Endo-Noetherian rings

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Abstract

A commutative ring R is said to be endo-Noetherian if the chain of annihilators $\text{ann}(a_1) \subseteq \text{ann}(a_2) \subseteq \dots$ stabilizes for each sequence $(a_k)_k$ of elements of R (Ndiaye and Gueye in Int J Appl Math 86:871–881, 2013). In this paper, we give equivalent conditions for the power series (resp., polynomial) rings over an Armendariz ring to be endo-Noetherian. We also study several properties of an endo-Noetherian ring. The concept of the amalgamated duplication of R along an ideal I , $R \bowtie I$ is studied.

Keywords Endo-Noetherian rings · Armendariz rings · Power serieswise Armendariz rings

Mathematics Subject Classification 13E99 · 13F25 · 13F20

1 Introduction

All the rings considered are commutative with identity. An R -module M is called *endo-Noetherian* if it satisfies the ascending chain condition on endomorphic kernels, i.e. the chain $\text{Ker}(f_1) \subseteq \text{Ker}(f_2) \subseteq \dots$ stabilizes for each sequence $(f_k)_k$ of endomorphisms of M [17]. A commutative ring R is said to be *endo-Noetherian* if it is endo-Noetherian as an R -module. Equivalently, R is endo-Noetherian if the chain of annihilators $\text{ann}(a_1) \subseteq \text{ann}(a_2) \subseteq \dots$ stabilizes for each sequence $(a_k)_k$ of elements of R . The class of endo-Noetherian rings includes Noetherian rings and iso-Noetherian rings (Recall from [7] that an R -module M is *iso-Noetherian* if for every ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M , there exists an index $n \geq 1$ such that $M_n \simeq M_i$ for every $i \geq n$. We say that a ring R is *iso-Noetherian* if the R -module R is iso-Noetherian). The purpose of this paper is to characterize when the power series (resp., polynomial) rings over Armendariz rings are endo-Noetherian.

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In the first part of this paper, we study several properties of an endo-Noetherian ring. We show that if R is a commutative ring and S a multiplicative subset of R consisting of regular elements, then R is endo-Noetherian if and only if $S^{-1}R$ is endo-Noetherian. Also, we construct an example of a commutative ring R and a multiplicative subset S of R such that R_S is Noetherian (so endo-Noetherian) but R is not endo-Noetherian. We study when the direct product of rings is endo-Noetherian. We prove that if $(R_i)_{i \in I}$ is a family of commutative rings and $R = \prod_{i \in I} R_i$ the product of R_i , then R is endo-Noetherian if and only if I is a finite set and for each $i \in I$, R_i is endo-Noetherian. Let $(*)$ be the following property: every increasing sequence of the form $(I : a_1) \subseteq (I : a_2) \subseteq \cdots$ is stationary, where I is an ideal of R and $(a_k)_k$ a sequence of elements of R . Note that if R satisfies the property $(*)$, then R satisfies the *accr* condition (we say that R satisfies the *accr* condition if the ascending chain of colon ideals of the form $(I : a) \subseteq (I : a^2) \subseteq (I : a^3) \subseteq \cdots$ is stationary for each $a \in R$ and every ideal I of R). We show that R satisfies the property $(*)$ if and only if R/I is an endo-Noetherian ring for each ideal I of R . We also study the amalgamated duplication of R along an ideal I , $R \bowtie I$ to be endo-Noetherian. The amalgamated duplication of a ring R along an ideal I is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i) | r \in R, i \in I\}.$$

This construction has been studied, in the general case, and from the different points of view of pullbacks, by D'Anna and Fontana [5]. According to [6], an element of R is called a *regular element* if it is not a zerodivisor. Also, we say that I is a *regular ideal* if it contains a regular element of R . We prove that if I is a regular ideal of R , then R is endo-Noetherian if and only if $R \bowtie I$ is endo-Noetherian. In the second part of this paper, we give equivalent conditions for the power series (resp., polynomial) rings over an Armendariz ring to be endo-Noetherian. First, we need to collect some necessary notions. Recall that a ring R is called an *Armendariz ring* if whenever the polynomials $f = \sum_{i=0}^m a_i X^i$ and $g = \sum_{i=0}^n b_i X^i \in R[X]$ satisfy $fg = 0$ we have $a_i b_j = 0$ for every i and j [14]. Also, R is said *power serieswise Armendariz* if for each $f = \sum_{i \geq 0} a_i X^i$ and $g = \sum_{i \geq 0} b_i X^i \in R[[X]]$ such that $fg = 0$, then $a_i b_j = 0$ for each i and j . We prove that if R is an Armendariz commutative ring, then the following are equivalent:

1. $R[X]$ is endo-Noetherian.
2. R satisfies the acc on annihilators of finite subset.
3. R satisfies the acc on annihilators of finitely generated ideals of R .

