A hyperkähler submanifold of the monopole moduli space

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Abstract

We discuss a $4[k/2]$ -dimensional complete hyperkähler submanifold of the $(4k - 4)$ -dimensional moduli space of strongly centred $SU(2)$ -monopoles of charge k .

Keywords Hyperkaehler geometry · Monopole moduli spaces · Transverse Hilbert schemes

Mathematics Subject Classification 53C26 · 81T13

The isometry group of $\mathbb{R}^3$ acts isometrically on the moduli space M_k of Euclidean $SU(2)$ -monopoles of charge k , and consequently, a fixed point set of any subgroup of this group is a totally geodesic submanifold. Similar statement holds for any subgroup of the orthogonal group $O(3)$ acting on the submanifold M_k^0 consisting of *strongly centred* monopoles [8]. (These are monopoles with the centre at the origin and total phase equal to 1.) Houghton and Sutcliffe [9] have shown that the submanifold of M_3^0 consisting of monopoles symmetric about the origin is isometric to the Atiyah–Hitchin manifold (i.e. the moduli space of centred monopoles of charge 2). Surprisingly, monopoles of higher charges invariant under the reflection $x \mapsto -x$ seem not to have been considered in the literature. This reflection is particularly interesting since it extends to a reflection $\tau : (x, t) \mapsto (-x, t^{-1})$ on $\mathbb{R}^3 \times S^1$ which preserves the hyperkähler structure of $\mathbb{R}^3 \times S^1$. It is then easy to deduce that the submanifold N_k of strongly centred monopoles symmetric about the origin is a (complete) hyperkähler submanifold of M_k^0 for any charge k . We have stumbled upon this hyperkähler manifold (for even k) in a completely different context in [3] and realised only *a posteriori*, by identifying the twistor space, that it must be a submanifold of M_k^0 . In the present paper, we describe the submanifold N_k in terms of Nahm’s equations. Since N_k is $SO(3)$ -invariant, all of its complex structures are equivalent and can be identified with a complex submanifold of based rational maps of degree k . This is straightforward, given that the involution τ acts on rational maps via $p(z)/q(z) \mapsto \bar{p}(z)/q(-z)$, where $\bar{p}(-z)p(z) - 1 = 0 \pmod{q(z)}$, but we also show this directly using Nahm’s equations. We then show that N_k is biholomorphic to the *transverse Hilbert scheme* of n points [3, §5] on the D_1 -surface if $k = 2n$ and on the D_0 -surface if $k = 2n + 1$. This allows us to conclude that N_{2n} is simply connected, while

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N_{2n+1} has fundamental group of order 2. In Sect. 2, we present an alternative construction, which also gives a description of hyperkähler deformations of N_{2n} .

1 Description in terms of Nahm's equations

The moduli space M_k^0 of strongly centred $SU(2)$ -monopoles of charge k is isomorphic to the moduli space of $\mathfrak{su}(k)$ -valued solutions to Nahm's equations on $(0, 2)$ with $T_1(t)$, $T_2(t)$, $T_3(t)$ having simple poles at $t = 0, 2$, the residues of which define the standard k -dimensional irreducible representation of $\mathfrak{su}(2)$, i.e.

$$\begin{aligned} \text{Res } T_1(t) &= i \text{diag}(k-1, k-3, \dots, 3-k, 1-k), \\ \text{Res}(T_2 + iT_3)(t)_{ij} &= \begin{cases} \sqrt{j(k-j)} & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The corresponding representation of $\mathfrak{sl}_2(\mathbb{C})$ is given by the action $y\partial_x, x\partial_y, x\partial_x - y\partial_y$ on binary forms of degree $k-1$ with a basis

$$v_i = \binom{k-1}{i-1}^{1/2} x^{k-i} y^{i-1}, \quad i = 1, \dots, k. \quad (1.1)$$

Write V for the standard two-dimensional $\mathfrak{sl}_2(\mathbb{C})$ -module, so that the symmetric product $S^{k-1}V$ is the standard k -dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.

The equivariant isomorphisms

$$\Lambda^2(S^{2n+1}V^*) \simeq \bigoplus_{i=0}^n S^{4i}V^*, \quad S^2(S^{2n}V^*) \simeq \bigoplus_{i=0}^n S^{4i}V^*,$$

imply that, if k is even (resp. if k is odd), then there exists a unique (up to scaling) $\mathfrak{sl}_2(\mathbb{C})$ -invariant skew-symmetric (resp. symmetric) bilinear form on $S^{k-1}V$.

