Existence and concentration of positive solutions for p -fractional Schrödinger equations

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Abstract

We deal with the existence and concentration of positive solutions for the following p -fractional Schrödinger equation:

$$\varepsilon^{sp}(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(u) + \gamma|u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a small parameter, $s \in (0, 1)$, $p \in (1, \infty)$, $N > sp$, $\gamma \in \{0, 1\}$, $p_s^* = \frac{Np}{N-sp}$ is the fractional critical Sobolev exponent, $(-\Delta)_p^s$ is the fractional p -Laplacian operator, V is a continuous positive potential having a local minimum and f is a superlinear continuous function with subcritical growth. The main results are obtained by using penalization techniques and suitable variational arguments.

Keywords Fractional Schrödinger equation · Fractional p -Laplacian operator · Variational methods · Critical exponent

Mathematics Subject Classification 47G20 · 35R11 · 35A15 · 58E05

1 Introduction

This paper is concerned with the existence and concentration of positive solutions for the following fractional p -Laplacian problem

$$\begin{cases} \varepsilon^{sp}(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(u) + \gamma|u|^{p_s^*-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

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where $\varepsilon > 0$ is a small parameter, $s \in (0, 1)$, $p \in (1, \infty)$, $N > sp$, $p_s^* = \frac{Np}{N-sp}$ is the fractional critical Sobolev exponent, $\gamma \in \{0, 1\}$, $W^{s,p}(\mathbb{R}^N)$ is the space of functions $u \in L^p(\mathbb{R}^N)$ such that

$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < \infty,$$

endowed with the natural norm

$$\|u\|_{s,p}^p = [u]_{s,p}^p + \|u\|_p^p.$$

The main operator $(-\Delta)_p^s$ is the fractional p -Laplacian which (up to normalization factors) may be defined for every function $u \in C_c^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_p^s u(x) = 2 \lim_{r \rightarrow 0} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy \quad (x \in \mathbb{R}^N).$$

Throughout the paper we will assume that $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous potential satisfying the following assumptions due to del Pino and Felmer [16]:

- (V₁) there exists $V_1 > 0$ such that $V_1 = \inf_{x \in \mathbb{R}^N} V(x)$,
 (V₂) there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$0 < V_0 = \inf_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x),$$

and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $f(t) = 0$ for $t \leq 0$ and fulfills the following conditions if $\gamma = 0$:

- (f₁) $f(t) = o(t^{p-1})$ as $t \rightarrow 0^+$,
 (f₂) there exists $q \in (p, p_s^*)$ such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t^{q-1}} = 0,$$

- (f₃) there exists $\vartheta \in (p, p_s^*)$ such that $0 < \vartheta F(t) := \vartheta \int_0^t f(\tau) d\tau \leq t f(t)$ for all $t > 0$,
 (f₄) the map $t \mapsto \frac{f(t)}{t^{p-1}}$ is increasing in $(0, \infty)$,

while in the case $\gamma = 1$ we require that f satisfies (f₁), (f₃), (f₄) and the following technical condition:

- (f'₂) there exist $q, \sigma \in (p, p_s^*)$ and $\lambda > 0$ such that

$$f(t) \geq \lambda t^{q-1} \quad \forall t > 0, \quad \lim_{t \rightarrow \infty} \frac{f(t)}{t^{\sigma-1}} = 0,$$

where λ is such that

- $\lambda > 0$ if either $N > sp^2$, or $sp < N < sp^2$ and $p_s^* - \frac{p}{p-1} < q < p_s^*$,
- λ is sufficiently large if $sp < N < sp^2$ and $p < q \leq p_s^* - \frac{p}{p-1}$.

When $p = 2$, Eq. (1.1) boils down to a fractional Schrödinger equation of the type

$$\varepsilon^{2s} (-\Delta)^s u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^N \quad (1.2)$$

for which several contributions regarding existence, multiplicity, regularity and asymptotic behavior of solutions have been obtained by different mathematicians; see for example [3, 5–7, 19, 20, 22, 26, 28, 40] and the references therein. We recall that one of the main reasons of studying (1.2) is related to find standing waves solutions, that is solutions of the form $\psi(x, t) = u(x)e^{-\frac{iEt}{\varepsilon}}$, where E is a constant, for the time-dependent fractional Schrödinger equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s \psi + (V(x) + E)\psi - f(x, \psi) \quad \text{for } (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

proposed by Laskin in the study of fractional Quantum Mechanics; see [13, 30] for a more physical background.

More generally, in the last few years, a great attention has been focused on the study of nonlocal operators since they find applications in several fields such as, for instance, game theory, finance, population dynamics, image processing, Lévy processes and optimization; see [18, 29, 33] and the references therein. On the other hand, from the mathematical point of view, the fractional p -Laplacian has received a great interest because both nonlocal and nonlinear issues appear in it. We mention [8, 9, 17, 23–25, 27, 32, 34, 36, 38, 41] and the references therein for various and interesting existence, multiplicity and regularity results involving $(-\Delta)_p^s$. For instance, Franzina and Palatucci [25] considered fractional eigenvalues problems driven by integro-differential operators whose model is $(-\Delta)_p^s$. Mosconi et al. [34] (see also [32]) studied the Brezis–Nirenberg problem for the fractional p -Laplacian operator. Di Castro et al. [17] obtained interior Hölder regularity results for fractional p -minimizers. Fiscella and Pucci [23] dealt with Kirchhoff type equations on the whole space $\mathbb{R}^N$ driven by the p -fractional Laplace operator, involving critical Hardy–Sobolev nonlinearities and nonnegative potentials. Recently, in [8] (see also [9]), the existence and multiplicity of solutions to (1.1) under the following global condition due to Rabinowitz [39] have been investigated:

$$V_\infty = \liminf_{|x| \rightarrow \infty} V(x) > V_1 = \inf_{x \in \mathbb{R}^N} V(x) > 0.$$