Finally, if R is a power serieswise Armendariz ring, then the following statements are equivalent:

1. $R[[X]]$ is endo-Noetherian.
2. R satisfies the acc on annihilators of countably subset.
3. R satisfies the acc on annihilators of countably generated ideals of R .
4. For each sequence $(f_k = \sum_{j \geq 0} a_{k,j} X^j)_k$ of elements of $R[[X]]$ such that $\text{ann}_{R[[X]]}(f_1) \subseteq \text{ann}_{R[[X]]}(f_2) \subseteq \cdots$, there exists an $n \in \mathbb{N}$ such that for each $k \geq n$, $\cap_{j \geq 0} \text{ann}_R(a_{k,j}) = \cap_{j \geq 0} \text{ann}_R(a_{n,j})$.

2 General properties

Let R be a commutative ring. According to [17], the ring R is said to be *endo-Noetherian* if the increasing sequence $\text{ann}(a_1) \subseteq \text{ann}(a_2) \subseteq \cdots$ stabilizes for any sequence $(a_k)_k$ of elements of R . In this section, we study several properties of the endo-Noetherian ring.

Example 2.1 1. Every Noetherian ring is an endo-Noetherian ring.

2. The converse of (1) is not true in general. Let R be a commutative ring with identity and M an R -module. The *idealization* of M in R (or *trivial extension* of R by M) is a commutative ring $R(+M) = \{(r, m) \mid r \in R \text{ and } m \in M\}$ under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+M)$. Now, let K be a field and V a K -vector space, then $K(+V)$ is an endo-Noetherian ring, because for each $(k, m) \in K \times V \setminus (0, 0)$,

$$\text{ann}_{K(+V)}((k, m)) = \begin{cases} (0, 0) & \text{if } k \neq 0 \\ 0(+V) & \text{if } k = 0 \end{cases}$$

Note that if V has an infinite dimensional, then by [2, Theorem 4.8], $K(+V)$ is not Noetherian.

3. Recall from [7] that an R -module M is *iso-Noetherian* if for every ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M , there exists an index $n \geq 1$ such that $M_n \simeq M_i$ for every $i \geq n$. We say that a ring R is iso-Noetherian if the R -module R is iso-Noetherian. By [7, Lemma 5.4(1)], every iso-Noetherian ring is endo-Noetherian.

Recall that a commutative ring R is said to be *strongly Hopfian* if the chain of annihilators $\text{ann}(a) \subseteq \text{ann}(a^2) \subseteq \cdots$ is stationary for each $a \in R$ [3,9,12,13]. Note that every endo-Noetherian ring satisfies the strongly Hopfian property. In Example 2.7, we provide a strongly Hopfian ring which is not endo-Noetherian.

Proposition 2.2 *Let R be a commutative ring and S a multiplicative subset of R consists of regular elements. Then, the following assertions are equivalent:*

1. R is endo-Noetherian.
2. $S^{-1}R$ is endo-Noetherian.

Proof (1) $\Rightarrow$ (2). Let $(\frac{a_k}{s_k})_k$ be a sequence of elements of $S^{-1}R$ such that $\text{ann}_{S^{-1}R}(\frac{a_1}{s_1}) \subseteq \text{ann}_{S^{-1}R}(\frac{a_2}{s_2}) \subseteq \cdots$. We will show that $\text{ann}_R(a_k) \subseteq \text{ann}_R(a_{k+1})$ for each $k \in \mathbb{N}$. Let $b \in \text{ann}_R(a_k)$. Since $\frac{b}{1} \in \text{ann}_{S^{-1}R}(\frac{a_k}{s_k})$, $\frac{b}{1} \in \text{ann}_{S^{-1}R}(\frac{a_{k+1}}{s_{k+1}})$. This implies that $b \in \text{ann}_R(a_{k+1})$, because S consists of regular elements.

Now, as R is endo-Noetherian, there exists a positive integer n such that for each $k \geq n$, $\text{ann}(a_k) = \text{ann}(a_n)$. Let $k \geq n$ and let $\frac{b}{t} \in \text{ann}_{S^{-1}R}(\frac{a_k}{s_k})$. Then, $ba_k = 0$; so $ba_n = 0$ which implies that $\frac{b}{t} \in \text{ann}_{S^{-1}R}(\frac{a_n}{s_n})$. Thus, the chain $\text{ann}_{S^{-1}R}(\frac{a_1}{s_1}) \subseteq \text{ann}_{S^{-1}R}(\frac{a_2}{s_2}) \subseteq \cdots$ is stationary, and hence the ring $S^{-1}R$ satisfies the endo-Noetherian property.