This invariant form is a classical object and is called *transvectant*. In the basis (1.1), the transvectant $T(f, g)$ of $f = \sum_{i=1}^k a_i v_i$, $g = \sum_{i=1}^k b_i v_i$ is given [10, Ex.2.7]

$$T(f, g) = \sum_{i=0}^{k-1} (-1)^i a_{i+1} b_{k-i}.$$

We conclude that the residues of T_1, T_2, T_3 belong to a symplectic subalgebra of $\mathfrak{su}(k)$ if k is even and to an orthogonal subalgebra if k is odd. These subalgebras are defined as

$$\left\{ A \in \mathfrak{su}(k); \quad AJ + JA^T = 0 \right\}, \quad (1.2)$$

where J is an antidiagonal matrix with $J_{i, k+1-i} = (-1)^{i-1}$. In other words, they are the fixed point sets of the involution

$$\sigma(A) = -JA^T J^{-1}. \quad (1.3)$$

We denote by $\mathfrak{su}(k)^\sigma$ the subalgebra (1.2) and by $SU(k)^\sigma$ the corresponding subgroup of $SU(k)$ ($SU(k)^\sigma \simeq Sp(n)$ if $k = 2n$ and $SU(k)^\sigma \simeq SO(2n+1)$ if $k = 2n+1$).

We consider the space $\mathcal{A}^\sigma$ of $\mathfrak{su}(k)^\sigma$ -valued solutions to Nahm's equations on $[0, 2]$. Let $\mathcal{G}$ denote the group of $SU(k)$ -valued gauge transformations which are identity at $t = 0, 2$, and let $\mathcal{G}^\sigma$ be its subgroup of $SU(k)^\sigma$ -valued gauge transformations. It is easy to verify that two

$\mathcal{G}$ -equivalent elements of $\mathcal{A}^\sigma$ are also $\mathcal{G}^\sigma$ -equivalent. Thus the natural map $\mathcal{A}^\sigma/\mathcal{G}^\sigma \rightarrow M_k^0$ is an embedding and we view $N_k = \mathcal{A}^\sigma/\mathcal{G}^\sigma$ as a submanifold of M_k^0 . N_k is the fixed point set of an involution σ which sends each $T_i(t)$ to $\sigma(T_i(t))$ (and acts the same way on gauge transformations) and therefore a complete hyperkähler submanifold of M_k^0 .

Proposition 1.1 *With respect to any complex structure N_k is biholomorphic to the space of based rational maps $\frac{p(z)}{q(z)}$ of degree k such that*

- (i) *if $k = 2n$, then $q(z) = \tilde{q}(z^2)$ for a monic polynomial $\tilde{q}$ of degree n and $p(z)p(-z) \equiv 1 \pmod{q(z)}$;*
- (ii) *if $k = 2n + 1$, then $q(z) = z\tilde{q}(z^2)$ for a monic polynomial $\tilde{q}$ of degree n , $p(0) = 1$ and $p(z)p(-z) \equiv 1 \pmod{q(z)}$.*

In particular, $\dim_{\mathbb{R}} N_{2n} = \dim_{\mathbb{R}} N_{2n+1} = 4n$.

Remark 1.2 If the roots of $q(z)$ are distinct, then the above condition means that $p(w)p(-w) = 1$ for each root w of $q(z)$ (and $p(0) = 1$ if k is odd). The full N_k is then the closure of this set inside the space of all rational maps.

Proof Recall [6] that M_k is biholomorphic to the space of based (i.e. $f(\infty) = 0$) rational maps of degree k and M_k^0 to the submanifold consisting of rational maps $p(z)/q(z)$ such that the sum of the poles z_i is equal to 0 and $\prod_{i=1}^k p(z_i) = 1$. From the point of view of Nahm's equations, the rational map is obtained by applying a (singular) complex gauge transformation g to the complex Nahm equation $\beta = [\beta, \alpha]$ in order to make β a constant matrix of the form

$$S = \begin{pmatrix} 0 & \dots & \dots & 0 & s_n \\ 1 & \ddots & & 0 & s_{n-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & \ddots & 0 & s_2 \\ 0 & \dots & \dots & 1 & s_1 \end{pmatrix}. \quad (1.4)$$