Anyway, it seems that in the literature only a few papers concern with p -fractional Schrödinger equations and the aim of this paper is to give a further result in this direction. Particularly motivated by this fact and by the articles [3, 5, 8, 9, 26], in this paper we focus our attention on the existence and concentration of solutions to (1.1) under local conditions (V_1) – (V_2) on the potential V . More precisely, our main result is the following:

Theorem 1.1 *Assume that (V_1) – (V_2) hold, and suppose that (f_1) – (f_4) hold if $\gamma = 0$, and (f_1) , (f'_2) , (f_3) , (f_4) hold with $\gamma = 1$. Then, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) admits a positive solution. Moreover, if $x_\varepsilon \in \mathbb{R}^N$ denotes a global maximum point of u_ε , then*

$$\lim_{\varepsilon \rightarrow 0} V(x_\varepsilon) = V_0,$$

and there exists $C > 0$ such that

$$0 < u_\varepsilon(x) \leq \frac{C\varepsilon^{N+sp}}{\varepsilon^{N+sp} + |x - x_\varepsilon|^{N+sp}} \quad \forall x \in \mathbb{R}^N.$$

The proof of the above results relies on suitable variational arguments. Due to the lack of informations about the behavior of V at infinity, we adapt the penalization argument introduced in [16] (see also [1, 2, 21]) which consists in modifying appropriately the nonlinearity f outside Λ in such a way that the energy functional of the modified problem satisfies the

Palais–Smale condition. We note that a similar approach has been used in [3,5,26] to study (1.1) with $p = 2$. Anyway, in contrast with [3,5,26], the operator $(-\Delta)_p^s$ is not linear when $p \neq 2$, so we can not make use of the s -harmonic extension by Caffarelli and Silvestre [12] and some standard variational arguments used in the literature (see [3,5,22,26]) to study (1.2) are not so easy to adapt in our context due to the non-Hilbertian structure of the involved fractional Sobolev spaces $W^{s,p}$. For this reason, we take advantage of some technical results recently established in [8]. We stress that in the case $\gamma = 1$, the calculations performed to recover compactness are much more involved with respect to the case $\gamma = 0$ due to the presence of the critical exponent. Moreover, differently from [26] and the quasilinear local case in [21], we do not use concentration-compactness arguments [19,31,37], but we provide some technical results which allow us to obtain the existence of a nontrivial solution for the modified problem; see Lemmas 3.2 and 3.4. Finally, in order to prove that, for ε small enough, the solution of the penalized problem is indeed a solution of the original one, we combine a Moser iteration argument [35] with a compactness argument for the solutions of the modified problem and the Hölder continuity result established for $(-\Delta)_p^s$ (see [17,27]). We also provide a power-type decay estimate of solutions to (1.1) borrowing some ideas contained in [8,15].

The paper is structured as follows. In Sect. 2, we give the notations and collect some useful results for fractional Sobolev spaces. In Sect. 3, we introduce the modified problem and we prove the existence of a positive solution for it. In Sect. 4, we give the proof of Theorem 1.1.

2 Preliminaries

In this preliminary section, we fix the notations and we recall some facts about the fractional Sobolev spaces.

We denote by $\mathcal{B}_R(x)$ the ball of radius R and center at x . When $x = 0$, we write $\mathcal{B}_R = \mathcal{B}_R(0)$. Let $1 \leq r \leq \infty$ and $A \subset \mathbb{R}^N$. We denote by $\|u\|_{L^r(A)}$ the $L^r(A)$ -norm of a function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ belonging to $L^r(A)$, and by $\|u\|_q$ its $L^q(\mathbb{R}^N)$ -norm. We define $\mathcal{D}^{s,p}(\mathbb{R}^N)$ as the closure of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to

$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy.$$

Let us indicate by $W^{s,p}(\mathbb{R}^N)$ the set of functions $u \in L^p(\mathbb{R}^N)$ such that $[u]_{s,p} < \infty$, endowed with the natural norm

$$\|u\|_{s,p}^p = [u]_{s,p}^p + \|u\|_p^p.$$

We have the following well-known embeddings (see [18]).

Theorem 2.1 [18] *Let $s \in (0, 1)$ and $p \in [1, \infty)$ be such that $sp < N$. Then there exists a constant $C_* := C_*(N, s, p) > 0$ such that, for any $u \in \mathcal{D}^{s,p}(\mathbb{R}^N)$, we have*

$$\|u\|_{p_s^*}^p \leq C_* [u]_{s,p}^p.$$

Moreover, $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^]$ and compactly in $L_{loc}^q(\mathbb{R}^N)$ for any $q \in [1, p_s^*)$.*

We will often use the following compactness-Lions-type result (see Lemma 2.2 in [8]).

Lemma 2.1 [8] *Let $N > sp$ and $r \in [p, p_s^*)$. If $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and if*

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^r dx = 0,$$

where $R > 0$, then $u_n \rightarrow 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p_s^)$.*

We also recall the following technical lemma (see Lemma 2.3 in [8]).

Lemma 2.2 [8] *Let $u \in W^{s,p}(\mathbb{R}^N)$ and $\phi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ in B_1 and $\phi = 0$ in $\mathbb{R}^N \setminus B_2$. Set $\phi_r(x) = \phi(\frac{x}{r})$. Then*

$$\lim_{r \rightarrow \infty} [u\phi_r - u]_{s,p} = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \|u\phi_r - u\|_p = 0.$$

In what follows, we provide some useful estimates which will be needed to overcome the difficulty coming from the critical exponent in the case $\gamma = 1$.