(2) $\Rightarrow$ (1). Assume that $S^{-1}R$ is endo-Noetherian and let $(a_k)_k$ be a sequence of elements of R such that $\text{ann}(a_1) \subseteq \text{ann}(a_2) \subseteq \cdots$. Since S consists of regular elements, for each $k \in \mathbb{N}^*$, $\text{ann}_{S^{-1}R}(\frac{a_k}{1}) \subseteq \text{ann}_{S^{-1}R}(\frac{a_{k+1}}{1})$. Thus by hypothesis, there exists a nonzero positive integer n such that for each $k \geq n$, $\text{ann}_{S^{-1}R}(\frac{a_n}{1}) = \text{ann}_{S^{-1}R}(\frac{a_k}{1})$. Now, let $k \geq n$, and $\alpha \in \text{ann}(a_k)$, then $\frac{\alpha}{1} \in \text{ann}_{S^{-1}R}(\frac{a_k}{1}) = \text{ann}_{S^{-1}R}(\frac{a_n}{1})$; so $\alpha a_n = 0$, because S consists of regular elements. This implies that $\text{ann}(a_n) = \text{ann}(a_k)$ for each $k \geq n$. Hence, R is endo-Noetherian. $\square$

We next construct an example of a commutative ring R and a multiplicative subset S of R such that R_S is Noetherian (so endo-Noetherian) but R is not endo-Noetherian. To prove this, we need to collect some necessary notions. Let R be a commutative ring and S be a multiplicative subset of R . Recall from [1] (or [10]) that an ideal I of R is called *S-finite* (resp., *S-principal*), if $sI \subseteq J \subseteq I$ for some finitely generated (resp., principal) ideal J of R and some $s \in S$. Also, R is called *S-Noetherian* if each ideal of R is *S-finite*. In [1], the authors showed that R is *S-Noetherian* if and only if every prime ideal is *S-finite*.

Example 2.3 Let F be a field, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over F , I the ideal of $F[\mathbf{X}]$ generated by the set $\{X_0^j X_j \mid j \geq 1\}$ and $R = F[\mathbf{X}]/I$. For each $i \in \mathbb{N}$, let $\overline{X}_i$ denote the homomorphic image of X_i in R . Let $S = \{\overline{X}_0^i \mid i \in \mathbb{N}\}$ and $\overline{\mathbf{X}} = \{\overline{X}_i \mid i \in \mathbb{N}\}$. Then, S is a multiplicative subset of R .

1. R is not strongly Hopfian. Indeed, let $a = \overline{X}_0$, then for each $n \in \mathbb{N}^*$, $\overline{X}_n \in \text{ann}(a^n) \setminus \text{ann}(a^{n-1})$. So R is not endo-Noetherian.
2. Let P be the prime ideal of R generated by the set $\{\overline{\mathbf{X}}\}$. Then, P is the only prime ideal of R . Since $\overline{X}_0 \in P$, $\overline{X}_0 P \subseteq (\overline{X}_0) \subseteq P$; so P is an *S-finite* ideal of R . This implies that R is an *S-Noetherian* ring, and hence R_S is Noetherian [1, Proposition 2(f)]. In particular, R is an *S-Noetherian* ring which is not Noetherian.

Let R be a commutative ring and S a multiplicative subset of R . Our next result gives a sufficient condition for the localization of R , R_S satisfies the endo-Noetherian property implies that R is endo-Noetherian.

Proposition 2.4 Let R be a quasi-local ring with maximal ideals $M_1, \dots, M_n$. If $(R \setminus M_i)^{-1} R$ is an endo-Noetherian ring for each $1 \leq i \leq n$, then R is endo-Noetherian.

Proof Let $(a_k)_k$ be a sequence of elements of R such that $\text{ann}_R(a_1) \subseteq \text{ann}_R(a_2) \subseteq \dots$. It is easy to show that $\text{ann}_{(R \setminus M_i)^{-1} R}(\frac{a_1}{1}) \subseteq \text{ann}_{(R \setminus M_i)^{-1} R}(\frac{a_2}{1}) \subseteq \dots$ for each $1 \leq i \leq n$. Note that for each $1 \leq i \leq n$, there exists a positive integer m_i such that for each $k \geq m_i$, $\text{ann}_{(R \setminus M_i)^{-1} R}(\frac{a_k}{1}) = \text{ann}_{(R \setminus M_i)^{-1} R}(\frac{a_{m_i}}{1})$. Let $m = \max\{m_i \mid 1 \leq i \leq n\}$. Then, for each $1 \leq i \leq n$ and each $k \geq m$, $\text{ann}_{(R \setminus M_i)^{-1} R}(\frac{a_k}{1}) = \text{ann}_{(R \setminus M_i)^{-1} R}(\frac{a_m}{1})$.

Now, we will prove that for each $k \geq m$, $\text{ann}_R(a_k) = \text{ann}_R(a_m)$. Let $k \geq m$ and $b \in \text{ann}_R(a_k)$. Then $ba_k = 0$, and for each $1 \leq i \leq n$, $\frac{b}{1} \frac{a_k}{1} = 0$ in $(R \setminus M_i)^{-1} R$; so for each $1 \leq i \leq n$, $\frac{b}{1} \frac{a_m}{1} = 0$. This implies that for each $1 \leq i \leq n$, there exists an $s_i \in R \setminus M_i$ such that $s_i ba_m = 0$. As the ideal generated by $\{s_1, \dots, s_n\}$ is equal to R , we deduce that $ba_m = 0$, and hence R is endo-Noetherian. $\square$

Theorem 2.5 Let $(R_i)_{i \in I}$ be a family of commutative rings and $R = \prod_{i \in I} R_i$ the product of R_i . Then, R is endo-Noetherian if and only if I is finite and for each $i \in I$, R_i is endo-Noetherian.