The value of the complex gauge transformation $g(t)$ at $t = 2$ (modulo a fixed singular gauge transformation) is then an element u of the centraliser of S in $GL(k, \mathbb{C})$. The pair (S, u) corresponds to the rational map $\text{tr } u(z - S)^{-1}$. (This is the description given in [2]). The complex structures of M_k^0 and of N_k are obtained by assuming that S and u belong to the appropriate subalgebra and subgroup (i.e. to $\mathfrak{sl}(k, \mathbb{C})$, $SL(n, \mathbb{C})$ for M_k^0 and to $\mathfrak{sl}(k, \mathbb{C})^\sigma$, $SL(n, \mathbb{C})^\sigma$ for N_k). It is enough to consider the subset where the poles of the rational map, i.e. the eigenvalues of S , are distinct (since N_k is the closure of this set in the space of all rational maps). A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(k, \mathbb{C})^\sigma$ is given by the diagonal matrices h satisfying $h_{ii} + h_{k+1-i, k+1-i} = 0$, $i = 1, \dots, k$. It is then immediate that if (S, u) is conjugate to an element of $\mathfrak{h} \times H$ ($H = \exp \mathfrak{h}$), then the corresponding rational map satisfies the conditions in the statement. $\square$

In [3, Ex. 5.4], we have identified the complex manifold described in the above proposition for $k = 2n$ as the Hilbert scheme of n points on the D_1 -surface $x^2 - zy^2 = 1$ transverse to the projection $(x, y, z) \mapsto z$. (Similarly, the space of all rational maps of degree k is the Hilbert scheme of k points on $\mathbb{C}^* \times \mathbb{C}$ transverse to the projection onto the second factor [1, Ch. 6].)

It turns out that for odd k the complex structure of N_k is that of the transverse Hilbert scheme of points on the D_0 -surface $x^2 - zy^2 - y = 0$:

Proposition 1.3 (i) *With respect to any complex structure, N_{2n} is biholomorphic to the Hilbert scheme of n points on the D_1 -surface $x^2 - zy^2 = 1$ transverse to the projection $(x, y, z) \mapsto z$.*

(ii) *With respect to any complex structure, N_{2n+1} is biholomorphic to the Hilbert scheme of n points on the D_0 -surface $x^2 - zy^2 - y = 0$ transverse to the projection $(x, y, z) \mapsto z$.*

Remark 1.4 The fact that N_3 is biholomorphic to the D_0 -surface has been observed by Houghton and Sutcliffe [9].

Proof Part (i) has already been shown in [3, Ex. 5.4]. For part (ii), recall from [3] that the Hilbert scheme of n points on the D_0 -surface $x^2 - zy^2 + y = 0$ transverse to the projection $(x, y, z) \mapsto z$ is an affine variety in $\mathbb{C}^{3n}$ given by the same equation, but for polynomials. More precisely its points are polynomials $x(z), y(z), r(z)$ with degrees of x and y at most $n - 1$ and $r(z)$ a monic polynomial of degree n , satisfying the condition

$$x(z)^2 - zy(z)^2 - y(z) = 0 \pmod{r(z)}.$$

As in [3] write $z = u^2$ and rewrite the above equation as

$$(x(u^2) + uy(u^2))(x(u^2) - uy(u^2)) - \frac{(x(u^2) + uy(u^2)) - (x(u^2) - uy(u^2))}{u} = 0 \pmod{r(u^2)}.$$

Let us write $f(u) = x(u^2) + uy(u^2)$, $p(u) = 1 + uf(u)$, $q(u) = ur(u^2)$. Then it is easy to see that if the roots of q are distinct, then the last equation is equivalent to the condition $p(w)p(-w) = 1$ for any nonzero root w of $q(u)$. Since this last equation is polynomial in the coefficients of p and q , it describes a closed affine subvariety inside the affine variety of all rational maps. Since the two closed affine subvarieties, namely the transverse Hilbert scheme of points on the D_0 -surface and the variety described in Proposition 1.1, have a common open dense subset, they must coincide. $\square$

We can now compute the fundamental group of N_k :

Proposition 1.5

$$\pi_1(N_k) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ \mathbb{Z}_2 & \text{if } k \text{ is odd.} \end{cases}$$

Proof The fundamental groups of the D_0 - and D_1 -surface (i.e. the Atiyah–Hitchin manifold and its double cover) are well known [1] and equal to $\mathbb{Z}_2$ and to 1, respectively. The result follows from $\square$