Let

$$S_* := \inf_{u \in \mathcal{D}^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}.$$

As shown in [10], there exists a radially symmetric nonnegative decreasing minimizer $U = U(r)$ for S_* such that $(-\Delta)_p^s U = U^{p_s^*-1}$ in $\mathbb{R}^N$, and $[U]_{s,p}^p = \|U\|_{p_s^*}^p = S_*^{\frac{N}{p_s^*}}$. Moreover, $U \in L^\infty(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$,

$$\lim_{|x| \rightarrow \infty} |x|^{\frac{N-sp}{p-1}} U(x) = U_\infty \in \mathbb{R} \setminus \{0\}$$

and verifies the following decay estimate:

Lemma 2.3 [10] *There exist constants $c_1, c_2 > 0$ and $\theta > 1$ such that for all $r \geq 1$,*

$$\frac{c_1}{r^{\frac{N-sp}{p-1}}} \leq U(r) \leq \frac{c_2}{r^{\frac{N-sp}{p-1}}} \quad (2.1)$$

and

$$\frac{U(\theta r)}{U(r)} \leq \frac{1}{2}.$$

For any $\varepsilon > 0$, we consider the following family of minimizers for S_* given by

$$U_\varepsilon(x) := \frac{1}{\varepsilon^{\frac{N-sp}{p}}} U\left(\frac{|x|}{\varepsilon}\right).$$

For $\varepsilon, \delta > 0$, set

$$m_{\varepsilon,\delta} := \frac{U_\varepsilon(\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)},$$

and define

$$g_{\varepsilon,\delta}(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq U_\varepsilon(\theta\delta) \\ m_{\varepsilon,\delta}^p(t - U_\varepsilon(\theta\delta)) & \text{if } U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta) \\ t + U_\varepsilon(\delta)(m_{\varepsilon,\delta}^{p-1} - 1) & \text{if } t \geq U_\varepsilon(\delta), \end{cases}$$

and

$$G_{\varepsilon,\delta}(t) := \int_0^t (g'_{\varepsilon,\delta}(\tau))^{\frac{1}{p}} d\tau = \begin{cases} 0 & \text{if } 0 \leq t \leq U_\varepsilon(\theta\delta) \\ m_{\varepsilon,\delta}(t - U_\varepsilon(\theta\delta)) & \text{if } U_\varepsilon(\theta\delta) \leq t \leq U_\varepsilon(\delta) \\ t & \text{if } t \geq U_\varepsilon(\delta). \end{cases}$$

Let us observe that $g_{\varepsilon,\delta}$ and $G_{\varepsilon,\delta}$ are nondecreasing and absolutely continuous functions. Now, we consider the radially symmetric nonincreasing function

$$u_{\varepsilon,\delta}(r) = G_{\varepsilon,\delta}(U_\varepsilon(r)),$$

which, in view of the definition of $G_{\varepsilon,\delta}$, satisfies

$$u_{\varepsilon,\delta}(r) = \begin{cases} U_\varepsilon(r) & \text{if } r \leq \delta \\ 0 & \text{if } r \geq \theta\delta. \end{cases} \quad (2.2)$$

Thus, we have the following useful estimates established in Lemma 2.7 in [34]:

Lemma 2.4 [34] *There exists $C = C(N, p, s) > 0$ such that for any $\varepsilon \leq \frac{\delta}{2}$ the following estimates hold*

$$\begin{aligned} |u_{\varepsilon,\delta}|_{s,p}^p &\leq S_*^{\frac{N}{sp}} + C \left(\left(\frac{\varepsilon}{\delta} \right)^{\frac{N-sp}{p-1}} \right), \\ |u_{\varepsilon,\delta}|_{p_s^*}^{p_s^*} &\geq S_*^{\frac{N}{sp}} - C \left(\left(\frac{\varepsilon}{\delta} \right)^{\frac{N}{p-1}} \right). \end{aligned}$$

Next, we establish the following L^r -estimates, with $r \in \{p, q\}$, for $u_{\varepsilon,\delta}$.

Lemma 2.5 *There exists a constant $C = C(N, p, s) > 0$ such that for any $\varepsilon \leq \frac{\delta}{2}$*

$$|u_{\varepsilon,\delta}|_p^p \leq \begin{cases} C\varepsilon^{sp} & \text{if } N > sp^2 \\ C\varepsilon^{sp} \log\left(\frac{\delta}{\varepsilon}\right) & \text{if } N = sp^2 \\ C\varepsilon^{\frac{N-sp}{p-1}} \delta^{\frac{sp^2-N}{p-1}} - C\varepsilon^{sp} & \text{if } N < sp^2. \end{cases}$$

Proof Firstly, we assume that $N > sp^2$. By the definition of $u_{\varepsilon,\delta}$ we obtain that

$$\begin{aligned} \int_{\mathbb{R}^N} u_{\varepsilon,\delta}^p dx &= \int_{\mathcal{B}_\delta} u_{\varepsilon,\delta}^p dx + \int_{\mathcal{B}_{\theta\delta} \setminus \mathcal{B}_\delta} u_{\varepsilon,\delta}^p dx + \int_{\mathbb{R}^N \setminus \mathcal{B}_{\theta\delta}} u_{\varepsilon,\delta}^p dx \\ &= \int_{\mathcal{B}_\delta} u_{\varepsilon,\delta}^p dx + \int_{\mathcal{B}_{\theta\delta} \setminus \mathcal{B}_\delta} u_{\varepsilon,\delta}^p dx =: I + II. \end{aligned} \quad (2.3)$$

