The field of moduli of singular K3 surfaces

Roberto Laface¹

Received: 28 September 2018 / Accepted: 26 June 2019 / Published online: 18 July 2019

© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag GmbH Germany, part of Springer Nature 2019

Abstract

We study the field of moduli of singular K3 surfaces. We discuss both the field of moduli over the CM field and over $\mathbb{Q}$. As a by-product, we show non-finiteness of isomorphism classes of singular K3 surfaces with field of moduli of bounded degree. Finally, we provide an explicit description of the field of moduli in terms of the 2-torsion of the Galois group of the ring class field over which the K3 surface is defined.

Keywords K3 surfaces · Abelian surfaces · Elliptic curves · Complex multiplication · Class field theory · Field of moduli

Mathematics Subject Classification 14J28 · 14K15 · 14K22

1 Introduction

In algebraic geometry, it is sometimes convenient to work with general objects inside a moduli space, as the genericity assumption often seems to grant more control on the geometry. For instance, a general hypersurface of degree $d \geq 4$ in $\mathbb{P}^3$ has Picard group of rank one, generated by the class of a hyperplane section.

On the other extreme, one might want to consider objects with extra structure, and this often leads to very interesting arithmetic phenomena. As an example, the generic elliptic curve will have endomorphism ring isomorphic to $\mathbb{Z}$, but it is well known that elliptic curves with complex multiplication (CM), meaning those whose endomorphism ring is strictly bigger than the ring of integers, have attracted many researchers in algebraic and arithmetic geometry. In the moduli space of elliptic curves, those with CM form a countable set, and it can be shown that they correspond to quadratic imaginary numbers (see, for instance, [4, Ch. IV, Sec. 4]).

Abelian surfaces and K3 surfaces, which are the natural generalization of elliptic curves to dimension two, are no exception: the groundbreaking work of Shioda and Mitani [17] has revealed a deep connection between the geometry of *singular* abelian surfaces (abelian surfaces with maximum Picard number) and the arithmetic of quadratic forms. Also, this

✉ Roberto Laface
laface@ma.tum.de

¹ Technische Universität München, Zentrum Mathematik-M11, 85748 Garching bei München, Germany

connection extends to singular K3 surfaces, as it was shown by Shioda and Inose [16], by means of what Morrison later called Shioda–Inose structures [10].

The arithmetic data of a singular abelian surface are encoded in its transcendental lattice, and a Shioda–Inose structure associates with it a singular K3 surface with the same transcendental lattice, thus preserving the arithmetic information. This has been employed, for instance, by Schütt in the study of the field of definition of singular K3 surfaces [11]: he proved that a singular K3 surface X always admits a model over a ring class field H/K , K being the field $K = \mathbb{Q}(\text{disc } T(X))$, generalizing the previous results of Inose [5]. Moreover, generalizing previous work of Shimada [14], he describes the conjugate varieties of X (modulo $\mathbb{C}$ -isomorphism) under the action of $\text{Aut}(\mathbb{C}/K)$: this is done by looking at the corresponding transcendental lattices, and it is best understood in the language of genus theory of quadratic forms. This suggests that, given a good notion of field of moduli, the degree of the field of moduli should be exactly the number of Galois conjugates of X . This paper aims at explicitly describing the field of moduli of singular K3 surfaces.

By using an idea of Šafarevič [13], we reduce the problem of studying the field of moduli of a singular K3 surface X to the study of the analogous field of a singular abelian surface A with transcendental lattice $T(A) = T(X)$. (This condition can always be achieved by means of Shioda–Inose structure.) Since the transcendental lattice of X and A is isomorphic, they will have the same field of moduli. Our main tools are Galois theory, the theory of complex multiplication on elliptic curves and the theory of quadratic forms.

Throughout the paper, we stress the analogy between CM elliptic curves and singular K3/abelian surfaces. For instance, singular K3/abelian surfaces can always be defined over a number field (see, for example, [12, 16]). It turns out that, in the case of K3/abelian surfaces, we recover a very similar picture to the case of CM elliptic curves and their field of moduli.

We give a short overview of the results achieved in the present article. We start by giving a notion of *field of K -moduli*. Let K be a number field and X a variety defined over $\mathbb{C}$. The field of K -moduli of X is the subfield M_K of $\mathbb{C}$ fixed by the group

$$G := \{\sigma \in \text{Aut}(\mathbb{C}/K) \mid X^\sigma \in [X]\},$$

where $[X]$ here denotes the $\mathbb{C}$ -isomorphism class of X . Typically, we will consider the cases where K is the CM field of a singular K3 surface (see Sect. 3.2) and the case $K = \mathbb{Q}$.

Given a singular K3 surface X with CM field K , we show that the field M_K of K -moduli of X is a Galois extension of K whose degree is determined by the genus of the transcendental lattice of X , seen as a binary quadratic form. For a more precise and complete statement, we refer to Theorem 5.4.

Theorem A *Let X be a singular K3 surface. Then, the field of K -moduli is a Galois extension of K of degree g , g being the order of the genus of the transcendental lattice of X .*

Then, in a similar fashion, we study the field $M_{\mathbb{Q}}$ of $\mathbb{Q}$ -moduli of X , also called the *absolute field of moduli*.

Theorem B (Theorem 6.2) *Let X be a singular K3 surface. Its absolute field of moduli $M_{\mathbb{Q}}$ is an index-two subfield of the field of K -moduli M_K . Moreover, $M_{\mathbb{Q}}$ is an extension of $\mathbb{Q}$ of degree*

$$[M_{\mathbb{Q}} : \mathbb{Q}] = [M_K : K] = g,$$

g being the order of the genus of $T(X)$. In general, it is not a Galois extension of $\mathbb{Q}$.

In Example 6.4, we exhibit a singular K3 surface whose field of $\mathbb{Q}$ -moduli is not a Galois extension of the rational numbers.

Afterwards, we investigate non-finiteness of singular K3 surfaces with respect to the field of moduli, and how the field of K -moduli changes as we vary the singular K3 surface: as an example, we prove that the field of moduli of a singular K3 surface is independent of the index of primitivity of the transcendental lattice. For more results and precise statements, we defer the reader to Sect. 7.

2 Preliminaries

2.1 Singular surfaces

We start by recalling some basics on singular surfaces, and, to this end, let us work over the field $\mathbb{C}$ of complex numbers. If X is a smooth algebraic surface, we can define the Néron–Severi lattice of X : it is the group of divisors on X , modulo algebraic equivalence, namely

$$\mathrm{NS}(X) := \mathrm{Div}(X) / \sim_{\mathrm{alg}},$$

together with the restriction of the intersection form on $H^2(X, \mathbb{Z})$. Its rank $\rho(X) := \mathrm{rank} \, \mathrm{NS}(X)$ is called *Picard number* of X ; the Picard number measures how many different curves lie on a surface. By the Lefschetz theorem on $(1, 1)$ -classes, we have the bound

$$\rho(X) \leq h^{1,1}(X) = b_2(X) - 2p_g(X),$$

where $b_2(X) := \mathrm{rank} \, H^2(X, \mathbb{Z})$ and $p_g(X) := \dim_{\mathbb{C}} H^0(X, \omega_X)$.

We can consider the lattice

$$H^2(X, \mathbb{Z})_{\mathrm{free}} := H^2(X, \mathbb{Z}) / (\mathrm{torsion}),$$

and since $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$, also $\mathrm{NS}(X)_{\mathrm{free}} \subset H^2(X, \mathbb{Z})_{\mathrm{free}}$; $\mathrm{NS}(X)_{\mathrm{free}}$ is a lattice of signature $(1, \rho(X) - 1)$. Its orthogonal complement $T(X) \subset H^2(X, \mathbb{Z})_{\mathrm{free}}$ is called the *transcendental lattice* of X , and it has signature

$$(2p_g(X), h^{1,1}(X) - \rho(X)).$$

A smooth algebraic surface with maximum Picard number, i.e. $\rho(X) = h^{1,1}(X)$, is called a *singular surface*. In this case, the transcendental lattice acquires the structure of a positive definite lattice of rank $2p_g(X)$. Both in the case of singular abelian surfaces and singular K3 surfaces, $T(X)$ is a positive definite rank-two lattice. By taking cohomology of the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0,$$

we obtain a long exact sequence that yields a map

$$p_X : H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X^\times) \cong \mathbb{C}^*$$

the map p_X is called *the period of X* . For further details on singular surfaces, see [1, 17].

2.2 Class group theory

In this section, we recall some rudiments of the theory of binary quadratic forms, and we refer the reader to [3] for a detailed account. Let Q be a quadratic form, namely

$$Q(x, y) = ax^2 + bxy + cy^2.$$

The form Q is said to be *primitive* if Q is not an integral multiple of any other form. Equivalently, Q is primitive if $m := \gcd(a, b, c) = 1$; the quantity m is called *index of primitivity* of Q . It follows that we can define the *primitive part* of a form Q as the quadratic form Q_0 such that $mQ_0 = Q$. A quadratic form Q as above will be denoted in short by $Q = (a, b, c)$.

