Wave-breaking phenomena and persistence properties for the nonlocal rotation-Camassa–Holm equation

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Received: 2 April 2019 / Accepted: 7 June 2019 / Published online: 14 June 2019

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Abstract

Consideration in the present paper is a mathematical model proposed as an equation of long-crested shallow water waves propagating in one direction with the effect of Earth's rotation. This model equation is analogous to the Camassa–Holm approximation of the two-dimensional incompressible and irrotational Euler equations, and its solution corresponding to physically relevant initial perturbations is more accurate on a much longer timescale. The effects of the Coriolis force caused by the Earth rotation and nonlocal higher nonlinearities on the blow-up criteria and wave-breaking phenomena in the periodic setting are investigated. Moreover, working with moderate weight functions that are commonly used in time–frequency analysis, some persistence results to the equation are illustrated.

Keywords Rotation-Camassa–Holm equation · Blow-up · Wave breaking · Persistence

Mathematics Subject Classification 35B30 · 35G25 · 35Q53

1 Introduction

It is known that near the Equator the ocean waves propagate mainly along the Equator as two-dimensional waves, due to the joint action of the wind and the Coriolis force [11, 12]. We study here a model equation with the Coriolis effect derived from the incompressible

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and irrotational two-dimensional shallow water in the equatorial region [22]. This model equation called the rotation-Camassa–Holm(R-CH) equation has a cubic and even quartic nonlinearities and a formal Hamiltonian structure. More precisely, the motion of the fluid is described by the scalar equation in the following form:

$$\begin{aligned} u_t - \beta \mu u_{xxt} + cu_x + 3\alpha \varepsilon u u_x - \beta_0 \mu u_{xxx} + \omega_1 \varepsilon^2 u^2 u_x + \omega_2 \varepsilon^3 u^3 u_x \\ = \alpha \beta \varepsilon \mu (2u_x u_{xx} + uu_{xxx}), \end{aligned} \quad (1.1)$$

where u represents the horizontal velocity field at height z_0 , the constants appearing in the equation are defined by $c = \sqrt{1 + \Omega^2} - \Omega$ with the parameter Ω which is the constant rotational frequency due to the Coriolis effect, $\alpha = \frac{c^2}{1+c^2}$, $\beta_0 = \frac{c(c^4+6c^2-1)}{6(c^2+1)^2}$, $\beta = \frac{3c^4+8c^2-1}{6(c^2+1)^2}$ and $\omega_1 = \frac{-3c(c^2-1)(c^2-2)}{2(1+c^2)^3}$ and $\omega_2 = \frac{(c^2-2)(c^2-1)^2(8c^2-1)}{2(1+c^2)^5}$ satisfying $c \rightarrow 1$, $\beta \rightarrow \frac{5}{12}$, $\beta_0 \rightarrow \frac{1}{4}$, $\omega_1, \omega_2 \rightarrow 0$ and $\alpha \rightarrow \frac{1}{2}$ when $\Omega \rightarrow 0$. Using the rescaling, it is required that $0 \leq z_0 \leq 1$, where

$$z_0^2 = \frac{1}{2} - \frac{2}{3} \frac{1}{(c^2+1)} + \frac{4}{3} \frac{1}{(c^2+1)^2}.$$

Since it is also natural to require that the constant $\beta > 0$, it must be the case

$$0 \leq \Omega \leq \sqrt{\frac{1}{6}(1+2\sqrt{19})} \approx 1.273,$$

and

$$\frac{1}{\sqrt{2}} \leq z_0 < \sqrt{\frac{61-2\sqrt{19}}{54}} \approx 0.984.$$

Equation (1.1) is derived in [22] by showing that after a double asymptotic expansion with respect to ε and μ , the free surface $\eta = \eta(\tau, \xi)$ under the field variable (η, ξ) defined in 2D Euler's dynamics is governed by the following equation:

$$\begin{aligned} 2(\Omega + c)\eta_\tau + 3c^2\eta\eta_\xi + \frac{c^2}{3}\mu\eta_{\xi\xi\xi} + A_1\varepsilon\eta^2\eta_\xi + A_2\varepsilon^2\eta^3\eta_\xi + A_0\varepsilon^3\eta^4\eta_\xi \\ = \varepsilon\mu(A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi}) + o(\varepsilon^4, \mu^2), \end{aligned}$$

where the constants A_0, A_1, A_2, A_3 and A_4 satisfy that

$$\begin{aligned} A_0 &= \frac{c^2(c^2-2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{12(c^2+1)^6}, \quad A_1 = \frac{3c^2(c^2-2)}{(c^2+1)^2}, \\ A_2 &= -\frac{c^2(2-c^2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4}, \quad A_3 = \frac{-c^2(9c^4+16c^2-2)}{3(c^2+1)^2}, \quad \text{and} \\ A_4 &= \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^2}. \end{aligned}$$

The free surface η with respect to the horizontal component of the velocity u at $z = z_0$ under the CH regime $\varepsilon = O(\sqrt{\mu})$ is also given by

$$\eta = \frac{1}{c}u + \gamma_1\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4 + \gamma_4\varepsilon\mu u_{\xi\xi} + o(\varepsilon^4, \mu^2),$$

where the constants in the expression are given by

$$\gamma_1 = \frac{2 - c^2}{2c^2(c^2 + 1)}, \quad \gamma_2 = \frac{(c^2 - 1)(c^2 - 2)(2c^2 + 1)}{2c^3(c^2 + 1)^3},$$

$$\gamma_3 = -\frac{(c^2 - 1)^2(c^2 - 2)(21c^4 + 16c^2 + 4)}{8c^4(c^2 + 1)^5}, \quad \text{and} \quad \gamma_4 = \frac{-(3c^4 + 6c^2 - 5)}{12c(c^2 + 1)^2}.$$

Denote $m = (1 - \beta\mu\partial_x^2)u$. One can rewrite the above equation in terms of the evolution of the momentum density m , namely

$$\partial_t m + \alpha\varepsilon(um_x + 2mu_x) + cu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x = 0. \quad (1.2)$$

In the case that the Coriolis effect vanishes ($\Omega = 0$), the coefficients in the higher-power nonlinearities correspond to $\omega_1 = 0$ and $\omega_2 = 0$. In this case, using the scaling transformation $u(t, x) \rightarrow \alpha\varepsilon u(\sqrt{\beta\mu}t, \sqrt{\beta\mu}x)$ and then the Galilean transformation $u(t, x) \rightarrow u(t, x - \frac{3}{4}t) + \frac{1}{4}$, the R-CH equation (1.2) is then reduced to the classical CH equation:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xx},$$

which was derived [5,18] (see also the rigorous derivation in [13]) as a model describing the unidirectional propagation of shallow water waves. It was later shown to model the propagation of axially symmetric waves in hyperelastic rods [15] and also derived by applying tri-Hamiltonian duality to the bi-Hamiltonian structure of the Korteweg–de Vries (KdV) equation [27]. In contrast to the KdV equation, the CH equation has many remarkable distinctive properties: The CH equation is completely integrable for a large class of initial data, for which it can be solved by the inverse scattering method [5,8,14,17], it can describe wave-breaking phenomenon, the solution remains bounded while its slope becomes infinite in finite time [5,9], it has peaked solutions, which are nonanalytic solitary waves that are global weak solutions, and interact cleanly like solitons [1,5], and it has a variety of interesting geometric formulations [10,26]. On the other hand, if we take formally $\beta = 0$ and $\omega_2 = 0$ in (1.2), then we get the following integrable Gardner equation [20]:

$$u_t + cu_x + 3\alpha\varepsilon uu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x = 0.$$

Note that the R-CH equation (1.2) has the following three conserved quantities:

$$I(u) = \int u dx, \quad E(u) = \frac{1}{2} \int u^2 + \beta\mu u_x^2 dx$$

and

$$F(u) = \frac{1}{2} \int \left(cu^2 + \alpha\varepsilon u^3 + \beta_0\mu u_x^2 + \frac{\omega_1\varepsilon^2}{6} u^4 + \frac{\omega_2\varepsilon^3}{10} u^5 + \alpha\beta\varepsilon\mu u u_x^2 \right) dx.$$

Define that

$$B_1 = \partial_x(1 - \beta\mu\partial_x^2),$$

and

$$B_2 = \partial_x \left(\alpha\varepsilon m + \frac{c}{2} \right) + \left(\alpha\varepsilon m + \frac{c}{2} \right) \partial_x - \beta_0\mu\partial_x^3 + \frac{2}{3}\omega_1\varepsilon^2\partial_x(u\partial_x^{-1}(u\partial_x))$$

$$+ \frac{5}{8}\omega_2\varepsilon^3\partial_x \left(u^{\frac{3}{2}}\partial_x^{-1} \left(u^{\frac{3}{2}}\partial_x \right) \right).$$

A simple calculation then reveals that the R-CH equation (1.2) can be written as

$$m_t = -B_1 \frac{\delta F}{\delta m} = -B_2 \frac{\delta E}{\delta m},$$

where B_1 and B_2 are two skew-symmetric differential operators.