Now, we deal with each integral on the right hand side of (2.3). Using a change of variable, Lemma 2.3 and the fact that $\varepsilon \leq \frac{\delta}{2}$, we can deduce that

$$\begin{aligned} I &= \int_{\mathcal{B}_\delta} U_\varepsilon^p(x) dx = \varepsilon^{sp} \int_{\mathcal{B}_\delta} U^p(x) dx \\ &= \varepsilon^{sp} \omega_{N-1} \int_1^{\frac{\delta}{\varepsilon}} U^p(r) r^{N-1} dr \\ &\leq c\varepsilon^{sp} \int_1^{\frac{\delta}{\varepsilon}} r^{-\frac{(N-sp)p}{p-1} + N-1} dr \end{aligned}$$

$$= \frac{C\varepsilon^{sp}}{\frac{N-sp^2}{p-1}} \left[1 - \left(\frac{\varepsilon}{\delta} \right)^{\frac{N-sp^2}{p-1}} \right] \leq C\varepsilon^{sp}, \quad (2.4)$$

where C is a positive constant.

Since U_ε is radially nonincreasing, for any $\delta \leq r \leq \theta\delta$, we get

$$0 \leq m_{\varepsilon,\delta}(U_\varepsilon(r) - U_\varepsilon(\theta\delta)) = U_\varepsilon(\delta) \left[\frac{U_\varepsilon(r) - U_\varepsilon(\theta\delta)}{U_\varepsilon(\delta) - U_\varepsilon(\theta\delta)} \right] \leq U_\varepsilon(\delta).$$

Taking into account the definition of U_ε , $\frac{\delta}{\varepsilon} \geq 2$ and Lemma 2.3 we can see that

$$\begin{aligned} II &= \int_{\mathcal{B}_{\theta\delta} \setminus \mathcal{B}_\delta} u_{\varepsilon,\delta}^p dx = \int_{\mathcal{B}_{\theta\delta} \setminus \mathcal{B}_\delta} [m_{\varepsilon,\delta}(U_\varepsilon(r) - U_\varepsilon(\theta\delta))]^p dx \\ &< \int_{\mathcal{B}_{\theta\delta} \setminus \mathcal{B}_\delta} U_\varepsilon^p(\delta) dx \\ &= |U_\varepsilon(\delta)|^p |\mathcal{B}_{\theta\delta} \setminus \mathcal{B}_\delta| \\ &= C \frac{\delta^N}{\varepsilon^{N-sp}} \left| U\left(\frac{\delta}{\varepsilon}\right) \right|^p \\ &\leq C \frac{\delta^N}{\varepsilon^{N-sp}} \left(\frac{\delta}{\varepsilon} \right)^{-\frac{(N-sp)p}{p-1}} c_2^p \\ &= C \delta^{-\frac{N-sp^2}{p-1}} \varepsilon^{\frac{N-sp}{p-1}} \\ &\leq C \varepsilon^{-\frac{N-sp^2}{p-1} + \frac{N-sp}{p-1}} = C\varepsilon^{sp}. \end{aligned} \quad (2.5)$$

Putting together (2.3)–(2.5) we obtain the thesis.

Next, we consider the case $N = sp^2$. Then, it is easy to see that

$$I \leq c_2^p \varepsilon^{sp} \int_1^{\frac{\delta}{\varepsilon}} r^{-1} dr = c_2^p \varepsilon^{sp} \log\left(\frac{\delta}{\varepsilon}\right)$$

and

$$II \leq C\varepsilon^{sp}.$$

Since for all $\varepsilon \leq \frac{\delta}{2}$ it holds that $\log(\frac{\delta}{\varepsilon}) \geq \log(2)$, we can infer that

$$|u_{\varepsilon,\delta}|_p^p \leq \varepsilon^{sp} \log\left(\frac{\delta}{\varepsilon}\right) \left[1 + \frac{1}{\log\left(\frac{\delta}{\varepsilon}\right)} \right] \leq C\varepsilon^{sp} \log\left(\frac{\delta}{\varepsilon}\right).$$

Finally, we consider the case $N < sp^2$. Then, arguing as in (2.4) and (2.5), we get

$$\begin{aligned} I + II &\leq \left(C_1 \varepsilon^{\frac{N-sp}{p-1}} \delta^{\frac{sp^2-N}{p-1}} - C_2 \varepsilon^{sp} \right) + C_3 \delta^{-\frac{N-sp^2}{p-1}} \varepsilon^{\frac{N-sp}{p-1}} \\ &\leq C_4 \varepsilon^{\frac{N-sp}{p-1}} \delta^{\frac{sp^2-N}{p-1}} - C_2 \varepsilon^{sp}. \end{aligned}$$

□

Lemma 2.6 *There exists a constant $C = C(N, q, s) > 0$ such that for any $\varepsilon \leq \frac{\delta}{2}$*

$$|u_{\varepsilon, \delta}|_q^q \geq \begin{cases} C\varepsilon^{N - \frac{(N-sp)}{p}q} & \text{if } q > \frac{N(p-1)}{N-sp} \\ C\varepsilon^{N - \frac{(N-sp)}{p}q} |\log \varepsilon| & \text{if } q = \frac{N(p-1)}{N-sp} \\ C\varepsilon^{\frac{(N-sp)q}{p(p-1)}} & \text{if } q < \frac{N(p-1)}{N-sp}. \end{cases}$$