Two forms $Q = (a, b, c)$ and $Q' = (a', b', c')$ are equivalent (properly equivalent, respectively) if they differ only up to conjugation by an element in $\mathrm{GL}_2(\mathbb{Z})$ ($\mathrm{SL}_2(\mathbb{Z})$, respectively), or equivalently if there exists a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ ($\mathrm{SL}_2(\mathbb{Z})$, respectively) such that

$$Q(px + qy, rx + sy) = Q'(x, y).$$

The integer $D := b^2 - 4ac$ is the *discriminant* of Q . When $D < 0$, the quadratic forms are either positive or negative definite; in this paper, we will deal with positive definite quadratic forms only. For a fixed value D of the discriminant, the (form) *class group of discriminant D* is the set $C(D)$ of proper equivalence classes of primitive forms of discriminant D . The class of a form Q inside $C(D)$ will be denoted by $[Q]$; the class group is a group with respect to the Dirichlet composition of forms (see [3, §3A] for a definition), and its zero element is the class of the *principal form* P , i.e. of the quadratic form

$$P = \begin{cases} x^2 - \frac{D}{4}y^2, & D \equiv 0 \pmod{4} \\ x^2 + xy + \frac{1-D}{4}y^2, & D \equiv 1 \pmod{4} \end{cases}.$$

The theory of fractional ideals parallels the theory of binary quadratic forms: we can define a notion of ideal class group which is compatible with the form class group. Let K be a quadratic imaginary field, and let $\mathcal{O}$ be an *order* in K , i.e. a subring (with unity) of K that is also a rank-two free $\mathbb{Z}$ -module. Every order $\mathcal{O}$ has a unique decomposition

$$\mathcal{O} = \mathbb{Z} + f w_K \mathbb{Z}, \quad w_K := \frac{d_K + \sqrt{d_K}}{2}, \quad d_K := \mathrm{disc} \mathcal{O}_K, \quad f \in \mathbb{Z}^+.$$

Here, the discriminant $\mathrm{disc} \mathcal{O}_K$ is defined as follows: choose a $\mathbb{Q}$ -basis $\{\alpha_1, \dots, \alpha_n\}$ of K ($K/\mathbb{Q}$ is a finite extension, by definition of number field), and set $\mathrm{disc} \mathcal{O}_K := \det((\alpha_i \alpha_j)_{i,j})$. Moreover, the integer f is called *conductor* of $\mathcal{O}$ and it characterizes uniquely an order in K . Therefore, we will write $\mathcal{O}_{K,f}$ for the order in K of conductor f .

For an order $\mathcal{O}$ in a quadratic field K , it is possible to define a class group $C(\mathcal{O})$: letting $I(\mathcal{O})$ denote the group of proper fractional ideals, meaning those whose CM ring is $\mathcal{O}$ itself, and letting $P(\mathcal{O})$ be the subgroup generated by the principal ones, we set $C(\mathcal{O}) := I(\mathcal{O})/P(\mathcal{O})$, and we call it the *ideal class group of $\mathcal{O}$* . The ideal class group $C(\mathcal{O})$ is a group with respect to the usual multiplication of ideals. An important result in algebraic number theory states that if $\mathrm{disc} \mathcal{O} = D$, then $C(D) \cong C(\mathcal{O})$ (see, for example [3, Theorem 7.7]); from now on, we will use interchangeably the two class groups to our convenience. We will denote by $*$ both the Dirichlet composition of quadratic forms in $C(D)$ and the multiplication of ideal classes in $C(\mathcal{O})$.

2.3 Two interesting spaces of singular surfaces

Let us denote by Σ^{Ab} the set of moduli of singular abelian surfaces, i.e. the set of isomorphism classes of singular abelian surfaces modulo $\mathbb{C}$ -isomorphism. Shioda and Mitani [17] used the

transcendental lattice to describe Σ^{Ab} and relate it to quadratic binary forms up to $\text{SL}_2(\mathbb{Z})$ -equivalence.

An ordered basis $\underline{t} = \{t_1, t_2\}$ of $T(A)$ is positive if

$$\text{Im}(p_A(t_1)/p_A(t_2)) > 0,$$

and $T(A)$ with a choice of a positive basis $\underline{t}$ is said to be positively oriented. The transcendental lattice $T(A)$ is an even lattice, and thus, after choosing a positive basis, one has that

$$T(A) \cong \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad a, c > 0, \quad b^2 - 4ac < 0.$$

In other words, we can associate with $T(A)$ an integral binary quadratic form modulo $\text{SL}_2(\mathbb{Z})$ -equivalence, which we will denote by $[T(A)] = [(a, b, c)] \in \mathcal{Q}^+ / \text{SL}_2(\mathbb{Z})$.

We can associate with any quadratic form $Q = (a, b, c)$ an abelian surface A_Q . In order to describe the correspondence, we set

$$\tau(Q) := \frac{-b + \sqrt{D}}{2a}, \quad D := \text{disc } Q = b^2 - 4ac,$$

and we will denote by E_τ the elliptic curve $\mathbb{C}/\Lambda_\tau$, Λ_τ being the lattice $\mathbb{Z} + \tau\mathbb{Z}$. The abelian surface associated with Q is the surface

$$A_Q := E_\tau \times E_{a\tau+b},$$

where $\tau = \tau(Q)$. We will be calling A_Q the *Shioda–Mitani abelian surface* associated with Q .

The mapping $Q \mapsto A_Q$ realizes a 1:1 correspondence between $\text{SL}_2(\mathbb{Z})$ -conjugacy classes of binary forms and isomorphism classes of singular abelian surfaces, namely

$$\Sigma^{\text{Ab}} \longleftrightarrow \mathcal{Q}^+ / \text{SL}_2(\mathbb{Z}),$$

$\mathcal{Q}^+$ being the set of positive definite integral binary quadratic forms. Its inverse is the map $[A] \mapsto [T(A)]$ associating with an isomorphism class of singular abelian surfaces its transcendental lattice as an oriented lattice (i.e. an integral binary quadratic form modulo $\text{SL}_2(\mathbb{Z})$ -equivalence). If A is a singular abelian surface whose transcendental lattice is $[T(A)] = [Q]$ (as a quadratic form), then $A \cong A_Q$, and we will be saying that A_Q is the *Shioda–Mitani model* of A . As a consequence, we get that every singular abelian surface A is isomorphic to the product of two isogenous elliptic curves with complex multiplication.

By forgetting the orientation of the transcendental lattice, we get a generically 2:1 map $\Sigma^{\text{Ab}} \rightarrow \mathcal{Q}^+ / \text{GL}_2(\mathbb{Z})$, which is just taking the transcendental lattice of an abelian surface without orientation:

$$\Sigma^{\text{Ab}} \ni [A] \mapsto [T(A)] \bmod \text{GL}_2(\mathbb{Z}) \in \mathcal{Q}^+ / \text{GL}_2(\mathbb{Z}).$$

Under this map, the classes $[A_Q]$ and $[A_{Q'}]$ in Σ^{Ab} , where

$$Q' := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

have the same image in $\mathcal{Q}^+ / \text{GL}_2(\mathbb{Z})$, although A_Q might not be isomorphic to $A_{Q'}$ (equivalently, their classes have distinct images in $\mathcal{Q}^+ / \text{SL}_2(\mathbb{Z})$). This means that the ramification locus of the map $\Sigma^{\text{Ab}} \rightarrow \mathcal{Q}^+ / \text{GL}_2(\mathbb{Z})$ consists of all those singular abelian surfaces that are invariant under complex conjugation (see [17, §3] for details).

One can build the analogous space Σ^{K3} of singular K3 surfaces and ask for its structure. In their paper, Shioda and Mitani [17] showed that by taking the Kummer surface of a singular abelian surface, one is able to recover all singular K3 surfaces whose transcendental lattice has primitivity index which is divisible by 2. Later, Shioda and Inose [16] proved the surjectivity of the period map for singular K3 surfaces by means of Shioda–Inose structures (as Morrison called them in [10]): if A is an abelian surface, a Shioda–Inose structure associated with A is a K3 surface $X = SI(A)$, which is a 2:1 cover of $Km(A)$ and has the property that $[T(X)] = [T(A)]$.

$$A \xrightarrow{2:1} Km(A) \xleftarrow{2:1} X$$

It turns out that also (isomorphism classes of) singular K3 surfaces are uniquely characterized by their transcendental lattice, and thus, $\Sigma^{Ab} \cong \Sigma^{K3}$ [16]; in particular, this implies that two singular abelian surfaces are isomorphic if and only if the corresponding K3 surfaces via a Shioda–Inose structure are. This amounts to saying that we can interchangeably consider singular abelian surfaces and singular K3 surfaces when studying the field of moduli.

2.4 CM theory of elliptic curves

We recall a couple of elementary facts about the CM theory of elliptic curves; for a reference, see [19, Ch. 2]. Let $\mathcal{E}ll(\mathcal{O})$ be the set of $\mathbb{C}$ -isomorphism classes of elliptic curves with CM by the order $\mathcal{O} \subset K$. Since a proper $\mathcal{O}$ -ideal is also a lattice, quotienting by $\mathcal{O}$ -ideals induces a map

$$C(\mathcal{O}) \longrightarrow \mathcal{E}ll(\mathcal{O}), \quad \bar{a} \longmapsto [\mathbb{C}/a],$$

which is an isomorphism. Multiplication of ideal classes and lattices gives a simply transitive action

$$C(\mathcal{O}) \times \mathcal{E}ll(\mathcal{O}) \longrightarrow \mathcal{E}ll(\mathcal{O}), \quad (\bar{a}, [\mathbb{C}/\Lambda]) \longmapsto \bar{a} * [\mathbb{C}/\Lambda] := [\mathbb{C}/a^{-1}\Lambda].$$

Another action on $\mathcal{E}ll(\mathcal{O})$ is given by the absolute Galois group $\text{Gal}(\bar{K}/K)$:

$$\text{Gal}(\bar{K}/K) \times \mathcal{E}ll(\mathcal{O}) \longrightarrow \mathcal{E}ll(\mathcal{O}), \quad (\sigma, [E]) \longmapsto [E^\sigma].$$

Now, let us fix $[E] \in \mathcal{E}ll(\mathcal{O})$; given $\sigma \in \text{Gal}(\bar{K}/K)$, we can form $[E^\sigma]$, and, by using the action of $C(\mathcal{O})$, there exists a unique $\bar{a} \in C(\mathcal{O})$ such that $\bar{a} * [E] = [E^\sigma]$. This correspondence defines a surjective homomorphism

$$F : \text{Gal}(\bar{K}/K) \longrightarrow C(\mathcal{O}), \quad \sigma \longmapsto F(\sigma) : F(\sigma) * [E] = [E^\sigma].$$

One of the properties of this map is that it is independent of the curve $[E]$ chosen to define it, and thus, we have that $F(\sigma) * [E] = [E^\sigma]$, $\forall \sigma \in \text{Gal}(\bar{K}/K)$ and $\forall [E] \in \mathcal{E}ll(\mathcal{O})$.