Proof The “only if” part assume that $R = \prod_{i \in I} R_i$ is endo-Noetherian. If I is not finite, then the sequence $\text{ann}_R((0, 1, 1, \dots)) \subseteq \text{ann}_R((0, 0, 1, 1, \dots)) \subseteq \dots$ does not stabilize. It is easy to prove that if $R = \prod_{i=1}^n R_i$ is endo-Noetherian, then for each $1 \leq i \leq n$, R_i is endo-Noetherian.

For “if” part, let $R_1, \dots, R_n$ be endo-Noetherian rings. We will show that $R = \prod_{i=1}^n R_i$ is endo-Noetherian. To prove this, it sufficient to show the proof in the case $n = 2$ and conclude by induction on n .

Let $R = A \times B$ and $(a_1, b_1), \dots \in R$ such that $\text{ann}_R((a_1, b_1)) \subseteq \dots$. It is easy to show that $\text{ann}_A(a_1) \subseteq \text{ann}_A(a_2) \subseteq \dots$ and $\text{ann}_B(b_1) \subseteq \text{ann}_B(b_2) \subseteq \dots$. Since A and B are

endo-Noetherian rings, there exists a positive integer n such that for each $k \geq n$, $\text{ann}_A(a_k) = \text{ann}_A(a_n)$ and $\text{ann}_B(b_k) = \text{ann}_B(b_n)$. This implies that for each $k \geq n$, $\text{ann}_R((a_k, b_k)) = \text{ann}_R((a_n, b_n))$, and hence R is endo-Noetherian. $\square$

Corollary 2.6 *Let $(R_i)_{1 \leq i \leq n}$ be a finite family of commutative rings and $R = \prod_{i=1}^n R_i$. Then, R is endo-Noetherian if and only if R_i is endo-Noetherian for each $1 \leq i \leq n$.*

Example 2.7 Let $(F_i)_{i \in I}$ be an infinite family of fields and $R = \prod_{i \in I} F_i$. Since R is a zero-dimensional ring, by [12, Remark 1.20], R is strongly Hopfian. But by Theorem 2.5, R is not an endo-Noetherian ring.

Remark 2.8 Note that if R is endo-Noetherian and I an ideal of R , then R/I is not endo-Noetherian in general. In fact, let K be a field and $R = K[X_0, \dots]$, then R is endo-Noetherian. Let I be the ideal of R generated by the elements $X_0X_1, X_0^2X_2, \dots$. Then, R/I is not strongly Hopfian, because $\overline{X_n} \in \text{ann}_{R/I}(\overline{X_0^n}) \setminus \text{ann}_{R/I}(\overline{X_0^{n-1}})$. In particular, R/I is not endo-Noetherian.

Let R be a commutative ring and let $(*)$ be the following property: every increasing sequence of the form $(I : a_1) \subseteq (I : a_2) \subseteq \dots$ is stationary, where I is an ideal of R and $(a_k)_k$ a sequence of elements of R . Note that if R satisfies the property $(*)$, then R satisfies the accr condition (we say that R satisfies the accr condition if the ascending chain of colon ideals of the form $(I : a) \subseteq (I : a^2) \subseteq (I : a^3) \subseteq \dots$ is stationary for each $a \in R$ and every ideal I of R). It is easy to show that if R satisfies the property $(*)$, then R is an endo-Noetherian ring and the converse does not hold in general. It suffices to take an integral domain which is not satisfied accr. Indeed, let F be a field, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over F and $R = F[\mathbf{X}]$. Let Y be an indeterminate over R . Since R is not Noetherian, by [16, Theorem 2], $R[[Y]]$ does not satisfy accr, in particular $R[[Y]]$ does not satisfy $(*)$. Note that $R[[Y]]$ is endo-Noetherian, because it is an integral domain.

Theorem 2.9 *The following assertions are equivalent for a commutative ring R .*

1. R/I is an endo-Noetherian ring for each ideal I of R .
2. R satisfies the property $(*)$.

Proof (1) $\Rightarrow$ (2). Let I be an ideal of R and $a_1, a_2, \dots \in R$ such that $(I : a_1) \subseteq (I : a_2) \subseteq \dots$. We show that $\text{ann}_{R/I}(\overline{a_1}) \subseteq \text{ann}_{R/I}(\overline{a_2}) \subseteq \dots$. Let $\overline{\alpha} \in \text{ann}_{R/I}(\overline{a_k})$, then $\alpha a_k \in I$; so $\alpha a_{k+1} \in I$. Therefore, $\overline{\alpha} \in \text{ann}_{R/I}(\overline{a_{k+1}})$.