It is observed that the consideration of the Coriolis effect gives rise to a higher-power nonlinear term into the R-CH model, which has interesting implications for the fluid motion, particularly in the relation to the wave-breaking phenomena and the permanent waves. It is noted that in the process of the derivation of the asymptotic model equation, the rotation parameter Ω is treated as a fixed $O(1)$ constant relevant to the nonlinearity parameter ε and the shallowness parameter μ . As is pointed out in [6,22], the motivation for such a fixed Ω in the asymptotic expansion is to retain higher-degree nonlinearities in the asymptotic equation so that in the analytical aspect is studied with an emphasis of investigating whether rotation effects can defer or enhance the formation of singularity. For the other small parameter Ω due to the Coriolis force in the shallow water wave propagation regime, it is referred to recent works in [21,24] for the geophysical applications.

One of our goals in the present paper is to investigate from this model in the periodic setting how the higher-power nonlinearities affect the wave-breaking phenomena and what conditions can ensure the occurrence of the wave-breaking phenomena or permanent waves. The dynamics of the blow-up quantity along the characteristics in the R-CH equation actually involves the interaction among three parts: a local nonlinearity, a nonlocal term and a term stemming from the weak Coriolis forcing. It is known that the nonlocal (smoothing) effect can help maintain the regularity while waves propagate and hence prevent them from blowing up, even when dispersion is weak or absent. See, for example, the Benjamin–Bona–Mahoney (BBM) equation [2]. As the local nonlinearity becomes stronger and dominates over the dispersion and nonlocal effects, singularities may occur in the sense of wave breaking. Examples can be found in the Whitham equation [9] and the Camassa–Holm (CH) equation [5,13]. It is also found that the Coriolis effect will spread out waves and make them decay in time, delaying the onset of wave breaking. Understanding the wave-breaking mechanism such as when a singularity can form and what the nature of it is not only presents fundamental importance from mathematical point of view but also is of great physical interest, since it would help provide a key mechanism for localizing energy in conservative systems by forming one or several small-scale spots. For instance, in fluid dynamics, the possible phenomenon of finite-time breakdown for the incompressible Euler equations signifies the onset of turbulence in high Reynolds number flows.

The R-CH equation with a nonlocal structure can be reformulated in a weak form of nonlinear nonlocal transport type. From the transport theory, the blow-up criteria assert that singularities are caused by the focusing of characteristics, which involve the information on the gradient u_x . The dynamics of the wave-breaking quantity along the characteristics is established by the Riccati-type differential inequality. Inspired by the idea in study of the classical CH equation to show how local structure of the solution affects the blow-ups [4], the argument in our case is then approached by a refined analysis on evolution of the solution u and its gradient u_x . It is noted that the method in [4] relies heavily on the fact that the convolution terms are quadratic and positively definite. As for the R-CH equation, the convolution contains cubic even quartic nonlinearities which do not have a lower bound in terms of the local terms. Hence, the higher-power nonlinearities in the equation make it difficult to obtain a purely local condition on the initial data that can generate finite-time wave breaking. In our case, the blow-up can be deduced by the interplay between u and u_x . More precisely, this motivates us to carry out a refined analysis of the characteristic dynamics

of $M = \gamma u - u_x$ and $N = \gamma u + u_x$. The estimates of M and N can be closed in the form of

$$M'(t) \geq -CMN + K_1, \quad N'(t) \leq CMN + K_2,$$

where the nonlocal terms K_1, K_2 can be bounded in terms of certain order conservation laws. From these Riccati-type differential inequalities, the monotonicity of M and N can be established, and hence the finite-time wave-breaking follows.

Another purpose of the present paper is to find a large class of weight functions φ such that

$$\sup_{t \in [0, T)} (\|\varphi u(t)\|_p + \|\varphi u_x(t)\|_p) < \infty$$

where $\|\cdot\|_p$ denotes the usual L_p norm. We obtain a persistence result on solutions u in the weighted L^p spaces $L_{p,\varphi} : L_p(\mathbb{R}, \varphi^p dx)$. As a consequence and an application, we determine the spatial asymptotic behavior of certain solutions to the R-CH equation. We will work with moderate weight functions which appear with regularity in the theory of time–frequency analysis [19].

The remainder of the paper is organized as follows. In Sect. 2, some preliminary estimates and results for the periodic R-CH equation are recalled and presented. Section 3 is devoted to proof the breakdown mechanisms to the R-CH equation. In Sect. 4, some fundamentals concerning moderate weight functions and the functional analytic setting for the R-CH equation are presented and persistence results for the R-CH equation to its supersymmetric extension are given.

2 Preliminaries and dynamics along the characteristics

In this section, we will recall some useful properties of solutions to (1.2) in the periodic setting. Using the scaling transformation $u(t, x) \rightarrow \alpha \varepsilon u(\sqrt{\beta \mu} t, \sqrt{\beta \mu} x)$, equation (1.2) can be written as the following R-CH equation:

$$\begin{cases} u_t - u_{xxt} + cu_x + 3uu_x - \frac{\beta_0}{\beta} u_{xxx} + \frac{\omega_1}{\alpha^2} u^2 u_x + \frac{\omega_2}{\alpha^3} u^3 u_x = 2u_x u_{xx} + uu_{xxx}, \\ u(t, x) = u(t, x + 1), \quad \forall t > 0, x \in \mathbb{R}. \end{cases} \quad (2.1)$$

Then, we convert the above equation into the following weak form:

$$\begin{cases} u_t + uu_x + \frac{\beta_0}{\beta} u_x + G * \partial_x \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right) = 0, \\ u(t, x) = u(t, x + 1), \quad \forall t > 0, x \in \mathbb{R}, \end{cases} \quad (2.2)$$

where $G(x) = \frac{\cosh(x - [x] - \frac{1}{2})}{2 \sinh(\frac{1}{2})}$ (here $[x]$ represents the largest integer part of x), which is the fundamental solution of $(1 - \partial_x^2)^{-1}$ on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, that is for any $x \in \mathbb{S}$,

$$\begin{aligned} (1 - \partial_x^2)^{-1} f(x) &= (G * f)(x) = \int_0^1 \frac{\cosh((x - y) - [x - y] - \frac{1}{2})}{2 \sinh(\frac{1}{2})} f(y) dy \\ &= \int_0^x \frac{\cosh(x - y - \frac{1}{2})}{2 \sinh(\frac{1}{2})} f(y) dy + \int_x^1 \frac{\cosh(x - y + \frac{1}{2})}{2 \sinh(\frac{1}{2})} f(y) dy. \end{aligned} \quad (2.3)$$

The following conservation law is useful to establish the result of wave breaking.

Lemma 2.1 *Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, and T be the maximal existence time of the solution u in the R-CH equation (2.2) with initial value $u(0) = u_0$. Then, for all $t \in [0, T)$, we have*

$$E(u) = \frac{1}{2} \int_{\mathbb{S}} (u^2 + u_x^2) dx = \frac{1}{2} \int_{\mathbb{S}} (u_0^2 + u_{0,x}^2) dx \equiv E_0.$$

Proof To see this, it is observed that

$$\int_{\mathbb{S}} u^2 u_{xxx} dx = - \int_{\mathbb{S}} 2uu_x u_{xx} dx$$

and

$$\frac{\beta_0}{\beta} \int_{\mathbb{S}} uu_{xxx} dx = \frac{\beta_0}{\beta} \int_{\mathbb{S}} u du_{xx} = -\frac{\beta_0}{2\beta} \int_{\mathbb{S}} (u_x^2)_x dx = 0.$$

Multiplying the first equation of (2.1) by u and integrating by parts then yields the required conservation law, which implies that for any $t \in (0, T)$, $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$. $\square$

One of important ingredients in our development is a local existence theory for the initial-value problem for the periodic R-CH equation (2.2), which may be similarly obtained as in [13, 16] (up to a slight modification) and the proof is omitted.

Lemma 2.2 *Let $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$. Then, there exist a positive time $T_{u_0}^* > 0$ and a unique solution $u \in C([0, T_{u_0}^*]; H^s(\mathbb{S})) \cap C^1([0, T_{u_0}^*]; H^{s-1}(\mathbb{S}))$ to the periodic R-CH equation (2.2) with $u(0) = u_0$. Moreover, the life span $T_{u_0}^*$ does not depend on the regularity index s of the initial data u_0 and the solution u depends continuously on the initial value u_0 .*

Now, we return to the original R-CH equation (1.1), and let

$$\|u\|_{X_{\mu}^{s+1}}^2 = \|u\|_{H^s}^2 + \mu\beta \|\partial_x u\|_{H^s}^2.$$

For some $\mu_0 > 0$ and $M > 0$, we define the Camassa–Holm regime

$$P_{\mu_0, M} := (\varepsilon, \mu) : 0 < \mu \leq \mu_0, 0 < \varepsilon \leq M\sqrt{\mu}.$$

By following exact approach in [13], we have the local well-posedness result without proof.