Notice that the action

$$C(\mathcal{O}) \times \mathcal{E}ll(\mathcal{O}) \longrightarrow \mathcal{E}ll(\mathcal{O})$$

can be interpreted in terms of quadratic forms. Indeed, to any $[E] \in \mathcal{E}ll(\mathcal{O})$, one can associate a quadratic form Q such that $j(\tau(Q)) = j(E)$. Given a quadratic form Q , we will be denoting $E_Q := E_{\tau(Q)}$. Then, the action is isomorphic to the action

$$C(\mathcal{O}) \times C(\mathcal{O}) \longrightarrow C(\mathcal{O}), \quad (\bar{a}, \bar{b}) \longmapsto \bar{a}^{-1}\bar{b}.$$

Also, by class group theory, we can phrase everything in terms of the corresponding classes of quadratic forms, where now multiplication of ideal classes corresponds to the Dirichlet

composition. Under this interpretation, the map $F : \text{Gal}(\bar{K}/K) \rightarrow C(\mathcal{O})$ sends an element σ to the element $F(\sigma)$ satisfying the condition

$$[Q^\sigma] = [E^\sigma] = F(\sigma) * [E] = F(\sigma) * [\mathbb{C}/\mathfrak{a}] = [\mathbb{C}/F(\sigma)^{-1}\mathfrak{a}] = [F(\sigma)]^{-1} * [Q],$$

where $[Q]$ (respectively, $[Q^\sigma]$) corresponds to $[E]$ (respectively, $[E^\sigma]$), and $[F(\sigma)]$ is meant as the class of quadratic forms corresponding to $F(\sigma) \in C(\mathcal{O})$.

2.5 The group of idèles

Given a number field K , we denote by I_K the group of fractional ideals in K . We can define the so-called *group of idèles* by setting

$$\mathbb{I}_K := \left\{ (a_v) \in \prod_v K_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for all but finitely many } v \right\}.$$

There is a canonical surjective homomorphism

$$\mathbb{I}_K \longrightarrow I_K, \quad (a_v) \longmapsto \prod_{v \text{ finite}} p_v^{\text{ord}_{p_v}(a_v)},$$

which associates with every idèle an element of I_K . There is also a canonical injective (diagonal) homomorphism

$$K^\times \longrightarrow \mathbb{I}_K, \quad a \longmapsto (a, a, a, \dots),$$

with discrete image.

The statement of the main theorems of class field theory in terms of ideals is very explicit. However, it has the big disadvantage of working for a fixed modulus $\mathfrak{m}$ at the time, and so it describes only the abelian extensions whose conductor divides $\mathfrak{m}$. On the other hand, the statements in terms of idèles allow one to consider infinite abelian extensions, or equivalently all finite abelian extensions simultaneously. It also relates local and global class field theory, namely the global Artin map to its local components.

Proposition 2.1 *There exists a unique continuous surjective homomorphism $\phi_K : \mathbb{I}_K \rightarrow \text{Gal}(K^{\text{ab}}/K)$ with the following property: for any $L \subset K^{\text{ab}}$ finite over K and any place w of L lying over a place v of K , the diagram*

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\phi_v} & \text{Gal}(L_w/K_v) \\ \downarrow & & \downarrow \\ \mathbb{I}_K & \xrightarrow{\phi_{L/K}} & \text{Gal}(L/K) \end{array}$$

where the bottom map sends $a \in \mathbb{I}_K$ to $\phi_K(a)|_L$. Here, ϕ_v is the local component of the Artin map at the place v .

For any finite extension L of K which is contained in K^{ab} , ϕ_K gives rise to a commutative diagram

$$\begin{array}{ccc} \mathbb{I}_K & \xrightarrow{\phi_K} & \text{Gal}(K^{\text{ab}}/K) \\ & \searrow \phi_{L/K} & \downarrow |_L \\ & & \text{Gal}(L/K) \end{array}$$

We recall that the Main Theorem of Complex Multiplication makes use of the group of idèles $\mathbb{I}_K$ to control the Galois conjugates of an elliptic curve with CM in K . Let K be an imaginary quadratic field and E an elliptic curve with CM in K ; then, there exist an order $\mathcal{O} \subset K$ and a fractional ideal $\mathfrak{a} \subset \mathcal{O}$ such that $E \cong \mathbb{C}/\mathfrak{a}$, and thus, E has CM in the order $\mathcal{O}$.

Theorem 2.2 (Main Theorem of Complex Multiplication, Theorem 5.4 of [15]) *Let $E = \mathbb{C}/\Lambda$ be an elliptic curve with CM by an order in K . Let $\sigma \in \text{Gal}(\bar{K}/K)$ and $s \in \mathbb{I}_K$ such that $\sigma = \phi_K(s)$ on K^{ab} . Then, there exists an isomorphism*

$$E^\sigma \cong \mathbb{C}/s^{-1}\Lambda.$$

This result allows to describe the action of $\text{Gal}(\bar{K}/K)$ on $\mathcal{E}ll(\mathcal{O})$ by means of the group of idèles $\mathbb{I}_K$ acting on lattices. We briefly recall how $\mathbb{I}_K$ acts on fractional ideals, as it will be a crucial tool in the proofs, but we refer the reader to [15, Ch. 5, Section 2] for more details. If $\mathfrak{a}$ is a fractional ideal in K and $x = (x_p)_p \in \mathbb{I}_K$ is an idèle, we denote by $x\mathfrak{a}$ the fractional ideal such that $(x\mathfrak{a})_p = x_p \mathfrak{a}_p$ for all p prime in K . In other words, we first multiply the local components, and then, we glue them back together.

3 The field of K -moduli

3.1 A new definition

We define the *field of K -moduli* M_K of a variety X/k , where K is a given field having the same prime field as k . In order to do so, we recall that a universal domain¹ Ω is an algebraically closed field with infinite transcendence degree over its prime field. We now fix a universal domain Ω , and all fields in the following will be subfields of Ω (later we will specialize to $\Omega = \mathbb{C}$).

A variety X/k is said to be defined over a field L if there exists a variety Y/L such that X is Ω -isomorphic to Y (i.e. $X \times_{\text{Spec } k} \text{Spec } \Omega \cong Y \times_{\text{Spec } L} \text{Spec } \Omega$). If $\sigma \in \text{Aut}(\Omega)$, the conjugate variety of X/k under σ is the scheme X^σ over $\sigma(k)$ obtained by base change along $\sigma^*: \text{Spec } \sigma(k) \rightarrow \text{Spec } k$, namely $X^\sigma = X \times_{\text{Spec } k} \text{Spec } \sigma(k)$ (in the end, we will always consider its extension to the universal domain Ω).

Given a field K , Matsusaka first introduced the *relative field of moduli* (or *field of moduli* over K , see [8]) as the intersection of all fields of definition of X which contain K , in other words

$$M_K := \bigcap_{\substack{X \text{ defined over } L \\ L \supset K}} L.$$

Later, Koizumi [6] adjusted the definition to positive characteristic geometry by adding the extra condition that for an automorphism $\sigma \in \text{Aut}(\Omega/K)$,

$$\sigma \in G := \{\sigma \in \text{Aut}(\Omega/K) \mid X^\sigma \in [X]\} \iff \sigma|_{M_K} = \text{id}_{M_K},$$

where $[X]$ denotes the Ω -isomorphism class of X and X^σ is the conjugate variety of X under σ . For our purposes, it is best to introduce the following

¹ Universal domains were the fundamental object algebraic geometry was based on before the advent of Grothendieck. More details can be found in the foundational book of Weil [20]; this uses notions very much different from the modern language of schemes and it is quite hard to read at times.

Definition 3.1 Let K be a number field, and X a variety defined over $\mathbb{C}$. The *field of K -moduli* of X is the subfield M_K of $\mathbb{C}$ fixed by the group

$$G := \{\sigma \in \text{Aut}(\mathbb{C}/K) \mid X^\sigma \in [X]\},$$

where $[X]$ here denotes the $\mathbb{C}$ -isomorphism class of X .

In practice, we are dropping Matsusaka's condition and keeping the one Koizumi introduced. Notice that, unlike in the case of Koizumi's definition [6], our field of moduli always exists and it is unique by Galois theory. Following [6], if the characteristic of the ground field is zero, then M_K is contained in any field of definition for X which contains K , and thus, we have the following extension

$$M_K \subset \bigcap_{\substack{X \text{ defined over } L \\ L \supset K}} L,$$

which in fact is algebraic and Galois. We remark that the right-hand side of this inclusion is quite a mysterious object in general.

If X is a variety, by the *absolute field of moduli* of X we will mean the field of $\mathbb{Q}$ -moduli, i.e. the field $M_{\mathbb{Q}}$ such that for all automorphisms $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$,

$$X^\sigma \in [X] \iff \sigma \text{ acts trivially on } M_{\mathbb{Q}};$$

equivalently, it is defined as the fixed field of the group

$$G := \{\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \mid X^\sigma \in [X]\}.$$

Galois theory once again guarantees that this field is unique for a given variety X .