Proof Taking into account the definitions of $u_{\varepsilon, \delta}$ and U_ε , Lemma 2.1 and using polar coordinates, we can see that for any $\varepsilon \leq \frac{\delta}{2}$ it holds

$$\begin{aligned} \int_{\mathbb{R}^N} u_{\varepsilon, \delta}^q dx &\geq \int_{B_\delta} u_{\varepsilon, \delta}^q dx = \int_{B_\delta} U_\varepsilon^q dx \\ &= \varepsilon^{N - \frac{(N-sp)}{p}q} \int_{B_{\frac{\delta}{\varepsilon}}(0)} U^q dx \\ &\geq \varepsilon^{N - \frac{(N-sp)}{p}q} \omega_{N-1} \int_1^{\frac{\delta}{\varepsilon}} U(r)^q r^{N-1} dr \\ &\geq c_1^q \varepsilon^{N - \frac{(N-sp)}{p}q} \omega_{N-1} \int_1^{\frac{\delta}{\varepsilon}} r^{N - \frac{(N-sp)}{p}q - 1} dr \\ &\geq \begin{cases} C\varepsilon^{N - \frac{(N-sp)}{p}q} & \text{if } q > \frac{N(p-1)}{N-sp} \\ C\varepsilon^{N - \frac{(N-sp)}{p}q} |\log \varepsilon| & \text{if } q = \frac{N(p-1)}{N-sp} \\ C\varepsilon^{\frac{(N-sp)q}{p(p-1)}} & \text{if } q < \frac{N(p-1)}{N-sp}. \end{cases} \end{aligned}$$

□

3 Variational setting

Using the change of variable $x \mapsto \varepsilon x$, we can see that the study of (1.1) is equivalent to investigate the following problem

$$\begin{cases} (-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = f(u) + \gamma|u|^{p_s^*-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N. \end{cases} \quad (3.1)$$

Now, we introduce a penalized function in the spirit of [16]. First of all, without loss of generality, we assume that

$$0 \in \Lambda \quad \text{and} \quad V(0) = V_0 = \inf_{\Lambda} V.$$

Take $K > \frac{\vartheta}{\vartheta-p} > 1$ and $a > 0$ such that $f(a) + \gamma a^{p_s^*-1} = \frac{V_1}{K} a^{p-1}$, and we define

$$\tilde{f}(t) = \begin{cases} f(t) + \gamma(t^+)^{p_s^*-1} & \text{if } t \leq a \\ \frac{V_1}{K} t^{p-1} & \text{if } t > a, \end{cases}$$

and

$$g(x, t) = \begin{cases} \chi_\Lambda(x)(f(t) + \gamma t^{p_s^*-1}) + (1 - \chi_\Lambda(x))\tilde{f}(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

It is easy to check that g satisfies the following properties:

- (g₁) $\lim_{t \rightarrow 0} \frac{g(x, t)}{t^{p-1}} = 0$ uniformly with respect to $x \in \mathbb{R}^N$,
 (g₂) $g(x, t) \leq f(t) + \gamma t^{p_s^*-1}$ for all $x \in \mathbb{R}^N$, $t > 0$,
 (g₃) (i) $0 < \vartheta G(x, t) := \vartheta \int_0^t g(x, \tau) d\tau < g(x, t)t$ for all $x \in \Lambda$ and $t > 0$,
 (ii) $0 \leq pG(x, t) < g(x, t)t \leq \frac{V_1}{K} t^p$ for all $x \in \mathbb{R}^N \setminus \Lambda$ and $t > 0$,
 (g₄) for each $x \in \Lambda$ the function $\frac{g(x, t)}{t^{p-1}}$ is increasing in $(0, \infty)$, and for each $x \in \mathbb{R}^N \setminus \Lambda$ the function $\frac{g(x, t)}{t^{p-1}}$ is increasing in $(0, a)$.

Then, we consider the following modified problem

$$\begin{cases} (-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = g(\varepsilon x, u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (3.2)$$

In view of the definition of g , we will look for weak solutions to (3.2) having the property

$$|u(x)| \leq a \quad \text{for any } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon = \Lambda/\varepsilon$. In order to study (3.2), we seek the critical points of the following functional

$$\mathcal{J}_\varepsilon(u) = \frac{1}{p} \|u\|_\varepsilon^p - \int_{\mathbb{R}^N} G(\varepsilon x, u) dx,$$

which is well-defined for all $u : \mathbb{R}^N \rightarrow \mathbb{R}$ belonging to the following fractional space

$$\mathcal{W}_\varepsilon := \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_\varepsilon^p := [u]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^p dx.$$

Standard arguments show that $\mathcal{J}_\varepsilon \in \mathcal{C}^1(\mathcal{W}_\varepsilon, \mathbb{R})$ and that its differential is given by

$$\begin{aligned} \langle \mathcal{J}'_\varepsilon(u), v \rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dx dy \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{p-2}uv dx - \int_{\mathbb{R}^N} g(\varepsilon x, u)v dx \end{aligned}$$

for any $u, v \in \mathcal{W}_\varepsilon$. Let us note that $\mathcal{J}_\varepsilon$ possesses a mountain pass geometry [4]:

Lemma 3.1 *The functional $\mathcal{J}_\varepsilon$ has a mountain pass geometry:*

- (a) *there exist $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$;*
 (b) *there exists $e \in \mathcal{W}_\varepsilon$ such that $\|e\|_\varepsilon > \rho$ and $\mathcal{J}_\varepsilon(e) < 0$.*

Proof (a) By (g₁), (g₂), (f₂) and (f₂'), we can see that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|g(x, t)| \leq \xi |t|^{p-1} + C_\xi |t|^{p_s^*-1} \quad \text{for any } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Therefore,

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{p} \|u\|_\varepsilon^p - \int_{\mathbb{R}^N} G(\varepsilon x, u) \, dx \geq \frac{1}{p} \|u\|_\varepsilon^p - \xi C \|u\|_\varepsilon^p - C_\xi C \|u\|_\varepsilon^{p_s^*},$$

and we can find $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ with $\|u\|_\varepsilon = \rho$.

