Bilinear estimates and uniqueness for Navier–Stokes equations in critical Besov-type spaces

Lucas C. F. Ferreira¹ · Jhean E. Pérez-López²

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Abstract

We show bilinear estimates for the Navier–Stokes equations in critical Besov-weak-Morrey (BWM) spaces that contain the so-called Besov–Morrey (BM) spaces. Our estimates employ only the norm of the natural persistence space and do not use auxiliary norms like, e.g., Kato time-weighted norms. As a corollary, we obtain the uniqueness of mild solutions in the class of continuous functions from $[0, \infty)$ to critical BWM-spaces and, in particular, to BM-spaces. For our purposes, we need to show interpolation properties, heat semigroup estimates, and a characterization of preduals (of BWM-spaces) that are Besov-type spaces based on Lorentz-block ones. Another ingredient is a product estimate in our setting.

Keywords Navier–Stokes equations · Bilinear estimates · Uniqueness · Mild solutions · Besov-type spaces

Mathematics Subject Classification 35Q30 · 35A02 · 76D03 · 76D05 · 35C15 · 42B35

1 Introduction

We consider the free Navier–Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \mathbb{P}(u \cdot \nabla u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) , \\ u(0) = u_0 & \text{in } \mathbb{R}^n \end{cases} \quad (1.1)$$

✉ Lucas C. F. Ferreira
lcff@ime.unicamp.br

Jhean E. Pérez-López
jhean.perez@uis.edu.co

¹ Universidade Estadual de Campinas, Departamento de Matemática-IMECC, Rua Sérgio Buarque de Holanda, 651, Campinas, SP, CEP 13083-859, Brazil

² Universidad Industrial de Santander, Escuela de Matemáticas A.A. 678, Bucaramanga, Colombia

where $n \geq 3$, $u = (u_j)_{j=1}^n$ is the velocity field, u_0 is an initial data such that $\nabla \cdot u_0 = 0$ and $\mathbb{P}$ is the so-called Leray–Hopf projector defined as $(\mathbb{P}_{i,j})_{n \times n}$ where $\mathbb{P}_{i,j} = \delta_{i,j} + \mathcal{R}_i \mathcal{R}_j$ and $\mathcal{R}_i = (-\Delta)^{-1/2} \partial_i$ is the i -th Riesz transform.

By means of Duhamel's principle, the problem (1.1) can be formally converted to the integral equation

$$u(t) = U(t)u_0 + \mathcal{B}(u, u)(t), \quad (1.2)$$

where $U(t) = e^{t\Delta}$ is the heat semigroup and $\mathcal{B}(\cdot, \cdot)$ is the bilinear operator

$$\mathcal{B}(u, v)(t) = - \int_0^t \nabla_x \cdot U(t - \tau) \mathbb{P}(u \otimes v)(\tau) d\tau. \quad (1.3)$$

We say that u is a mild solution for (1.1) if u satisfies (1.2) and $\nabla \cdot u = 0$. An important property of (1.1) is that if u is a solution then $u_\lambda := \lambda u(\lambda^2 x, \lambda t)$ is also a solution with initial data $u_{0\lambda}(x) = \lambda u_0(\lambda x)$ for all $\lambda > 0$. So, we have the scaling map $u \rightarrow u_\lambda$ and the one for initial data

$$u_0(x) \rightarrow \lambda u_0(\lambda x). \quad (1.4)$$

In turn, a solution of (1.1) that is invariant under the scaling $u \rightarrow u_\lambda$, i.e., $u = u_\lambda$ for all $\lambda > 0$, is called a self-similar one of (1.1). Notice that self-similar solutions correspond to initial data homogeneous of degree -1 .

A Banach space X has scaling degree equal to $k \in \mathbb{R}$ if $\|f(\lambda x)\|_X \approx \lambda^k \|f\|_X$ holds for all $\lambda > 0$ and $f \in X$. Motivated by (1.4), a space X is said to be critical for (1.1) whether it has scaling degree equal to -1 . Throughout this paper, spaces of scalar and vector functions are denoted in the same way, namely we write $f \in X$ instead of $f \in (X)^n$. Furthermore, except for Sects. 2 and 3, elements u of functional spaces are assumed to satisfy $\nabla \cdot u = 0$ in $\mathcal{S}'$ (divergence-free condition).

There is by now a number of works about existence of global-in-time mild solutions for (1.1) with small initial data in critical spaces. Here, we mention some of them and refer the reader to the book [24] for a nice review. For example, we have global existence results in homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ [8], Lebesgue space $L^n(\mathbb{R}^n)$ [16], Marcinkiewicz space $L^{n,\infty}(\mathbb{R}^n)$ [1,31], pseudomeasure space PM^{n-1} [5,23], homogeneous Besov spaces $\dot{B}_{p,\infty}^{n-1}(\mathbb{R}^n)$ $p > n$ [4], Morrey spaces $\mathcal{M}_q^n(\mathbb{R}^n)$ [17,29], $\nabla \times u_0 \in \mathcal{M}_1^{3/2}(\mathbb{R}^3)$ [10] (for the vorticity formulation in 3D), weak-Morrey $\mathcal{M}_{(p,\infty)}^n(\mathbb{R}^n)$ [7,25,28], modulation spaces $M_{n,\sigma}^{n-1-n/\sigma}$ with $1 \leq \sigma < \infty$ [14], homogeneous Fourier–Besov spaces $F\dot{B}_{p,\infty}^{n-1-n/p}$ $1 < p \leq \infty$ [15,19], homogeneous Fourier–Herz spaces $\mathcal{B}_r^{-1} = F\dot{B}_{1,r}^{-1}$ with $r \in [1, 2]$ [6,15,22], $FM_0^{-1} = \text{div}[(FM_0)^3]$ (divergence of 3×3 -fields of Fourier transformed finite Radon measures with no point mass at the origin) [11], homogeneous weak-Herz space $W\dot{K}_{n,\infty}^0(\mathbb{R}^n)$ [30], homogeneous Besov–Morrey spaces $\mathcal{N}_{l,p,\infty}^{n/l-1}$ with $l > n/2$ [20] (see also [26]), BMO^{-1} [18]. Up to now, BMO^{-1} and $\mathcal{N}_{l,p,\infty}^{n/l-1}$ are considered to be maximal critical spaces for (1.1) in the sense that larger critical spaces are not available in the global existence literature.

Most of the global-in-time existence results in critical spaces are obtained by means of the so-called Kato approach that consists in a fixed-point argument in a suitable time-dependent space Z whose norm is the sum of two parts. The first is the norm of the persistence space $L^\infty((0, \infty); X)$ (for $u_0 \in X$) and the second is an auxiliary norm of the type

$$\sup_{t>0} t^\rho \|u\|_Y \quad \text{with } \rho \neq 0 \text{ and } Y \text{ a Banach space.} \quad (1.5)$$

We refer the reader to [24] for other examples of auxiliary norms. In general, the second norm helps to handle some integral terms in estimates for the bilinear operator $\mathcal{B}(\cdot, \cdot)$. This approach usually does not allow to obtain uniqueness in the natural class $C([0, T]; \tilde{X})$ where $0 < T \leq \infty$ and $\tilde{X}$ stands for the maximal closed subspace of X in which the heat semigroup $\{U(t)\}_{t \geq 0}$ is continuous. In [9], the authors showed uniqueness in critical Lebesgue and Morrey spaces by making use of auxiliary norms with $\rho = 0$ and considering Y as suitable Besov spaces.

A natural way to get the uniqueness in $C([0, T]; \tilde{X})$ is by means of the bilinear estimate

$$\|\mathcal{B}(u, v)\|_{L^\infty((0, T); X)} \leq C \|u\|_{L^\infty((0, T); X)} \|v\|_{L^\infty((0, T); X)}, \quad (1.6)$$

where $C > 0$ is a universal constant. Notice that in (1.6) one uses only the norm of the persistence space. In general, this estimate is more difficult to obtain than the ones with auxiliary norms and involves more subtle arguments. As far as we know, it has been proved in $L^{3, \infty}(\mathbb{R}^3)$ [27, 31], PM^{n-1} [5, 23], $F\dot{B}_{p, \infty}^{2-\frac{3}{p}}(\mathbb{R}^3)$ with $p > 3$ [19], $W\dot{K}_{n, \infty}^0(\mathbb{R}^n)$ [30], $\mathcal{M}_{(p, \infty)}^n(\mathbb{R}^n)$ with $2 < p \leq n$ [7, 25], and $\dot{B}_{p, \infty}^{\frac{n}{p}-1}$ with $2 \leq p < n$ [24]. We are not aware about previous results containing the estimate (1.6) and uniqueness in $C([0, T]; \tilde{X})$ for X equal to Besov–Morrey spaces $\mathcal{N}_{l, p, \infty}^{n/l-1}$, BMO^{-1} or other larger critical spaces.

As pointed out above, Kozono and Yamazaki [20] employed the Kato approach in order to show global-in-time well-posedness of small solutions for (1.1) in $\mathcal{N}_{l, p, \infty}^{n/l-1}$. More precisely, they considered the critical class

$$Z = \left\{ u \in BC\left((0, \infty); \mathcal{N}_{l, p, \infty}^{n/l-1}\right); \sup_{t>0} t^\rho \|u(\cdot, t)\|_{\mathcal{M}_{2p}^{2l}(\mathbb{R}^n)} < \infty \right\},$$

where $n/2 < l < \infty$ and $\rho = 1/2 - n/4l$. In the statement of their result, they assumed $n < l < \infty$ but a more detailed checking shows that in fact their proof works well for $n/2 < l < \infty$ (see [21, Theorem 3]). This solution u is continuous in the weak*-topology of $\dot{B}_{\infty, \infty}^{-1}$ at $t = 0^+$. Moreover, the solution $u \in BC([0, \infty); \mathcal{N}_{l, p, \infty}^{n/l-1})$ provided that the initial data $u_0 \in \tilde{\mathcal{N}}_{l, p, \infty}^{n/l-1}$.