Corollary 2.3 *Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ and $\mu_0 > 0$. Then, there exist $T > 0$ and a unique family of solutions $(u_{\varepsilon, \mu})_{(\varepsilon, \mu) \in P_{\mu_0, M}}$ in $C([0, \frac{T}{\varepsilon}]; X^{s+1}(\mathbb{S})) \cap C^1([0, \frac{T}{\varepsilon}]; X^s(\mathbb{S}))$ to the following periodic initial-value problem:*

$$\begin{cases} \partial_t u - \beta\mu \partial_t u_{xx} + cu_x + 3\alpha\varepsilon uu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ \quad = \alpha\beta\varepsilon\mu(2u_x u_{xx} + uu_{xxx}), & t > 0, x \in \mathbb{S}, \\ u(0, x) = u_0(x), & x \in \mathbb{S}. \end{cases} \quad (2.4)$$

In order to study the dynamics along the characteristics of the R-CH equation (1.1), we introduce the associated Lagrangian scales as

$$\begin{cases} \frac{dq(t, x)}{dt} = u(t, q(t, x)), \\ q(0, x) = x, \end{cases} \quad x \in \mathbb{S}, \quad t \in [0, T), \quad (2.5)$$

where $u \in C^1([0, T), H^{s-1}(\mathbb{S}))$ is the solution to Eq. (2.2).

Lemma 2.4 [22] Suppose $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$, and let $T > 0$ be the maximal existence time of the strong solution u to the corresponding the periodic R-CH equation (2.2). Then, equation (2.5) has a unique solution $q \in C^1([0, T]; \mathbb{S})$ such that $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{S}$ for all $(t, x) \in [0, T) \times \mathbb{S}$ with

$$q_x(t, x) = \exp\left(2 \int_0^t u_x(s, q(s, x)) ds\right) > 0.$$

The following Sobolev estimate is crucial to obtain the optimal blow-up result.

Lemma 2.5 [28] For every $f \in H^1(\mathbb{S})$, we have

$$\max_{x \in [0, 1]} f^2(x) \leq \kappa_1 \int_{\mathbb{S}} (f^2 + \epsilon^2 f_x^2) dx,$$

where

$$\kappa_1 = \frac{\cosh\left(\frac{1}{2\epsilon}\right)}{2\epsilon \sinh\left(\frac{1}{2\epsilon}\right)},$$

and κ_1 is the optimal constant which is obtained by the associated Green function defined by

$$G_1 = \frac{\cosh\left(\frac{x}{\epsilon} - \frac{[x]}{\epsilon} - \frac{1}{2\epsilon}\right)}{2\epsilon \sinh\left(\frac{1}{2\epsilon}\right)}.$$

In particular, when the parameter $\epsilon = 1$, the constant $\kappa_1 = \frac{e+1}{2(e-1)}$ is sharp.

3 Wave-breaking criterion and wave-breaking data

Using the energy estimates, it enables us to obtain the following wave-breaking criterion to the R-CH equation. The wave-breaking phenomena could be illustrated by choosing certain initial data.

Theorem 3.1 Let $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$, and $T_{u_0}^* > 0$ be the maximal existence time of the solution u to (2.2) with initial data u_0 . Then, the corresponding solution u blows up in finite time if and only if

$$\lim_{t \rightarrow T_{u_0}^*} \inf_{x \in \mathbb{S}} u_x(t, x) = -\infty, \quad (3.1)$$

that is, the solution breaks down in finite time $T_{u_0}^*$.

Proof Applying Lemma 2.2 and a simple density argument, we only need to show that the theorem holds for some $s \geq 3$. Here, we assume $s = 3$ to prove the above theorem. Multiplying the equation in (2.1) by $-u_{xx}$ and integrating by parts again, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_x\|_{H^1}^2 &= -\frac{3}{2} \int_{\mathbb{S}} u_x(u_x^2 + u_{xx}^2) dx + \int_{\mathbb{S}} \left(\frac{\omega_1}{\alpha^2} u^2 u_x + \frac{\omega_2}{\alpha^3} u^3 u_x \right) u_{xx} dx \\ &\leq -\frac{3}{2} \int_{\mathbb{S}} u_x(u_x^2 + u_{xx}^2) dx + \int_{\mathbb{S}} \left| \frac{\omega_1}{2\alpha^2} u^2 + \frac{\omega_2}{2\alpha^3} u^3 \right| (u_x^2 + u_{xx}^2) dx. \end{aligned} \quad (3.2)$$

Assume that $T_{u_0}^* < +\infty$ and there exists $m_0 > 0$ such that

$$u_x(t, x) \geq -m_0, \quad \forall (t, x) \in [0, T_{u_0}^*) \times \mathbb{S}. \quad (3.3)$$

It then follows from Lemmas 2.1 and 2.5 that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{S}} (u^2 + 2u_x^2 + u_{xx}^2) dx \\ & \leq \left(\frac{3}{2}M + \frac{|\omega_1|}{2\alpha^2} \|u\|_{L^\infty}^2 + \frac{|\omega_2|}{2|\alpha|^3} \|u\|_{L^\infty}^3 \right) \int_{\mathbb{S}} (u_x^2 + u_{xx}^2) dx \\ & \leq C(1 + m_0 + \|u_0\|_{H^1}^3) \int_{\mathbb{S}} (u_x^2 + u_{xx}^2) dx, \end{aligned} \quad (3.4)$$

where we used the Sobolev embedding theorem $H^s(\mathbb{S}) \hookrightarrow L^\infty(\mathbb{S})$ (with $s > \frac{1}{2}$) in the last inequality. Applying Gronwall's inequality to (3.4) yields for every $t \in [0, T_{u_0}^*)$

$$\begin{aligned} \|u(t)\|_{H^2}^2 & \leq 2\|u_0\|_{H^2(\mathbb{S})}^2 e^{Ct(1+m_0+\|u_0\|_{H^1}^3)} \\ & \leq 2\|u_0\|_{H^2(\mathbb{S})}^2 e^{CT_{u_0}^*(1+m_0+\|u_0\|_{H^1}^3)}. \end{aligned} \quad (3.5)$$

Differentiating the first equation in (2.1) with respect to x , and multiplying the result equation by $-u_{xxx}$, then integrating by parts, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}} (u_{xx}^2 + u_{xxx}^2) dx \\ & = -\frac{15}{2} \int_{\mathbb{S}} u_x u_{xx}^2 dx - \frac{5}{2} \int_{\mathbb{S}} (u_x u_{xxx}^2) dx + \int_{\mathbb{S}} \left(\frac{\omega_1}{\alpha^2} u^2 u_x + \frac{\omega_2}{\alpha^3} u^3 u_x \right)_x u_{xxx} dx \\ & \leq C(1 + m_0 + \|u\|_{L^\infty}^3) \int_{\mathbb{S}} (u_{xx}^2 + u_{xxx}^2) dx + C(\|u\|_{L^\infty}^2 + \|u\|_{L^\infty}^4) \|u_x\|_{L^4}^4, \end{aligned}$$

where use has been made of the assumption in (3.3). It now follows from the Sobolev embedding theorem and the interpolation inequality $\|f\|_{L^4(\mathbb{S})} \leq C\|f\|_{L^2(\mathbb{S})}^{\frac{3}{4}}\|f_x\|_{L^4(\mathbb{S})}^{\frac{1}{4}}$ that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{S}} (u_{xx}^2 + u_{xxx}^2) dx \leq C(1 + m_0 + \|u_0\|_{H^1}^3) \int_{\mathbb{S}} (u_{xx}^2 + u_{xxx}^2) dx \\ & \quad + C\|u_0\|_{H^1}^5 (1 + \|u_0\|_{H^1}^2) \|u_{xx}\|_{L^2} \\ & \leq C(1 + m_0 + \|u_0\|_{H^1}^{14}) \int_{\mathbb{S}} (u_{xx}^2 + u_{xxx}^2) dx. \end{aligned} \quad (3.6)$$

Hence, applying Gronwall's inequality to (3.6) implies that for every $t \in [0, T_{u_0}^*)$

$$\int_{\mathbb{S}} (u_{xx}^2 + u_{xxx}^2) dx \leq e^{C(1+m_0+\|u_0\|_{H^1}^{14})T_{u_0}^*} \int_{\mathbb{S}} (u_{0xx}^2 + u_{0xxx}^2) dx.$$

Combining the above estimate together with (3.5) then yields

$$\|u(t)\|_{H^3(\mathbb{S})}^2 \leq 3\|u_0\|_{H^3(\mathbb{S})}^2 e^{C(1+m_0+\|u_0\|_{H^1}^{14})T_{u_0}^*}.$$

This contradicts the assumption on the maximal existence time $T_{u_0}^* < +\infty$. Conversely, the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$, ($s > \frac{1}{2}$) implies that if (3.1) holds, the corresponding solution u blows up in finite time. This completes the proof of Theorem 3.1. $\square$

It is also found that the solution of (2.2) cannot break up in any time because of the conserved energy.