(b) Using (g_3) -(i), we can deduce that for any $u \in C_c^\infty(\mathbb{R}^N)$ such that $u \geq 0$, $u \not\equiv 0$ and $\text{supp}(u) \subset \Lambda_\varepsilon$

$$\begin{aligned} \mathcal{J}_\varepsilon(\tau u) &\leq \frac{\tau^p}{p} \|u\|_\varepsilon^p - \int_{\Lambda_\varepsilon} G(\varepsilon x, \tau u) \, dx \\ &\leq \frac{\tau^p}{p} \|u\|_\varepsilon^p - C_1 \tau^\vartheta \int_{\Lambda_\varepsilon} u^\vartheta \, dx + C_2 \quad \text{for any } \tau > 0, \end{aligned}$$

for some positive constants C_1 and C_2 . Since $\vartheta \in (p, p_s^*)$, we get $\mathcal{J}_\varepsilon(\tau u) \rightarrow -\infty$ as $\tau \rightarrow +\infty$. $\square$

Invoking a variant of the mountain pass theorem without Palais–Smale condition (see [42]), we can see that there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{W}_\varepsilon$ such that

$$\mathcal{J}_\varepsilon(u_n) \rightarrow c_\varepsilon \quad \text{and} \quad \mathcal{J}'_\varepsilon(u_n) \rightarrow 0,$$

where

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0, 1]} \mathcal{J}_\varepsilon(\gamma(t)) \quad \text{and} \quad \Gamma_\varepsilon = \{v \in \mathcal{W}_\varepsilon : \mathcal{J}_\varepsilon(0) = 0, \mathcal{J}_\varepsilon(\gamma(1)) \leq 0\}.$$

As in [42], we can use the equivalent characterization of c_ε more appropriate to our aim given by

$$c_\varepsilon = \inf_{u \in \mathcal{W}_\varepsilon \setminus \{0\}} \max_{t \geq 0} \mathcal{J}_\varepsilon(tu).$$

Moreover, from the monotonicity of g , it is easy to check that for any non-negative $u \in \mathcal{W}_\varepsilon \setminus \{0\}$ there exists a unique $t_0 = t_0(u) > 0$ such that

$$\mathcal{J}_\varepsilon(t_0 u) = \max_{t \geq 0} \mathcal{J}_\varepsilon(tu).$$

In order to obtain the existence of a nontrivial solution to (3.2) in the case $\gamma = 1$, we need to prove the next fundamental result.

Lemma 3.2 *Assume that $\gamma = 1$. Then there exists $v \in \mathcal{W}_\varepsilon \setminus \{0\}$ such that*

$$\max_{t \geq 0} \mathcal{J}_\varepsilon(tv) < \frac{S}{N} S_*^{\frac{N}{sp}}.$$

In particular $c_\varepsilon < \frac{S}{N} S_^{\frac{N}{sp}}$.*

Proof We follow [9]. Let $u_{h,\delta}$ be the function defined in (2.2) such that $\text{supp}(u_{h,\delta}) \subset \mathcal{B}_{\theta\delta} \subset \Lambda_\varepsilon$. For simplicity, we take $\delta = 1$ and we set $u_h := u_{h,1}$. Then, using (f_2') , we can see that

$$\begin{aligned} \mathcal{J}_\varepsilon(tu_h) &= \frac{t^p}{p} \|u_h\|_\varepsilon^p - \int_{\mathbb{R}^N} F(tu_h) \, dx - \frac{t^{p_s^*}}{p_s^*} |u_h|_{p_s^*}^{p_s^*} \\ &\leq \frac{t^p}{p} ([u_h]_{s,p}^p + |V|_{L^\infty(\Lambda)} |u_h|_p^p) - \lambda t^q |u_h|_q^q \, dx - \frac{t^{p_s^*}}{p_s^*} |u_h|_{p_s^*}^{p_s^*} \rightarrow -\infty \quad \text{as } t \rightarrow \infty, \end{aligned}$$

so there exists $t_h > 0$ such that

$$\mathcal{J}_\varepsilon(t_h u_h) = \max_{t \geq 0} \mathcal{J}_\varepsilon(t u_h).$$

Let us show that there exist $A, B > 0$ such that

$$A \leq t_h \leq B \quad \text{for } h > 0 \text{ sufficiently small.} \quad (3.3)$$

Since $\langle \mathcal{J}'_\varepsilon(t_h u_h), u_h \rangle = 0$, we deduce that

$$(t_h)^{p-1} \|u_h\|_\varepsilon^p = \int_{\mathbb{R}^N} f(t_h u_h) u_h \, dx + (t_h)^{p_s^*-1} |u_h|_{p_s^*}^{p_s^*}. \quad (3.4)$$

If $t_{h_n} \rightarrow \infty$ as $h_n \rightarrow 0$, by (3.4) it follows that

$$(t_{h_n})^{p-1} \|u_{h_n}\|_\varepsilon^p \geq (t_{h_n})^{p_s^*-1} |u_{h_n}|_{p_s^*}^{p_s^*},$$

which gives a contradiction in view of $p_s^* > p$ and Lemma 2.4.