Motivated by the above considerations, we arrive at the following uniqueness problem: *are mild solutions unique in the class $C([0, T]; \mathcal{N}_{l, p, \infty}^{n/l-1})$ with data $u_0 \in \tilde{\mathcal{N}}_{l, p, \infty}^{n/l-1}$ and $0 < T \leq \infty$?*

The main goal of the present paper is to give a positive answer to the above uniqueness question (with some restrictions on l and p) by showing the bilinear estimate (1.6) in a suitable space containing $\mathcal{N}_{l, p, \infty}^{n/l-1}$. In view of the above-mentioned works in $L^{3, \infty}(\mathbb{R}^3)$ [27, 31], $W\dot{K}_{n, \infty}^0(\mathbb{R}^n)$ [30] and $\mathcal{M}_{(p, \infty)}^n(\mathbb{R}^n)$ [7, 25], it seems to be natural to consider a family of Besov spaces containing $\mathcal{N}_{l, p, \infty}^{n/l-1}$ and with underlying spaces having suitable preduality and real interpolation properties. In this direction, we introduce Besov spaces based on Lorentz–Morrey spaces $\dot{B}\mathcal{M}_{(p, d), r}^{l, s}$ (see Definition 3.1 in Section 3 for more details), in particular the Besov–weak–Morrey $\dot{B}W\mathcal{M}_{(p, \infty), r}^{l, s} = \dot{B}\mathcal{M}_{(p, \infty), r}^{l, s}$, and show the bilinear estimate (1.6) with X equal to the critical space

$$\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{n}{l}-1} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \|f\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{n}{l}-1}} = \sup_{j \in \mathbb{Z}} 2^{j(\frac{n}{l}-1)} \|\Delta_j f\|_{\mathcal{M}_{(p, \infty)}^l} < \infty \right\}, \quad (1.7)$$

which is a Besov-type space based on weak-Morrey spaces. As a corollary, we obtain the uniqueness of solutions in $C([0, T]; \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})$ for $u_0 \in \dot{B}\tilde{W}\mathcal{M}_{p,\infty}^{l,n/l-1}$ and $0 < T \leq \infty$. Since $\mathcal{N}_{l,p,\infty}^{n/l-1} \subset \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}$, the uniqueness also holds in $\mathcal{N}_{l,p,\infty}^{n/l-1}$ -spaces.

Comparing with more standard critical spaces, we have the continuous inclusions

$$L^n(\mathbb{R}^n) \hookrightarrow L^{n,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{(p,\infty)}^n(\mathbb{R}^n) \hookrightarrow \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}, \quad l \geq n > 2.$$

Moreover, we have

$$\dot{H}^{n/2-1}(\mathbb{R}^n) \hookrightarrow \dot{H}_p^{n/p-1}(\mathbb{R}^n) \hookrightarrow \dot{B}_{p,\infty}^{n/p-1}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{l,p,\infty}^{n/l-1} \hookrightarrow \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}, \quad 2 < p < n.$$

In order to reach our aims, we need to show interpolation properties, heat semigroup estimates, and a characterization of the predual space of $\dot{B}W\mathcal{M}_{p,\infty}^{l,s}$ which is a Besov-type space based on Lorentz-block spaces. Another key ingredient is a product estimate in the framework of Besov-weak-Morrey spaces (see Lemma 3.14).

Our results read as follows.

Theorem 1.1 (Bilinear estimate) *Let $n \geq 3$, $0 < T \leq \infty$, $2 < p \leq l < n$ and $n/2 < l$. Then, there exists a constant $K > 0$ (independent of T) such that*

$$\|\mathcal{B}(u, v)\|_{L^\infty((0,T);\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})} \leq K \|u\|_{L^\infty((0,T);\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})} \|v\|_{L^\infty((0,T);\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})}, \quad (1.8)$$

for all $u, v \in L^\infty((0, T); \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})$.

Corollary 1.2 (Uniqueness) *Let $n \geq 3$, $0 < T \leq \infty$, $2 < p \leq l < n$, and $n/2 < l$. If u and v are two mild solutions of (1.1) in $C([0, T]; \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})$ with the same initial data $u_0 \in \dot{B}\tilde{W}\mathcal{M}_{p,\infty}^{l,n/l-1}$, then $u(\cdot, t) = v(\cdot, t)$ in $\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}$ for all $t \in [0, T]$. In particular, there is a unique solution $u \in C([0, T]; \mathcal{N}_{l,p,\infty}^{n/l-1})$ with initial data $u_0 \in \tilde{\mathcal{N}}_{l,p,\infty}^{n/l-1}$.*

Let us point out that the uniqueness result in Corollary 1.2 holds for $T = \infty$ and does not assume smallness conditions. Considering small solutions and using (1.8), one can remove the time-continuity condition, as well as the tilde $\sim$ in initial data spaces and get uniqueness in the class $L^\infty((0, \infty); \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})$ (resp. $L^\infty((0, \infty); \mathcal{N}_{l,p,\infty}^{n/l-1})$) for initial data $u_0 \in \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}$ (resp. $\mathcal{N}_{l,p,\infty}^{n/l-1}$), which covers the persistence class of Kozono-Yamazaki solutions.

In what follows, we make some comments about well-posedness in $\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}$ and self-similarity. In view of the bilinear estimate (1.8) and heat semigroup estimates in Lemma 3.10, one can employ a fixed-point argument in order to obtain well-posedness of global-in-time solutions $u \in BC((0, \infty); \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})$ under a smallness condition on initial data norms $\|u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}}$. These solutions are time-weakly continuous at $t = 0^+$. In fact, a well-posedness result for small solutions in the critical class $\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}$ seems not to be available in the previous literature. Since $\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}$ contains homogeneous functions of degree -1 , a scaling argument for the associated fixed-point Picard sequence yields the existence of self-similar solutions. Moreover, for $u_0 \in \dot{B}\tilde{W}\mathcal{M}_{p,\infty}^{l,n/l-1}$, we can replace the smallness of $\|u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1}}$ by the one of the existence time $T > 0$ in order to get a unique solution $u \in C([0, T]; \dot{B}W\mathcal{M}_{p,\infty}^{l,n/l-1})$ that can be large.

The outline of this paper is as follows. Section 2 is devoted to some preliminaries about sequence spaces, Lorentz–Morrey and Lorentz-block spaces. In Sect. 3, we introduce Besov–Lorentz–Morrey spaces $\dot{B}\mathcal{M}_{(p,d),r}^{l,s}$ and Besov–Lorentz-block spaces $\dot{B}\mathcal{E}_{(p,d),r}^{l,s}$ and show properties for them such as interpolation, duality, heat semigroup estimates, product estimates, among others. With these properties in hands, in Sect. 4, we prove Theorem 1.1 and Corollary 1.2.

2 Preliminaries

This section contains some preliminaries about sequence spaces, Lorentz–Morrey, and Lorentz-block spaces. The subjects presented here are known except for Lemmas 2.5, 2.7, and 2.10. In fact, these lemmas seem to be also known, however, their proofs are included for the reader convenience since we have not been able to locate complete statements in the literature for all range $1 \leq d \leq \infty$ and $\theta \in L^1$.

2.1 Sequence spaces

Here, we recall some definitions and results of the interpolation theory in sequence spaces. For $1 \leq r \leq \infty$, $s \in \mathbb{R}$ and a Banach space E , we define $\dot{l}_r^s(E)$ as the space of all sequences $a = (a_k)_{k \in \mathbb{Z}}$ with $a_k \in E$ for all k and such that

$$\|a\|_{\dot{l}_r^s(E)} = \left(\sum_{k=-\infty}^{\infty} 2^{ksr} \|a_k\|_E^r \right)^{1/r} < \infty.$$

As we will show later, the spaces in which we are interested behave like sequence spaces. In what follows, we present two lemmas about real interpolation in this kind of spaces (see e.g. [2]).

Lemma 2.1 *Assume that $1 \leq r_0, r_1, r \leq \infty$ and $s_0 \neq s_1$. Then, we have that*

$$(\dot{l}_{r_0}^{s_0}(E), \dot{l}_{r_1}^{s_1}(E))_{\theta,r} = \dot{l}_r^s(E),$$

where $0 < \theta < 1$ and $s = (1 - \theta)s_0 + \theta s_1$. If $s = s_0 = s_1$, it follows that

$$(\dot{l}_{r_0}^s(E), \dot{l}_{r_1}^s(E))_{\theta,r} = \dot{l}_r^s(E)$$

provided that

$$\frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}.$$

Lemma 2.2 *Let $1 \leq r_0, r_1 < \infty$ and $s_0 \neq s_1$. Then*

$$(\dot{l}_{r_0}^{s_0}(E_0), \dot{l}_{r_1}^{s_1}(E_1))_{\theta,r} = \dot{l}_r^s((E_0, E_1)_{\theta,r}),$$

where $0 < \theta < 1$, $s = (1 - \theta)s_0 + \theta s_1$ and $\frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}$.

The lemma below contains a basic duality property of sequence spaces (see [13]).

Lemma 2.3 *Let $1 \leq r < \infty$ and $s \in \mathbb{R}$. Then*

$$(\dot{l}_r^s(E))' = (\dot{l}_{r'}^{-s}(E')).$$

2.2 Lorentz–Morrey spaces

In this subsection, we recall some facts concerning Lorentz–Morrey spaces (see e.g. [7] for more details). We start with the definition of these spaces.

Definition 2.4 Let $1 < p \leq l \leq \infty$ and $1 \leq d \leq \infty$ ($d = \infty$ if $p = \infty$). The homogeneous Lorentz–Morrey space $\mathcal{M}_{(p,d)}^l = \mathcal{M}_{(p,d)}^l(\mathbb{R}^n)$ is the one of all measurable functions such that

$$\|f\|_{\mathcal{M}_{(p,d)}^l} := \sup_{x_0 \in \mathbb{R}^n} \sup_{R>0} R^{\frac{n}{l}-\frac{n}{p}} \|f\|_{L^{p,d}(D(x_0,R))} < \infty, \quad (2.1)$$

where $L^{p,d}$ stands for Lorentz spaces and $D(x_0, R) = \{x \in \mathbb{R}^n; |x - x_0| < R\}$.

For the definition and properties about $L^{p,d}$ -spaces, we refer the reader to [12] and references therein (see also [2]). The pair $(\mathcal{M}_{(p,d)}^l, \|\cdot\|_{\mathcal{M}_{(p,d)}^l})$ are Banach spaces. In [7], the space $\mathcal{M}_{(p,d)}^l$ is denoted as $\mathcal{M}_{r,d,\lambda}$ with $-\frac{\lambda}{r} = \frac{n}{l} - \frac{n}{p}$. The space $\mathcal{M}_{(p,\infty)}^l$ is called weak-Morrey space, and here, we denote it by $W\mathcal{M}_p^l$. In [28], $W\mathcal{M}_p^l$ is denoted as $\mathcal{M}_{p,\lambda}^*$.