Suppose we want to study the field of L -moduli, for some number field L , and denote by $G(X)$ (respectively, $G(X^\tau)$) the group fixing the modulus of X (respectively, X^τ) and by $M(X)$ (respectively, $M(X^\tau)$) the field of L -moduli. Then, one can show that:

- (1) $G(X)$ only depends on the isomorphism class of X ;
- (2) $G(X^\tau) = \tau \cdot G(X) \cdot \tau^{-1}$;
- (3) $M(X^\tau) = \tau(M(X))$.

3.2 A little motivation

Let X be a singular K3 surface, and let

$$T(X) \cong \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = (a, b, c)$$

be its transcendental lattice, where the left-hand side equality identifies $T(X)$ with the corresponding quadratic form (after the choice of a positive basis of $T(X)$). Then, X can be defined over a number field: this follows from the results in [16, 17], together with the fact that elliptic curves with CM can always be defined over a number field.

To $T(X)$, one can associate the integer disc $T(X)$, when $T(X)$ is regarded as a quadratic form. Clearly, $\text{disc } T(X) < 0$, and thus, $K := \mathbb{Q}(\sqrt{\text{disc } T(X)})$ is a quadratic imaginary field. We will call K the *CM field*² of X . If X is the K3 surface associated with a singular abelian

² There are other notions of CM field currently in use. For example, a number field K is a CM field if it is a totally imaginary quadratic extension of a totally real field. In fact, the CM field of a singular K3 surface is also a CM field in the latter sense.

surface A via a Shioda–Inose structure (so that, in particular, $T(A) = T(X)$), we will say that K is the CM field of A as well. Generalizing a previous result of Shimada [14], Schütt was able to prove the following result

Theorem 3.2 (Theorem 5.2 in [11]) *Let X be a singular K3 surface, and let $T(X)$ be its transcendental lattice. Assume that X is defined over a Galois extension L/K . Then, the action of the Galois group $\text{Gal}(L/K)$ spans the genus of $T(X)$, i.e.*

$$(\text{genus of } T(X)) = \{[T(X^\sigma)] : \sigma \in \text{Gal}(L/K)\}.$$

For the definition of genus in general, we refer the reader to [2, Ch. 15, Sect. 7]. Set $L := H(\text{disc } T(X))$ in Theorem 3.2, where $H(D)$ denotes the ring class field of the order in K of discriminant D , for $D < 0$. Galois theory tells us that

$$\text{Gal}(L/\mathbb{Q}) \cong \text{Gal}(L/K) \rtimes \text{Gal}(K/\mathbb{Q}),$$

where $\text{Gal}(K/\mathbb{Q})$ accounts for the complex conjugation (for a reference, see [3, Ch. 9]). But complex conjugation has the effect of sending a singular K3 surface of transcendental lattice (a, b, c) to the singular K3 surface with transcendental lattice $(a, -b, c)$, so it acts as inversion on the corresponding class group (see [11, 17]). By observing that a form and its inverse lie in the same genus, we conclude that

$$\begin{aligned} (\text{genus of } T(X)) &= \{[T(X^\sigma)] \mid \sigma \in \text{Gal}(L/K)\} \\ &= \{[T(X^\sigma)] \mid \sigma \in \text{Gal}(L/\mathbb{Q})\}. \end{aligned}$$

This observation suggests a connection between the field of moduli of a singular K3 surface and the genus of its transcendental lattice, even in the case of the field of $\mathbb{Q}$ -moduli.

The classification of decompositions of a singular abelian surface [7] allows us to tell something more about the field of moduli of X containing K . Recall that M_K is contained in the intersection of all possible fields of definition for X . Then, by means of Shioda–Inose structures, we can study X by means of those abelian surfaces A whose transcendental lattice equals $T(X)$. Let A be such a surface and consider all product surfaces $E_1 \times E_2$ isomorphic to A (which we know explicitly by [7]); if $j_k := j(E_k)$, by work of Schütt [11], X admits a model over $\mathbb{Q}(j_1 j_2, j_1 + j_2)$. Therefore, considering all admissible pairs (E_1, E_2) as above, we see that

$$M_K \subseteq \bigcap_{X \text{ defined over } L} L \subseteq \bigcap_{j_1, j_2 \text{ as above}} \mathbb{Q}(j_1 j_2, j_1 + j_2).$$

We deduce a slightly clearer picture of what M_K looks like, as we know where it has to sit as an extension of $\mathbb{Q}$, namely M_K lies in right-hand side above, which is theoretically clear. In practice, describing it is a hard task, as this involves the computation of j -invariants.

3.3 The case of elliptic curves

Our toy example is the case of an elliptic curve E , for which one always has a Weierstraß model

$$y^2 = x^3 + Ax + B,$$

for some $A, B \in \mathbb{C}$. It can be proven (see [18, Ch. 1]) that an elliptic curve E can be defined over the field $\mathbb{Q}(j_E)$; moreover, the field of $\mathbb{Q}$ -moduli of E is again $\mathbb{Q}(j_E)$.

Let now E be a CM elliptic curve. The theory of complex multiplication tells us (see [18, Ch. 2]) that $j_E \in \overline{\mathbb{Q}}$, i.e. the j -invariant of a CM elliptic curve is always an algebraic number. Suppose that E has CM by an order $\mathcal{O}$ in $K = \mathbb{Q}(\sqrt{D})$. Then, by means of class field theory, one can show that there exists a commutative diagram of field extensions,

$$\begin{array}{ccc} & H = K(j_E) & \\ & \swarrow \quad \searrow & \\ K & & \mathbb{Q}(j_E) \\ & \swarrow \quad \searrow & \\ & \mathbb{Q} & \end{array}$$

where $H := H(D)$ is the ring class field corresponding to the order $\mathcal{O}$ (for details, consult [15]).

We would like to let the reader notice that $K(j_E)$ is indeed the field of K -moduli of E . Our study of the field of moduli in the rest of the chapter will reveal that this very picture carries over to singular K3 surfaces (and singular abelian surfaces).

3.4 An alternative definition of M_K

As a singular K3 surface has a model defined over a number field by a result of Inose [5], when studying the field of moduli one would like to consider the field

$$\bar{K}^{G'}, \quad G' := \{\sigma \in \text{Aut}(\bar{K}/K) \mid X^\sigma \in [X]\},$$

rather than $\mathbb{C}^G$, as we defined it above. In fact, one has that $\mathbb{C}^G = \bar{K}^{G'}$; also this is independent of the fact that we are working on a singular K3 surface, as the following more general result shows.

Proposition 3.3 *Let X be a variety defined over a number field containing K . Then, the fields $\mathbb{C}^G$ and $\bar{K}^{G'}$ coincide.*

Proof If X is defined over a number field containing K , then $\text{Aut}(\mathbb{C}/\bar{K}) \subseteq G$, and thus, $\mathbb{C}^G \subseteq \mathbb{C}^{\text{Aut}(\mathbb{C}/\bar{K})} = \bar{K}$ (see [9, Theorem 9.29]). This immediately implies that $\mathbb{C}^G \subseteq \bar{K}^{G'}$. The reverse inclusion follows from the surjectivity of the restriction map $|_K : G \rightarrow G'$. $\square$

As every singular K3 surface can be defined over a number field, we can define the field of K -moduli of a singular K3 surface to be the field

$$M_K := \bar{K}^{G_K}, \quad G_K := \{\sigma \in \text{Gal}(\bar{K}/K) \mid X^\sigma \in [X]\}.$$

In the following, we will be concerned with finding explicitly the group G_K , as it characterizes uniquely, thanks to Galois theory, the field of moduli.

4 Characterization in the primitive case

4.1 Statement of the result

Let X be a singular K3 surface, with transcendental lattice $T(X) = Q = mQ_0$ (Q_0 being the primitive part of $T(X)$), and discriminant $\text{disc } T(X) = D = m^2D_0$ (D_0 being the discriminant

of Q_0). Recall that we can always find a singular abelian surface A such that X is obtained from A by means of the Shioda–Inose structure, and in particular such that $T(A) = T(X)$. In light of this, notice that determining the field of moduli of X is equivalent to determining the field of moduli of any such A , so that we can reduce to considering the problem for singular abelian surfaces.

We will now proceed in giving a different characterization of G_K . In what follows, let us assume additionally that $m = 1$, which is to say that the transcendental lattice $T(X)$ is primitive. Under this assumption, for any decomposition $A \cong E_1 \times E_2$, the quadratic forms Q_1 and Q_2 corresponding to the elliptic curves E_1 and E_2 both lie in $C(D) \cong C(\mathcal{O})$, $\mathcal{O}$ being the order of discriminant D . Observe that, if we fix a decomposition of $A \cong E_1 \times E_2$, then

$$X^\sigma \in [X] \iff A^\sigma \in [A] \iff E_1^\sigma \times E_2^\sigma \cong E_1 \times E_2.$$

In the remainder on this section, we will prove the following

Theorem 4.1 *Let X be a singular K3 surface with primitive transcendental lattice, and let H be the ring class field of $\mathcal{O}$, the order of discriminant $\text{disc } T(X)$. Then, the field of K -moduli is*

$$M_K = \bar{K}^{G_K}, \quad G_K = (|_H)^{-1} \text{Gal}(H/K)[2];$$

it is a Galois extension of K of degree g , g being the order of the genus of the transcendental lattice of X .