Now, assume that there exists $t'_{h_n} \rightarrow 0$ as $h_n \rightarrow 0$. From (f_1) and (f'_2) , we can see that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} f(t'_{h_n} u_{h_n}) u_{h_n} \, dx &\leq \xi (t'_{h_n})^{p-1} |u_{h_n}|_p^p + C_\xi (t'_{h_n})^{p_s^*-1} |u_{h_n}|_{p_s^*}^{p_s^*} \\ &\leq \frac{\xi}{V_1} (t'_{h_n})^{p-1} \|u_{h_n}\|_\varepsilon^p + C_\xi (t'_{h_n})^{p_s^*-1} |u_{h_n}|_{p_s^*}^{p_s^*}. \end{aligned} \quad (3.5)$$

Choosing $\xi = \frac{V_1}{2}$, and using (3.4) and (3.5), we obtain

$$\frac{(t'_{h_n})^{p-1}}{2} \|u_{h_n}\|_\varepsilon^p \leq C_\xi (t'_{h_n})^{p_s^*-1} |u_{h_n}|_{p_s^*}^{p_s^*} + (t'_{h_n})^{p_s^*-1} |u_{h_n}|_{p_s^*}^{p_s^*}$$

which is impossible because $p_s^* > p$. Therefore, (3.3) holds true.

Thus, recalling that for $C, D > 0$ it holds

$$\frac{t^p}{p} C - \frac{t^{p_s^*}}{p_s^*} D \leq \frac{s}{N} \left(\frac{C}{D^{\frac{N-sp}{N}}} \right)^{\frac{N}{sp}} \quad \text{for all } t \geq 0,$$

and using (3.3), we can see that

$$\begin{aligned} \mathcal{J}_\varepsilon(t_h u_h) &\leq \frac{t_h^p}{p} \|u_h\|_\varepsilon^p - \lambda t_h^q |u_h|_q^q - \frac{t_h^{p_s^*}}{p_s^*} |u_h|_{p_s^*}^{p_s^*} \\ &\leq \frac{s}{N} \left(\frac{[u_h]_{s,p}^p + |V|_{L^\infty(\Lambda)} |u_h|_p^p}{|u_h|_{p_s^*}^p} \right)^{\frac{N}{sp}} - \lambda A^q |u_h|_q^q. \end{aligned}$$

Now, in view of the following elementary inequality

$$(a+b)^r \leq a^r + r(a+b)^{r-1}b \quad \text{for all } a, b > 0, r \geq 1,$$

and gathering the estimates in Lemmas 2.4 and 2.5, we get

$$\mathcal{J}_\varepsilon(t_h u_h) \leq \begin{cases} \frac{s}{N} S_*^{\frac{N}{sp}} + C_1 \left(h^{\frac{(N-sp)}{p-1}} \right) + C_2 h^{sp} - \lambda A^q |u_h|_q^q & \text{if } N > sp^2 \\ \frac{s}{N} S_*^{\frac{N}{sp}} + C_3 (h^{sp} (1 + |\log h|)) - \lambda A^q |u_h|_q^q & \text{if } N = sp^2 \\ \frac{s}{N} S_*^{\frac{N}{sp}} + C_4 \left(h^{\frac{(N-sp)}{p-1}} \right) - C_5 h^{sp} - \lambda A^q |u_h|_q^q & \text{if } N < sp^2. \end{cases}$$

Let $N > sp^2$. Then $q > p > \frac{N(p-1)}{N-sp}$ and using Lemma 2.6 we have

$$\mathcal{J}_\varepsilon(t_h u_h) \leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_1 \left(h^{\frac{(N-sp)}{p-1}} \right) + C_2 h^{sp} - C_6 \lambda h^{N - \frac{(N-sp)}{p} q}.$$

Since

$$N - \frac{(N-sp)}{p} q < sp < \frac{N-sp}{p-1},$$

thanks to $q > p$ and $N > sp^2$, we can infer that

$$\mathcal{J}_\varepsilon(t_h u_h) < \frac{s}{N} S_*^{\frac{N}{sp}},$$

provided that $h > 0$ is sufficiently small.

Assume that $N = sp^2$. Thus, $q > p = \frac{N(p-1)}{N-sp}$ and in view of Lemma 2.6 we obtain

$$\mathcal{J}_\varepsilon(t_h u_h) \leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_3 (h^{sp} (1 + |\log h|)) - C_7 \lambda h^{sp^2 - s(p-1)q}.$$

Observing that $q > p$ yields

$$\lim_{h \rightarrow 0} \frac{h^{sp^2 - s(p-1)q}}{h^{sp} (1 + |\log h|)} = \infty,$$

we get the conclusion for h small enough.

Finally, we consider the case $N < sp^2$. Suppose that $p_s^* - \frac{p}{p-1} < q < p_s^*$. Then,

$$q > p_s^* - \frac{p}{p-1} > \frac{N(p-1)}{N-sp}$$

from which

$$\mathcal{J}_\varepsilon(t_h u_h) \leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_4 \left(h^{\frac{(N-sp)}{p-1}} \right) - C_5 h^{sp} - C_8 \lambda h^{N - \frac{(N-sp)}{p} q}.$$

Using the fact that

$$N - \frac{(N-sp)}{p} q < \frac{(N-sp)}{p-1} < sp,$$

we have for $h > 0$ small enough

$$\mathcal{J}_\varepsilon(t_h u_h) < \frac{s}{N} S_*^{\frac{N}{sp}}.$$

Now, we assume that $N < sp^2$ and we deal with the case $p < q \leq p_s^* - \frac{p}{p-1}$. For this purpose, we distinguish the following cases:

$$p < q < \frac{N(p-1)}{N-sp}, \quad q = \frac{N(p-1)}{N-sp} \quad \text{and} \quad \frac{N(p-1)}{N-sp} < q \leq p_s^* - \frac{p}{p-1}.$$

Let us observe that $N < sp^2$ implies that $\frac{(N-sp)}{p-1} < sp$.