Hölder inequality holds true in the framework of homogeneous Lorentz–Morrey spaces. To be more specific, let $1 < p, p_0, p_1 \leq \infty$, $1 \leq d_0, d_1 \leq \infty$, $p \leq l$, and $p_i \leq l_i$ (for $i = 0, 1$) be such that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $\frac{1}{l} = \frac{1}{l_0} + \frac{1}{l_1}$, and $\frac{1}{d_0} + \frac{1}{d_1} \geq \frac{1}{d}$. Then, there exists a universal constant $C > 0$ such that

$$\|fg\|_{\mathcal{M}_{(p,d)}^l} \leq C \|f\|_{\mathcal{M}_{(p_0,d_0)}^{l_0}} \|g\|_{\mathcal{M}_{(p_1,d_1)}^{l_1}}. \quad (2.2)$$

In the next lemma, we provide an estimate for convolution operators in homogeneous Lorentz–Morrey spaces that will be useful for our ends.

Lemma 2.5 (Convolution in Lorentz–Morrey spaces) *Let $1 < p \leq l \leq \infty$, $1 \leq d \leq \infty$, and $\theta \in L^1(\mathbb{R}^n)$. Then, there exists $C > 0$ (independent of θ) such that*

$$\|\theta * f\|_{\mathcal{M}_{(p,d)}^l} \leq C \|\theta\|_{L^1} \|f\|_{\mathcal{M}_{(p,d)}^l}, \quad (2.3)$$

for all $f \in \mathcal{M}_{(p,d)}^l$.

Proof Let $x_0 \in \mathbb{R}^n$ and $R > 0$. Using Minkowski-type inequality in Lorentz spaces, we get

$$\begin{aligned} \|\theta * f\|_{L^{p,d}(D(x_0,R))} &= \|(\theta * f) \chi_{D(x_0,R)}\|_{L^{p,d}(\mathbb{R}^n)} = \left\| \int_{\mathbb{R}^n} \theta(y) f(\cdot - y) \chi_{D(x_0,R)}(\cdot) dy \right\|_{L^{p,d}(\mathbb{R}^n)} \\ &\leq C \int_{\mathbb{R}^n} \|\theta(y) f(\cdot - y) \chi_{D(x_0,R)}(\cdot)\|_{L^{p,d}(\mathbb{R}^n)} dy \\ &\leq C \int_{\mathbb{R}^n} |\theta(y)| \|f\|_{L^{p,d}(D(x_0-y,R))} dy \\ &\leq C \|\theta\|_{L^1(\mathbb{R}^n)} R^{\frac{n}{p}-\frac{n}{l}} \|f\|_{\mathcal{M}_{(p,d)}^l}, \end{aligned}$$

and thus

$$R^{\frac{n}{l}-\frac{n}{p}} \|\theta * f\|_{L^{p,d}(D(x_0,R))} \leq C \|\theta\|_{L^1(\mathbb{R}^n)} \|f\|_{\mathcal{M}_{(p,d)}^l}. \quad (2.4)$$

Now, we can conclude by taking the supremum in the left-hand side of (2.4). $\square$

We recall that for each $t > 0$ the heat semigroup $U(t)$ is defined as

$$U(t)f = e^{t\Delta}f = g(x, t) * f,$$

for all $f \in \mathcal{S}'$, where $g(x, t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ is the heat kernel. For a multi-index β , we have that

$$|(\partial_x^\beta g)(x, t)| \leq \bar{h}(x, t) = h(|x|, t),$$

where $\bar{h}(x, t) = t^{-\frac{|\beta|+n}{2}} \bar{h}\left(\frac{x}{\sqrt{t}}, 1\right)$, $h(\rho, t) \in \mathcal{S}(\mathbb{R})$, and $\partial_\rho h(\rho, t) = t^{-\frac{|\beta|+n+1}{2}} (\partial_\rho h)\left(\frac{\rho}{\sqrt{t}}, 1\right)$ for all $t > 0$.

The next lemma can be found in [7] and contains an estimate for the heat semigroup in Lorentz–Morrey spaces. It is a generalization of [17, Lemma 2.1] (case $p = d$) and [28, Lemma 3.1] (case $d = \infty$).

Lemma 2.6 *Let β be a multi-index, $1 \leq d_0 \leq d \leq \infty$, $1 < p_0, p \leq \infty$, $p_0 \leq l_0$, $p \leq l$, and $\frac{1}{l} \leq \frac{1}{l_0}$. Assume further that $\frac{p}{l} = \frac{p_0}{l_0}$ when $p_0 \leq p$. Then, there exists a constant $C > 0$ such that*

$$\|\partial_x^\beta U(t)f\|_{\mathcal{M}_{(p,d)}^l} \leq Ct^{-\frac{|\beta|}{2} - \frac{n}{2}\left(\frac{1}{l_0} - \frac{1}{l}\right)} \|f\|_{\mathcal{M}_{(p_0,d_0)}^{l_0}}, \quad (2.5)$$

for all $f \in \mathcal{M}_{(p_0,d_0)}^{l_0}$.

In what follows, φ denotes a radially symmetric function such that

$$\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\}), \quad \text{supp } \varphi \subset \left\{x; \frac{3}{4} \leq |x| \leq \frac{8}{3}\right\},$$

and

$$\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{where } \varphi_j(\xi) := \varphi(\xi 2^{-j}).$$

Recall the localization operators Δ_j and S_j defined by

$$\Delta_j f = \varphi_j(D)f = (\varphi_j)^\vee * f \quad \text{and} \quad S_k f = \sum_{j \leq k} \Delta_j f.$$

One can check easily the identities

$$\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \text{ and } \Delta_j (S_{k-2} g \Delta_k f) = 0 \text{ if } |j - k| \geq 5.$$

Moreover, we have the Bony's decomposition (see [3])

$$fg = T_f g + T_g f + R(fg), \quad (2.6)$$

where

$$T_f g = \sum_{j \in \mathbb{Z}} S_{j-2} f \Delta_j g, \quad R(fg) = \sum_{j \in \mathbb{Z}} \Delta_j f \tilde{\Delta}_j g \quad \text{and} \quad \tilde{\Delta}_j g = \sum_{|j-j'| \leq 1} \Delta_{j'} g.$$

We also denote $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and $\tilde{D}_j = D_{j-1} \cup D_j \cup D_{j+1}$ where $j \in \mathbb{Z}$ and $D_j = \{x; \frac{3}{4}2^j \leq |x| \leq \frac{8}{3}2^j\}$. Notice that $\tilde{\varphi}_j = 1$ in D_j .

Lemma 2.7 *Let $1 < p \leq l \leq \infty$, $1 \leq d \leq \infty$, and $m \in \mathbb{R}$. Let P be a C^n -function on $\tilde{D}_j$ such that $\left| \partial_\xi^\beta P(\xi) \right| \leq C 2^{j(m-|\beta|)}$ for all $\xi \in \tilde{D}_j$ and $|\beta| \leq n$. Then,*

$$\left\| \left(P \hat{f} \right)^\vee \right\|_{\mathcal{M}_{(p,d)}^l} \leq C 2^{jm} \|f\|_{\mathcal{M}_{(p,d)}^l},$$

for all $f \in \mathcal{M}_{(p,d)}^l$ such that $\text{supp } \hat{f} \subset D_j$.

Proof Let us define $K(x) = (P \tilde{\varphi}_j)^\vee$. Since $\text{supp } \hat{f} \subset D_j$, we have $P(\xi) \hat{f}(\xi) = P(\xi) \tilde{\varphi}_j(\xi) \hat{f}(\xi)$, and therefore $(P \hat{f})^\vee = (P \tilde{\varphi}_j \hat{f})^\vee = K * f$. It follows from Lemma 2.5 that

$$\left\| \left(P \hat{f} \right)^\vee \right\|_{\mathcal{M}_{(p,d)}^l} \leq C \|K\|_{L^1} \|f\|_{\mathcal{M}_{(p,d)}^l}.$$

Now, we are going to show that $\|K\|_{L^1} \leq C 2^{mj}$. Taking $N \in \mathbb{N}$ such that $\frac{n}{2} < N \leq n$, we can estimate

$$\begin{aligned} \|K\|_{L^1} &= \int_{D(0,2^{-j})} K(y) + \int_{|y| \geq 2^{-j}} K(y) \\ &\leq \left(\int_{D(0,2^{-j})} 1 \right)^{1/2} \left(\int_{D(0,2^{-j})} |K(y)|^2 \right)^{1/2} + \left(\int_{|y| \geq 2^{-j}} |y|^{-2N} \right)^{1/2} \\ &\quad \times \left(\int_{|y| \geq 2^{-j}} |y|^{2N} |K(y)|^2 \right)^{1/2} \\ &\leq C 2^{-j \frac{n}{2}} \|P \tilde{\varphi}_j\|_{L^2} + C 2^{-j(-N+\frac{n}{2})} \sum_{|\beta|=N} \|(\cdot)^\beta K\|_{L^2} \\ &\leq C 2^{-j \frac{n}{2}} \|P \tilde{\varphi}_j\|_{L^2} + C 2^{-j(-N+\frac{n}{2})} \sum_{|\beta|=N} \|\partial^\beta (P \tilde{\varphi}_j)\|_{L^2} \\ &\leq C 2^{-j \frac{n}{2}} C 2^{mj} 2^{j \frac{n}{2}} + C 2^{-j(-N+\frac{n}{2})} C 2^{j(m-N)} 2^{j \frac{n}{2}} \\ &\leq C 2^{mj}. \end{aligned}$$

□

The following lemma is a Bernstein-type inequality in Lorentz–Morrey spaces. We omit its proof here because it is similar to the proof of Lemma 2.6 that can be found in [7]. In fact, it is sufficient to note that for $f \in \mathcal{S}'$ with $\text{supp } \hat{f} \subset D_j$ we have $f = \tilde{\Delta}_j f = 2^{jn} (\tilde{\varphi}_0)^\vee (2^j \cdot) * f$ and the function $2^{jn} (\tilde{\varphi}_0)^\vee (2^j \cdot)$ and heat kernel $g(x, 2^{-j})$ have a similar behavior.