The proof is divided into two steps. First, we will prove that G_K restricts to the subgroup of 2-torsion elements of $\text{Gal}(H/K)$, and thus, it is a closed and normal subgroup of $\text{Gal}(\bar{K}/K)$ with respect to the Krull topology. Afterwards, we will use these facts to study the field extension M_K/K and hence to prove Theorem 4.1.

4.2 The group G_K

By the previous discussions, it follows that

$$\begin{aligned} G_K &= \{\sigma \in \text{Gal}(\bar{K}/K) \mid X^\sigma \in [X]\} \\ &= \{\sigma \in \text{Gal}(\bar{K}/K) \mid E_1^\sigma \times E_2^\sigma \cong E_1 \times E_2\}. \end{aligned}$$

We will now proceed in giving a different characterization of G_K .

Proposition 4.2 $G_K = F^{-1}(C(\mathcal{O})[2])$.

Proof Let Q_i be the form corresponding to E_i ($i = 1, 2$), and let Q_i^σ be the one corresponding to E_i^σ ($i = 1, 2$). By use of the map

$$F : \text{Gal}(\bar{K}/K) \longrightarrow C(\mathcal{O}),$$

we get that

$$[Q_1^\sigma] = [F(\sigma)]^{-1} * [Q_1] \quad \text{and} \quad [Q_2^\sigma] = [F(\sigma)]^{-1} * [Q_2],$$

where here we make use of the fact that F is independent of the elliptic curve (and thus of the quadratic form) we use to define it. By [7, Proposition 3.2], we see that

$$\begin{aligned} E_1^\sigma \times E_2^\sigma \cong E_1 \times E_2 &\iff Q_1^\sigma * Q_2^\sigma = Q_1 * Q_2 \\ &\iff F(\sigma)^2 = 1. \end{aligned}$$

□

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(\bar{K}/K) & \xrightarrow{F} & C(\mathcal{O}) \\ \downarrow |_H & \nearrow \cong & \\ \mathrm{Gal}(H/K) & & \end{array} \quad (\dagger)$$

where $H := H(\mathcal{O})$, which follows from class group theory and says that F is an isomorphism on the restriction of the elements of $\mathrm{Gal}(\bar{K}/K)$ to H . In particular, $C(\mathcal{O})[2] \cong \mathrm{Gal}(H/K)[2]$, and thus,

$$G_K = \{\sigma \in \mathrm{Gal}(\bar{K}/K) : (\sigma|_H)^2 = \mathrm{id}_H\}.$$

This implies the following

Corollary 4.3 $G_K = (|_H)^{-1}(\mathrm{Gal}(H/K)[2]).$

We can now describe G_K as a topological subgroup of $\mathrm{Gal}(\bar{K}/K)$, by using Galois theory.

Proposition 4.4 G_K is a closed normal subgroup of $\mathrm{Gal}(\bar{K}/K)$ with respect to the Krull topology.

4.3 The extension M_K/K

We can now use our knowledge of G_K to give a proof of Theorem 4.1.

Proof of Theorem 4.1 As G_K is closed and normal in $\mathrm{Gal}(\bar{K}/K)$, we have that

$$\mathrm{Gal}(\bar{K}/M_K) = \mathrm{Gal}(\bar{K}/\bar{K}^{G_K}) = G_K$$

and M_K/K is a (finite) Galois extension. The exact sequence

$$0 \rightarrow G_K \rightarrow \mathrm{Gal}(\bar{K}/K) \rightarrow C(\mathcal{O})/C(\mathcal{O})[2] \rightarrow 0$$

tells us that $\mathrm{Gal}(M_K/K) \cong C(\mathcal{O})/C(\mathcal{O})[2]$, from which we can now cook up the following short exact sequence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Gal}(H/K)[2] & \longrightarrow & \mathrm{Gal}(H/K) & \longrightarrow & \mathrm{Gal}(M_K/K) \longrightarrow 0 \\ & & \parallel \text{iso} & & \nearrow & & \\ & & \mathrm{Gal}(H/M_K) & & & & \end{array}$$

By genus theory (see [3, §3B]), there is a short exact sequence

$$0 \rightarrow C(D)[2] \rightarrow C(D) \rightarrow C(D)^2 \rightarrow 0,$$

where $C(D)^2$ is the group of squares in the class group $C(D)$ (in fact, it is the principal genus). As $\mathrm{Gal}(H/K) \cong C(D)$, we deduce that

$$\mathrm{Gal}(M_K/K) \cong C(D)^2,$$

and in particular that $\#\mathrm{Gal}(M_K/K) = g$, where $g = \#C(D)^2$ is the order of the genus of the transcendental lattice. $\square$

Example 4.5 Let $D = -23$ and $K = \mathbb{Q}(\sqrt{D})$. The class group of discriminant D is

$$C(D) = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & 6 \end{pmatrix} \right\}.$$

There is only one genus in $C(D)$ (of order 3); thus, we expect a field of moduli of degree 3 over K .

Let X be the singular K3 surface whose transcendental lattice is

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}.$$

A Shioda–Inose structure starting from the self-product of E , E being the elliptic curve corresponding to the principal form P in $C(D)$, reveals that X has a model over $\mathbb{Q}(j(P))$. We now show that the field of K -moduli is $M_K = K(j(P)) = H(\mathcal{O}_K)$, which is a degree 3 extension of K by class field theory. Indeed, as X is realized starting from the self-product of E , where E corresponds to the principal form P , then the transcendental lattice of the conjugate surface by $\sigma \in \mathrm{Gal}(\bar{K}/K)$ is given by

$$P^\sigma * P^\sigma = F(\sigma)^{-2},$$

and this is trivial if and only if $F(\sigma)$ is 2-torsion. However, as $\#\mathrm{Gal}(H/K) = 3$, it follows that $F(\sigma)$ is necessarily trivial, and thus, $G_K = \ker F = \mathrm{Gal}(\bar{K}/H)$. Therefore, we have proven that $M_K = H$.

We can also look at the K3 surface Y whose transcendental lattice is

$$Q := \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}.$$

By means of a Shioda–Inose structure, Y has a model over $\mathbb{Q}(j_1, j_2)$, where $j_1 := j(P)$ and $j_2 := j(Q)$; notice that

$$\mathbb{Q}(j_1, j_2) = K(j_1, j_2) = K(j_2),$$

as Y comes from a Shioda–Inose structure associated with the Shioda–Mitani model $E_P \times E_Q$ (plus some class field theory considerations). It follows that $H = K(j_2)$, which is a degree 3 extension of K , and thus, we have that $M_K = H = K(j_2)$. $\square$

5 Generalization to the imprimitive case

5.1 A first look at G_K

We will now treat the case of a singular K3 surface X with imprimitive transcendental lattice $T(X) = Q = mQ_0$ ($m > 1$). As in the primitive case, we see that it is enough to choose a decomposition of A and to compute the field of moduli in that case. Thus, we now fix a decomposition $A \cong E_1 \times E_2$.

We would like to mimic the techniques used in the primitive case to give an analogous characterization of the field of moduli. The issue at hand is that given a decomposition $A \cong E_1 \times E_2$, the quadratic forms Q_1 and Q_2 corresponding to E_1 and E_2 must necessarily

lie in class groups with different discriminant by [7]. Therefore, we need to use the Dirichlet composition in its generalized sense in order to compute transcendental lattices.

We briefly recall the definition of the generalized Dirichlet composition, following [7, Section 3.2]. If $[Q_1] \in C(D_1)$ and $[Q_2] \in C(D_2)$, with $D_1 = f_1^2 d_K$ and $D_2 = f_2^2 d_K$, we can set $f := \text{lcm}(f_1, f_2)$. Putting

$$D := f^2 d_K, \quad d_1 := f/f_1, \quad d_2 := f/f_2,$$

it is straightforward to see that $\text{disc } d_1 Q_1 = \text{disc } d_2 Q_2$. After possibly replacing Q_1 and Q_2 by a suitable representative in their proper equivalence classes, one can assume that $d_1 Q_1$ and $d_2 Q_2$ have coprime leading coefficients, and hence, we can use the usual Dirichlet composition.

The form $(d_1 Q_1) * (d_2 Q_2)$ has then index of primitivity $d_1 d_2$ and discriminant $D = (d_1 d_2)^2 \gcd(f_1, f_2)^2 d_K$. Therefore, its primitive part Q_0 is a form in $C(D_0)$, where $D_0 = f_0^2 d_K$ and $f_0 = \gcd(f_1, f_2)$. This shows that the generalized Dirichlet composition is a family of maps

$$\circledast : C(D_1) \times C(D_2) \longrightarrow C(D_0),$$

with the following property: given $[Q_1] \in C(D_1)$ and $[Q_2] \in C(D_2)$, $[Q_1] \circledast [Q_2]$ is the form in $C(D_0)$ with the property that

$$(d_1 d_2)([Q_1] \circledast [Q_2]) = [d_1 Q_1] * [d_2 Q_2].$$

When dealing with decompositions, it is always useful to keep in mind the diagram of orders,

and the corresponding one of class groups,

where f_0, f_1, f_2, f are such that

$$\text{lcm}(f_1, f_2) = f, \quad \gcd(f_1, f_2) = f_0, \quad f^2 d_K = \text{disc } T(A),$$

and also $[Q_1] \in C(\mathcal{O}_{K,f_1})$ and $[Q_2] \in C(\mathcal{O}_{K,f_2})$. The maps between the above class group are the one induced by extension of ideals; in terms of quadratic forms, these correspond to multiplication by the principal form of the target order: for instance, the reduction map

$$\text{red}_0 : C(\mathcal{O}_{K,f}) \longrightarrow C(\mathcal{O}_{K,f_0})$$

sends $[Q]$ to $[Q] \circledast [P_0]$, where $\circledast$ is the generalized Dirichlet composition (see [7] for definition and properties), and P_0 is the principal form in $C(\mathcal{O}_{K,f_0})$. As before, there are maps

$$F_i : \text{Gal}(\bar{K}/K) \longrightarrow C(\mathcal{O}_{K,f_i}) \quad (i = 0, 1, 2),$$

such that

$$[Q_i^\sigma] = [F_i(\sigma)]^{-1} \circledast [Q_i] \quad (i = 0, 1, 2).$$

By use of the generalized Dirichlet composition $\circledast$ (see [7, Proposition 3.2]) and the maps F_i ($i = 1, 2$), we see that

$$E_1^\sigma \times E_2^\sigma \cong E_1 \times E_2 \iff Q_1^\sigma \circledast Q_2^\sigma = Q_1 \circledast Q_2 \iff F_1(\sigma) \circledast F_2(\sigma) = P_0.$$

The discussion above can be rephrased as follows:

Lemma 5.1 $G_K = \{\sigma \in \text{Gal}(\bar{K}/K) \mid F_1(\sigma) \circledast F_2(\sigma) = P_0\}$.

