Bounds on the dimension of the Brill–Noether schemes of rank two bundles

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Abstract

The aim of this note is to find upper bounds on the dimension of Brill–Noether locus inside the moduli space of rank two vector bundles on a smooth algebraic curve. We deduce some consequences of these bounds.

Keywords Brill–Noether theory · Petri map · Vector bundles

Mathematics Subject Classification 14H51 · 14H60

1 Introduction

Let C be a projective smooth algebraic curve of genus g . For nonnegative integers n and d , we denote by $U(n, d)$ the moduli space of stable vector bundles of rank n and degree d , which is an irreducible scheme of dimension $n^2(g - 1) + 1$. For an integer k with $1 \leq k \leq n + \frac{d}{2}$, the subset

$$B_{n,d}^k = \{E \in U(n, d) \mid h^0(E) \geq k\}$$

of $U(n, d)$ inherits the structure of a closed sub-scheme of $U(n, d)$. With these notations, $B_{1,d}^k$ is the scheme of line bundles of degree d with the space of global sections of dimension at least k , which is denoted commonly in the literature by W_d^{k-1} . In the case of its non-emptiness, $B_{n,d}^k$ is expected to be of dimension $n^2(g - 1) + 1 - k(k - d + n(g - 1))$. As well, for a fixed line bundle $\mathcal{L}$ of degree d , we denote the sub-scheme of $U(n, d)$ parameterizing stable bundles $E \in B_{n,d}^k$ with determinant $\mathcal{L}$, by $B_{n,\mathcal{L}}^k$.

The schemes $B_{n,d}^k$, being as natural generalization of the Brill–Noether spaces of line bundles, as well as the spaces $B_{n,\mathcal{L}}^k$, have received wide attention from various authors. However, in contrast with extensive results concerning these schemes, specifically the results

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on the non-emptiness and existence of components with minimum dimension, there are not, to our knowledge, systematic studies about upper bounds for their dimensions, when $n \geq 2$.

We study this problem for Brill–Noether schemes of rank two bundles, and we obtain upper bounds for $\dim B_{2,d}^k$ and $\dim B_{2,K}^k$, where K denotes the canonical line bundle on C .

The significant point in the rank two case is that a general element in a component of some $B_{2,d}^k$, which violates the upper bound and under some specified circumstances, might be assumed to be globally generated. Under the globally generated assumption, a result of Michael Atiyah is applicable. Based on the mentioned result, a globally generated vector bundle can be represented as an extension of a line bundle by the trivial line bundle. Then, using the structure of tangent spaces of $B_{2,d}^k$, we relate the kernels of the Petri maps of appropriate bundles in suitable exact sequences. See Theorem 1. As a by-product, we obtain a Mumford type classification result. See Corollary 1.

As for the schemes $B_{2,K}^k$, we use an unpublished result of B. Feinberg, which might be considered as a refined version of Atiyah's result. See proposition 1 and lemma 1.

By proving that for an arbitrary smooth curve C , a specific component $X \subset B_{2,d}^k$ with prescribed circumstances, would be generically smooth of expected dimension; our results push the results of Teixidor [10] and Flamini et al. [5], one step further. See Remark 1(c).

Similar problems, as the problems studied in this paper, have been studied for schemes of Secant Loci in [2–4] by the author.

2 Preliminaries

For $E \in B_{n,d}^k \setminus B_{n,d}^{k+1}$, the Petri map associated with E controls the tangent vectors of $B_{n,d}^k$ at E . Indeed, the orthogonal of the image of the Petri map

$$\mu_E^2 : H^0(E) \otimes H^0(K \otimes E^*) \rightarrow H^0(K \otimes E \otimes E^*), \quad (1)$$

identifies the tangent space of $B_{n,d}^k$ at E . Similarly, the tangent space for $B_{2,K}^k$ is parameterized by the orthogonal of the image of the symmetric Petri map

$$\mu_{s,E}^0 : S^2 H^0(E) \rightarrow H^0(S^2 E). \quad (2)$$

See for example [7].