Theorem 3.2 Suppose $u_0 \in H^s$ with $s > \frac{3}{2}$. Let $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ of (2.2) with the initial data u_0 . Then, for any $t \in [0, T)$

$$\sup_{x \in \mathbb{S}} u_x(t, x) \leq \|\partial_x u_0\|_{L^\infty} + (2K)^{\frac{1}{2}},$$

where the constant K is given by $K = K_0 + \frac{e^{\frac{1}{2}}}{2} K_1 + \frac{3e^{\frac{3}{2}}}{2} K_1$ with

$$\begin{aligned} E_0 &= \frac{1}{2} \int_{\mathbb{S}} (u_0^2 + u_{0,x}^2) dx, \\ K_0 &= \left(\frac{e+1}{2(e-1)} \right) E_0 + \left| c - \frac{\beta_0}{\beta} \right| \left(\frac{e+1}{2(e-1)} \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \\ &\quad + \frac{|\omega_1|}{3\alpha^2} \left(\frac{e+1}{2(e-1)} \right)^{\frac{3}{2}} E_0^{\frac{3}{2}} + \frac{|\omega_2|}{4\alpha^3} \left(\frac{e+1}{2(e-1)} \right)^2 E_0^2, \quad \text{and} \\ K_1 &= \left| c - \frac{\beta_0}{\beta} \right| \left(\frac{e+1}{2(e-1)} \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} + E_0 \\ &\quad + \frac{|\omega_1|}{3\alpha^2} \left(\frac{e+1}{2(e-1)} \right)^{\frac{1}{2}} E_0^{\frac{3}{2}} + \frac{\omega_2}{4\alpha^3} \left(\frac{e+1}{2(e-1)} \right) E_0^2. \end{aligned}$$

Proof Applying the translation $u(t, x) \rightarrow u(t, x - \frac{\beta_0}{\beta} t)$ to Eq. (2.2) yields the equation in the form

$$u_t + uu_x + G_x * \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right) = 0.$$

Taking the derivative ∂_x to the above equation, we have

$$\begin{aligned} u_{xt} + uu_{xx} &= -\frac{1}{2} u_x^2 + u^2 + \left(c - \frac{\beta_0}{\beta} \right) u + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \\ &\quad - G * \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right). \end{aligned} \quad (3.7)$$

Let $M_0(t, x) = u_x(t, q(t, x))$ with the characteristic curve $q(t, x)$ defined in (2.5). Then, we obtain the following equation:

$$\begin{aligned} \frac{\partial M_0(t, q)}{\partial t} &= -\frac{1}{2} M_0^2(t, q(t, x)) + u^2 + \left(c - \frac{\beta_0}{\beta} \right) u + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4(q) \\ &\quad - G * \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right)(q), \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} (G * f)(x) &= \frac{e^{x-\frac{1}{2}}}{2} \int_0^x e^{-y} f(y) dy + \frac{e^{\frac{1}{2}-x}}{2} \int_0^x e^y f(y) dy \\ &\quad + \frac{e^{x+\frac{1}{2}}}{2} \int_x^1 e^{-y} f(y) dy + \frac{e^{-\frac{1}{2}-x}}{2} \int_x^1 e^y f(y) dy. \end{aligned} \quad (3.9)$$

Let

$$H(u(t, q(t, x))) = u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4(q) - G * \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)(q).$$

Then, we have

$$\frac{\partial M_0(t, q)}{\partial t} = -\frac{1}{2}M_0^2(t, q(t, x)) + H(u(t, q(t, x))).$$

On the other hand, a direct computation reveals that

$$\begin{aligned} & \left| u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right| \\ & \leq \|u\|_{L^\infty}^2 + \left| c - \frac{\beta_0}{\beta} \right| \|u\|_{L^\infty} + \frac{|\omega_1|}{3\alpha^2} \|u\|_{L^\infty}^3 + \frac{|\omega_2|}{4\alpha^3} \|u\|_{L^\infty}^4 \\ & \leq \left(\frac{e+1}{2(e-1)} \right) (2E_0) + \left| c - \frac{\beta_0}{\beta} \right| \left(\frac{e+1}{2(e-1)} \right)^{\frac{1}{2}} (2E_0)^{\frac{1}{2}} + \frac{|\omega_1|}{3\alpha^2} \left(\frac{e+1}{2(e-1)} \right)^{\frac{3}{2}} (2E_0)^{\frac{3}{2}} \\ & \quad + \frac{|\omega_2|}{4\alpha^3} \left(\frac{e+1}{2(e-1)} \right)^2 (2E_0)^2 \\ & = \frac{(e+1)E_0}{e-1} + \left| c - \frac{\beta_0}{\beta} \right| \left(\frac{(e+1)E_0}{e-1} \right)^{\frac{1}{2}} + \frac{|\omega_1|}{3\alpha^2} \left(\frac{(e+1)E_0}{e-1} \right)^{\frac{3}{2}} + \frac{|\omega_2|}{4\alpha^3} \left(\frac{(e+1)E_0}{e-1} \right)^2 \\ & = K_0 \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} & \left| \frac{e^{x-\frac{1}{2}}}{2} \int_0^x e^{-y} \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) dy \right| \\ & \leq \frac{e^{\frac{1}{2}}}{2} \left(\left| c - \frac{\beta_0}{\beta} \right| \left(\frac{(e+1)E_0}{e-1} \right)^{\frac{1}{2}} + 2E_0 + \frac{|\omega_1|}{3\alpha^2} \left(\frac{(e+1)E_0}{e-1} \right)^{\frac{3}{2}} \right. \\ & \quad \left. + \frac{|\omega_2|}{4|\alpha|^3} \left(\frac{(e+1)E_0}{e-1} \right)^2 \right) \\ & = \frac{e^{\frac{1}{2}}}{2} K_1. \end{aligned} \quad (3.11)$$

Similarly,

$$\begin{aligned} & \left| \frac{e^{\frac{1}{2}-x}}{2} \int_0^x e^y \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) dy \right| \leq \frac{e^{\frac{3}{2}}}{2} K_1, \\ & \left| \frac{e^{x+\frac{1}{2}}}{2} \int_x^1 e^{-y} \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) dy \right| \leq \frac{e^{\frac{3}{2}}}{2} K_1, \quad \text{and} \\ & \left| \frac{e^{-x-\frac{1}{2}}}{2} \int_x^1 e^y \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) dy \right| \leq \frac{e^{\frac{3}{2}}}{2} K_1. \end{aligned} \quad (3.12)$$

Putting Eqs. (3.10), (3.11) and (3.12) into (3.8), it follows that

$$H(u(t, q(t, x))) \leq K,$$

where the constant K is defined by

$$K = K_0 + \frac{e^{\frac{1}{2}}}{2} K_1 + \frac{3e^{\frac{3}{2}}}{2} K_1.$$

Hence, there appears the relation

$$-\frac{1}{2} M_0^2(t, q(t, x)) - K \leq \frac{\partial M_0(t, q)}{\partial t} \leq -\frac{1}{2} M_0^2(t, q(t, x)) + K. \quad (3.13)$$

For any given $x \in \mathbb{S}$, we now consider a function $P(t)$ defined by

$$P(t) = M_0(t) - \|u_{0,x}\|_{L^\infty} - (2K)^{\frac{1}{2}}.$$

Note that $P(t)$ is a C^1 -differential function in $[0, T)$ and satisfies

$$P(0) = M_0(0) - \|u_{0,x}\|_{L^\infty} - (2K)^{\frac{1}{2}} \leq 0.$$

We now claim that $P(t) \leq 0$, for any $t \in [0, T)$, by using the argument of continuous deduction. Indeed, since $M_0(t)$ is continuous on $[0, T)$, if the above inequality does not hold, we can find a $t_0 \in (0, T)$ such that $P(t_0) > 0$.

Denote now that

$$t_1 = \max\{t < t_0 : P(t) = 0\}.$$

Then, $P(t_1) = 0$ and $P'(t_1) \geq 0$. Consequently, $M'_0(t_1) \geq 0$. On the other hand, in view of (3.13), we have

$$\begin{aligned} M'_0(t_1) &\leq -\frac{1}{2} M_0^2(t_1) + K \\ &\leq -\frac{1}{2} \left(\|u_{0,x}\|_{L^\infty} + (2K)^{\frac{1}{2}} \right)^2 + K < 0, \end{aligned}$$

which yields a contradiction. Consequently, it is concluded that for any $t \in [0, T)$

$$\sup_{x \in \mathbb{S}} u_x(t, x) = \sup_{x \in \mathbb{S}} M_0(t, q(t, x)) \leq \|\partial_x u_0\|_{L^\infty} + (2K)^{\frac{1}{2}}.$$

The proof of Theorem 3.2 is thus complete. $\square$

Attention is now given to searching initial data to illustrate the wave-breaking phenomena. It is known that a local-in-space type of blow-up mechanism in [4] (see also [7]) considerably simplifies the classical results and characterizes how the local structure of the solution both in periodic and in non-periodic cases can affect the formation of singularities. It is our purpose to study the issue that under what kind of condition the particular periodic initial data could generate the wave-breaking solution. We first present some notations and useful results obtained in [4] before stating our wave-breaking result.