Lemma 2.8 (Bernstein-type inequality in Lorentz–Morrey spaces) *Assume that p, p_0, l, l_0, d, d_0 are as in Lemma 2.6. Then, we have the estimate*

$$\|f\|_{\mathcal{M}_{(p,d)}^l} \leq C 2^{jn(\frac{1}{l_0} - \frac{1}{l})} \|f\|_{\mathcal{M}_{(p_0,d_0)}^{l_0}}, \quad (2.7)$$

for all $f \in \mathcal{M}_{(p_0,d_0)}^{l_0}$ such that $\text{supp } \hat{f} \subset D_j$.

2.3 Lorentz-block spaces

In this section, we address Lorentz-block spaces that are block spaces based on Lorentz ones. We recall some properties found in the literature and prove some others that will be needed for our purpose.

Let $1 < l \leq p \leq \infty$ and $1 \leq d \leq \infty$ and assume that $d = \infty$ when $p = \infty$. We say that a measurable function b is a (l, p, d) -block if there exist $a \in \mathbb{R}^n$ and $\rho > 0$ such that $\text{supp}(b) \subset D(a, \rho)$ and

$$\rho^{\frac{n}{l} - \frac{n}{p}} \|b\|_{L^{p,d}(D(a,\rho))} \leq 1. \quad (2.8)$$

Next, let $\mathcal{E}_{(p,d)}^l$ stand for the space of all measurable functions ϕ that can be decomposed as

$$\phi(x) = \sum_{k=1}^{\infty} \alpha_k b_k(x), \text{ for a.e. } x \in \mathbb{R}^n, \quad (2.9)$$

where b_k is a (l, p, d) -block and $\sum_{k=1}^{\infty} |\alpha_k| < \infty$. The decomposition (2.9) is called a representation of ϕ in $\mathcal{E}_{(p,d)}^l$. The space $\mathcal{E}_{(p,d)}^l$ endowed with the norm

$$\|h\|_{\mathcal{E}_{(p,d)}^l} = \inf \left\{ \sum_{k=1}^{\infty} |\alpha_k| < \infty; h = \sum_{k=1}^{\infty} \alpha_k b_k \text{ where } b_k \text{'s are } (l, p, d) \text{-blocks} \right\} \quad (2.10)$$

is a Banach space. In [7], the space $\mathcal{E}_{(p,d)}^l$ is denoted as $\mathcal{PD}_{(p,d),\kappa}$ where $\frac{\kappa}{p} = \frac{n}{l} - \frac{n}{p}$.

The following lemma gives a characterization of the predual of Lorentz–Morrey spaces (see [7]).

Lemma 2.9 *If $1 < p \leq l < \infty$ and $1 \leq d \leq \infty$, then*

$$\left(\mathcal{E}_{(p',d')}^{l'} \right)' = \mathcal{M}_{(p,d)}^l.$$

Now, we present a convolution lemma in Lorentz-block spaces.

Lemma 2.10 (Convolution in Lorentz-block spaces) *Let $1 < p \leq l < \infty$, $1 \leq d \leq \infty$, and $\theta \in L^1(\mathbb{R}^n)$. Then, there exists $C > 0$ (independent of θ) such that*

$$\|\theta * f\|_{\mathcal{E}_{(p,d)}^l} \leq C \|\theta\|_{L^1} \|f\|_{\mathcal{E}_{(p,d)}^l}, \quad (2.11)$$

for all $f \in \mathcal{E}_{(p,d)}^l$.

Proof Let $\tilde{\theta}(x) = \theta(-x)$. Using duality and Lemma 2.5, we can estimate

$$\begin{aligned} \|\theta * f\|_{\mathcal{E}_{(p,d)}^l} &= \sup_{\|h\|_{\mathcal{M}_{(p',d')}^{l'}} \leq 1} |\langle \theta * f, h \rangle| = \sup_{\|h\|_{\mathcal{M}_{(p',d')}^{l'}} \leq 1} \left| \langle f, \tilde{\theta} * h \rangle \right| \\ &\leq \|f\|_{\mathcal{E}_{(p,d)}^l} \sup_{\|h\|_{\mathcal{M}_{(p',d')}^{l'}} \leq 1} \|\tilde{\theta} * h\|_{\mathcal{M}_{(p',d')}^{l'}} \\ &\leq C \|\theta\|_{L^1} \|f\|_{\mathcal{E}_{(p,d)}^l} \sup_{\|h\|_{\mathcal{M}_{(p',d')}^{l'}} \leq 1} \|h\|_{\mathcal{M}_{(p',d')}^{l'}} \\ &\leq C \|\theta\|_{L^1} \|f\|_{\mathcal{E}_{(p,d)}^l}, \end{aligned}$$

as required. $\square$

The lemma below contains some results about interpolation in Lorentz-block spaces (see [7]).

Lemma 2.11 *Let $1 < l \leq p < \infty$, $1 < l_i \leq p_i \leq \infty$, and $1 \leq d, d_i \leq \infty$ (for $i = 0, 1$) be such that $p_0 \neq p_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{l} = \frac{1-\theta}{l_0} + \frac{\theta}{l_1}$ with $\theta \in (0, 1)$. Assume that X_0 and X_1 are Banach spaces and $\mathcal{T}$ is a continuous linear map from $\mathcal{E}_{(p_0, d_0)}^{l_0}$ to X_0 and from $\mathcal{E}_{(p_1, d_1)}^{l_1}$ to X_1 with operator norm C_0 and C_1 , respectively. Then, we have that $\mathcal{T} : \mathcal{E}_{(p, d)}^l \longrightarrow (X_0, X_1)_{\theta, d}$ is a continuous linear map with operator norm bounded from above by $\tilde{C} = (C_0)^{1-\theta} (C_1)^\theta$.*

Remark 2.12 Notice that in the conditions of Lemma 2.11 we have the continuous inclusion

$$\mathcal{E}_{(p, d)}^l \hookrightarrow \left(\mathcal{E}_{(p_0, d_0)}^{l_0}, \mathcal{E}_{(p_1, d_1)}^{l_1} \right)_{\theta, d}. \quad (2.12)$$

We finish this subsection by recalling estimates for the heat semigroup in Lorentz-block spaces (see [7]).

Lemma 2.13 *Let β be a multi-index, $1 < l_i \leq p_i < \infty$, $1 \leq d_i \leq \infty$ for $i = 1, 2$ and $l_1 \leq l_2$. Assume also that $\frac{p'_1}{l'_1} = \frac{p'_2}{l'_2}$ when $p_1 \leq p_2$. Then, there exists a constant $C > 0$ such that*

$$\left\| \partial_x^\beta U(t) f \right\|_{\mathcal{E}_{(p_2, d_2)}^{l_2}} \leq C t^{-\frac{|\beta|}{2} - \frac{n}{2} \left(\frac{1}{l_1} - \frac{1}{l_2} \right)} \|f\|_{\mathcal{E}_{(p_1, d_1)}^{l_1}}, \quad (2.13)$$

for all $f \in \mathcal{E}_{(p_1, d_1)}^{l_1}$.

3 Besov–Lorentz–Morrey and Besov–Lorentz-Block spaces

We start with a general definition of Besov-type spaces based on a Banach space E . This definition has been used by some authors, see e.g. [9, 24, 26].

Definition 3.1 Let E be a Banach space, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. The homogeneous Besov- E space $\dot{B}E_r^s$ is defined as

$$\dot{B}E_r^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \|f\|_{\dot{B}E_r^s} < \infty \right\},$$

where

$$\|f\|_{\dot{B}E_r^s} := \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j f\|_E^r \right)^{\frac{1}{r}} & \text{if } r < \infty \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_E & \text{if } r = \infty. \end{cases} \quad (3.1)$$

The space $\dot{B}E_r^s$ is also denoted by $\dot{B}_E^{s, r}$ or $\dot{B}_{E, r}^s$ (see [24]). If $E = \mathcal{M}_{(p, p)}^p = L^p$ then $\dot{B}E_r^s = \dot{B}_{p, r}^s$ is the usual homogeneous Besov space. Taking $E = \mathcal{M}_{(p, p)}^l = \mathcal{M}_p^l$, we obtain the homogeneous Besov–Morrey space $\mathcal{N}_{l, p, r}^s$ introduced in [20]. Here, we use $E = \mathcal{M}_{(p, d)}^l$ that corresponds to the homogeneous Besov–Lorentz–Morrey space $\dot{B}E_r^s = \dot{B}\mathcal{M}_{(p, d), r}^{l, s}$. When $d = \infty$, we obtain the Besov-weak-Morrey space $\dot{B}\mathcal{M}_{(p, \infty), r}^{l, s} = \dot{B}W\mathcal{M}_{p, r}^{l, s}$.

Now, we prove an auxiliary lemma that has already been showed in the case of homogeneous Besov spaces $\dot{B}_{p,r}^s = \dot{B}_{(p,p),r}^{p,s}$ (see e.g. [2]) and homogeneous Besov–Morrey spaces $\mathcal{N}_{l,p,r}^s = \dot{B}_{(p,p),r}^{l,s}$ (see e.g. [26]).