Assume that $E \in U(2, d)$ and

$$0 \longrightarrow G \longrightarrow E \longrightarrow L \longrightarrow 0, \quad (3)$$

is an exact sequence of bundles, with $G, L \in \text{Pic}(C)$. Then, there exists a chain of bundles $S^2(E) \supset E^1 \supset E^2 \supset 0$, such that

$$\frac{S^2(E)}{E^1} = 2L, \quad \frac{E^1}{E^2} = G \otimes L, \quad E^2 = 2G. \quad (4)$$

See [8, p. 127]. So, one has two exact sequences

$$0 \longrightarrow 2G \longrightarrow S^2 E \longrightarrow \frac{S^2 E}{2G} \longrightarrow 0, \quad (5)$$

$$0 \longrightarrow G \otimes L \longrightarrow \frac{S^2 E}{2G} \longrightarrow 2L \longrightarrow 0. \quad (6)$$

An unpublished result of B. Feinberg, Lemma 1, is the key tool in the proof of Theorem 4. The lemma is a direct consequence of a characterization result, attributed to B. Feinberg. We quote Teixidor's statement, [11, Lemma 1.1], of this characterizing result in Proposition 1. The proof we present for proposition 1, is quoted from Feinberg's unpublished work in [6].

Proposition 1 Denote by F the greatest common divisor of the zeroes of the sections of E . Then, either there is a section of $E(-F)$ without zeroes or all sections of E are sections of a line sub-bundle of E .

Proof The assertion is an immediate consequence of the following,

Claim 1 Assume that $s_1, \dots, s_{r+1}$ are base point free linearly independent sections of E such that the space $\langle s_1, \dots, s_{r+1} \rangle$ does not contain a nowhere vanishing section. Then, there exists a line sub-bundle L of E such that $\langle s_1, \dots, s_{r+1} \rangle$ is contained in $H^0(C, L)$.

Proof of Claim 1 Set $V := \langle s_1, \dots, s_{r+1} \rangle$ and consider the evaluation map

$$e_V : C \times V \rightarrow E, \quad e_V(p, s) := s(p). \quad (7)$$

We show that $\ker(e_V)$ is a vector bundle of rank r and consequently the saturation of the image of e_V is a line bundle. Observe that the hypothesis of being base point free is equivalent to the fact that the dimension of $\ker(e_V)_p$ is at most r for all p in C . If, on the other hand, the rank of $\ker(e_V)$ is generically less than r , then the dimension of the image of $\ker(e_V)$ under the composition:

$$\ker(e_V) \rightarrow C \times V \rightarrow V$$

is at most r . This, however, would imply that V has a nowhere vanishing section, which is a contradiction. Therefore, $\ker(e_V)$ is a vector bundle of rank r and e_V surjectively maps onto a line sub-bundle in E . This completes the proof of Claim 1. $\square$

Lemma 1 Any vector bundle E with $h^0(E) = k \geq 2$ admits an extension as

$$0 \longrightarrow \mathcal{O}(D) \xrightarrow{i} E \xrightarrow{\pi} L \longrightarrow 0, \quad (8)$$

where D is an effective divisor and either $h^0(\mathcal{O}(D)) = 1$ or $h^0(\mathcal{O}(D)) = k$.

Motivated by Lemma 1, two types of bundles with sections are distinguishable.

Definition 1 A vector bundle E with $h^0(E) = k \geq 1$ will be said of first type if it admits an extension as (8) with $h^0(\mathcal{O}(D)) = k$. Otherwise, we call E of second type.

3 Main results

Theorem 1 Let k, d be integers with $3 \leq d \leq 2g - 2$, $2 \leq k \leq 2 + \frac{d}{2}$. Then,

$$\dim B_{2,d}^k \leq 2(g - 1) + d - 2k + 1. \quad (9)$$

Proof Observe first that if a general element E of an irreducible component X of $B_{2,d}^k$ satisfies $h^0(E) \geq k + 1$, then we can consider X as a component of $B_{2,d}^{k+1}$. Therefore for general $E \in X$ one may assume $h^0(E) = k$. Assume that d is a minimum integer such that for some suitable k there exists a component X of $B_{2,d}^k$ with $\dim X \geq 2(g - 1) + d - 2k + 2$. Then, a general element E in X is globally generated. Indeed otherwise we obtain $\dim B_{2,d-2}^{k-1} \geq 2(g - 1) + d - 2k + 2$, which is impossible by minimality of d . Therefore, by [1, Theorem 2], a general element E in X has a trivial line bundle as its line sub-bundle. Furthermore E admits a representation as