For any real constant γ , define $I(\gamma) \geq -\infty$ by

$$I(\gamma) = \inf \left\{ \int_0^1 (G + \gamma G_x) * (2u^2 + u_x^2) dx \mid u \in H^1(\mathbb{S}), u(0) = u(1) = 1 \right\},$$

and the quantity $\gamma^* \in [0, +\infty)$ by

$$\gamma^* = \inf \{ \gamma \in (0, +\infty) \mid \gamma^2 + I(\gamma) - 2 \geq 0 \}$$

which the usual convention that $\gamma^* = +\infty$ if the infimum is taken on the empty set.

It is known in [4] that $I(\gamma)$ is even with respect to the variable $\gamma \in \mathbb{R}$ and $I(\gamma) > -\infty$ if and only if

$$-\frac{e+1}{e-1} \leq \gamma \leq \frac{e+1}{e-1}.$$

Particularly, if $|\gamma| < \frac{e+1}{e-1}$, then $I(\gamma)$ is in fact a minimum with only one minimizer $u \in H^1(\mathbb{S})$ with $u(0) = u(1) = 1$. In addition, γ^* was computed numerically as the zero point of the function $\gamma \rightarrow \gamma^2 + I(\gamma) - 2$ by

$$\gamma^* = 0.513 \dots \quad (3.14)$$

The following convolution estimate is crucial for the blow-up analysis.

Lemma 3.3 [4] *For any $\gamma \in \mathbb{R}$ and all $u \in H^1(\mathbb{S})$, the following convolution estimate holds*

$$(G \pm \gamma G_x) * (2u^2 + u_x^2)(x) \geq I(\gamma)u^2(x), \quad \forall x \in \mathbb{S},$$

and $I(\gamma)$ is the best possible constant.

We are now in a position to state the wave-breaking result to the R-CH equation (2.2).

Theorem 3.4 *Suppose that $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$. Let $T_0 > 0$ be the maximal time of existence of the corresponding solution u to (2.2) with the initial data u_0 . Assume there is a point $x_0 \in \mathbb{S}$ such that*

$$u_{0,x}(x_0) < -\left|\gamma^* \left(u_0(x_0) - \frac{k}{2}\right)\right| - \sqrt{2}C_0,$$

where the constant $k = \frac{\beta_0}{\beta} - c$, γ^* is defined in (3.14) and $C_0 > 0$ is defined by

$$C_0^2 = \left(\frac{1}{2} + (|\gamma^*| + 1)(e^{\frac{1}{2}} + e^{\frac{3}{2}})\right) \left(\frac{|\omega_1|}{3\alpha^2} \left(\frac{(e+1)E_0}{e-1}\right)^{\frac{3}{2}} + \frac{|\omega_2|}{2|\alpha|^3} \left(\frac{(e+1)E_0}{e-1}\right)^2\right)$$

with

$$E_0 = \frac{1}{2} \int_{\mathbb{S}} (u_0^2 + u_x^2) dx.$$

Then, the solution $u(t, x)$ of the R-CH equation (2.2) breaks down such that

$$\lim_{t \rightarrow T_0^-} \inf_{x \in \mathbb{S}} \{u_x(t, x)\} = -\infty$$

at the time

$$T_0 \leq \frac{2}{\sqrt{u_{0,x}^2(x_0) - \left(\gamma^* \left(u_0(x_0) - \frac{k}{2}\right)\right)^2 - \sqrt{2}C_0}} < \infty.$$

Remark 3.5 In the case of the rotation frequency $\Omega = 0$, or the wave speed $c = 1$, the corresponding constant C_0 in the theorem must be zero, because the parameters ω_1 and ω_2 vanish. The assumption on the wave breaking is then back to the case of the classical CH equation.

Proof In view of (3.7), we have

$$\begin{aligned} u_{xt} + uu_{xx} &= -\frac{1}{2}u_x^2 + u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - G * \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\ &= -\frac{1}{2}u_x^2 + \left(u - \frac{k}{2}\right)^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - G * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned}$$

Along with the trajectory of $q(t, x)$, then using (3.7) and the above equation gives

$$\begin{aligned} \frac{\partial u(t, q)}{\partial t} &= -G_x * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right), \\ \frac{\partial u_x(t, q)}{\partial t} &= -\frac{1}{2}u_x^2 + \left(u - \frac{k}{2}\right)^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - G * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned}$$

Denote now two functions $M(t)$ and $N(t)$ at $(t, q(t, x_0))$ by

$$\begin{aligned} M(t) &= \gamma^* \left(u(t, q(t, x_0)) - \frac{k}{2}\right) - u_x(t, q(t, x_0)) \quad \text{and} \\ N(t) &= \gamma^* \left(u(t, q(t, x_0)) - \frac{k}{2}\right) + u_x(t, q(t, x_0)). \end{aligned}$$

It then follows from expressions $M(t, q(t, x_0))$ and $N(t, q(t, x_0))$ that

$$\begin{aligned} \frac{dM}{dt} &= \frac{1}{2}u_x^2 - \left(u - \frac{k}{2}\right)^2 - \frac{\omega_1}{3\alpha^2}u^3 - \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - \gamma^* G_x * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\ &\quad + G * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right), \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \frac{dN}{dt} &= -\frac{1}{2}u_x^2 + \left(u - \frac{k}{2}\right)^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ &\quad - \gamma^* G_x * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\ &\quad - G * \left(\left(u - \frac{k}{2}\right)^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned}$$

Then, using Lemma 3.3, we have

$$\begin{aligned}
 \frac{dM}{dt} &= \frac{1}{2} \left(u_x^2 - 2 \left(u - \frac{k}{2} \right)^2 - \frac{2\omega_1}{3\alpha^2} u^3 - \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &\quad - \frac{1}{2} \gamma^* G_x * \left(2 \left(u - \frac{k}{2} \right)^2 + u_x^2 + \frac{2\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &\quad + \frac{1}{2} G * \left(2 \left(u - \frac{k}{2} \right)^2 + u_x^2 + \frac{2\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &\geq \frac{1}{2} \left(u_x^2 - 2 \left(u - \frac{k}{2} \right)^2 + I(\gamma^*) \left(u - \frac{k}{2} \right)^2 - \frac{2\omega_1}{3\alpha^2} u^3 - \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &\quad + \frac{1}{2} (G - \gamma^* G_x) * \left(\frac{2\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{2\alpha^3} u^4 \right). \tag{3.16}
 \end{aligned}$$

Because $I(\gamma^*) = 2 - (\gamma^*)^2$, it must be the case

$$\begin{aligned}
 \frac{dM}{dt} &\geq \frac{1}{2} \left(u_x^2 - (\gamma^*)^2 \left(u - \frac{k}{2} \right)^2 - \frac{2\omega_1}{3\alpha^2} u^3 - \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &\quad + \frac{1}{2} (G - \gamma^* G_x) * \left(\frac{2\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &= -\frac{1}{2} \left(\gamma^* \left(u - \frac{k}{2} \right) - u_x \right) \left(\gamma^* \left(u - \frac{k}{2} \right) + u_x \right) + \frac{1}{2} \left(-\frac{2\omega_1}{3\alpha^2} u^3 - \frac{\omega_2}{2\alpha^3} u^4 \right) \\
 &\quad + \frac{1}{2} (G - \gamma^* G_x) * \left(\frac{2\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{2\alpha^3} u^4 \right). \tag{3.17}
 \end{aligned}$$

In view of (3.9) and applying Lemma 2.5, there appears the relation

$$\begin{aligned}
 &\left| \frac{1}{2} \left(-\frac{2\omega_1}{3\alpha^2} u^3 - \frac{\omega_2}{2\alpha^3} u^4 \right) + \frac{1}{2} (G - \gamma^* G_x) * \left(\frac{2\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{2\alpha^3} u^4 \right) \right| \\
 &\leq \left(\frac{1}{2} + (|\gamma^*| + 1) \left(e^{\frac{1}{2}} + e^{\frac{3}{2}} \right) \right) \left(\frac{|\omega_1|}{3\alpha^2} \|u\|_{L^\infty}^3 + \frac{|\omega_2|}{2|\alpha|^3} \|u\|_{L^\infty}^4 \right) \\
 &= \left(\frac{1}{2} + (|\gamma^*| + 1) \left(e^{\frac{1}{2}} + e^{\frac{3}{2}} \right) \right) \left(\frac{|\omega_1|}{3\alpha^2} \left(\frac{(e+1)E_0}{e-1} \right)^{\frac{3}{2}} + \frac{|\omega_2|}{2|\alpha|^3} \left(\frac{(e+1)E_0}{e-1} \right)^2 \right) \\
 &= C_0^2. \tag{3.18}
 \end{aligned}$$

