E_j}^h = 1$ and $\beta_1 \leq \beta_2 \leq \beta_3$, we establish the following estimate for (29)

$$\begin{aligned} & \sum_{j=1,3} \left(\sqrt{1+L^2}|\beta_j - \beta_2| - \alpha_j \right) \int_{\Omega_\epsilon^*} |D\phi_{E_j}^h| - \alpha_2 \int_{\Omega_\epsilon^*} |D\phi_{E_2}^h| \\ &= \left(\sqrt{1+L^2}(\beta_2 - \beta_1) - \alpha_1 \right) \int_{\Omega_\epsilon^*} |D(1 - \phi_{E_2}^h - \phi_{E_3}^h)| - \alpha_2 \int_{\Omega_\epsilon^*} |D\phi_{E_2}^h| \end{aligned}$$

$$+ \left(\sqrt{1 + L^2}(\beta_3 - \beta_2) - \alpha_3 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_3^h}| \quad (30)$$

$$\leq \left(\sqrt{1 + L^2}(\beta_2 - \beta_1) - \alpha_1 - \alpha_2 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_2^h}| \\ + \left(\sqrt{1 + L^2}(\beta_3 - \beta_1) - \alpha_1 - \alpha_3 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_3^h}|. \quad (31)$$

Then, our assumption that $\alpha_i + \alpha_j \geq \sqrt{1 + L^2}|\beta_i - \beta_j|$ implies that the end of our estimate is nonpositive. The case when $\alpha_3 < \sqrt{1 + L^2}|\beta_3 - \beta_2|$ is slightly different. Our estimate is

$$\sum_{j=1,3} \left(\sqrt{1 + L^2}|\beta_j - \beta_2| - \alpha_j \right) \int_{\Omega_\epsilon^*} |D\phi_{E_j^h}| - \alpha_2 \int_{\Omega_\epsilon^*} |D\phi_{E_2^h}| \\ \leq \left(\sqrt{1 + L^2}(\beta_2 - \beta_1) - \alpha_1 - \alpha_3 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_1^h}| \\ + \left(\sqrt{1 + L^2}(\beta_3 - \beta_2) - \alpha_2 - \alpha_3 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_2^h}| \\ \leq \left(\sqrt{1 + L^2}(\beta_3 - \beta_1) - \alpha_1 - \alpha_3 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_1^h}| \\ + \left(\sqrt{1 + L^2}(\beta_3 - \beta_2) - \alpha_2 - \alpha_3 \right) \int_{\Omega_\epsilon^*} |D\phi_{E_2^h}|, \quad (32)$$

and the same conclusion holds. The remaining terms are

$$\sum_{j=1,3} \sum_{k=1}^K \left(\sqrt{1 + L^2}|\tau_{p_k(j)k} - \tau_{p_k(2)k}| - \alpha_j \right) \int_{\mathfrak{B}_{k\epsilon}^{h*}} |D\phi_{E_{p_k(j)}^h}| - \alpha_2 \int_{\mathfrak{B}_\epsilon^{h*}} |D\phi_{E_2^h}|, \quad (33)$$

and, upon splitting the second integral using $\mathfrak{B}_\epsilon^{h*} = \cup_k \mathfrak{B}_{k\epsilon}^{h*}$, the above proof applies in the same manner. Therefore, $\mathcal{A}$ is lower semicontinuous. $\square$

Next, we include the gravity terms to treat the whole energy functional.

Lemma 2.6 *Suppose that the hypotheses of Lemma 2.3 are satisfied. Then,*

$$\mathcal{F}(\mathfrak{B}, \mathfrak{E}) \leq \liminf_h \mathcal{F}(\mathfrak{B}^h, \mathfrak{E}^h). \quad (34)$$