Now, since R/I is endo-Noetherian, there exists a positive integer n such that $\text{ann}_{R/I}(\overline{a_k}) = \text{ann}_{R/I}(\overline{a_n})$ for each $k \geq n$. We will prove that $(I : a_k) = (I : a_n)$ for each $k \geq n$. Let $b \in (I : a_k)$. Then, $\overline{b a_k} = \overline{0}$ in R/I . Therefore, $\overline{b} \in \text{ann}_{R/I}(\overline{a_k}) \subseteq \text{ann}_{R/I}(\overline{a_n})$; so $b a_n \in I$.

(2) $\Rightarrow$ (1). Let $a_1, a_2, \dots \in R$ such that $\text{ann}_{R/I}(\overline{a_1}) \subseteq \text{ann}_{R/I}(\overline{a_2}) \subseteq \dots$. We show that $(I : a_1) \subseteq (I : a_2) \subseteq \dots$. Let $b \in (I : a_k)$. Then, $\overline{b} \in \text{ann}_{R/I}(\overline{a_k}) \subseteq \text{ann}_{R/I}(\overline{a_{k+1}})$. Thus, $b \in (I : a_{k+1})$.

Now, by hypothesis, there exists a positive integer n such that $(I : a_k) = (I : a_n)$ for each $k \geq n$. This implies that $\text{ann}_{R/I}(\overline{a_k}) = \text{ann}_{R/I}(\overline{a_n})$ for each $k \geq n$. $\square$

Proposition 2.10 *Let R, T be two commutative rings and let $f : R \rightarrow T$ be a ring homomorphism. If R satisfies $(*)$, then $f(R)$ satisfies $(*)$. In particular, $f(R)$ is an endo-Noetherian ring.*

Proof Let I be an ideal of $f(R)$ and $a_1, a_2, \dots \in R$ such that $(I :_{f(R)} f(a_1)) \subseteq (I :_{f(R)} f(a_2)) \subseteq \dots$. We show that $(f^{-1}(I) :_R a_1) \subseteq (f^{-1}(I) :_R a_2) \subseteq \dots$. Let $x \in R$ such that $xa_k \in f^{-1}(I)$; so $f(x) \in (I :_{f(R)} f(a_k))$. Therefore, $f(x) \in (I :_{f(R)} f(a_{k+1}))$, and hence $xa_{k+1} \in f^{-1}(I)$.

Now, since R satisfies $(*)$, there exists a positive integer n such that $(f^{-1}(I) :_R a_k) = (f^{-1}(I) :_R a_n)$ for each $k \geq n$. We show that $(I :_{f(R)} f(a_k)) = (I :_{f(R)} f(a_n))$ for each $k \geq n$. Let $y \in f(R)$ such that $yf(a_k) \in I$. There exists $x \in R$ such that $y = f(x)$. Then, $f(xa_k) \in I$. This implies that $xa_k \in f^{-1}(I)$. Thus, $xa_n \in f^{-1}(I)$. Hence, $yf(a_n) \in I$. $\square$

Recall the following definitions: A ring R is called a *zero divisor ring* (a *ZD-ring*), if the set of zero divisors on the R -module R/I is a finite union of prime ideals for each ideal I of R . A *Laskerian ring* is one for which every ideal is a finite intersection of primary ideals, and finally, a ring R is *strongly Laskerian* if it is Laskerian and for each ideal I of R there exists $k \in \mathbb{N}$ such that $(\sqrt{I})^k \subseteq I$. Clearly, every Laskerian ring is a ZD-ring. Heinzer and Ohm [11], proved that a polynomial ring $R[X]$ is a ZD-ring if and only if R is Noetherian, and hence the conditions Noetherian, Strongly Laskerian, Laskerian, and ZD are equivalent in $R[X]$. Similarly, Gilmer and Heinzer [8] proved that a power series ring $R[[X]]$ is Laskerian if and only if R is Noetherian (but R need not be Noetherian if $R[[X]]$ is a ZD-ring). We remark that Lu [16] gave a very simple proof to show that if $R[X]$ (resp. $R[[X]]$) satisfies *accr*, then R is Noetherian. Using this result, we prove the following proposition.

Proposition 2.11 $R[X]$ (resp., $R[[X]]$) satisfies $(*)$ if and only if R is Noetherian.

Proof Since $R[X]$ satisfies $(*)$, $R[X]$ satisfies *accr*; so by [16, Theorem 2], R is Noetherian. The other implication is obvious. $\square$

Remark 2.12 By Proposition 2.11, the conditions $(*)$, Noetherian, Strongly Laskerian, Laskerian, and ZD are equivalent in $R[X]$. This shows that the property $(*)$ does not pass to the polynomial (resp., formal power series) rings. Indeed, we will construct a commutative ring R satisfies the condition $(*)$, but R is not Noetherian. Let K be a field and V an infinite-dimensional K -vector space. By Example 2.1(2), $R = K(+)M$ is not Noetherian. On the other hand, by [2, Corollary 3.4], every proper ideal of R has the form $I = 0(+)N$ for some submodule N of M . Since $R/I = (K(+)M)/(0(+)N) \simeq K(+)(M/N)$, by Example 2.1(2), R/I is endo-Noetherian for every ideal I of R . Thus by Theorem 2.9, R satisfies $(*)$.