In order to go any further, we need to understand the interaction of the maps F_i ($i = 0, 1, 2$). As the class groups are abelian groups, these maps factor through the Galois group of K^{ab} , the maximal abelian extension of K . We get maps (again called F_i by abuse of notation)

$$F_i : \text{Gal}(K^{\text{ab}}/K) \longrightarrow C(\mathcal{O}_{K,f_i}).$$

Here is where the theory of idèles comes into play, picturing the behaviour of these maps in their totality.

5.2 Compatibility condition for the maps F_i

The idea is inspired by a paper of Schütt [11]: given a singular abelian surface A , among all decompositions that we can choose, there is one that behaves better than the others, namely the decomposition that Shioda and Mitani used to prove the surjectivity of the period map for singular abelian surfaces [17].

To the reader's convenience, we briefly recall this construction. Letting A be a singular abelian surface of transcendental lattice $T(A) \cong (a, b, c)$, Shioda and Mitani showed that $A \cong E_\tau \times E_{a\tau+b}$, where

$$\tau := \tau(Q) = \frac{-b + \sqrt{D}}{2a}.$$

In particular, $E_{a\tau+b}$ always corresponds to the principal form in the class group of discriminant $D = \text{disc } T(A)$, and E_τ instead corresponds to the quadratic form $T(A)_0$, the primitive part of $T(A)$.

Let us assume $A \cong E_1 \times E_2$ is the Shioda–Mitani model of A : if $T(A) = Q = mQ_0$, then E_1 corresponds to the quadratic form $Q_0 \in C(D_0)$ and E_2 corresponds to the principal form $P \in C(D)$. Notice that we also have $A \cong \mathbb{C}/\mathfrak{a} \times \mathbb{C}/\mathcal{O}_{K,f}$, for $\mathfrak{a} \in C(\mathcal{O}_{K,f_0})$, and thus, the proof of [11, Theorem 5.4] shows in particular that, for $\sigma \in \text{Gal}(\bar{K}/K)$

$$A^\sigma \cong E_1^\sigma \times E_2^\sigma \cong \mathbb{C}/s^{-1}\mathfrak{a} \times \mathbb{C}/s^{-1}\mathcal{O}_{K,f} \cong \mathbb{C}/s^{-2}\mathfrak{a} \times \mathbb{C}/\mathcal{O}_{K,f},$$

where, as σ varies in $\text{Gal}(\bar{K}/K)$ (or s corresponding to σ varies in $\mathbb{I}_K$), $(s^{-1}\mathfrak{a})^2 = (s^{-1}Q_0)^2$ spans the whole genus of Q_0 in $C(D_0)$. This ultimately suggests that we look at elements of

the form $s^{-1}\mathcal{O}$, as their squares span the principal genus of a class group and characterize the transcendental lattice as it moves in its genus.

To do so, suppose we are given an order $\mathcal{O} \subset K$, the map

$$F : \text{Gal}(\bar{K}/K) \longrightarrow C(\mathcal{O})$$

factorizes through a map

$$F : \text{Gal}(K^{\text{ab}}/K) \longrightarrow C(\mathcal{O}).$$

For $\sigma \in \text{Gal}(K^{\text{ab}}/K)$, $F(\sigma)$ has the property $[E^\sigma] = [\mathbb{C}/F(\sigma)^{-1} \cdot \mathfrak{a}]$ independently of the chosen $E = \mathbb{C}/\mathfrak{a} \in \mathcal{E}ll(\mathcal{O})$. By the Main Theorem of CM, there exists an idèle $s \in \mathbb{I}_K$ such that $\phi_K(s) = \sigma$ and

$$[E^\sigma] = [\mathbb{C}/F(\sigma)^{-1} \cdot \mathfrak{a}] = [\mathbb{C}/s^{-1}\mathfrak{a}].$$

As $s^{-1}\mathfrak{a} = (s^{-1}\mathcal{O}) \cdot \mathfrak{a}$, we can identify $[s\mathcal{O}] = [F(\sigma)]$. Now, let $\mathcal{O}_0$ be another order in K , $\mathcal{O} \subset \mathcal{O}_0 \subset K$, and consider the following diagram.

$$\begin{array}{ccc} \mathbb{I}_K & \xrightarrow{\phi_K} & \text{Gal}(K^{\text{ab}}/K) \\ & & \swarrow F \quad \searrow F_0 \\ & & C(\mathcal{O}) \quad C(\mathcal{O}_0) \\ & & \downarrow \text{red}_0 \\ & & C(\mathcal{O}_0) \end{array}$$

We would like to show that the triangle on the right-hand side is indeed commutative. For $\sigma \in K^{\text{ab}}$, we have the identifications

$$[F(\sigma)] = [s\mathcal{O}] \quad \text{and} \quad [F_0(\sigma)] = [s\mathcal{O}_0],$$

which are a consequence of the Main Theorem of CM. Notice that this uses the fact that the Main Theorem of CM holds for all elliptic curves with CM in any order in K at once. By looking at every rational prime p , one checks that $(s\Lambda) \cdot \Lambda' = s(\Lambda \cdot \Lambda')$, for two lattices Λ and Λ' in K (see [15]). In particular, after noticing that Λ and $s\Lambda$ have the same endomorphism ring, we get $[s\mathcal{O}] \otimes [\mathcal{O}_0] = [s\mathcal{O}_0]$. We have proven the following compatibility condition

Lemma 5.2 *Under the assumptions above,*

$$[F_0(\sigma)] = [F(\sigma)] \otimes [P_0],$$

or equivalently $\text{red}_0 \circ F = F_0$.

This proves the commutativity of the triangle in the diagram above, and thus, we are now ready to prove a characterization theorem for the field of K -moduli also in the imprimitive case.

5.3 Completion of the proof

In Lemma 5.1, we showed that

$$E_1^\sigma \times E_2^\sigma \cong E_1 \times E_2 \iff F_1(\sigma) \otimes F_2(\sigma) = P_0.$$

Now, $F_1(\sigma) \otimes F_2(\sigma)$ lives in $C(\mathcal{O}_{K,f_0})$ so we can multiply by the principal form P_0 , and, by commutativity and Lemma 5.2, the last condition above is equivalent to $F_0(\sigma)^2 = P_0$, i.e. $F_0(\sigma) \in C(\mathcal{O}_{K,f_0})[2]$. Therefore, we get, in analogy to the primitive case:

Proposition 5.3 $G_K = F_0^{-1}(C(\mathcal{O}_{K,f_0})[2])$.

Now, the same argument used in the primitive case (replacing every occurrence of H with H_0 , the ring class field of $\mathcal{O}_{K,f_0}$), yields the following result, which extends Theorem 4.1 to the imprimitive case.

Theorem 5.4 *Let X be a singular K3 surface with transcendental lattice $T(X) = Q = mQ_0$, and let H_0 be the ring class field of $\mathcal{O}_{K,f_0}$, the order of discriminant disc Q_0 . Then, the field of K -moduli is*

$$M_K = \bar{K}^{G_K}, \quad G_K = (|_{H_0})^{-1} \text{Gal}(H_0/K)[2];$$

it is a Galois extension of K of degree g , g being the order of the genus of the transcendental lattice of X .

6 The absolute field of moduli

So far, we have studied the field of K -moduli of a singular K3 surface X , K being the CM field of X . Now, we want to move our attention to the *absolute field of moduli* $M_{\mathbb{Q}}$, by which we mean the field of $\mathbb{Q}$ -moduli. We will proceed as in the case of M_K .

Let us recall that the absolute field of moduli of X is the field $M_{\mathbb{Q}} := \mathbb{C}^{G_{\mathbb{Q}}}$, where

$$G_{\mathbb{Q}} = \{\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \mid X^{\sigma} \in [X]\}.$$

The proof of Lemma 3.3 shows that we can equivalently define the field of moduli $M_{\mathbb{Q}}$ to be the subfield of $\bar{\mathbb{Q}}$ which is fixed by the group

$$G_{\mathbb{Q}} = \{\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid X^{\sigma} \in [X]\}.$$

As G_K is the subgroup of elements of $G_{\mathbb{Q}}$ whose restriction to K is trivial, we have the following commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_K & \longrightarrow & G_{\mathbb{Q}} & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Gal}(\bar{\mathbb{Q}}/K) & \longrightarrow & \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{|_K} & \text{Gal}(K/\mathbb{Q}) \longrightarrow 0 \end{array}$$

where C is simply the quotient group $G_{\mathbb{Q}}/G_K$ (notice that G_K is normal in $G_{\mathbb{Q}}$). We have the following:

Proposition 6.1 $C \cong \text{Gal}(K/\mathbb{Q})$.