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{i} E \xrightarrow{\pi} L \longrightarrow 0, \quad (10)$$

with the property that the sections of L belonging to the image of $H^0(\pi)$ have at most one number of base points. Indeed, if L has the points p, q as its base points, then $h^0(E(-p-q)) \geq k-2$. This implies that

$$\dim B_{2,d-4}^{k-2} \geq 2(g-1) + d - 2k + 2,$$

which is absurd again by minimality of d . Take an extension as (10) and consider the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_C) \xrightarrow{H^0(i)} H^0(E) \xrightarrow{H^0(\pi)} V \longrightarrow 0, \quad (11)$$

where V is the image of the map $H^0(\pi) : H^0(E) \longrightarrow H^0(L)$. The exact sequence (11) together with various Petri maps gives rise to a commutative diagram as

$$\begin{array}{ccccc} H^0(\mathcal{O}_C) \otimes H^0(K \otimes E^*) & \xrightarrow{f_1} & H^0(E) \otimes H^0(K \otimes E^*) & \xrightarrow{g_1} & V \otimes H^0(K \otimes E^*) \\ \downarrow \mu & & \downarrow \mu_E^2 & & \downarrow \mu_{L,V} \\ H^0(\mathcal{O}_C \otimes K \otimes E^*) & \xrightarrow{f_2} & H^0(K \otimes E \otimes E^*) & \xrightarrow{g_2} & H^0(K \otimes L \otimes E^*), \end{array} \quad (12)$$

in which the maps f_1 and f_2 are injective and g_1 is surjective. Observe furthermore that the map μ is an isomorphism. The snake lemma applied to this situation implies that

$$\dim \ker \mu_E^2 = \dim \ker \mu_{L,V}. \quad (13)$$

According to the assumption concerning dimension of X , we obtain

$$\dim \ker \mu_{L,V} \geq (k-1)(2g-2-d+k-1).$$

Assuming $V = \langle v_1, \dots, v_{k-1} \rangle$ and setting

$$V_i := \langle v_1, \dots, v_i \rangle, \quad i = 2, \dots, k-1,$$

we would have $\dim \ker \mu_{L,V_i} - \dim \ker \mu_{L,V_{i-1}} \leq h^0(K \otimes E^*)$. These together with the base-point-free pencil trick applied to the map

$$\mu_{L,V_2} : V_2 \otimes H^0(K \otimes E^*) \rightarrow H^0(K \otimes L \otimes E^*),$$

implies $h^0(K \otimes E^* \otimes L^*(B)) \geq 2(2g-2-d+k) - (k-1)$, where B is the base locus of the sections of V_2 . Note also that $0 \leq \deg(B) \leq 1$. Therefore,

$$h^0(K \otimes E^* \otimes L^*) \geq 2(2g-2-d+k) - k. \quad (14)$$

If $\frac{4g-3}{3} \leq d$ then, as $\deg(K \otimes E^* \otimes L^*) < 0$ and $K \otimes E^* \otimes L^*$ is stable, one has $h^0(K \otimes E^* \otimes L^*) = 0$, which is in contradiction with inequality (14).

Recall that $h^0(E \otimes L) = h^0(K \otimes E^* \otimes L^*) + 3d - 2(g-1) \geq 2(g-1) + d + k$. Now if $d \leq \frac{4g-4}{3}$, then $\mu(E \otimes L) \leq 2g-3$. Observe furthermore that $E \otimes L$ is stable. As a consequence of Propositions 3 and 4 of [9], the Clifford theorem for vector bundles for such a this situation asserts that $h^0(E \otimes L) \leq \frac{\deg(E \otimes L) + \text{rk}(E \otimes L)}{2}$, by which we obtain $2(g-1) + d + k \leq 1 + \frac{3d}{2}$. Consequently, we get $d + k \leq 0$, which is absurd. $\square$

Theorem 2 *If $g \geq 5$, then*

$$\dim B_{n,d}^2 \leq n(n-1)(g-1) + d - 3. \quad (15)$$

Proof Assume that X is an irreducible component of $B_{n,d}^2$ and E is a general element of X . Assume moreover, as in theorem 1, that a general element $E \in X$ satisfies $h^0(E) = 2$. Observe that, using a diagram as in diagram (12), we can obtain an equality as (13), by which, if E turns out to be of second type, then μ_E^2 would be injective. So X has to be generically smooth, and it has to have the expected dimension, which is certainly smaller than the claimed bound.