In [4], D'Anna and Fontana considered a different type of construction obtained involving a ring R and an ideal $I \subseteq R$ that is denoted by $R \bowtie I$, called *amalgamated duplication*, and it is defined as the following subring of $R \times R$:

$$R \bowtie I = \{(r, r + i) \mid r \in R, i \in I\}.$$

According to [6], an element of R is called a *regular element* if it is not a zerodivisor. Also, we say that I is a *regular ideal* if it contains a regular element of R . We end this section by the following result.

Theorem 2.13 Let R be a commutative ring and I a regular ideal of R . Then, the following assertions are equivalent:

1. R is endo-Noetherian.
2. $R \times R$ is endo-Noetherian.
3. $R \bowtie I$ is endo-Noetherian.

Proof (1) $\Rightarrow$ (2). It follows from Theorem 2.5.

(2) $\Rightarrow$ (3). Let $(r_1, r_1 + i_1), (r_2, r_2 + i_2), \dots \in R \bowtie I$ such that $\text{ann}_{R \bowtie I}((r_1, r_1 + i_1)) \subseteq \text{ann}_{R \bowtie I}((r_2, r_2 + i_2)) \subseteq \dots$. We will show that $\text{ann}_{R \times R}((r_k, r_k + i_k)) \subseteq \text{ann}_{R \times R}((r_{k+1}, r_{k+1} + i_{k+1}))$ for each $k \geq 1$. Let $(a, b) \in \text{ann}_{R \times R}((r_k, r_k + i_k))$ and let i be a regular element of I . We have $(i, i)(a, b) = (ia, ia + i(b - a)) \in R \bowtie I$. Moreover, $(i, i)(a, b)(r_k, r_k + i_k) = (0, 0)$, then $(i, i)(a, b) \in \text{ann}_{R \bowtie I}((r_{k+1}, r_{k+1} + i_{k+1}))$. Thus, $(a, b) \in \text{ann}_{R \times R}((r_{k+1}, r_{k+1} + i_{k+1}))$, because (i, i) is regular in $R \times R$. Now, by hypothesis, there exists a positive integer n such that $\text{ann}_{R \times R}((r_k, r_k + i_k)) = \text{ann}_{R \times R}((r_n, r_n + i_n))$ for each $k \geq n$. This implies that for each $k \geq n$, $(R \bowtie I) \cap \text{ann}_{R \times R}((r_k, r_k + i_k)) = (R \bowtie I) \cap \text{ann}_{R \times R}((r_n, r_n + i_n))$; so $\text{ann}_{R \bowtie I}((r_k, r_k + i_k)) = \text{ann}_{R \bowtie I}((r_n, r_n + i_n))$ for each $k \geq n$.

(3) $\Rightarrow$ (1). Note that this implication is always true and does not need the assumption that I contains a regular element.

Let $r_1, r_2, \dots \in R$ such that $\text{ann}_R(r_1) \subseteq \text{ann}_R(r_2) \subseteq \dots$. We will prove that $\text{ann}_{R \bowtie I}((r_k, r_k)) \subseteq \text{ann}_{R \bowtie I}((r_{k+1}, r_{k+1}))$ for each $k \geq 1$. Let $(\alpha, \alpha + i) \in R \bowtie I$ such that $(\alpha, \alpha + i)(r_k, r_k) = (0, 0)$. Then, α and $\alpha + i$ are in $\text{ann}_R(r_k)$; so $(\alpha, \alpha + i) \in \text{ann}_{R \bowtie I}((r_{k+1}, r_{k+1}))$. Since $R \bowtie I$ is endo-Noetherian, there exists a positive integer n such that $\text{ann}_{R \bowtie I}((r_k, r_k)) = \text{ann}_{R \bowtie I}((r_n, r_n))$ for each $k \geq n$. Now, we show that $\text{ann}_R(r_k) = \text{ann}_R(r_n)$ for each $k \geq n$. Let $b \in \text{ann}_R(r_k)$. Since $(b, b) \in \text{ann}_{R \bowtie I}((r_k, r_k))$, $(b, b)(r_n, r_n) = (0, 0)$ thus $br_n = 0$, and hence $b \in \text{ann}_R(r_n)$. $\square$

3 The case of Armendariz and power serieswise Armendariz rings

Recall that a ring R is called an *Armendariz ring* if whenever the polynomials $f = \sum_{i=0}^m a_i X^i$ and $g = \sum_{i=0}^n b_i X^i \in R[X]$ satisfy $fg = 0$ we have $a_i b_j = 0$ for every i and j [14]. We next give a necessary and sufficient condition for the polynomial ring $R[X]$ to be endo-Noetherian, where R is an Armendariz ring.

Theorem 3.1 *Let R be an Armendariz commutative ring. Then, the following statements are equivalent:*

1. $R[X]$ is endo-Noetherian.
2. R satisfies the acc on annihilators of finite subset.
3. R satisfies the acc on annihilators of finitely generated ideals of R .