Lemma 3.2 *Let $1 < p \leq l \leq \infty$, $1 \leq d, r \leq \infty$, and $s \in \mathbb{R}$. Then, the space $\dot{B}\mathcal{M}_{(p,d),r}^{l,s}$ is a retract of $\dot{I}_r^s(\mathcal{M}_{(p,d)}^l)$.*

Proof For each $f \in \mathcal{S}'$, we define the operator $\mathcal{I}$ as $(\mathcal{I}(f))_j := \Delta_j f$. It is clear that

$$\mathcal{I} : \dot{B}\mathcal{M}_{(p,d),r}^{l,s} \longrightarrow \dot{I}_r^s\left(\mathcal{M}_{(p,d)}^l\right).$$

We also define the operator $\mathcal{L}$ by $\mathcal{L}(\psi) = \sum_{k=-\infty}^{\infty} \tilde{\Delta}_j \psi_j$, for all $\psi = (\psi_j) \in \dot{I}_r^s\left(\mathcal{M}_{(p,d)}^l\right)$. Because $\varphi_j = \tilde{\varphi}_j \varphi_j$, it follows that $\Delta_j \mathcal{L}(\psi) = \Delta_j \psi_j$ and

$$\begin{aligned} \|\mathcal{L}(\psi)\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s}} &= \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j \psi_j\|_{\mathcal{M}_{(p,d)}^l}^r \right)^{\frac{1}{r}} \leq C \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\psi_j\|_{\mathcal{M}_{(p,d)}^l}^r \right)^{\frac{1}{r}} \\ &\leq C \|\psi\|_{\dot{I}_r^s(\mathcal{M}_{(p,d)}^l)}. \end{aligned}$$

Therefore, $\mathcal{L} : \dot{I}_r^s\left(\mathcal{M}_{(p,d)}^l\right) \longrightarrow \dot{B}\mathcal{M}_{(p,d),r}^{l,s}$ and $\mathcal{L} \circ \mathcal{I}$ is the identity in $\dot{B}\mathcal{M}_{(p,d),r}^{l,s}$, as required. $\square$

Corollary 3.3 *Let $1 < p \leq l \leq \infty$, $1 \leq d, r, r_0, r_1 \leq \infty$, and $s, s_0, s_1 \in \mathbb{R}$ be such that $s = (1 - \theta)s_0 + \theta s_1$ with $\theta \in (0, 1)$. Then,*

$$\left(\dot{B}\mathcal{M}_{(p,d),r_0}^{l,s_0}, \dot{B}\mathcal{M}_{(p,d),r_1}^{l,s_1} \right)_{\theta,r} = \dot{B}\mathcal{M}_{(p,d),r}^{l,s}.$$

Proof It follows from Lemma 2.1 that

$$\left(\dot{I}_{r_0}^{s_0}\left(\mathcal{M}_{(p,d)}^l\right), \dot{I}_{r_1}^{s_1}\left(\mathcal{M}_{(p,d)}^l\right) \right)_{\theta,r} = \dot{I}_r^s\left(\mathcal{M}_{(p,d)}^l\right).$$

Now, employing Lemma 3.2, we are done. $\square$

Remark 3.4 It is proved in [20] that $\dot{B}\mathcal{M}_{(p,p),r}^{l,s} = \mathcal{N}_{l,p,r}^s \subset \dot{B}_{\infty,r}^{s-n/l}$. Proceeding in a similar way, one also can show that

$$\dot{B}\mathcal{M}_{(p,d),r}^{l,s} \subset \dot{B}W\mathcal{M}_{p,r}^{l,s} \subset \dot{B}_{\infty,r}^{s-n/l}. \quad (3.2)$$

In particular, it follows that

$$\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{\eta}{l}-1} \subset \dot{B}_{\infty,\infty}^{-1}$$

and

$$\dot{B}W\mathcal{M}_{p,1}^{l,\frac{\eta}{l}} \subset \dot{B}_{\infty,1}^0 \subset L^\infty. \quad (3.3)$$

Remark 3.5 Let $s - n/l < 0$ and $r > 1$, or $s - n/l \leq 0$ and $r = 1$. In view of (3.2), for $f \in \dot{B}\mathcal{M}_{(p,d),r}^{l,s}$, we have that $\sum_{j=-\infty}^{\infty} \Delta_j f$ converges in $\mathcal{S}'$ to a representative of f in $\mathcal{S}'/\mathcal{P}$ (see e.g. [24]). Then, the space $\dot{B}\mathcal{M}_{(p,d),r}^{l,s}$ can be seen as a subspace of $\mathcal{S}'$. From now on,

we say that $f \in \mathcal{S}'$ belongs to $\dot{B}\mathcal{M}_{(p,d),r}^{l,s}$ when $f = \sum_{j=-\infty}^{\infty} \Delta_j f$ in $\mathcal{S}'$, that is, if f is the canonical representative of its class in $\mathcal{S}'/\mathcal{P}$.

In our next result, we present a multiplier theorem of Mihlin–Hörmander type in homogeneous Besov–Lorentz–Morrey spaces. This can be seen as an extension of [20, Theorem 2.9] to our setting.

Lemma 3.6 *Let $1 < p \leq l \leq \infty$, $1 \leq d, r \leq \infty$, and $m, s \in \mathbb{R}$. Let P be a $C^n(\mathbb{R}^n \setminus \{0\})$ function such that $|\partial_{\xi}^{\beta} P| \leq C |\xi|^{(m-|\beta|)}$ for all multi-index β satisfying $|\beta| \leq n$. Then, there exists a constant $C > 0$ such that*

$$\|P(D)f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s-m}} \leq C \|f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s}},$$

for all $f \in \dot{B}\mathcal{M}_{(p,d),r}^{l,s}$.

Proof Note that $|\xi|^{m-|\beta|} \leq C 2^{j(m-|\beta|)}$ for all $\xi \in \tilde{D}_j$ and $j \in \mathbb{Z}$. Thus, $|\partial_{\xi}^{\beta} P| \leq C 2^{j(m-|\beta|)}$. Moreover, since $\text{supp } \widehat{\Delta_j f} \subset D_j$, we can use Lemma 2.7 in order to get

$$\|\Delta_j(P(D)f)\|_{\mathcal{M}_{(p,d)}^l} = \|P(D)(\Delta_j f)\|_{\mathcal{M}_{(p,d)}^l} \leq C 2^{jm} \|\Delta_j f\|_{\mathcal{M}_{(p,d)}^l}. \quad (3.4)$$

Multiplying (3.4) by $2^{j(s-m)}$ and after taking the l^r -norm, we obtain the result. $\square$

The following lemma is similar to Lemma 3.2, but now in the framework of Besov–Lorentz-block spaces. We omit its proof because it follows by proceeding in a parallel way to that of Lemma 3.2.

Lemma 3.7 *Let $1 < l \leq p < \infty$, $1 \leq d, r \leq \infty$, and $s \in \mathbb{R}$. Then, the space $\dot{B}\mathcal{E}_{(p,d),r}^{l,s}$ is a retract of $\dot{I}_r^s(\mathcal{E}_{(p,d)}^l)$.*

In our next lemma, we obtain a key inclusion involving Besov–Lorentz-block spaces and their interpolation spaces.

Lemma 3.8 *Let $1 < l_i \leq p_i \leq \infty$, $1 \leq d_i \leq \infty$, $1 \leq r_i < \infty$, and $s_i \in \mathbb{R}$ (for $i = 0, 1$). Assume also that $p_1 \neq p_0$ and $r, s, p, l \in \mathbb{R}$ are such that $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{l} = \frac{1-\theta}{l_0} + \frac{\theta}{l_1}$ and $s = (1-\theta)s_0 + \theta s_1$ with $\theta \in (0, 1)$. Then*

$$\dot{B}\mathcal{E}_{p,r}^{l,s} \hookrightarrow \left(\dot{B}\mathcal{E}_{(p_0,d_0),r_0}^{l_0,s_0}, \dot{B}\mathcal{E}_{(p_1,d_1),r_1}^{l_1,s_1} \right)_{\theta,r}.$$

In particular, we have that

$$\dot{B}\mathcal{E}_{(p,1),1}^{l,s} \hookrightarrow \left(\dot{B}\mathcal{E}_{(p_0,1),1}^{l_0,s_0}, \dot{B}\mathcal{E}_{(p_1,1),1}^{l_1,s_1} \right)_{\theta,1}. \quad (3.5)$$

Proof By using Lemma 2.2, we obtain that

$$i_r^s \left(\left(\mathcal{E}_{(p_0,d_0)}^{l_0}, \mathcal{E}_{(p_1,d_1)}^{l_1} \right)_{\theta,r} \right) = \left(i_{r_0}^{s_0} \left(\mathcal{E}_{(p_0,d_0)}^{l_0} \right), i_{r_1}^{s_1} \left(\mathcal{E}_{(p_1,d_1)}^{l_1} \right) \right)_{\theta,r}. \quad (3.6)$$

It follows from (3.6) and (2.12) that

$$i_r^s \left(\mathcal{E}_{p,r}^l \right) \subset \left(i_{r_0}^{s_0} \left(\mathcal{E}_{(p_0,d_0)}^{l_0} \right), i_{r_1}^{s_1} \left(\mathcal{E}_{(p_1,d_1)}^{l_1} \right) \right)_{\theta,r}.$$

Now, we can conclude the proof by employing Lemma 3.7. $\square$

Collecting the above results, we are in position to show the duality between Besov–Lorentz-block and Besov–Lorentz–Morrey spaces. It will be crucial in the proofs of some estimates (see Lemmas 3.10 and 3.13 below).

Lemma 3.9 *Let $1 < p \leq l < \infty$, $1 \leq d \leq \infty$, $1 < r \leq \infty$ and $s \in \mathbb{R}$. Then*

$$\left(\dot{B}\mathcal{E}_{(p',d'),r'}^{l',-s} \right)' = \dot{B}\mathcal{M}_{(p,d),r}^{l,s}.$$

Proof From Lemma 2.3, we get

$$\left(i_{r'}^{-s} \left(\mathcal{E}_{(p',d')}^{l'} \right) \right)' = i_r^s \left(\mathcal{M}_{(p,d)}^l \right).$$

Using Lemmas 3.2 and 3.7, we are done. $\square$

In order to estimate the bilinear operator $\mathcal{B}(\cdot, \cdot)$, we need to deal with the action of the heat semigroup $U(t)$. The following lemma contains estimates for $U(t)$ in Besov–Lorentz–Morrey and Besov–Lorentz-block spaces. The item (i) extends [20, Theorem 3.1] to our setting.

Lemma 3.10 *Let $s, \sigma \in \mathbb{R}$, $s \leq \sigma$ and $1 \leq d, r \leq \infty$.*

(i) *If $1 < p \leq l \leq \infty$ then there exists $C > 0$ (independent of t) such that*

$$\|U(t)f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,\sigma}} \leq Ct^{(s-\sigma)/2} \|f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s}}, \quad (3.7)$$

for all $f \in \dot{B}\mathcal{M}_{(p,d),r}^{l,s}$. Moreover, if $s < \sigma$ then

$$\|U(t)f\|_{\dot{B}\mathcal{M}_{(p,d),1}^{l,\sigma}} \leq Ct^{(s-\sigma)/2} \|f\|_{\dot{B}\mathcal{M}_{(p,d),\infty}^{l,s}}, \quad (3.8)$$

for all $f \in \dot{B}\mathcal{M}_{(p,d),\infty}^{l,s}$.