If a general element of X turns to be of first type, then

$$\dim X \leq n(n-1)(g-1) + d - 4.$$

Indeed, if a general element $E \in X$ admits a presentation as

$$0 \rightarrow H \rightarrow E \rightarrow F \rightarrow 0,$$

where H is a line bundle with $h^0(H) = 2$, $\deg(H) = d_1$ and $\text{rk}(F) = n-1$, then since the stable bundles deform to non-stable ones, we can assume in counting that F is stable as well. So the dimension of the set of bundles as F , is bounded by $\dim U(n-1, d-d_1) = (n-1)^2(g-1) + 1$. Meanwhile, the line bundles as H would vary in a subset $\mathcal{H}$ of B_{1,d_1}^2 and the Martens' theorem asserts that $\dim \mathcal{H} \leq d_1 - 2$ ($\dim \mathcal{H}$ can be $d_1 - 2$ if C is hyper-elliptic and $\dim \mathcal{H} \leq d_1 - 3$ otherwise). Therefore, the dimension of X would be bounded by

$$[d_1 - 2] + [(n-1)^2(g-1) + 1] + (h^1(H \otimes F^*) - 1).$$

Observe that $h^1(H \otimes F^*) = (n-1)(g-1) + d - nd_1$ by Riemann–Roch. Moreover, $d_1 \geq 2$ and so

$$\dim X \leq n(n-1)(g-1) + d - 2 - d_1(n-1) \leq n(n-1)(g-1) + d - 4,$$

as required. $\square$

Motivated by [5, Theorem 1.2], one can sharpen the bound in Theorem 1 under some restrictions on the numbers r, d , as

Theorem 3 *Let k, d be integers with $3 \leq d \leq 2g - 2 - \frac{k}{2}$, $2 \leq k \leq 2 + \frac{d}{2}$. Then, if $k \geq 3$, then $\dim B_{2,d}^k \leq 2g + d - 4k$. While for $k = 2$, the integer d can vary in the set $\{3, \dots, 2g - 5\}$ with the same bound for $\dim B_{2,d}^2$.*

Proof The argument of proof of Theorem 1 goes through to deduce the result. Notice that the further restriction on d in the case $k = 2$ was needed to be imposed, because the quantity $2g + d - 4k$ turns out to be smaller than the expected dimension for $2g - 4 \leq d \leq 2g - 2$. $\square$

3.1 The case of canonical determinant

Theorem 4 *For an integer k with $2 \leq k \leq g + 1$, any irreducible component X of $B_{2,K}^k$ satisfies*

$$\dim X \leq 3g - 2k - 2. \quad (16)$$

Proof Let X be an irreducible component of $B_{2,K}^k$ and a general element E of X satisfies $h^0(E) = k$. Assume that a general member $E \in X$ is of second type and set $\gamma := \dim X$. Then, one has

$$3g - 3 - (\dim \text{im } \mu_{s,E}^0) \geq \gamma, \quad (17)$$

where $\mu_{s,E}^0$ is the symmetric Petri map associated with E as in (2). So

$$\dim \ker \mu_{s,E}^0 \geq \gamma + \frac{k(k+1)}{2} + 3 - 3g. \quad (18)$$

The exact sequence $0 \rightarrow \mathcal{O}(2D) \rightarrow S^2 E \rightarrow \frac{S^2 E}{\mathcal{O}(2D)} \rightarrow 0$, arising from the exact sequence (5), gives rise to a commutative diagram as

$$\begin{array}{ccccccc} 0 & \rightarrow & S^2 H^0(\mathcal{O}(D)) & \longrightarrow & S^2 H^0(E) & \longrightarrow & \frac{S^2 H^0(E)}{S^2 H^0(\mathcal{O}(D))} \rightarrow 0 \\ & & \downarrow \mu_{s,D}^0 & & \downarrow \mu_{s,E}^0 & & \downarrow \mu \\ 0 & \rightarrow & H^0(\mathcal{O}(2D)) & \longrightarrow & H^0(S^2 E) & \longrightarrow & H^0\left(\frac{S^2 E}{\mathcal{O}(2D)}\right) \rightarrow \dots \end{array} \quad (19)$$