Proof (1) $\Rightarrow$ (2). Let $S_1, S_2, \dots$ be a sequence of finite subset such that $\text{ann}_R(S_1) \subseteq \text{ann}_R(S_2) \subseteq \dots$. Put $S_i = \{a_{0,i}, a_{1,i}, \dots, a_{n_i,i}\}$ for each i and let $P_i = \sum_{j=0}^{n_i} a_{j,i} X^j$. We will show that $\text{ann}_{R[X]}(P_1) \subseteq \text{ann}_{R[X]}(P_2) \subseteq \dots$. Let $k \in \mathbb{N}^*$ and let $f = \sum_{j=0}^d b_j X^j$ such that $fP_k = 0$. Since R is an Armendariz ring, for each $0 \leq j \leq d$ and $0 \leq l \leq n_k$, $b_j a_{l,k} = 0$. Thus for each $0 \leq j \leq d$, $b_j \in \text{ann}_R(S_k)$, which implies that for each $0 \leq j \leq d$, $b_j \in \text{ann}_R(S_{k+1})$; so $b_j a_{l,k+1} = 0$ for each $0 \leq j \leq d$ and $0 \leq l \leq n_{k+1}$. This implies that $fP_{k+1} = 0$ and we conclude that $\text{ann}_{R[X]}(P_1) \subseteq \text{ann}_{R[X]}(P_2) \subseteq \dots$. Now, since $R[X]$ is endo-Noetherian, there exists a positive integer n such that $\text{ann}_{R[X]}(P_k) = \text{ann}_{R[X]}(P_n)$ for each $k \geq n$. It is easy to show that $\text{ann}_R(S_k) = \text{ann}_R(S_n)$ for each $k \geq n$. Hence, R satisfies the acc on annihilators of finite subset.

(2) $\Rightarrow$ (1) Let $(P_k)_k$ be a sequence of elements of $R[X]$ such that $\text{ann}_{R[X]}(P_1) \subseteq \text{ann}_{R[X]}(P_2) \subseteq \dots$. For each $i \geq 1$, let $P_i = \sum_{j=0}^{n_i} a_{j,i} X^j$ and put $S_i = \{a_{0,i}, a_{1,i}, \dots, a_{n_i,i}\}$. We will prove that $\text{ann}_R(S_k) \subseteq \text{ann}_R(S_{k+1})$ for each $k \geq 1$. Let $\alpha \in R$ such that $\alpha S_k = 0$. Then by construction of S_k , we have $\alpha P_k = 0$; so $\alpha P_{k+1} = 0$ which implies that $\alpha S_{k+1} = 0$.

Since R satisfies the acc on annihilators of finite subset, there exists a positive integer n such that $\text{ann}_R(S_k) = \text{ann}_R(S_n)$ for each $k \geq n$. Now, let $Q = \sum_{i=0}^r q_i X^i \in R[X]$ such that $Q P_k = 0$. Then $q_i S_k = 0$ for each $0 \leq i \leq r$; so $q_i S_n = 0$ for each $0 \leq i \leq r$. This implies that $Q P_n = 0$, and hence $\text{ann}_{R[X]}(P_n) = \text{ann}_{R[X]}(P_k)$ for each $k \geq n$.

(2) $\Leftrightarrow$ (3). It follows from the fact that for each finitely generated ideal $I = (a_1, \dots, a_n)$ of R , $\text{ann}_R(I) = \text{ann}_R(S) = \cap_{k=1}^n \text{ann}_R(a_k)$, where $S = \{a_k \mid 1 \leq k \leq n\}$. $\square$

Let R be a ring. According to [12], R is said *power serieswise Armendariz* if for each $f = \sum_{i \geq 0} a_i X^i$ and $g = \sum_{i \geq 0} b_i X^i \in R[[X]]$ such that $fg = 0$, then $a_i b_j = 0$ for each i and j . Next, we characterize when the power series rings over power serieswise Armendariz rings are endo-Noetherian. First, we need to recall some results. Note that those results can be found in [12] or [15].

Notation 3.2 Let R be a commutative ring. We note $\text{ann}_R(2^R) = \{\text{ann}_R(U) \mid U \subseteq R\}$, and $\text{ann}_{R[[X]]}(2^{R[[X]]}) = \{\text{ann}_{R[[X]]}(V) \mid V \subseteq R[[X]]\}$. If $f \in R[[X]]$, we note C_f the set of all coefficients of f and for $V \subseteq R[[X]]$, we note $C_V = \cup_{f \in V} C_f$.

Remark that $\text{ann}_{R[[X]]}(V) \cap R = \text{ann}_R(V) = \text{ann}_R(C_V)$ for $V \subseteq R[[X]]$. So we can define the application

$$\begin{aligned} \psi : \text{ann}_{R[[X]]}(2^{R[[X]]}) &\rightarrow \text{ann}_R(2^R) \\ I &\mapsto I \cap R. \end{aligned}$$

Then, ψ is surjective.