Lemma 1.6 *Let X be a smooth complex surface and $\pi : X \rightarrow C$ a holomorphic submersion onto a connected Riemann surface C . Suppose further that, over an open dense subset of C , π is a locally trivial fibration with connected fibres. Then the fundamental group of the transverse Hilbert scheme $X_\pi^{[n]}$ of n points, $n \geq 2$, is equal to $H_1(X, \mathbb{Z})$.*

We first observe that if X is a smooth complex surface and $n \geq 2$, then the fundamental group of the full Hilbert scheme $X^{[n]}$ of n points, $n \geq 2$, is equal to $H_1(X, \mathbb{Z})$. This follows by combining two facts: 1) a classical result of Dold and Puppe [5, Thm. 12.15] which says that $\pi_1(S^n X) \simeq H_1(X, \mathbb{Z})$ for $n \geq 2$ and 2) a theorem of Kollár [11, Thm. 7.8] which implies that $\pi_1(X^{[n]}) \simeq \pi_1(S^n X)$.

We now aim to show that $\pi_1(X_\pi^{[n]}) \simeq \pi_1(X^{[n]})$. Let Y denote a submanifold of $X^{[n]}$ consisting of $D \in X^{[n]}$ with $\pi(D) = z_0 + E$, where E has length $n - 2$ in C , $z_0 \notin \text{supp } E$,

and $D \cap \pi^{-1}(z_0)$ consists of two distinct points. Clearly $Y \cap X_\pi^{[n]} = \emptyset$. Denote by U a tubular neighbourhood of Y and let $Z = U \cup X_\pi^{[n]}$, $W = U \cap X_\pi^{[n]}$. Since the complement of Z in $X^{[n]}$ has (complex) codimension 2, the fundamental groups of Z and of $X^{[n]}$ coincide. Since W is a punctured disc bundle over Y , the long exact sequence of homotopy groups implies that the map $\pi_1(W) \rightarrow \pi_1(U)$ is surjective and its kernel consists of at most the “meridian loop” around Y . Owing to the assumption, we can choose a point y of Y so that a neighbourhood of the fibre containing two points is isomorphic to $B \times F$, where B is a disc $\{z \in \mathbb{C}; |z| < 1 + \epsilon\}$ and F is the generic fibre of π . The intersection of a neighbourhood of y in Z with W is then of the form $S_\pi^2(B \times F) \times V$, where the subscript π means that the pair of points $\{x_1, x_2\} \in B \times F$ satisfies $\pi(x_1) \neq \pi(x_2)$ and V is an open subset in $X_\pi^{[n-2]}$. Let ρ be the projection $(B \times F) \times (B \times F) \rightarrow S^2(B \times F)$.

A “meridian loop” in W around Y can be then chosen to be

$$t \mapsto (\rho(e^{\pi it}, f, e^{-\pi it}, f), v), \quad t \in [0, 1]$$

for constant f and v . This loop is contractible in $X_\pi^{[n]}$: the homotopy

$$H(r, t) = (\rho(re^{\pi it}, f, re^{-\pi it}, f), v), \quad t, r \in [0, 1]$$

contracts it to (D, v) where D is the double point $(\{z^2 = 0\}, f)$ in $B \times F$.

To recapitulate, we have shown that the map $\pi_1(W) \rightarrow \pi_1(U)$ is surjective and its kernel has trivial image in $\pi_1(X_\pi^{[n]})$. It follows that the amalgamated free product $\pi_1(U) *_{\pi_1(W)} \pi_1(X_\pi^{[n]})$ is isomorphic to $\pi_1(X_\pi^{[n]})$, and hence, van Kampen’s theorem implies that $\pi_1(Z) \simeq \pi_1(X_\pi^{[n]})$. $\square$

Remark 1.7 The D_0 -surface X is the quotient of the D_1 -surface $\tilde{X}$ by a free action of $\mathbb{Z}_2$ given by $(x, y, z) \mapsto (-x, -y, z)$. This induces a free $\mathbb{Z}_2$ -action on the transverse Hilbert scheme $\tilde{X}_\pi^{[n]}$ of n points for any n , but $\tilde{X}_\pi^{[n]}/\mathbb{Z}_2 \not\simeq X_\pi^{[n]}$, unless $n = 1$. Certainly, there is a surjective holomorphic map $\tilde{X}_\pi^{[n]} \rightarrow X_\pi^{[n]}$ for any n , which, in the description of these spaces provided in Proposition 1.1, sends a rational function $p(z)/q(z) \in \tilde{X}_\pi^{[n]}$ to $\bar{p}(z)/zq(z)$, where $\bar{p}(z) = p(z)^2 \bmod zq(z)$. This map is constant on $\mathbb{Z}_2$ -orbits, but it is generically 2^n -to-1 and not a covering. (The preimage of a point consists of 2^m points, where $2m$ is the number of distinct roots of $q(z)$.)