(ii) *If $1 < l \leq p < \infty$ then we have the estimate*

$$\|U(t)f\|_{\dot{B}\mathcal{E}_{(p,d),r}^{l,\sigma}} \leq Ct^{(s-\sigma)/2} \|f\|_{\dot{B}\mathcal{E}_{(p,d),r}^{l,s}}, \quad (3.9)$$

for all $f \in \dot{B}\mathcal{E}_{(p,d),r}^{l,s}$.

Proof of (i) Note that for each multi-index β , there exists a polynomial $p_\beta(\cdot)$ of degree $|\beta|$ such that

$$\partial_\xi^\beta (e^{-t|\xi|^2}) = t^{|\beta|/2} p_\beta(\sqrt{t}\xi) (e^{-t|\xi|^2}).$$

Therefore, it follows that

$$\left| \partial_\xi^\beta (e^{-t|\xi|^2}) \right| \leq Ct^{-m/2} |\xi|^{-m-|\beta|},$$

for some $C > 0$. So, by using Lemma 3.6 we obtain

$$\|U(t)f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s-m}} \leq Ct^{-m/2} \|f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s}}.$$

Taking $m = s - \sigma$, we get the inequality (3.7).

Next, we turn to (3.8). Since $s < \sigma$, we get from (3.7) with $r = \infty$ that

$$\|U(t)f\|_{\dot{B}\mathcal{M}_{(p,d),\infty}^{l,2\sigma-s}} \leq Ct^{s-\sigma} \|f\|_{\dot{B}\mathcal{M}_{(p,d),\infty}^{l,s}}$$

and

$$\|U(t)f\|_{\dot{B}\mathcal{M}_{(p,d),\infty}^{l,s}} \leq C \|f\|_{\dot{B}\mathcal{M}_{(p,d),\infty}^{l,s}}.$$

These last estimates together with Corollary 3.3 yield

$$U(t) : \dot{B}\mathcal{M}_{(p,d),\infty}^{l,s} \longrightarrow \left(\dot{B}\mathcal{M}_{(p,d),\infty}^{l,2\sigma-s}, \dot{B}\mathcal{M}_{(p,d),\infty}^{l,s} \right)_{\frac{1}{2},1} = \dot{B}\mathcal{M}_{(p,d),1}^{l,\sigma},$$

with $\|U(t)\|_{\dot{B}\mathcal{M}_{(p,d),\infty}^{l,s} \rightarrow \dot{B}\mathcal{M}_{(p,d),1}^{l,\sigma}} \leq Ct^{(s-\sigma)/2}$, as required. $\square$

Proof of (ii) Here, we use duality and item (i) in order to estimate

$$\begin{aligned} \|U(t)f\|_{\dot{B}\mathcal{E}_{(p,d),r}^{l,\sigma}} &= \sup_{\|h\|_{\dot{B}\mathcal{M}_{(p',d'),r'}^{l',-\sigma}} \leq 1} |\langle U(t)f, h \rangle| = \sup_{\|h\|_{\dot{B}\mathcal{M}_{(p',d'),r'}^{l',-\sigma}} \leq 1} |\langle f, U(t)h \rangle| \\ &\leq \sup_{\|h\|_{\dot{B}\mathcal{M}_{(p',d'),r'}^{l',-\sigma}} \leq 1} \|f\|_{\dot{B}\mathcal{E}_{(p,d),r}^{l,s}} \|U(t)h\|_{\dot{B}\mathcal{M}_{(p',d'),r'}^{l',-\sigma}} \\ &\leq C \sup_{\|h\|_{\dot{B}\mathcal{M}_{(p',d'),r'}^{l',-\sigma}} \leq 1} \|f\|_{\dot{B}\mathcal{E}_{(p,d),r}^{l,s}} t^{(s-\sigma)/2} \|h\|_{\dot{B}\mathcal{M}_{(p',d'),r'}^{l',-\sigma}} \\ &= Ct^{(s-\sigma)/2} \|f\|_{\dot{B}\mathcal{E}_{(p,d),r}^{l,s}}, \end{aligned}$$

which gives the desired estimate. $\square$

Remark 3.11 Note that for p, p_0, d, d_0, l, l_0 and β in the conditions of Lemma 2.6, $s \in \mathbb{R}$ and $1 \leq r \leq \infty$, we also get the estimate

$$\|\partial_x^\beta U(t)f\|_{\dot{B}\mathcal{M}_{(p,d),r}^{l,s}} \leq Ct^{-\frac{|\beta|}{2} - \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|f\|_{\dot{B}\mathcal{M}_{(p_0,d_0),r}^{l_0,s}}. \quad (3.10)$$

In fact, using Lemma 2.6, for every $j \in \mathbb{Z}$ we have that

$$\|\Delta_j(\partial_x^\beta U(t)f)\|_{\mathcal{M}_{(p,d)}^l} = \|\partial_x^\beta U(t)(\Delta_j f)\|_{\mathcal{M}_{(p,d)}^l} \leq Ct^{-\frac{|\beta|}{2} - \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|\Delta_j f\|_{\mathcal{M}_{(p_0,d_0)}^{l_0}} \quad (3.11)$$

Now, we can conclude by multiplying (3.11) by 2^{js} and after taking the l' -norm.

Similarly, for $p_1, p_2, d_1, d_2, l_1, l_2$ and β as in Lemma 2.13, $s \in \mathbb{R}$ and $1 \leq r \leq \infty$, we obtain the estimate in homogeneous Besov–Lorentz-block spaces

$$\|\partial_x^\beta U(t)f\|_{\dot{B}\mathcal{E}_{(p_2,d_2),r}^{l_2,s}} \leq Ct^{-\frac{|\beta|}{2} - \frac{n}{2} \left(\frac{1}{l_1} - \frac{1}{l_2} \right)} \|f\|_{\dot{B}\mathcal{E}_{(p_1,d_1),r}^{l_1,s}}. \quad (3.12)$$

The two following lemmas are extensions of [7, Lemmas 5.1 and 5.2] to Besov–Lorentz-block and Besov-weak-Morrey spaces, respectively. The first one can be regarded as a version in $\dot{B}\mathcal{E}_{(p,d),r}^{l,s}$ -spaces of the Yamazaki estimate that has been obtained by [31, pp. 648–650] in Lorentz spaces $L^{p,d}$.

Lemma 3.12 Let $1 < l_0 \leq p_0 < \infty$, $1 < l \leq p < \infty$, $p_0 < p$, and $s, s_0 \in \mathbb{R}$ be such that $\frac{p'}{l'} = \frac{p'_0}{l'_0}$ and $s_0 - 1 \leq s$. Then, there exists a constant $C > 0$ such that

$$\int_0^\infty s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|\nabla_x \cdot U(s) f\|_{\dot{B}\mathcal{E}_{(p,1),1}^{l,s}} ds \leq C \|f\|_{\dot{B}\mathcal{E}_{(p_0,1),1}^{l_0,s_0}},$$

for all $f \in \dot{B}\mathcal{E}_{(p_0,1),1}^{l_0,s_0}$.

Proof Consider

$$h_f(s) = s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|\nabla_x \cdot U(s) f\|_{\dot{B}\mathcal{E}_{(p,1),1}^{l,s}}.$$

We can choose $p_{01}, p_{02}, l_{01}, l_{02}$ such that $l_{0i} \leq p_{0i}$ ($i = 1, 2$), $\frac{1}{p_0} = \frac{\theta}{p_{01}} + \frac{1-\theta}{p_{02}}$, $\frac{1}{l_0} = \frac{\theta}{l_{01}} + \frac{1-\theta}{l_{02}}$ for some $\theta \in (0, 1)$, $1 < p_{01} < p_0 < p_{02} < p$, $\frac{p'_{02}}{l'_{02}} = \frac{p'}{l}$ and $\frac{n}{2} \left(\frac{1}{l_{02}} - \frac{1}{l_0} \right) + 1 > 0$. Using (3.12) in the previous remark and Lemma 3.10, we estimate

$$\begin{aligned} h_f(s) &= s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|\nabla_x \cdot U(s) f\|_{\dot{B}\mathcal{E}_{(p,1),1}^{l,s}} = s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|U(s) \nabla_x f\|_{\dot{B}\mathcal{E}_{(p,1),1}^{l,s}} \\ &= s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} \|U(s/2) U(s/2) \nabla_x f\|_{\dot{B}\mathcal{E}_{(p,1),1}^{l,s}} \\ &\leq C s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l} \right)} s^{-\frac{n}{2} \left(\frac{1}{l_{0i}} - \frac{1}{l} \right)} \|U(s/2) \nabla_x f\|_{\dot{B}\mathcal{E}_{(p_{0i},1),1}^{l_{0i},s}} \\ &\leq C s^{\frac{s-s_0-1}{2} + \frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l_{0i}} \right)} s^{\frac{s_0-1-s}{2}} \|\nabla f\|_{\dot{B}\mathcal{E}_{(p_{0i},1),1}^{l_{0i},s_0-1}} \leq C s^{\frac{n}{2} \left(\frac{1}{l_0} - \frac{1}{l_{0i}} \right) - 1} \|f\|_{\dot{B}\mathcal{E}_{(p_{0i},1),1}^{l_{0i},s_0}}. \end{aligned}$$

Next take z_1 and z_2 such that $\frac{1}{z_i} = \frac{n}{2} \left(\frac{1}{l_{0i}} - \frac{1}{l_0} \right) + 1$ and note that $0 < z_1 < 1 < z_2 < \infty$. For $i = 1, 2$, it follows that $h_f \in L^{z_i, \infty}(0, \infty)$ with the estimate $\|h_f\|_{L^{z_i, \infty}(0, \infty)} \leq C \|f\|_{\dot{B}\mathcal{E}_{(p_{0i},1),1}^{l_{0i},s_0}}$. Now, we can use interpolation in Lorentz spaces and the inclusion (3.5) in order to get

$$\|h_f\|_{L^1(0, \infty)} \leq C \|f\|_{\dot{B}\mathcal{E}_{(p_0,1),1}^{l_0,s_0}},$$

which finishes the proof. $\square$

The next lemma concerns with an estimate for the linear operator $\mathcal{T}(f) := \int_0^\infty \nabla_x \cdot U(s) f(\cdot, s) ds$. Notice that this estimate together with a product estimate morally give an estimate for (1.3).