Since $S^2 H^0(\mathcal{O}(D)) = \mathbb{C}$, the map $\mu_{s,D}^0$ turns to be injective. This together with the snake lemma gives an inequality as

$$\dim \ker \mu \geq \dim \ker \mu_{s,E}^0. \quad (20)$$

Therefore, using inequality (18), we obtain

$$\dim \ker \mu \geq \gamma + \frac{k(k+1)}{2} + 3 - 3g. \quad (21)$$

Let V be as in the proof of Theorem 1 and observe by effectiveness of D that the vector space V can be considered as a subspace of $H^0(\mathcal{O}(D) \otimes L)$. Similar to the previous argument, the exact sequence

$$0 \rightarrow \mathcal{O}(D) \otimes L \rightarrow \frac{S^2 E}{\mathcal{O}(2D)} \rightarrow 2L \rightarrow 0,$$

as well arising from the exact sequence (6), together with the equality

$$S^2 H^0(E) = S^2 V \oplus V \oplus \mathbb{C},$$

leads to the following commutative diagram of bundles

$$\begin{array}{ccccccc} 0 & \rightarrow & V & \longrightarrow & \frac{S^2 H^0(E)}{S^2 H^0(\mathcal{O}(D))} & \longrightarrow & S^2 V \rightarrow 0 \\ & & \downarrow \theta & & \downarrow \mu & & \downarrow \mu_{s,V,L}^0 \\ 0 & \rightarrow & H^0(\mathcal{O}(D) \otimes L) & \longrightarrow & H^0\left(\frac{S^2 E}{\mathcal{O}(2D)}\right) & \longrightarrow & H^0(2L) \rightarrow \dots, \end{array} \quad (22)$$

where $\mu_{s,V,L}^0$ is the symmetric Petri map of L restricted to $S^2 V$ and θ is the inclusion map. Once again, as a consequence of the injectivity of θ and the snake lemma, we obtain

$$\dim \ker \mu_{s,V,L}^0 \geq \dim \ker \mu, \quad (23)$$

by which together with (21) an inequality as

$$\dim \ker \mu_{s,V,L}^0 \geq \gamma + \frac{k(k+1)}{2} + 3 - 3g \quad (24)$$

would be obtained. This, in combination with $\dim \ker \mu_{s,V,L}^0 \leq \dim S^2 V - \dim V$, implies

$$\gamma \leq 3g - 2k - 2, \quad (25)$$

as required.

Finally, if $\dim X \geq 3g - 2k - 1$, then a general member E of X fails to be of first type. Indeed otherwise, assume that a general member $E \in X$ admits a presentation as

$$0 \longrightarrow \mathcal{O}(D) \xrightarrow{i} E \xrightarrow{\pi} K \otimes \mathcal{O}(-D) \rightarrow 0, \quad (26)$$

with $\deg(D) = t$. Then, the stability of E implies that $t \leq g - 2$, and we would have

$$\dim B_{1,t}^k + h^1(\mathcal{O}(D) \otimes L^{-1}) - 1 \geq 3g - 2k - 1.$$

This, since $h^0(\mathcal{O}(D) \otimes L^{-1}) = 0$ by stability of E , implies that

$$\dim B_{1,t}^k \geq 2t - 2k + 3,$$

which is absurd by Martens' theorem. $\square$

4 Remarks and Corollaries

Corollary 1 (Mumford's Theorem for rank two bundles) *If C is non-hyper-elliptic of genus $g \geq 19$ and if for some k, d with $0 < 2k - 2 \leq d \leq 2g - \frac{3}{2}k - \frac{7}{2}$ one had $\dim B_{2,d}^k = 2g + d - 4k$, then either C is trigonal, or bi-elliptic, or a smooth plane quintic.*

Proof Assume that X is an irreducible component of $B_{2,d}^k$ with $\dim X = 2g + d - 4k$. If a general element $E \in X$ is of first type and has k number of independent sections, then one has $\dim B_{1,t}^k \geq 2(t - 2k + 1)$ for some integer t with $0 < 2(k - 1) \leq t \leq g - 2$. This, by Mumford's theorem, might occur only if $t - 2k + 1 = 0$ by which the equality $\dim B_{1,2k-1}^k = 0$ holds. So $\mathcal{O}(D) \in B_{1,2k-1}^k$, which may happen only in the case that either C is trigonal, or bi-elliptic, or a smooth plane quintic.