Proof By assumption, each $\mathcal{B}_k(c_k, R_k)$ is a rigid object, and by hypothesis, we have convergent sequences $(c_k^h, R_k^h) \rightarrow (c_k^0, R_k^0)$. Thus, each $\mathcal{B}_k(c_k^h, R_k^h)$ converges uniformly to the corresponding $\mathcal{B}_k(c_k^0, R_k^0)$, so our $\rho_{\mathcal{B}_k}g$ terms are continuous by the uniform limit theorem. For the $\rho_{E_i}g$ terms, we will use Fatou's Lemma, which requires a nonnegative integrand. If $m := \min\{x_n : (x', x_n) \in \Omega\} \geq 0$, then those terms are lower semicontinuous. Otherwise, we add quantities to the integrals resulting in

$$g\rho_{E_i} \int_{\Omega \setminus \mathfrak{B}^h} x_n \phi_{E_i^h} dx - g\rho_{E_i} \int_{\Omega} m dx, = g\rho_{E_i} \int_{\Omega \setminus \mathfrak{B}^h} (x_n \phi_{E_i^h} - m) dx - g\rho_{E_i} \int_{\mathfrak{B}^h} m dx,$$

and then, we apply Fatou's lemma and then subtract the terms with m to return to the needed form of $\mathcal{F}$. Finally, we note that we have already shown the lower semicontinuity of $\mathcal{A}(\mathfrak{B}, \mathfrak{E})$ in Lemma 2.3. $\square$

Lastly, we have the main result.

Theorem 2.7 Let $\alpha_i, \rho_i, \rho_{\mathcal{B}_k} \in \mathbb{R}$ be such that $\rho_i \geq 0$, $\rho_{\mathcal{B}_k} \geq 0$, $\alpha_i \geq 0$, $\alpha_i + \alpha_j > 0$. We also require two conditions which include a Lipschitz constant L :

$$\alpha_i + \alpha_j \geq \sqrt{1 + L^2} |\beta_i - \beta_j|, \quad (35)$$

and

$$\alpha_i + \alpha_j \geq \sqrt{1 + L^2} |\tau_{ik} - \tau_{jk}|, \quad (36)$$

for each k . In addition suppose that we have a sequence $\{(\mathfrak{B}^h, \mathfrak{E}^h)\} \subset \mathcal{C}$ and a configuration $(\mathfrak{B}, \mathfrak{E}) \in \mathcal{C}$ with $(\mathfrak{B}^h, \mathfrak{E}^h) \rightarrow (\mathfrak{B}, \mathfrak{E})$ a.e. Then, there exists a $(\mathfrak{B}^0, \mathfrak{E}^0) \in \mathcal{C}$ such that for all $(\mathfrak{B}, \mathfrak{E}) \in \mathcal{C}$,

$$\mathcal{F}(\mathfrak{B}^0, \mathfrak{E}^0) \leq \mathcal{F}(\mathfrak{B}, \mathfrak{E}).$$

Proof By Lemma 2.2, $\mathcal{F}$ is bounded from below, and we can form a minimizing sequence $\{(\mathfrak{B}^h, \mathfrak{E}^h)\} \subset \mathcal{C}$. We may assume that $\mathcal{F}(\mathfrak{B}^h, \mathfrak{E}^h) \leq m_0 + 1$ where m_0 is the value of $\mathcal{F}$ at the first element of the sequence. If this inequality does not hold for some elements, discard them and form a subsequence by relabeling. Therefore, ϕ_{E_i} , $i = 1, 2, 3$, are bounded in $BV(\Omega)$.

The values of each R_k^h belong to a compact set, so each $|R_k^h|$ is bounded in $\mathbb{R}$ for all $h \in \mathbb{N}$. We also have bounds on each c_k , as each body satisfies $\mathcal{B}_k \subset \Omega$.

Therefore, our sequence is bounded in the sense that $\sum_k |c_k^h| + \sum_k |R_k^h| + \|\phi_{\mathfrak{E}}\|_{BV(\Omega)} \leq C$ for all $h \in \mathbb{N}$. Since our sequence was a minimizing sequence, we can form a minimizing convergent subsequence, which we again call $\{(\mathfrak{B}^h, \mathfrak{E}^h)\}$. By Lemma 2.6, $\mathcal{F}$ is lower semicontinuous with respect to this sequence; hence, the sequence converges in $\mathcal{C}$ to some $(\mathfrak{B}^0, \mathfrak{E}^0)$, as desired. $\square$

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