It is then inferred from (3.17) that

$$\frac{dM}{dt} \geq -\frac{1}{2} MN - C_0^2. \tag{3.19}$$

With the same approach, we have

$$\frac{dN}{dt} \leq \frac{1}{2} MN + C_0^2. \tag{3.20}$$

By the assumptions on $u_0(x_0)$, it is easy to see that

$$M(0) = \gamma^* \left(u_0(x_0) - \frac{k}{2} \right) - u_{0,x}(x_0) > 0, \quad N(0) = \gamma^* \left(u_0(x_0) - \frac{k}{2} \right) + u_{0,x}(x_0) < 0,$$

and

$$\frac{1}{2}M(0)N(0) + C_0^2 < 0.$$

By the continuity of $M(t)$ and $N(t)$, it then ensures that

$$\frac{dM}{dt} > 0, \quad \frac{dN}{dt} < 0, \quad \forall t \in [0, T).$$

This in turn implies that

$$M(t) > M(0) > 0, \quad N(t) < N(0) < 0, \quad \forall t \in [0, T).$$

Let $h(t) = \sqrt{-M(t)N(t)}$. It then follows that

$$\begin{aligned} \frac{dh}{dt} &= \frac{-M'(t)N(t) - M(t)N'(t)}{2h} \geq \frac{(-\frac{1}{2}MN - C_0^2)(-N) - M(\frac{1}{2}MN + C_0^2)}{2h} \\ &= \frac{M - N}{2h} \left(-\frac{1}{2}MN - C_0^2 \right). \end{aligned}$$

Using the estimate $\frac{M - N}{2h} \geq 1$ and the fact that $h + \sqrt{2}C_0 > h - \sqrt{2}C_0 > 0$, we obtain the following differential inequalities:

$$\frac{dh}{dt} \geq -\frac{1}{2}MN - C_0^2 = \frac{1}{2}(h - \sqrt{2}C_0)(h + \sqrt{2}C_0) \geq \frac{1}{2}(h - \sqrt{2}C_0)^2.$$

Hence, solving this inequality implies the wave breaks

$$\lim_{t \rightarrow T_0^-} h(t, q(t, x_0)) = \infty$$

at the wave-breaking time

$$T_0 \leq \frac{2}{\sqrt{u_{0,x}^2(x_0) - (\gamma^*(u_0(x_0) - \frac{k}{2}))^2 - \sqrt{2}C_0}} < \infty.$$

Using the fact that $-u_x(t, q(t, x_0)) = \frac{1}{2}(M - N) \geq h(t, q(t, x_0))$, this in turn implies there exists $T_0 < \infty$, such that

$$\lim_{t \rightarrow T_0^-} \inf_{x \in \mathbb{S}} \{u_x(t, x)\} = -\infty.$$

The proof of Theorem 3.4 is thus complete. $\square$

Remark 3.6 Returning to the original scale, our assumption for the wave breaking becomes

$$\sqrt{\beta\mu}u_{0,x}(\sqrt{\beta\mu}x_0) + \left| \gamma^* \left(u_0(\sqrt{\beta\mu}x_0) - \frac{k}{2} \right) \right| < -\frac{\sqrt{2}}{\alpha\varepsilon}C_1.$$

Note that when Ω increases, α and β decrease. It is then observed that with effect of the Earth rotation, a worse initial data $u_0(x_0)$ are required to make the breaking wave happen. On the other hand, in the original scale, we have the blow-up time T_1 bounded by

$$T_1 \leq \frac{2}{\alpha\varepsilon \left(\sqrt{\beta\mu}u_{0,x}(\sqrt{\beta\mu}x_0) - (\gamma^*(u_0(\sqrt{\beta\mu}x_0) - \frac{k}{2}))^2 - \frac{\sqrt{2}}{\alpha\varepsilon}C_1 \right)},$$

where

$$C_1^2 = \left(\frac{1}{2} + (|\gamma^*| + 1) \left(e^{\frac{1}{2}} + e^{\frac{3}{2}} \right) \right) \left(\frac{|\omega_1|}{3\alpha^2} \left(\frac{(e+1)\alpha^2 \varepsilon^2 E_0}{e-1} \right)^{\frac{3}{2}} + \frac{|\omega_2|}{2|\alpha|^3} \left(\frac{(e+1)\alpha^2 \varepsilon^2 E_0}{e-1} \right)^2 \right)$$

with

$$E(u_0) = \frac{1}{\alpha^2 \varepsilon^2} E_0(\alpha \varepsilon u_0(\sqrt{\beta \mu} x_0)),$$

which thus implies that a longer time is required for wave to break down when effect of the Earth rotation is considered.

The blow-up rate of the solution of (2.2) can be determined in the following result.

Corollary 3.7 *Let u be the solution of the periodic R-CH equation (2.2) with initial data $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$. Assume there is a point $x_0 \in \mathbb{S}$ such that*

$$u_{0,x}(x_0) < - \left| \gamma^* \left(u_0(x_0) - \frac{k}{2} \right) \right| - \sqrt{2} C_0.$$

Let $T_0 > 0$ be the blow-up time of existence of the corresponding solution $u(t, x)$ with the initial data u_0 . Then, the blow-up rate to the corresponding solution at T_0 is determined by

$$\lim_{t \rightarrow T_0^-} \left(\inf_{x \in \mathbb{S}} \{u_x(t, x)\} (T_0 - t) \right) = -2.$$

with the constant K defined in Theorem 3.2.

To prove Corollary 3.7, we need the following lemma due to Constantin and Escher [9]

Lemma 3.8 [9] *Let $T > 0$ and $v \in C^1([0, T]; H^2(\mathbb{S}))$. Then, for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{S}$ with*

$$I(t) := \inf_{x \in \mathbb{S}} \{v_x(t, x)\} = v_x(t, \xi(t)).$$

The function $I(t)$ is absolutely continuous on $(0, T)$ with

$$\frac{dI(t)}{dt} = v_{tx}(t, \xi(t)), \quad \text{a. e. on } (0, T).$$

Proof of Corollary 3.7 By Lemma 3.8, given $t \in [0, T_0)$, let $x_0(t) \in \mathbb{S}$ be such that

$$u_x(t, x_0(t)) = \inf_{x \in \mathbb{S}} \{u_x(t, x)\}$$

which implies $u_{xx}(t, x_0(t)) = 0$, a.e. on $(0, T_0)$. Let $M_1(t) = u_x(t, x_0(t))$. In view of (3.13), there appears the relation

$$-\frac{1}{2} M_1^2(t) - K \leq \frac{dM_1(t)}{dt} \leq -\frac{1}{2} M_1^2(t) + K, \quad \text{a. e. } t \in (0, T_0).$$

Choose $0 < \varepsilon < \frac{1}{2}$. Since $M_1(t) \rightarrow -\infty$ as $t \rightarrow T_0^-$, we can find $t_0 \in (0, T_0)$ such that

$$M_1(t_0) < -\sqrt{2K + \frac{K}{\varepsilon}} < -\sqrt{\frac{K}{\varepsilon}}, \quad t \in [t_0, T_0).$$

This in turn implies that

$$\frac{1}{2} - \varepsilon < \frac{d}{dt} \left(\frac{1}{M_1(t)} \right) < \frac{1}{2} + \varepsilon, \quad a. e. t \in [t_0, T_0).$$

Integrating the above relation on (t, T_0) with $t \in [t_0, T_0)$ and noticing that $M_1(t) \rightarrow -\infty$ as $t \rightarrow T_0^-$, we obtain

$$\left(\frac{1}{2} - \varepsilon \right) (T_0 - t) < -\frac{1}{M_1(t)} < \left(\frac{1}{2} + \varepsilon \right) (T_0 - t).$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, the above inequality implies the following equation:

$$\lim_{t \rightarrow T_0^-} (\inf_{x \in \mathbb{S}} \{u_x(t, x)\} (T_0 - t)) = -2.$$

This completes the proof of Corollary 3.7. $\square$

4 Persistence properties for the R-CH equation

Attention in this section is now turned to asymptotic persistence for (2.1). The Sobolev space $H^s = H^s(\mathbb{R})$, $s \geq 0$ on the line is the Hilbert space

$$H^s = \{u \in L_2(\mathbb{R}); \int_{\mathbb{R}} |\widehat{u}(\xi)|^2 (1 + \xi^2)^s d\xi < \infty\}$$

equipped with the norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}} |\widehat{u}(\xi)|^2 (1 + \xi^2)^s d\xi.$$

The pseudodifferential operator $(1 - \partial_x^2)^{-1}$ has the symbol $\frac{1}{1 + \xi^2}$ and hence defines an isomorphism $H^s \rightarrow H^{s+2}$ for any $s \geq 0$. Moreover, $(1 - \partial_x^2)^{-1} f(x) = (p * f)(x)$ with the kernel $p(x) = \frac{1}{2} e^{-|x|}$. One can rewrite Eq. (2.1) into the following weak form:

$$u_t + uu_x + \frac{\beta_0}{\beta} u_x + p * \partial_x \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right) = 0. \quad (4.1)$$

Some standard computations show that

$$(\partial_x p)(x) = -\frac{1}{2} e^{-x} 1_{x \geq 0} + \frac{1}{2} e^x 1_{x < 0} = -\frac{1}{2} \operatorname{sign}(x) e^{-|x|}$$

in the weak sense and that

$$\partial_x^2 p = p - \delta$$

with the Dirac distribution δ .