Claim 2 *If E fails to be of first type, then for general points $p_1, \dots, p_{\lfloor \frac{k-2}{2} \rfloor} \in C$, the stable vector bundle $E(-p_1 - \dots - p_{\lfloor \frac{k-2}{2} \rfloor})$ would fail to admit an extension of first type.*

Proof of Claim 2 Assume first that k is even. If the stable vector bundle $E(-p_1 - \dots - p_{\lfloor \frac{k-2}{2} \rfloor})$ turns to be of first type, then there exists a set of line bundles H with $h^0(H) \geq 2$ and $\deg H \leq g - 2$. Tensoring H with $\mathcal{O}(p_1 + \dots + p_t)$ for general points $p_1 + \dots + p_t$, if necessary, we can assume that $H \in B_{1,g-2}^2$. Therefore we obtain $\dim B_{1,g-2}^2 \geq 2g + d - 4k + \lfloor \frac{k-2}{2} \rfloor$. This by Martens' theorem implies that $7k - 8 \geq 2g + 2d$. On the other hand, the inequalities $2k - 2 \leq d$ and $2k - 2 \leq 2g - \frac{3}{2}k - \frac{7}{2}$ imply $4k \leq 2d + 4$ and $3k \leq \frac{12}{7}g - \frac{9}{7}$, respectively. Summing up all the inequalities we obtain $g \leq 18$, which is absurd. If k is an odd number, then the argument goes verbatim to prove the claim by replacing $B_{1,g-2}^3$ with $B_{1,g-2}^2$. So the Claim 2 is established.

If a general bundle $E \in X$ turns to be of second type and if $k = 2n$, then the scheme $B_{2,d-2\lfloor \frac{k-2}{2} \rfloor}^2$ contains a subset Y which is at least of dimension $2g + d - 4k + \lfloor \frac{k-2}{2} \rfloor$ and its general member is a vector bundle of second type. According to the work of M. Teixidor in [10], such a subset Y , if non-empty, is of expected dimension and the expected dimension is strictly smaller than $2g + d - 4k + \lfloor \frac{k-1}{2} \rfloor$ for d in the given range. This is a contradiction.

If $k = 2n + 1$, with similar assumption on E , the scheme $B_{2,d-2[\frac{k-2}{2}]}^3$ would contain a subset Y which is at least of dimension $2g + d - 4k + [\frac{k-2}{2}]$ and its general member is a vector bundle of second type. This possibility can be excluded by another work of M. Teixidor in [11]. $\square$

Corollary 2 *The scheme $B_{2,K}^2$ is reduced and irreducible of dimension $3g - 6$.*

Proof The upper bound $3g - 6$ on the dimension is obvious by theorem 4. If $E \in B_{2,K}^2 \setminus B_{2,K}^3$, then the petri map $\mu_{2,K,E}^0$ turns to be injective. Indeed, if E is a bundle of first type, then using diagram (19), since $\frac{S^2 H^0(E)}{S^2 H^0(\mathcal{O}(D))}$ vanishes, the Petri map $\mu_{s,E}^0$ would be injective. While if E is of second type, since $S^2 V$ is one dimensional, then $\mu_{s,V,L}^0$ is injective and so the map μ is injective by (23). This together with (20) implies that the Petri map $\mu_{s,E}^0$ is again injective. So we obtain

$$\text{Sing } B_{2,K}^2 = B_{2,K}^3. \quad (27)$$

Since $B_{2,K}^2$ is of expected dimension, so it might be reducible only if its singular locus is, by [12], of codimension ≤ 1 ; i.e., $\dim B_{2,K}^3 \geq 3g - 7$, by (27). This is a contradiction, because by Theorem (4) the locus $B_{2,K}^3$ is of dimension at most $3g - 8$.

Since, again by theorem 4, no irreducible component of $B_{2,K}^2$ is contained entirely in $B_{2,K}^3$, so $B_{2,K}^2$ would be reduced. $\square$

Using Lemma 2, the bound in theorem 4 can be sharpened for odd values of k .