Also, let $U \subseteq R$, then $\text{ann}_R(U)[[X]] \subseteq \text{ann}_{R[[X]]}(U)$. Conversely, let $f = \sum_{i \geq 0} a_i X^i$ such that $fU = 0$, then $a_i U = 0$ for each $i \in \mathbb{N}$; so $f \in \text{ann}_R(U)[[X]]$. Then, $\text{ann}_R(U)[[X]] = \text{ann}_{R[[X]]}(U)$. Thus, we can define the application

$$\begin{aligned} \phi : \text{ann}_R(2^R) &\rightarrow \text{ann}_{R[[X]]}(2^{R[[X]]}) \\ J &\mapsto J[[X]]. \end{aligned}$$

Then, ϕ is injective.

Proposition 3.3 [15] Let R be a ring. The following are equivalent:

1. R is power serieswise Armendariz.
2. ϕ is surjective.
3. ψ is injective.

Now, we are ready to prove the main result of this section. Note that the proof is inspired by [12, Proposition 2.28].

Theorem 3.4 Let R be a power serieswise Armendariz ring. Then, the following statements are equivalent:

1. $R[[X]]$ is endo-Noetherian.
2. R satisfies the acc on annihilators of countably subset.
3. R satisfies the acc on annihilators of countably generated ideals of R .
4. For each sequence $(f_k = \sum_{j \geq 0} a_{k,j} X^j)_k$ of elements of $R[[X]]$ such that $\text{ann}_{R[[X]]}(f_1) \subseteq \text{ann}_{R[[X]]}(f_2) \subseteq \dots$, there exists an $n \in \mathbb{N}$ such that for each $k \geq n$, $\cap_{j \geq 0} \text{ann}_R(a_{k,j}) = \cap_{j \geq 0} \text{ann}_R(a_{n,j})$.

Proof (1) $\Rightarrow$ (2). Let $S_1, S_2, \dots$ be a sequence of countably subset of R such that $\text{ann}_R(S_1) \subseteq \text{ann}_R(S_2) \subseteq \dots$. For each $i \in \mathbb{N}$, let $S_i = \{a_{j,i} \mid j \in \mathbb{N}\}$ and put $P_i = \sum_{j \geq 0} a_{j,i} X^j \in R[[X]]$. In the same way of the proof of Theorem 3.1, we can conclude that $\text{ann}(S_n) = \text{ann}(S_{n+1}) = \text{ann}(S_{n+2}) = \dots$ for some $n \in \mathbb{N}^*$. So R satisfies the acc on annihilators of countably subset.

(2) $\Rightarrow$ (1). Let $(P_k)_k$ be a sequence of elements of $R[[X]]$ such that $\text{ann}_{R[[X]]}(P_1) \subseteq \text{ann}_{R[[X]]}(P_2) \subseteq \dots$. For each $i \geq 1$, let $P_i = \sum_{j \geq 0} a_{j,i} X^j$ and put $S_i = \{a_{j,i} \mid j \in \mathbb{N}\}$. Now, in the same way of the proof of Theorem 3.1, we can conclude that $\text{ann}_{R[[X]]}(P_n) = \text{ann}_{R[[X]]}(P_{n+1}) = \text{ann}_{R[[X]]}(P_{n+2}) = \dots$ for some $n \in \mathbb{N}^*$. Hence, $R[[X]]$ is an endo-Noetherian ring.

(2) $\Leftrightarrow$ (3). Obvious.

(1) $\Leftrightarrow$ (4). By Proposition 3.3, R is power serieswise Armendariz if and only if ϕ is surjective if and only if ψ is injective. In that case, $\phi \circ \psi = \text{id}$ and $\psi \circ \phi = \text{id}$. Then for each $f \in R[[X]]$, $\text{ann}_{R[[X]]}(f) = \phi \circ \psi(\text{ann}_{R[[X]]}(f)) = \phi(\text{ann}_R(f)) = \text{ann}_R(f)[[X]]$. It is easy to prove that for each $f = \sum_{j \geq 0} a_j X^j \in R[[X]]$, $\text{ann}_R(f) = \cap_{j \geq 0} \text{ann}_R(a_j)$. Now, $R[[X]]$ is endo-Noetherian if and only if for each sequence $(f_k = \sum_{i \geq 0} a_{k,i} X^i)_k$ of elements of $R[[X]]$ such that $\text{ann}_{R[[X]]}(f_1) \subseteq \text{ann}_{R[[X]]}(f_2) \subseteq \dots$, there exists an $n \in \mathbb{N}$ such that for each $k \geq n$, $\text{ann}_{R[[X]]}(f_k) = \text{ann}_{R[[X]]}(f_n)$ if and only if $\text{ann}_R(f_k)[[X]] = \text{ann}_R(f_n)[[X]]$ if and only if $\cap_{j \geq 0} \text{ann}_R(a_{k,j}) = \cap_{j \geq 0} \text{ann}_R(a_{n,j})$. $\square$

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