Now let us first recall some standard definitions. In general, a function $v : \mathbb{R}^n \rightarrow \mathbb{R}$ is called sub-multiplicative if

$$v(x + y) \leq v(x)v(y), \quad \forall x, y \in \mathbb{R}^n.$$

Let v be a sub-multiplicative function. A positive function φ on $\mathbb{R}^n$ is called v -moderate if there exists a constant $c > 0$ such that

$$\varphi(x + y) \leq c v(x) \varphi(y), \quad \forall x, y \in \mathbb{R}^n.$$

We say that φ is moderate if it is v -moderate for some sub-multiplicative function v . Let us recall the most standard examples of such weights. Let

$$\varphi(x) = \varphi_{a,b,c,d}(x) = e^{|x|^b} (1 + |x|)^c \log(e + |x|)^d.$$

We have:

- (1) for $a, c, d > 0$ and $0 \leq b \leq 1$ such weight is sub-multiplicative.
- (2) If $a, c, d \in \mathbb{R}$ and $0 \leq b \leq 1$, then φ is moderate. More precisely, $\varphi_{a,b,c,d}$ is $\varphi_{\alpha,\beta,\gamma,\delta}$ -moderate for $|a| \leq \alpha$, $b \leq \beta$, $|c| \leq \gamma$ and $|d| \leq \delta$.

The interest of imposing the sub-multiplicativity condition on a weight function is also made clearly by the following proposition.

Proposition 4.1 *Let $v : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and $C_0 > 0$. Then, the following conditions are equivalent: (1) $\forall x, y : v(x + y) \leq C_0 v(x)v(y)$. (2) For all $1 \leq p, q, r \leq \infty$ and for any measurable functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{C}$ the weighted Young inequalities hold:*

$$\|(f_1 * f_2)v\|_r \leq C_0 \|f_1 v\|_p \|f_2 v\|_q, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Definition 4.2 We say that $\varphi : \mathbb{R} \rightarrow (0, \infty)$ is an admissible weight function for the R-CH equation if it is locally absolutely continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that for some $A > 0$ we have $|\phi'(x)| \leq A|\varphi(x)|$ and φ is v -moderate with a sub-multiplicative function v satisfying $\inf v > 0$ and

$$\int_{\mathbb{R}} v(x) e^{-|x|} dx < \infty.$$

We recall that a locally absolutely continuous function is a.e. differentiable in $\mathbb{R}$. Moreover, its a.e. derivative belongs to L^1_{loc} and agrees with its distributional derivative.

Theorem 4.3 *Let $s > \frac{3}{2}$ and $2 \leq p \leq \infty$. Let $u \in C([0, T], H^s)$ be the strong solution to (4.1) starting from u_0 so that $\varphi u_0, \varphi u_{0,x} \in L^p(\mathbb{R})$ for an admissible weight function φ of the R-CH equation (4.1). Let*

$$M^* = \sup_{t \in [0, T]} (\|u(t)\|_{\infty} + \|u_x(t)\|_{\infty}) < \infty.$$

Then, there is a constant $C > 0$ depending only on the weight φ such that

$$\|\varphi u(t)\|_p + \|\varphi u_x(t)\|_p \leq e^{CM^*t} (\|\varphi u_0\|_p + \|\varphi u_{0,x}\|_p)$$

for all $t \in [0, T]$.

Remark 4.4 The basic example of application of Theorem 4.2 is obtained by choosing the standard weights $\varphi = \varphi_{a,b,c,d}$ as in with the following conditions

$$a \geq 0, \quad c, d \in \mathbb{R}, \quad 0 \leq b \leq 1, \quad ab < 1.$$

(For $a < 0$, one has $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$: the conclusion of the theorem remains true but it is not interesting in the case.) The restriction $ab < 1$ guarantees the validity of condition for a multiplicative function $v(x) \geq 1$. The limit case $a = b = 1$ is not covered by Theorem 4.2. The result holds true, however, for the weight $\varphi = \varphi_{1,1,c,d}$ with $c < 0, d \in \mathbb{R}$ and $\frac{1}{|c|} < p \leq \infty$, or more generally when $(1 + |\cdot|)^c \log(e + |\cdot|)^d \in \mathbb{R}$.

Remark 4.5 Let us consider a few particular cases:

- (1) Let $\varphi = \varphi_{0,0,c,0}$, $c > 0$ and $p = \infty$ in Theorem 4.2. For this choice, Theorem 4.2 says that the algebraic decay of the initial datum u_0 ,

$$|u_0(x)| + |u_{0,x}(x)| \leq C(1 + |x|)^{-c},$$

for all $x \in \mathbb{R}$, is preserved by the solution u with u_0 on $[0, T)$, i.e.,

$$|u(x, t)| + |u_x(x, t)| \leq C'(1 + |x|)^{-c},$$

for all $(x, t) \in \mathbb{R} \times [0, T)$, where $C, C' > 0$ are constant.

- (2) Let $\varphi = \varphi_{a,1,0,0}1_{x \geq 0} + 1_{x < 0}$, $0 \leq a < 1$. By our definitions, φ is an admissible weight function for the R-CH. Let furthermore $p = \infty$ in Theorem 4.2. Then, one deduces that the R-CH preserves the point-wise decay $O(e^{-ax})$ of its solutions as $x \rightarrow \infty$, for any $t > 0$. Analogously, one concludes that, for $x \rightarrow -\infty$, the decay $O(e^{ax})$ is preserved during the evolution. Hence, our Theorem 4.2 encompasses also Theorem 1.2 of [23].

Now, let us prove Theorem 4.2 using the ideas of [25].

Proof Applying the translation $u(t, x) \rightarrow u(t, x - \frac{\beta_0}{\beta}t)$ to Eq. (4.1) yields the equation in the form

$$u_t + uu_x + p_x * \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right) = 0. \quad (4.2)$$

Let $F(u) = \left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4$ and assume that φ is v -moderate satisfying the conditions specified. For any $n \in \mathbb{N}$, let $\varphi_n(x) = \min\{\varphi(x), n\}$. Then, $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ is locally absolutely continuous, $\|\varphi_n\|_\infty \leq n$ and $\|\varphi'_n(x)\| \leq A|\varphi_n(x)|$ a.e. on $\mathbb{R}$. Moreover, as shown in [3], the n -truncations φ_n are again v -moderate. Let $p \in [2, \infty)$. We multiply the equation by $\varphi_n|\varphi_n u|^{p-2}\varphi_n u$ and integrate to obtain

$$\int_{\mathbb{R}} (\varphi_n u)_t \varphi_n u |\varphi_n u|^{p-2} dx + \int_{\mathbb{R}} |\varphi_n u|^p u_x dx + \int_{\mathbb{R}} \varphi_n (\partial_x p * F(u)) |\varphi_n u|^{p-2} \varphi_n u dx = 0. \quad (4.3)$$

We then denote the three terms on the left-hand side as I_1 , I_2 and I_3 and observe that

$$I_1 = \frac{1}{p} \int_{\mathbb{R}} \frac{d}{dt} \left[\sqrt{(\varphi_n u)^2} \right]^p dx = \frac{1}{p} \frac{d}{dt} \|\varphi_n u\|_p^p = \|\varphi_n u\|_p^{p-1} \frac{d}{dt} \|\varphi_n u\|_p.$$

Moreover, we have $|I_2| \leq M^* \|\varphi_n u\|_p^p$, where the M^* is related to E_0 . Applying Holder's inequality then gives

$$|I_3| \leq \|\varphi_n (\partial_x p * F(u))\|_p \|(\varphi_n u)^{p-1}\|_{\frac{1}{1-1/p}} = \|\varphi_n (\partial_x p * F(u))\|_p \|\varphi_n u\|_p^{p-1}.$$

Hence, (4.3) yields

$$\frac{d}{dt} \|\varphi_n u\|_p \leq M^* \|\varphi_n u\|_p + \|\varphi_n (\partial_x p * F(u))\|_p \quad (4.4)$$

and using the weighted Young inequality we have

$$\|\varphi_n (\partial_x p * F(u))\|_p \leq C_1 \|(\partial_x p)v\|_1 \|\varphi_n F(u)\|_p \leq C_2 \|\varphi_n F(u)\|_p, \quad (4.5)$$

and

$$\begin{aligned}\|\varphi_n F(u)\|_p &\leq \left| c - \frac{\beta_0}{\beta} \right| \|\varphi_n u\|_p + \|\varphi_n u^2\|_p + \frac{1}{2} \|\varphi_n u_x^2\|_p + \left| \frac{\omega_1}{3\alpha^2} \right| \|\varphi_n u^3\|_p \\ &\quad + \left| \frac{\omega_2}{4\alpha^3} \right| \|\varphi_n u^4\|_p \\ &\leq C_3 M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p),\end{aligned}\quad (4.6)$$

where C_1, C_2 and C_3 are independent of n . From equation (4.2), we conclude

$$\frac{d}{dt} \|\varphi_n u\|_p \leq (C_4 + 1) M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p), \quad (4.7)$$

where the constant $C_4 = C_2 C_3$ is independent of n . Multiplying the identity $u_{tx} + u_x^2 + uu_{xx} + \partial_x^2 p * F(u) = 0$ with $\varphi_n |\varphi_n u_x|^{p-2} \varphi_n u_x$, we can obtain that

$$\begin{aligned}\varphi_n u_{tx} |\varphi_n u_x|^{p-2} \varphi_n u_x + \varphi_n u_x^2 |\varphi_n u_x|^{p-2} \varphi_n u_x + \varphi_n uu_{xx} |\varphi_n u_x|^{p-2} \varphi_n u_x \\ + \varphi_n |\varphi_n u_x|^{p-2} \varphi_n u_x \partial_x^2 p * F(u) = 0.\end{aligned}\quad (4.8)$$