Lemma 2 *If $\mathcal{L}$ is a globally generated line bundle on C with $h^0(\mathcal{L}) = s + 1$, then the set of vector bundles of second type $E \in B_{2,d,\mathcal{L}}^k$ ($k = 1, 2$), if non-empty, is of dimension at most $s + d - 4$, (res. at most of dimension $\frac{d}{2} + 2(s - 3)$), if $k = 2$ (res. if $k = 3$).*

Proof For $k = 2$, with notations as in proof of [10, Page 124], the dimension of the set of vector bundles $E \in B_{2,d,\mathcal{L}}^k$ of second type is bounded by

$$\dim\{D\} + h^0(\mathcal{L}(-D)) - 1 + \dim\langle \acute{D} \rangle - (h^0(E) - 1)$$

where $\acute{D}$ is a divisor in the linear series $|\mathcal{L}(-D)|$ and $t = \deg(D)$. It is now an easy argument to see that this quantity is bounded by

$$t + (s - 1) + (d - t - 2) - 1 = d + s - 4.$$

If $k = 3$, then a close analysis in the proof of [11, Theorem 2], implies that the dimension of the bundles $E \in \dim B_{2,d,\mathcal{L}}^3$ which are of second type, is bounded by the quantity $\dim\{D\} + \dim \text{Grass}(2, \mathcal{L}(-D)) + \dim\langle \acute{D}_1 \cap \acute{D}_2 \rangle - (h^0(E) - 1) \leq \frac{d}{2} + 2(s - 3)$, as required. $\square$

Corollary 3 *If k is odd, then*

$$\dim B_{2,K}^k \leq 3g - 2k - 3. \quad (28)$$

Proof An irreducible component X of $B_{2,K}^k$ whose general member is a bundle of first type has dimension $\leq 3g - 2k - 3$, because otherwise one obtains $\dim B_{1,t}^k \geq 2t - 2k + 2$ for some k and t with $0 < 2k - 2 \leq t \leq g - 2$. This is obviously absurd.

Assume that $\dim X = 3g - 2k - 2$ and set $k - 1 = 2n$.

Claim 3 If a general $E \in X$ fails to be of first type, then for general points $p_1, \dots, p_i \in C$ with $1 \leq i \leq \frac{k-1}{2}$ the stable vector bundle $E(-p_1 - \dots - p_i)$ would fail to admit an extension of first type.

The proof of Claim 3 is similar to the proof of Claim 2 in corollary 1.

Lemma (2) together with Claim 3 implies that if a general element of X fails to be of first type then

$$3g - 2k - 2 \leq (n - 1) + \dim B_{2,2g-2n,K(-2p_1-\dots-2p_{n-1})}^3 \leq 3g - 4n - 5,$$

which is absurd. $\square$

Remark 1 (a) If C is an arbitrary 3-gonal curve, then Theorem 3 together with Theorem [5, Thm. 1.2(b)] imply $\dim B_{2,d}^2 = 2g + d - 8$. Indeed, Theorem [5, Thm. 1.2(b)] establishes this result for a general 3-gonal curve and so for non-general 3-gonal curves, one has $\dim B_{2,d}^2 \geq 2g + d - 8$. Now, Theorem 3 applied to such a non-generic curve implies the equality for any 3-gonal curve.

(b) According to Theorem 2, one immediately re-obtains $\dim B_{n,n(g-1)}^2 = n^2(g - 1) - 3$. Meanwhile, by the same theorem, an immediate prediction suggests the quantity $n(n - 1)(g - 1) + d - 2k + 1$ as a bound to the dimension of $B_{n,d}^k$ when $n \geq 3, k \geq 3$. A proof to this expectation is unknown to me. Such a bound re-obtains Marten’s bound on the dimension of the Brill–Noether schemes of line bundles.

(c) The proofs of Theorems 1 and 2 indicate that the Petri map is injective at the bundles $E \in B_{n,d}^2$ which are of second type. Therefore

$$\text{Sing } B_{n,d}^2 \subseteq B_{n,d}^3 \cup \mathcal{E}_1,$$

where $\mathcal{E}_1$ denotes the set of bundles $E \in B_{n,d}^2$ of first type. This reproves the generic smoothness of the locus’ introduced by Teixidor [10] and Flamini et al. [5].

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