Remark 1.8 It is instructive to compare spectral curves of monopoles in N_{2n+1} to those in N_{2n} . It follows from Proposition 1.1 that the spectral curve S of a monopole in N_{2n+1} is always singular and given by an equation of the form $\eta P(\zeta, \eta) = 0$, where ζ is the affine coordinate of $\mathbb{P}^1$, η is the induced fibre coordinate in $T\mathbb{P}^1$, and P is a polynomial of the form $\eta^{2n} + \sum_{i=1}^n a_i(\zeta)\eta^{2n-2i}$, $\deg a_i(\zeta) = 4i$. A spectral curve of a monopole satisfies [1] the condition $L^2|_S \simeq \mathcal{O}$, where L^2 is a line bundle on $T\mathbb{P}^1$ with transition function $\exp(2\eta/\zeta)$. It follows that the line bundle L^2 is also trivial on the curve $\tilde{S}$ defined by the equation $P(\zeta, \eta) = 0$. Conversely, the spectral curve $\tilde{S}$ of a monopole in N_{2n} admits a section $s(\zeta, \eta)$ of L^2 which satisfies $s(\zeta, \eta)s(\zeta, -\eta) \equiv 1 \bmod P(\zeta, \eta)$, where $P(\zeta, \eta) = 0$ is the equation of $\tilde{S}$.

It follows that $s(\zeta, 0) = \pm 1$ and if we set $\bar{s}(\zeta, \eta) = s^2(\zeta, \eta)$ on $\tilde{S}$ and $\bar{s}(\zeta, \eta) \equiv 1$ on $\eta = 0$, we obtain a nonvanishing section of L^4 on the curve $\eta P(\eta/\zeta, \zeta) = 0$, i.e. a section of L^2 on the curve S given by $\tilde{\eta} P(\tilde{\eta}/2, \zeta) = 0$, where $\tilde{\eta} = 2\eta$. In the case $n = 1$, Houghton and Sutcliffe [9] have shown that for $n = 1$ these maps $S \mapsto \tilde{S}$ and $\tilde{S} \mapsto S$ send spectral curves of monopoles in N_3 to spectral curves of monopoles in N_2 and vice versa¹, but for higher n this is

¹ Strictly speaking, the curve $\tilde{S}$ obtained from S must be rescaled via $\eta = 2\tilde{\eta}$ in order to be the spectral curve of a monopole in N_2 .

not the case. The reason is that Hitchin's [7] nonsingularity condition $H^0(S, L^t(k-2)) = 0$, $t \in (0, 2)$, is not satisfied for the resulting curves. Indeed, the curve S obtained from $\tilde{S}$ as above satisfies

$$\dim H^0(S, L(k-2)) = n^2 - n.$$

2 Deformations and coverings

Dancer [4] has shown that the D_1 -surface admits a 1-parameter family of deformations carrying complete hyperkähler metrics. As we observed in [3], the transverse Hilbert schemes of points on these deformations also admit natural complete hyperkähler metrics. We wish to describe these metrics as deformations of manifolds N_{2n} . We begin by describing N_k without reference to an embedding into M_k^0 .

Let G_k (resp. $\mathfrak{g}_k$) denote $Sp(n)$ (resp. $\mathfrak{sp}(n)$) if $k = 2n$ and $SO(2n+1)$ (resp. $\mathfrak{so}(k)$) if $k = 2n+1$.

The construction of the previous section shows that N_k is the moduli space of $\mathfrak{g}_k$ -valued solutions to Nahm's equations on $(0, 2)$ with simple poles at $t = 0, 2$ and residues defining the principal homomorphism $\mathfrak{su}(2) \rightarrow \mathfrak{g}_k$, modulo G_k -valued gauge transformations which are identity at $t = 0, 2$. This moduli space is, in turn, a finite-dimensional hyperkähler quotient of a simpler hyperkähler manifold.