Proof This is equivalent to G_K being an index-two subgroup of $G_{\mathbb{Q}}$. In fact, it is enough to show that $G_{\mathbb{Q}} \setminus G_K \neq \emptyset$. Indeed, assume there exists an element $\sigma \in G_{\mathbb{Q}} \setminus G_K$; thus, $X^{\sigma} \cong X$ and $\sigma|_K \neq \text{id}_K$. Also, notice that $\sigma^{-1} \in G_{\mathbb{Q}} \setminus G_K$. If $\tau \in G_{\mathbb{Q}} \setminus G_K$ is another such element, then $\sigma\tau^{-1} \in G_K$, which means $\bar{\sigma} = \bar{\tau} \in C$. This implies that $C \cong \mathbb{Z}/2\mathbb{Z}$, and thus, we can identify the quotient group C with $\text{Gal}(K/\mathbb{Q})$.

We now prove that $G_{\mathbb{Q}} \setminus G_K \neq \emptyset$. If $T(X)$ is 2-torsion, then the complex conjugation automorphism ι is an element in $G_{\mathbb{Q}} \setminus G_K$. Therefore, we can assume $T(X)$ is not 2-torsion.

Fig. 1 Relative and absolute field of moduli of moduli

In this case, we will build an element $\alpha \in G_{\mathbb{Q}} \backslash G_K$. Suppose X is given as the Shioda–Inose K3 surface of a product $E_1 \times E_2$, and let $Q_i \in C(\mathcal{O}_{K, f_i})$ be the corresponding quadratic forms (for $i = 1, 2$). Then, $T(X) = mQ_0$ with $Q_0 = Q_1 \otimes Q_2 \in C(\mathcal{O}_{K, f_0})$, $m \geq 1$ and $f_0 = \gcd(f_1, f_2)$. Then, for $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/K)$, consider $X^{\iota\sigma} = (X^\sigma)^{\iota}$: by looking at the transcendental lattice (as a quadratic form), we see that

$$T(X^{\iota\sigma}) = T(X^\sigma)^{-1} = [m(F_0(\sigma)^{-2} \otimes Q_1 \otimes Q_2)]^{-1} = m[F_0(\sigma)^2 \otimes (Q_1 \otimes Q_2)^{-1}].$$

Notice that we have used the fact that complex conjugation is insensitive to the index of primitivity of $T(X)$ and acts on it by inversion (i.e. by multiplication of the non-diagonal entries of $T(X)$ by -1 , after choosing a basis). At this point, we choose $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/K)$ such that $F_0(\sigma) = Q_0$. By Lemma 5.2, this choice is compatible with the reduction maps red_0 of quadratic forms. Such σ yields an element $\alpha := \iota\sigma$ as required. $\square$

As a consequence, $M_K/M_{\mathbb{Q}}$ is a Galois extension with group $C \cong \text{Gal}(K/\mathbb{Q})$, and thus, $M_K \supsetneq M_{\mathbb{Q}}$. By multiplicativity of degree, the extensions M_K/K and $M_{\mathbb{Q}}/\mathbb{Q}$ have the same degree.

We remark that K is not contained in $M_{\mathbb{Q}}$, and thus, $M_{\mathbb{Q}} \cap K = \mathbb{Q}$. Indeed, if this were the case, then we would have $G_{\mathbb{Q}} \subset \text{Gal}(\bar{\mathbb{Q}}/K)$. If $T(X)$ is 2-torsion, then $\iota \in G_{\mathbb{Q}} \backslash \text{Gal}(\bar{\mathbb{Q}}/K)$, and so we get a contradiction. If $T(X)$ is not 2-torsion, then any $\tau \in G_{\mathbb{Q}} \backslash G_K$ yields the same contradiction.

As a consequence of the discussion above, we have the following result.

Theorem 6.2 *Let X be a singular K3 surface. Its absolute field of moduli $M_{\mathbb{Q}}$ is an index-two subfield of the field of K -moduli M_K . Moreover, $M_{\mathbb{Q}}$ is an extension of $\mathbb{Q}$ of degree*

$$[M_{\mathbb{Q}} : \mathbb{Q}] = [M_K : K] = g,$$

g being the order of the genus of $T(X)$. In general, it is not a Galois extension of $\mathbb{Q}$.

In particular, we have the diagram of field extensions in Fig. 1, where all extensions are Galois, except possibly for $M_{\mathbb{Q}}/\mathbb{Q}$.

Remark 6.3 We would like to point out that Fig. 1 recovers the picture of the case of elliptic curves with CM in K . For an elliptic curve E with CM in an imaginary quadratic field K , $M_{\mathbb{Q}} = \mathbb{Q}(j(E))$ and $M_K = K(j(E)) = H$, H being the ring class field of the elliptic curve

E. (We are implicitly using the fact that elliptic curves correspond to quadratic forms.) The equality $M_K = H$ is explained by the fact that the field of K -moduli coincides with the minimal field of definition, for every elliptic curve with CM in K .

The final statement of Theorem 6.2 is that the extension $M_{\mathbb{Q}}/\mathbb{Q}$ is not Galois in general; the following example shows an occurrence of this phenomenon.

Example 6.4 (Example 4.5 reloaded) We compute the absolute field of moduli for the K3 surface X . Our results tell us that $M_{\mathbb{Q}}$ must be an extension of degree 3 of $\mathbb{Q}$. In this case, X has a model over $\mathbb{Q}(j(P))$, which is a degree 3 extension of $\mathbb{Q}$, so it follows that the absolute field of moduli $M_{\mathbb{Q}}$ is indeed $\mathbb{Q}(j(P))$ itself, which agrees with Theorem 6.2. Now, the class polynomial $H_{\mathcal{O}_K}(T)$ of the order $\mathcal{O}_K$ has $j(P)$, $j(Q)$ and $j(Q^{-1})$ as roots; $j(P)$ is real, while $j(Q) = \overline{j(Q^{-1})}$. It follows that the extension $\mathbb{Q}(j(P))/\mathbb{Q}$ cannot be Galois.

For a more interesting example, we look at the K3 surface Y , and we recall that

$$\mathbb{Q}(j_1, j_2) = K(j_1, j_2) = K(j_2), \quad [\mathbb{Q}(j_1, j_2) : \mathbb{Q}] = 6,$$

if we consider a Shioda–Inose model for Y . As the field of moduli $M_{\mathbb{Q}}$ is contained in every field of definition for Y and must have degree 3 by Theorem 6.2, we must find an element $\alpha \in \text{Gal}(H/\mathbb{Q})$ which leaves the modulus invariant. If ι denotes the (restriction of the) complex conjugation automorphism and $\sigma \in \text{Gal}(\bar{K}/K)$ is an element such that $F(\sigma) = Q^{-1}$, then $\alpha := \sigma\iota$ satisfies this condition. Therefore, $M_{\mathbb{Q}}$ is the subfield of H which is fixed by the group generated by α : this group has order 2, and thus, we get that $M_{\mathbb{Q}}$ is an extension of $\mathbb{Q}$ of degree 3, as expected. $\square$

7 Further questions

7.1 Non-finiteness of singular K3 surfaces

The following discussion is inspired by the following striking result of Šafarevič on the finiteness of singular K3 surfaces with bounded field of definition:

Theorem 7.1 (Theorem 1 of [13]) *Let n be a positive integer. There exist finitely many singular K3 surfaces with a model over a number field K of degree $[K : \mathbb{Q}] \leq n$.*

This result says that we can use the degree of the field of definition to stratify Σ^{K3} , and that each stratum contains finitely many elements only: the n^{th} stratum is defined as

$$\Sigma^{K3}(n) := \{[X] \in \Sigma^{K3} : X \text{ has a model over } K, [K : \mathbb{Q}] \leq n\}.$$

One might wonder whether a similar result holds for the field of moduli in place of the field of definition. We will now see that this is not the case.

Proposition 7.2 *Let X and Y be two singular K3 surfaces such that $T(X)$ is primitive (as a quadratic form) and $T(Y) = nT(X)$, for some $n \in \mathbb{N}$. Then, X and Y have the same field of K -moduli, K being the CM field of X and Y .*

Proof The argument used in proving Theorem 5.4 shows, in particular, that the ring class field H_0 only depends on the discriminant of the primitive part of the transcendental lattice. In the situation at hand, X and Y would both lead to the same ring class field, and the result is then a consequence of Theorem 5.4. $\square$

Proposition 7.3 *Let X and Y be two singular K3 surfaces whose transcendental lattices are primitive and lie in the same class group (as quadratic forms). Then, X and Y have the same field of K -moduli, K being the CM field of X and Y .*

Proof Same as for Proposition 7.2. $\square$

As a corollary, we get that

Corollary 7.4 *Let X and Y be two singular K3 surfaces such that the primitive parts of $T(X)$ and $T(Y)$ lie in the same class group (as quadratic forms). Then, X and Y have the same field of K -moduli, K being the CM field of X and Y .*

In particular, this shows that bounding the degree of the (relative) field of moduli is not enough to have a stratification of Σ^{K3} in strata containing finitely many elements only. In fact, we have shown that for each possible field of K -moduli, there exist infinitely many singular K3 surfaces with that field of K -moduli. This non-finiteness result holds true also if we replace the relative field of moduli with the absolute one: in fact, it is enough to fix a primitive quadratic form Q such that $\#C(\text{disc } Q) = 1$; then

$$\#\{[X] \in \Sigma^{K3} : T(X) = mQ, m \in \mathbb{N}\} = +\infty,$$

and all K3 surfaces in the set above have clearly $\mathbb{Q}$ as absolute field of moduli.