Lemma 3.13 *Let $n \geq 3$, $1 < p_0 \leq l_0 < \infty$, $1 < p \leq l < \infty$, $p_0 < p$, and $s_0 - 1 \leq s$ be such that $\frac{p}{l} = \frac{p_0}{l_0}$ and $s - s_0 - 1 + \frac{n}{l_0} - \frac{n}{l} = 0$. Then, there is $C > 0$ such that*

$$\|\mathcal{T}(f)\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \leq C \sup_{t>0} \|f(t)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}},$$

for all $f \in L^\infty((0, \infty); \dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0})$.

Proof Using duality, we can proceed as follows

$$\begin{aligned}
 \|\mathcal{T}(f)\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} &= C \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} |\langle \mathcal{T}(f), h \rangle| \\
 &\leq C \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \int_0^\infty |\langle \nabla_x \cdot U(s) f(s), h \rangle| ds \\
 &\leq C \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \int_0^\infty |\langle f(s), \nabla_x \cdot U(s) h \rangle| ds \\
 &\leq C \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \int_0^\infty \|f(s)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}} \|\nabla_x \cdot U(s) h\|_{\dot{B}\mathcal{E}'_{(p'_0,1),1}^{l'_0,-s_0}} ds.
 \end{aligned} \tag{3.13}$$

Now, after taking the supremum of $\|f(s)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}}$ over $s > 0$ within the integral, we can use Lemma 3.12 in order to estimate the right-hand side of (3.13) as

$$\begin{aligned}
 &\leq C \sup_{t>0} \|f(t)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}} \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \int_0^\infty \|\nabla_x \cdot U(s) h\|_{\dot{B}\mathcal{E}'_{(p'_0,1),1}^{l'_0,-s_0}} ds. \\
 &= C \sup_{t>0} \|f(t)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}} \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \int_0^\infty t^{\frac{s-s_0-1}{2} + \frac{n}{2}(\frac{1}{l_0} - \frac{1}{l})} \|\nabla_x \cdot U(s) h\|_{\dot{B}\mathcal{E}'_{(p'_0,1),1}^{l'_0,-s_0}} ds \\
 &= C \sup_{t>0} \|f(t)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}} \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \int_0^\infty t^{\frac{-s_0-(-s)-1}{2} + \frac{n}{2}(\frac{1}{l'} - \frac{1}{l_0})} \|\nabla_x \cdot U(s) h\|_{\dot{B}\mathcal{E}'_{(p'_0,1),1}^{l'_0,-s_0}} ds \\
 &\leq C \sup_{t>0} \|f(t)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}} \sup_{\|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}}^{l',-s}} \|h\|_{\dot{B}\mathcal{E}'_{(p',1),1}^{l',-s}} \\
 &\leq C \sup_{t>0} \|f(t)\|_{\dot{B}W\mathcal{M}_{p_0,\infty}^{l_0,s_0}}.
 \end{aligned}$$

□

We finish this section by providing a product estimate in Besov-weak-Morrey spaces.

Lemma 3.14 *Let $n \geq 3$, $2 < p \leq l < n$, and $n/2 < l$. Then, there exist $\beta_0 \in (\frac{1}{2}, 1)$ and $\rho_0 > 0$ such that for $\beta \in [\beta_0, 1]$, $\rho \in [0, \rho_0]$, $f \in \dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}$, and $g \in \dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1+\rho}$ we have that $fg \in \dot{B}W\mathcal{M}_{\beta p,\infty}^{\beta l, \frac{n}{\beta l}-2+\rho}$ with*

$$\|fg\|_{\dot{B}W\mathcal{M}_{\beta p,\infty}^{\beta l, \frac{n}{\beta l}-2+\rho}} \leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l, \frac{n}{l}-1}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l, \frac{n}{l}-1+\rho}},$$

where $C > 0$ is a universal constant.

Proof From the decomposition (2.6), we obtain

$$\begin{aligned}\Delta_j(fg) &= \sum_{|k-j|\leq 4} \Delta_j(S_{k-2}f\Delta_k g) + \sum_{|k-j|\leq 4} \Delta_j(S_{k-2}g\Delta_k f) + \sum_{k\geq j-2} \Delta_j(\Delta_k f\tilde{\Delta}_k g) \\ &= I_1^j + I_2^j + I_3^j.\end{aligned}\quad (3.14)$$

Next, we take $\beta_0 \in (0, 1)$ and ρ_0 such that $\max\{\frac{n}{2l}, \frac{1}{2}\} < \beta_0 < 1$ and $1 - \frac{n}{2\beta_0 l} - \rho_0 = 0$. Thus, for $\beta \in [\beta_0, 1]$ and $\rho \in [0, \rho_0]$, we get $1 - \frac{n}{2\beta l} > 0$ and $1 - \frac{n}{2\beta l} - \rho > 0$. We also denote $s = \frac{n}{l} - 1$ and $s_0 = \frac{n}{\beta l} - 2 + \rho$.

The parcel I_1^j can be estimated as follows

$$\begin{aligned}\|I_1^j\|_{W\mathcal{M}_{\beta p}^{\beta l}} &\leq C \sum_{|k-j|\leq 4} \|S_{k-2}f\|_{W\mathcal{M}_{2\beta p}^{2\beta l}} \|\Delta_k g\|_{W\mathcal{M}_{2\beta p}^{2\beta l}} \\ &\leq C \sum_{|k-j|\leq 4} \left(\sum_{m\leq k-2} \|\Delta_m f\|_{W\mathcal{M}_{2\beta p}^{2\beta l}} \right) \|\Delta_k g\|_{W\mathcal{M}_{2\beta p}^{2\beta l}} \\ &\leq C \sum_{|k-j|\leq 4} \left(\sum_{m\leq k-2} 2^{mn(\frac{1}{l}-\frac{1}{2\beta l})} \|\Delta_m f\|_{W\mathcal{M}_p^l} \right) 2^{kn(\frac{1}{l}-\frac{1}{2\beta l})} \|\Delta_k g\|_{W\mathcal{M}_p^l} \\ &\leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \sum_{|k-j|\leq 4} \left(\sum_{m\leq k-2} 2^{m(1-\frac{n}{2\beta l})} \right) 2^{k(1-\frac{n}{2\beta l}-\rho)},\end{aligned}$$

and so

$$\|I_1^j\|_{W\mathcal{M}_{\beta p}^{\beta l}} \leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} 2^{j(-s_0)}. \quad (3.15)$$

In order to estimate I_2^j , we proceed similarly to obtain

$$\begin{aligned}\|I_2^j\|_{W\mathcal{M}_{\beta p}^{\beta l}} &\leq C \sum_{|k-j|\leq 4} \|S_{k-2}g\|_{W\mathcal{M}_{2\beta p}^{2\beta l}} \|\Delta_k f\|_{W\mathcal{M}_{2\beta p}^{2\beta l}} \\ &\leq C \sum_{|k-j|\leq 4} \left(\sum_{m\leq k-2} 2^{mn(\frac{1}{l}-\frac{1}{2\beta l})} \|\Delta_m g\|_{W\mathcal{M}_p^l} \right) 2^{kn(\frac{1}{l}-\frac{1}{2\beta l})} \|\Delta_k f\|_{W\mathcal{M}_p^l} \\ &\leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \sum_{|k-j|\leq 4} \left(\sum_{m\leq k-2} 2^{m(1-\frac{n}{2\beta l}-\rho)} \right) 2^{k(1-\frac{n}{2\beta l})},\end{aligned}$$

which gives

$$\|I_2^j\|_{W\mathcal{M}_{\beta p}^{\beta l}} \leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} 2^{j(-s_0)}. \quad (3.16)$$

Now, we turn to I_3^j . Let $\tilde{l}$ and $\tilde{p}$ be such that $l < \tilde{l} < n$, $p < \tilde{p} \leq \tilde{l}$ and $\frac{\tilde{l}}{\tilde{p}} = \frac{l}{p}$. It follows that $\frac{1}{\tilde{l}} = \frac{1}{n} + \epsilon$ with $\epsilon > 0$. Thus, by using Lemma 2.8 we get

$$\begin{aligned}\|\Delta_k f\|_{W\mathcal{M}_{\tilde{p}}^{\tilde{l}}} &\leq C 2^{kn(\frac{1}{l}-\frac{1}{\tilde{l}})} \|\Delta_k f\|_{W\mathcal{M}_p^l} \leq C 2^{k(1-\frac{n}{\tilde{l}})} \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \\ &= C 2^{k(-n\epsilon)} \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}}.\end{aligned}\quad (3.17)$$

For $\|\Delta_k g\|_{W\mathcal{M}_{\tilde{p}}^{\tilde{l}}}$, similarly we have

$$\begin{aligned}\|\Delta_k g\|_{W\mathcal{M}_{\tilde{p}}^{\tilde{l}}} &\leq C 2^{kn\left(\frac{1}{\tilde{l}} - \frac{1}{\tilde{p}}\right)} \|\Delta_k g\|_{W\mathcal{M}_{\tilde{p}}^{l'}} \leq C 2^{k\left(1 - \frac{n}{\tilde{l}} - \rho\right)} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \\ &= C 2^{k(-n\epsilon - \rho)} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}}.\end{aligned}\quad (3.18)$$

It follows from (3.17) and (3.18) that

$$\begin{aligned}\|I_3^j\|_{W\mathcal{M}_{\tilde{p}/2}^{\tilde{l}/2}} &\leq C \sum_{k \geq j-2} \|\Delta_k f \tilde{\Delta}_k g\|_{W\mathcal{M}_{\tilde{p}/2}^{\tilde{l}/2}} \leq C \sum_{k \geq j-2} \|\Delta_k f\|_{W\mathcal{M}_{\tilde{p}}^{\tilde{l}}} \|\tilde{\Delta}_k g\|_{W\mathcal{M}_{\tilde{p}}^{\tilde{l}}} \\ &\leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \sum_{k \geq j-2} 2^{k(-2n\epsilon - \rho)} \\ &\leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} 2^{j(-2n\epsilon - \rho)}.\end{aligned}\quad (3.19)$$

Since $\frac{\tilde{l}}{2} < \beta l$, using again Lemma 2.8 and (3.19), we arrive at

$$\begin{aligned}\|I_3^j\|_{W\mathcal{M}_{\beta p}^{\beta l}} &\leq C 2^{jn\left(\frac{2}{\tilde{l}} - \frac{1}{\beta l}\right)} \|I_3^j\|_{W\mathcal{M}_{\tilde{p}/2}^{\tilde{l}/2}} \\ &\leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} 2^{jn\left(\frac{2}{\tilde{l}} - \frac{1}{\beta l}\right)} 2^{j(-2n\epsilon - \rho)} \\ &\leq C \|f\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|g\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} 2^{j(-s_0)}.\end{aligned}\quad (3.20)$$