Let W_k^- (resp. W_k^+) be the moduli space of $\mathfrak{g}_k$ -valued solutions to Nahm's equations on $(0, 1]$ (resp. $[1, 2)$) with the above boundary behaviour at $t = 0$ (resp. at $t = 2$) and regular at $t = 1$, modulo G_k -valued gauge transformations which are identity at $t = 0, 1$ (resp. at $t = 1, 2$). $W_k^\pm$ are hyperkähler manifolds (biholomorphic to $G_k^\mathbb{C} \times \mathbb{C}^n$ [2]) with an isometric and triholomorphic action of G_k obtained by allowing gauge transformations with an arbitrary value at $t = 1$. Then N_k is the hyperkähler quotient of $W_k^- \times W_k^+$ by the diagonal G_k .

We can also describe in a similar manner the universal (i.e. double) covering space of N_{2n+1} : it is given by the same construction, but with $G_{2n+1} = Spin(2n+1)$ instead of $SO(2n+1)$.

An alternative construction of the N_k proceeds as follows. Let G denote one of the groups $Sp(n)$, $SO(2n+1)$ or $Spin(2n+1)$, and let τ be an automorphic involution on G with fixed point set K . Consider the hyperkähler quotient Y^- of W_k^- by K (with zero-level set of the moment map). Let (T_0, T_1, T_2, T_3) be a solution to Nahm's equations corresponding to a point in Y^- . Modulo gauge transformations, we can assume that $T_0(1) = 0$. We can then extend this solution to a solution to Nahm's equations on $(0, 2)$ by setting

$$T_i(2-t) = -\tau(T_i(t)), \quad i = 0, 1, 2, 3. \quad (2.1)$$

This solution has the boundary behaviour of a solution in N_k , and we can describe the moduli space Y of such extended solutions as the space of solutions on $(0, 2)$ having the correct poles and residues at $t = 0, 2$ and satisfying (2.1), modulo G -valued gauge transformations $g(t)$ such that

$$g(0) = g(2) = 1, \quad g(2-t) = \tau(g(t)), \quad t \in [0, 2]. \quad (2.2)$$

The map $Y^- \rightarrow Y$ is a triholomorphic homothety with factor 2. For dimensional reasons, Y^- (and consequently Y) is empty unless $\mathfrak{k} = \mathfrak{u}(n)$ for $k = 2n$ or $\mathfrak{k} = \mathfrak{so}(n) \oplus \mathfrak{so}(n+1)$ for $k = 2n+1$. Thus there are the following three possibilities for the symmetric pair (G, K) :

- (i) $k = 2n$, $G = Sp(n)$ and $K = U(n)$;
- (ii) $k = 2n + 1$, $G = SO(2n + 1)$ and $K = S(O(n) \times O(n + 1))$;
- (iii) $k = 2n + 1$, $G = Spin(2n + 1)$ and K is the diagonal double cover of $SO(n) \times SO(n + 1)$ (i.e. $K = Spin(n) \times Spin(n + 1)/\{(1, 1), (-1, -1)\}$).

An easy computation shows that in each case $\dim Y = \dim N_k$. Moreover, the natural map $Y \rightarrow N_k$ is an isometric (and triholomorphic) immersion, and since both N_k and Y are complete ([3, Thm. A.1]), this map must be a covering. Thus it follows from Proposition 1.5 that Y is isometric to N_k in cases (i) and (ii), while in case (iii) Y is the universal cover of N_{2n+1} .

The above construction allows us easily to describe a family of deformations of N_{2n} . Indeed, the Lie algebra $\mathfrak{k} = \mathfrak{u}(n)$ has a nontrivial centre, and therefore, we can take hyperkähler quotients of W_{2n}^- by K at nonzero-level sets of the hyperkähler moment map. This produces a three-parameter family of hyperkähler deformations of N_{2n} . Arguments analogous to those in [3] show that these are the natural hyperkähler metrics on the transverse Hilbert schemes of points on Dancer's deformations of the D_1 -surface.

Remark 2.1 As already mentioned in Remark 1.7, N_{2n} admits a free action of $\mathbb{Z}_2$ for any n . This action is also isometric and triholomorphic, and hence, $N_{2n}/\mathbb{Z}_2$ is a hyperkähler manifold. This manifold can be described in the same way as N_{2n} but with $G_{2n} = \mathbb{P}Sp(n)$ rather than $Sp(n)$.

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