7.2 Explicit fields of K -moduli

We can still ask questions such as: which fields can appear as the field of K -moduli of a singular K3 surface? To answer such a question, Theorem 5.4 and its theoretical description of the field of moduli do not help us. The ideal situation would be to describe M_K as the subfield of a finite extension of K fixed by a (finite) group.

In consequence of Proposition 7.2, we can restrict ourselves to working with singular K3 surfaces whose transcendental lattice is primitive as a quadratic form; thus, let X be such a singular K3 surface. Then, Theorem 4.1 implies that:

Proposition 7.5 $M_K = H^{\text{Gal}(H/K)[2]}.$

This last result allows us to theoretically compute the field of moduli of a given singular K3 surface. More precisely, it says that we have an algorithm for computing M_K , as it is now described as the fixed field of a (well-known) number field by the action of a (well-known) finite group.

One can also describe, at least in theory, the possible fields of K -moduli for a given subset of Σ^{K3} . To this end, Proposition 7.2 enables us to project Σ^{K3} onto

$$\Sigma_{\text{prim}}^{K3} := \{[X] \in \Sigma^{K3} : T(X) \text{ is primitive}\},$$

by forgetting the index of primitivity of the transcendental lattice. Analogously to the situation of [17], there is a 1:1 correspondence

$$\Sigma_{\text{prim}}^{K3} \longleftrightarrow \mathcal{Q}_0^+ / \text{SL}_2(\mathbb{Z}),$$

where $\mathcal{Q}_0^+$ is the subset of $\mathcal{Q}^+$ containing primitive quadratic forms only. Class group theory implies that

$$\mathcal{Q}_0^+ / \text{SL}_2(\mathbb{Z}) \cong \bigsqcup_{\substack{K \text{ quadratic imaginary field} \\ f \in \mathbb{N}}} C(\mathcal{O}_{K,f}),$$

and thus, we can bound $\Sigma_{\text{prim}}^{\text{K3}}$, for example, by bounding the orders in the quadratic imaginary fields. This can be achieved, for instance, by bounding the discriminant or the class number. Such a constraint gives a stratification of $\Sigma_{\text{prim}}^{\text{K3}}$ whose strata contain finitely many elements only.

7.3 Invariance of the field of $\mathbb{Q}$ -moduli

Let X and Y be singular K3 surfaces such that their transcendental lattices have the same primitive part, i.e. $T(X) = mQ_0$, $T(Y) = m'Q'_0$ and $Q_0 = Q'_0$ (primitive). From previous discussions, we know that X and Y will have the same field of K -moduli, with $K = \mathbb{Q}(\sqrt{\text{disc } T(X)})$. We would like to study the analogous question for the field of $\mathbb{Q}$ -moduli, that is whether the field of $\mathbb{Q}$ -moduli is independent of the index of primitivity.

Proposition 7.6 *Let X and Y be two singular K3 surfaces such that $T(X)$ is primitive (as a quadratic form) and $T(Y) = mT(X)$, for some $m \in \mathbb{N}$. Then, X and Y have the same field of $\mathbb{Q}$ -moduli, i.e. $M_{\mathbb{Q}}(X) = M_{\mathbb{Q}}(Y)$.*

Proof The discussion in Sect. 6 shows that for $G_{\mathbb{Q}}(X) = \langle G_K(X), \alpha_X \rangle$ (as a group), where α_X is a suitable element in $\text{Gal}(\bar{\mathbb{Q}}/K)$. We will now show that $\alpha_X = \alpha_Y$, where α_Y is the analogous element of $\text{Gal}(\bar{\mathbb{Q}}/K)$ for Y .

Indeed, if we write $\alpha_X = \iota \circ \sigma_X$, with $\sigma_X \in \text{Gal}(\bar{\mathbb{Q}}/K)$ as in the proof of Proposition 6.1³, it is straightforward to see that

$$T(Y^{\alpha_X}) = T(Y^{\alpha_X})^{-1} = m[F_0(\sigma_X)^{-2} \otimes T(X)]^{-1} = m[F_0(\sigma_X)^2 \otimes T(X)^{-1}] = mT(X) = T(Y),$$

where we have used that $T(X)$ is primitive and that $F_0(\sigma_X) = T(X)$ (as a quadratic form—see proof of Proposition 6.1). $\square$

This result is clearly the analogue of Proposition 7.2 for the field of $\mathbb{Q}$ -moduli. Since conjugation by an automorphism acts as taking the conjugate of the field of moduli (see Sect. 3), we cannot have such an analogue for Proposition 7.3. However, the following analogue of Corollary 7.4 holds.

Corollary 7.7 *Let X and Y be two singular K3 surfaces. Assume that there exists $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $T(X^\sigma)_0 = T(Y)_0$ as quadratic forms, $T(Y)_0$ being the primitive part of $T(Y)$ (and similarly for X^σ). Then, $M_{\mathbb{Q}}(Y) = \sigma(M_{\mathbb{Q}}(X))$.*

Proof Use the fact that $M_{\mathbb{Q}}(X^\sigma) = \sigma(M_{\mathbb{Q}}(X))$ together with

$$T(X^\sigma) = mT(X^\sigma)_0 = m[F_0(\sigma)^{-2} \otimes T(X)_0].$$

$\square$

Finally, thanks to Proposition 7.5, it is straightforward to describe $M_{\mathbb{Q}}$ explicitly.

Proposition 7.8 $M_{\mathbb{Q}} = H^{\text{Gal}(H/K)[2] \times \text{Gal}(K/\mathbb{Q})}$.

Acknowledgements It is a pleasure to thank Matthias Schütt for fruitful conversations and for sharing his insights on this subject.

³ When $T(X)$ is 2-torsion (and thus also $T(Y)$ is), $\alpha_X = \alpha_Y = \iota$, ι being the complex conjugation automorphism, and thus, $\sigma_X = \text{id}_X$.

References

1. Beauville, A.: Some surfaces with maximal Picard number. *J. Éc. Polytech. Math.* **1**, 101–116 (2014)
2. Conway, J.H., Sloane, N.J.A.: *Sphere Packings, Lattices and Groups*, Volume 290 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 3rd edn. Springer, New York (1999). (With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov)
3. Cox, D.A.: *Pure and applied mathematics* (Hoboken). In: Cox, D.A. (ed.) *Primes of the Form $x^2 + ny^2$* , 2nd edn. Wiley, Hoboken, NJ (2013)
4. Hartshorne, R.: *Algebraic geometry*. Graduate Texts in Mathematics, vol. 52. Springer, New York (1977)
5. Inose, H.: Defining equations of singular $K3$ surfaces and a notion of isogeny. In: *Proceedings of the International Symposium on Algebraic Geometry* (Kyoto University, Kyoto, 1977), pp. 495–502. Kinokuniya Book Store, Tokyo (1978)
6. Koizumi, S.: The fields of moduli for polarized abelian varieties and for curves. *Nagoya Math. J.* **48**, 37–55 (1972)
7. Laface, R.: On decompositions of singular abelian surfaces. *Asian J. Math.* (to appear)
8. Matsusaka, T.: Polarized varieties, the fields of moduli and generalized Kummer varieties of abelian varieties. *Proc. Jpn. Acad.* **32**, 367–372 (1956)
9. Milne, J.S.: *Fields and Galois theory* (v4.51) (2015). Available at www.jmilne.org/math/. Accessed 20 Jan 2016
10. Morrison, D.R.: On $K3$ surfaces with large Picard number. *Invent. Math.* **75**(1), 105–121 (1984)
11. Schütt, M.: Fields of definition of singular $K3$ surfaces. *Commun. Number Theory Phys.* **1**(2), 307–321 (2007)
12. Serre, J.-P., Tate, J.: Good reduction of abelian varieties. *Ann. Math.* **2**(88), 492–517 (1968)
13. Shafarevich, I.R.: On the arithmetic of singular $K3$ -surfaces. In: *Algebra and Analysis* (Kazan, 1994), pp. 103–108. de Gruyter, Berlin (1996)
14. Shimada, I.: Transcendental lattices and supersingular reduction lattices of a singular $K3$ surface. *Trans. Am. Math. Soc.* **361**(2), 909–949 (2009)
15. Shimura, G.: *Introduction to the Arithmetic Theory of Automorphic Functions*, Volume 11 of Publications of the Mathematical Society of Japan. Princeton University Press, Princeton, NJ (1994). (Reprint of the 1971 original, Kanô Memorial Lectures, 1)
16. Shioda, T., Inose, H.: On singular $K3$ surfaces. In: Baily, W.L., Shioda, T. (eds.) *Complex Analysis and Algebraic Geometry*, pp. 119–136. Iwanami Shoten, Tokyo (1977)
17. Shioda, T., Mitani, N.: Singular abelian surfaces and binary quadratic forms. In: Popp, H. (ed.) *Classification of Algebraic Varieties and Compact Complex Manifolds. Lecture Notes in Mathematics*, vol. 412, pp. 259–287. Springer, Berlin (1974)
18. Silverman, J.H.: *Advanced topics in the Arithmetic of Elliptic Curves*, Volume 151 of Graduate Texts in Mathematics. Springer, New York (1994)
19. Silverman, J.H.: *The Arithmetic of Elliptic Curves*, Volume 106 of Graduate Texts in Mathematics, 2nd edn. Springer, Dordrecht (2009)
20. Weil, A.: *Foundations of Algebraic Geometry*, Volume 29 of American Mathematical Society Colloquium Publications. American Mathematical Society, New York (1946)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.