Computing the norm $\|\cdot\|_{W\mathcal{M}_{\beta p}^{\beta l}}$ in (3.14) and considering the estimates (3.15), (3.16) and (3.20), we get the result. $\square$

4 Proofs

4.1 Proof of Theorem 1.1

Let $0 < T \leq \infty$ and $t \in (0, T)$. The bilinear term $\mathcal{B}(u, v)$ can be written as

$$\mathcal{B}(u, v)(t) = - \int_0^t \nabla_x \cdot U(t-s) \mathbb{P}f(\cdot, s) ds = \mathcal{T}(f_t),$$

where $f_t(x, s)$ is defined by

$$\begin{aligned}f_t(\cdot, s) &= \mathbb{P}(u \otimes v)(\cdot, t-s), \text{ for a.e. } s \in (0, t), \\ f_t(\cdot, s) &= 0, \text{ for a.e. } s \in (t, \infty).\end{aligned}$$

Considering $\beta < 1$ as in Lemma 3.14, we have that $1 < \beta p < p$. From Lemma 3.13, we get

$$\|\mathcal{T}(f_t)\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l, \frac{n}{p}-1}} \leq C \sup_{t>0} \|f_t\|_{\dot{B}W\mathcal{M}_{\beta p,\infty}^{\beta l, \frac{n}{\beta p}-2}}.$$

Using Lemmas 3.6 and 3.14 with $\rho = 0$, we can estimate

$$\begin{aligned} \sup_{0 < s < T} \|f_t\|_{\dot{B}W\mathcal{M}_{\beta p, \infty}^{\beta l, \frac{\eta}{\beta l}-2}} &\leq C \sup_{0 < s < t < T} \|(u \otimes v)(\cdot, t-s)\|_{\dot{B}W\mathcal{M}_{\beta p, \infty}^{\beta l, \frac{\eta}{\beta l}-2}} \\ &\leq C \sup_{0 < s < t < T} \|u(\cdot, t-s)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \|v(\cdot, t-s)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \\ &\leq C \sup_{0 < s < T} \|u(\cdot, s)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \sup_{0 < s < T} \|v(\cdot, s)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}}. \end{aligned}$$

Thus, we can conclude that

$$\sup_{0 < t < T} \|\mathcal{B}(u, v)(t)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \leq K \sup_{0 < t < T} \|u(t)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \sup_{0 < t < T} \|v(t)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}}.$$

□

4.2 Proof of Corollary 1.2

With the bilinear estimate in hands, the uniqueness follows by adapting an argument due to Meyer [27].

Let u and v be two mild solutions of (1.1) in $C([0, T]; \dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1})$ with the same initial data $u_0 \in \dot{B}\tilde{W}\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}$. First, we prove that there exists $0 < T_1 < T$ such that $u(\cdot, t) = v(\cdot, t)$ in $\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}$ for all $t \in [0, T_1]$. Denoting $w = u - v$, $w_1 = U(t)u_0 - u$ and $w_2 = U(t)u_0 - v$, we have that

$$\begin{aligned} u \otimes u - v \otimes v &= w \otimes u + v \otimes w \\ &= w \otimes U(t)u_0 + U(t)u_0 \otimes w - w \otimes w_1 - w_2 \otimes w. \end{aligned}$$

Thus, we can estimate the difference $w(t)$ in $\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}$ as follows

$$\begin{aligned} \|w(t)\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} &= \left\| \int_0^t \nabla_x \cdot U(t-s) \mathbb{P}(u \otimes u - v \otimes v) ds \right\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \\ &\leq \left\| \int_0^t \nabla_x \cdot U(t-s) \mathbb{P}(w \otimes w_1 + w_2 \otimes w) ds \right\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \\ &\quad + \left\| \int_0^t \nabla_x \cdot U(t-s) \mathbb{P}(w \otimes U(s)u_0 + U(s)u_0 \otimes w) ds \right\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \\ &:= J_1(t) + J_2(t). \end{aligned}$$

Theorem 1.1 can be used in order to estimate

$$J_1(t) \leq K \sup_{0 < t < T_1} \|w\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \left(\sup_{0 < t < T_1} \|w_1\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} + \sup_{0 < t < T_1} \|w_2\|_{\dot{B}W\mathcal{M}_{p, \infty}^{l, \frac{\eta}{l}-1}} \right).$$

Next, we turn to $J_2(t)$. Set $s = \frac{n}{l} - 1$ and $s_0 = \frac{n}{\beta l} - 2 + \rho$ with $\beta < 1$ and $\rho > 0$ as in Lemma 3.14. First, we use Lemma 3.10 and estimate (3.10) in Remark 3.11 in order to obtain

$$\begin{aligned}
 J_2(t) &\leq C \int_0^t \|\nabla_x \cdot U(t-s) \mathbb{P}(w \otimes U(s)u_0 + U(s)u_0 \otimes w)\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}} ds \\
 &\leq C \int_0^t \|U((t-s)/2) U((t-s)/2) \mathbb{P}(w \otimes U(s)u_0 + U(s)u_0 \otimes w)\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}}} ds \\
 &\leq C \int_0^t (t-s)^{\left(\frac{n}{2l} - \frac{n}{2\beta l}\right)} \|U((t-s)/2) \mathbb{P}(w \otimes U(s)u_0 + U(s)u_0 \otimes w)\|_{\dot{B}W\mathcal{M}_{\beta p,\infty}^{l,\frac{n}{l}}} ds \\
 &\leq C \int_0^t (t-s)^{\left(\frac{s_0}{2} - \frac{n}{2\beta l}\right)} \|w \otimes U(s)u_0 + U(s)u_0 \otimes w\|_{\dot{B}W\mathcal{M}_{\beta p,\infty}^{l,s_0}} ds. \quad (4.1)
 \end{aligned}$$

Employing the product estimate in Lemma 3.14, we can estimate the right-hand side of (4.1) as

$$\begin{aligned}
 \text{R.H.S of (4.1)} &\leq C \int_0^t (t-s)^{\left(\frac{s_0}{2} - \frac{n}{2\beta l}\right)} \|w\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \|U(s)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} ds \\
 &\leq C \int_0^t (t-s)^{\left(-1+\frac{\rho}{2}\right)} \|w\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} s^{-\frac{\rho}{2}} s^{\frac{\rho}{2}} \|U(s)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} ds \\
 &\leq C \sup_{0 < t < T_1} \|w\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \sup_{0 < t < T_1} t^{\frac{\rho}{2}} \|U(t)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \\
 &\quad \times \int_0^1 (1-s)^{\left(-1+\frac{\rho}{2}\right)} s^{-\frac{\rho}{2}} ds. \quad (4.2)
 \end{aligned}$$

Choosing $\rho > 0$ small enough, we obtain that the integral in (4.2) is convergent and then

$$J_2(t) \leq C \sup_{0 < t < T_1} \|w\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \sup_{0 < t < T_1} t^{\frac{\rho}{2}} \|U(t)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}}.$$

Therefore

$$\sup_{0 < t < T_1} \|w(t)\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}} \leq CZ(T_1) \sup_{0 < t < T_1} \|w\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}},$$

where

$$Z(T_1) = \sup_{0 < t < T_1} \|w_1\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}} + \sup_{0 < t < T_1} \|w_2\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}} + \sup_{0 < t < T_1} t^{\frac{\rho}{2}} \|U(t)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}}.$$

Note that

$$\lim_{t \rightarrow 0^+} \|w_1\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}} = \lim_{t \rightarrow 0^+} \|w_2\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,\frac{n}{l}-1}} = 0,$$

because $U(t)u_0, u, v \rightarrow u_0$ as $t \rightarrow 0^+$. Now, we prove that

$$\limsup_{t \rightarrow 0^+} t^{\frac{\rho}{2}} \|U(t)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} = 0.$$

To do so, we define $u_{0k} = U\left(\frac{1}{k}\right)u_0$ for all $k \in \mathbb{N}$. In view of Lemma 3.10, it follows that $u_{0k} \in \dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}$. Moreover, $u_{0k} \rightarrow u_0$ in $\dot{B}W\mathcal{M}_{p,\infty}^{l,s}$ as $k \rightarrow \infty$. Then, we can estimate

$$\begin{aligned} \limsup_{t \rightarrow 0^+} t^{\frac{\rho}{2}} \|U(t)u_0\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} &\leq \limsup_{t \rightarrow 0^+} t^{\frac{\rho}{2}} \|U(t)(u_0 - u_{0k})\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \\ &\quad + \limsup_{t \rightarrow 0^+} t^{\frac{\rho}{2}} \|U(t)u_{0k}\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \\ &\leq C \|u_0 - u_{0k}\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} + C \|u_{0k}\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s+\rho}} \limsup_{t \rightarrow 0^+} t^{\frac{\rho}{2}} \\ &\leq C \|u_0 - u_{0k}\|_{\dot{B}W\mathcal{M}_{p,\infty}^{l,s}} \rightarrow 0, \text{ as } t \rightarrow \infty. \end{aligned}$$

Consequently, it is possible to choose $T_1 > 0$ small enough such that $CZ(T_1) < 1$ and then $w(t) = 0$ for all $t \in [0, T_1]$.

The remainder of the proof is to show that in fact $T_1 \in (0, \infty]$ can be arbitrary. Define

$$T_* = \sup \left\{ \tilde{T} ; 0 < \tilde{T} < T, u(t) = v(t) \text{ in } \dot{B}W\mathcal{M}_{p,\infty}^{l,s} \text{ for all } t \in [0, \tilde{T}] \right\}.$$

If $T_* = T$ we finish. If not, we have that $u(t) = v(t)$ for $t \in [0, T_*)$ which implies that $u(T_*) = v(T_*)$ because of the time-continuity of u and v . It follows from the first part of the proof that there exists $\sigma > 0$ small enough such that $u(t) = v(t)$ for $t \in [T_*, T_* + \sigma)$. Therefore, $u(t) = v(t)$ for $t \in [0, T_* + \sigma)$ and we obtain a contradiction with the definition of T_* . $\square$

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