As before, we will have

$$\begin{aligned}I_4 &= \int_{\mathbb{R}} \varphi_n u_x^2 |\varphi_n u_x|^{p-2} \varphi_n u_x dx \leq M^* \int_{\mathbb{R}} |\varphi_n u_x|^p dx = M^* |\varphi_n u_x|_p^p, \\ I_5 &= \int_{\mathbb{R}} \varphi_n uu_{xx} |\varphi_n u_x|^{p-2} \varphi_n u_x dx \leq M^* \|\varphi_n u\|_p \|\varphi_n u_x\|_p^{p-1}, \\ I_6 &= \int_{\mathbb{R}} \varphi_n |\varphi_n u_x|^{p-2} \varphi_n u_x \partial_x^2 p * F(u) dx \leq \|\varphi_n (\partial_x^2 p * F(u))\|_p \|\varphi_n u_x\|_p^{p-1}.\end{aligned}\quad (4.9)$$

Then,

$$\frac{d}{dt} \|\varphi_n u_x\|_p \leq M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p) + \|\varphi_n (\partial_x^2 p * F(u))\|_p. \quad (4.10)$$

Using that $\|vp\|_1 < \infty$, we have

$$\begin{aligned}\|\varphi_n (\partial_x^2 p * F(u))\|_p &\leq C_5 \|\varphi_n F(u)\|_p \leq C_5 C_3 M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p) \\ &= C_6 M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p).\end{aligned}\quad (4.11)$$

Moreover, the above inequalities, we can obtain that

$$\frac{d}{dt} \|\varphi_n u_x\|_p \leq (C_6 + 1) M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p). \quad (4.12)$$

Combining (4.7) and (4.12), there exists a constant C , only depending on $\alpha, \beta, \beta_0, \omega_1, \omega_2$ and u , such that

$$\frac{d}{dt} (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p) \leq C M^* (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p). \quad (4.13)$$

So that, by Gronwall's lemma, we obtain that

$$\|\varphi_n u\|_p + \|\varphi_n u_x\|_p \leq (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p) e^{CM^* t}, \quad \forall t \in [0, T].$$

Since $\varphi_n(x)$, a.e. as $n \rightarrow \infty$ and $\varphi u_0, \varphi u_{0x} \in L_p(\mathbb{R})$ the assertion of the theorem follows for the case $p \in [2, \infty)$. Since $\|\cdot\|_\infty = \lim_{p \rightarrow \infty} \|\cdot\|_p$, it is clear that the theorem also applies for $p = \infty$. $\square$

In the following corollary, however, we may choose $\varphi = \varphi_{1,1,c,d}$, if $c < 0, d \in \mathbb{R}$ and $\frac{1}{|c|} < p \leq \infty$.

Corollary 4.6 Let $2 \leq p \leq \infty$ and let $\varphi : \mathbb{R} \rightarrow (0, \infty)$ be a locally absolutely continuous and v -moderate weight function satisfying $|\varphi'(x)| \leq A|\varphi(x)|$ a.e., for some $A > 0$, $\inf v > 0$ and $ve^{-|\cdot|} \in L_p(\mathbb{R})$. For $s > 3$, let $u \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ be the strong solution to (4.1) starting from u_0 . Then,

$$\sup_{t \in [0, T]} (\|\varphi u(t)\|_p + \|\varphi u_x(t)\|_p)$$

and

$$\sup_{t \in [0, T]} (\|\varphi^{\frac{1}{2}} u(t)\|_p + \|\varphi^{\frac{1}{2}} u_x(t)\|_p)$$

are finite. For the particular choice $c = d = 0$ and $p = \infty$, we conclude from

$$|u_0(x)| + |u_{0,x}(x)| \leq C e^{-|x|}$$

for any $x \in \mathbb{R}$ that the unique solution u with $u(0)$ satisfies

$$|u(x)| + |u_x(x)| \leq C' e^{-|x|}$$

on $\mathbb{R} \times [0, T)$.

Proof As explained in [3], the function $\varphi^{\frac{1}{2}}$ is a $v^{\frac{1}{2}}$ moderate weight satisfying $|(\varphi^{\frac{1}{2}})'(x)| \leq \frac{A}{2} \varphi^{\frac{1}{2}}(x)$, $\inf v^{\frac{1}{2}} > 0$ and $v^{\frac{1}{2}} e^{-|\cdot|} \in L_1(\mathbb{R})$. We apply Theorem 4.2 with $p = 2$ to the weight $\varphi^{\frac{1}{2}}$ and obtain

$$\|\varphi^{\frac{1}{2}} u(t)\|_2 + \|\varphi^{\frac{1}{2}} u_x(t)\|_2 \leq (\|\varphi^{\frac{1}{2}} u_0\|_2 + \|\varphi^{\frac{1}{2}} u_{0x}\|_2) e^{CM^*t}. \quad (4.14)$$

Let φ_n be as in the proof of Theorem 4.2. Then, (4.14) holds equally with φ replaced by φ_n . By the definition of $F(u)$ and Eq. (4.14), there is a constant $\widetilde{C}_1$ depended only on φ and z_0 such that $\|\varphi_n F(u)\|_2 \leq \widetilde{C}_1 e^{2CM^*t}$. Using this estimate, we conclude that $\|\varphi_n(\partial_x p * F(u))\|_p \leq \widetilde{C}_2 e^{2CM^*t}$ and, that

$$\begin{aligned} \|\varphi_n(\partial_x^2 p * F(u))\|_p &\leq \|\varphi_n(p * F(u))\|_p + \|\varphi_n F(u)\|_p \\ &\leq \widetilde{C}_3 e^{2CM^*t} + \widetilde{C}_4 M^*(\|\varphi_n u\|_p + \|\varphi_n u_x\|_p). \end{aligned}$$

Using the same method in Theorem 4.2, this yields that

$$\frac{d}{dt} (\|\varphi_n u\|_p + \|\varphi_n u_x\|_p) \leq \widetilde{C}_5 M^*(\|\varphi_n u\|_p + \|\varphi_n u_x\|_p) + \widetilde{C}_6 e^{2CM^*t}, \quad (4.15)$$

and the constants $\widetilde{C}_j > 0$, $j = 1, \dots, 6$, depend only on φ and u_0 . Integrating this equation and letting $n \rightarrow \infty$, we obtain the result of the corollary for $p \in [2, \infty)$. The case $p = \infty$ is again obtained from a standard limit argument. This achieves the proof. $\square$

Remark 4.7 We can also apply this theorem choosing $\varphi(x) = \varphi_{1,1,0,0}(x) = e^{|x|}$ and $p = \infty$. It follows that if $|u_0(x)|$ and $|\partial_x u_0(x)|$ are both bounded by $C e^{-|x|}$, then the strong solution satisfies, uniformly in $[0, T]$

$$|u(x, t)| + |\partial_x u(x, t)| \leq C e^{-|x|}. \quad (4.16)$$

The peakon-like decay is the fastest possible decay that is possible to propagate for a nontrivial solution u . Indeed, arguing as in the proof of Theorem 1.1 in [23], it is not difficult to see that, for fast enough decaying data, the following asymptotic profiles hold:

$$\begin{aligned} u(x, t) &\sim u_0(x) + e^{-x} t \Phi(x), \quad x \rightarrow +\infty, \\ u(x, t) &\sim u_0(x) - e^x t \Psi(x), \quad x \rightarrow -\infty, \end{aligned} \quad (4.17)$$

where $\Phi(t) \neq 0$ and $\Psi(t) \neq 0$ for all $t \in [0, T]$ (unless $u \equiv 0$). Thus, if $u_0(x) = o(e^{-|x|})$, only the zero solution can decay faster than $e^{-|x|}$ at a later time $0 < t_1 \leq T$.

Acknowledgements The authors would like to thank the referees for constructive suggestions and comments. The work of Zhu is partially supported by the NSF of China under the Grant 11401309. The work of Liu is supported partially by the Simons Foundation Grant 499875. The work of Mi is partially supported by the NSF of China-11671055, the NSF of Chongqing-cstc2018jcyjAX0273, and the key project of science and technology research program of Chongqing Education Commission (KJZD-K20180140).

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