When $p < q < \frac{N(p-1)}{N-sp}$ then, for $h > 0$ small, it holds

$$\begin{aligned}\mathcal{J}_\varepsilon(t_h u_h) &\leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_4 \left(h^{\frac{(N-sp)}{p-1}} \right) - C_5 h^{sp} - C_9 \lambda h^{\frac{(N-sp)}{p(p-1)} q} \\ &\leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_{4,5} \left(h^{\frac{(N-sp)}{p-1}} \right) - C_9 \lambda h^{\frac{(N-sp)}{p(p-1)} q}\end{aligned}$$

and noting that

$$\frac{(N-sp)}{p-1} < \frac{(N-sp)}{p(p-1)} q$$

we can take $\lambda = h^{-\mu}$, with $\mu > \frac{(N-sp)}{p(p-1)} q - \frac{(N-sp)}{p-1}$, to get the thesis.

If $q = \frac{N(p-1)}{N-sp} (> p)$, then

$$\begin{aligned}\mathcal{J}_\varepsilon(t_h u_h) &\leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_4 \left(h^{\frac{(N-sp)}{p-1}} \right) - C_5 h^{sp} - C_{10} \lambda h^{\frac{N}{p}} |\log h| \\ &\leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_{4,5} \left(h^{\frac{(N-sp)}{p-1}} \right) - C_{10} \lambda h^{\frac{N}{p}} |\log h|\end{aligned}$$

and taking $\lambda = h^{-\mu}$, with $\mu > \frac{N}{p} - \frac{(N-sp)}{p-1}$, we can deduce the assertion.

Finally, when $\frac{N(p-1)}{N-sp} < q \leq p_s^* - \frac{p}{p-1}$, we have

$$\begin{aligned}\mathcal{J}_\varepsilon(t_h u_h) &\leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_4 \left(h^{\frac{(N-sp)}{p-1}} \right) - C_5 h^{sp} - C_{11} \lambda h^{N - \frac{(N-sp)}{p} q} \\ &\leq \frac{s}{N} S_*^{\frac{N}{sp}} + C_{4,5} \left(h^{\frac{(N-sp)}{p-1}} \right) - C_{11} \lambda h^{N - \frac{(N-sp)}{p} q}\end{aligned}$$

and choosing $\lambda = h^{-\mu}$, with $\mu > N - \frac{(N-sp)}{p} q - \frac{(N-sp)}{p-1}$, we have the conclusion. $\square$

In the next lemma, we prove that any Palais–Smale sequence of $\mathcal{J}_\varepsilon$ is bounded.

Lemma 3.3 *Let $\{u_n\}_{n \in \mathbb{N}}$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. Then, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{W}_\varepsilon$.*

Proof Let $\{u_n\}_{n \in \mathbb{N}}$ be a (PS) sequence at the level c , that is

$$\mathcal{J}_\varepsilon(u_n) \rightarrow c \quad \text{and} \quad \mathcal{J}'_\varepsilon(u_n) \rightarrow 0 \text{ in } \mathcal{W}_\varepsilon^*.$$

Using (g₃), we can deduce that

$$\begin{aligned}c + o_n(1) \|u_n\|_\varepsilon &= \mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta} \langle \mathcal{J}'_\varepsilon(u_n), u_n \rangle \\ &= \left(\frac{\vartheta - p}{p\vartheta} \right) \|u_n\|_\varepsilon^p + \frac{1}{\vartheta} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} [g(\varepsilon x, u_n) u_n - \vartheta G(\varepsilon x, u_n)] \, dx \\ &\quad + \frac{1}{\vartheta} \int_{\Lambda_\varepsilon} [g(\varepsilon x, u_n) u_n - \vartheta G(\varepsilon x, u_n)] \, dx \\ &\geq \left(\frac{\vartheta - p}{p\vartheta} \right) \|u_n\|_\varepsilon^p + \frac{1}{\vartheta} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} [g(\varepsilon x, u_n) u_n - \vartheta G(\varepsilon x, u_n)] \, dx\end{aligned}$$

$$\begin{aligned} &\geq \left(\frac{\vartheta - p}{p\vartheta} \right) \|u_n\|_\varepsilon^p - \left(\frac{\vartheta - p}{p\vartheta} \right) \frac{1}{K} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} V(\varepsilon x) |u_n|^p dx \\ &\geq \left(\frac{\vartheta - p}{p\vartheta} \right) \left(1 - \frac{1}{K} \right) \|u_n\|_\varepsilon^p. \end{aligned}$$

Since $\vartheta > p$ and $K > 1$, we can conclude that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{W}_\varepsilon$. $\square$

The lemma below will be very useful to obtain the existence of a nontrivial solution to (3.2).

Lemma 3.4 *There exist a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\int_{\mathcal{B}_R(x_n)} |u_n|^p dx \geq \beta. \quad (3.6)$$

Moreover, the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}^N$.

Proof Assume by contradiction that the conclusion does not hold. By Lemma 2.1, we can see that $u_n \rightarrow 0$ in $L^r(\mathbb{R}^N)$ for all $r \in (p, p_s^*)$. Firstly, we consider the case $\gamma = 0$. Then, in view of (g_1) , (g_2) , (f_2) and using $\langle \mathcal{J}'_\varepsilon(u_n), u_n \rangle = o_n(1)$, we can see that

$$\|u_n\|_\varepsilon^p = \int_{\mathbb{R}^N} g(\varepsilon x, u_n) u_n dx + o_n(1) = o_n(1)$$

which implies that $\mathcal{J}_\varepsilon(u_n) \rightarrow 0$ and this is an absurd thanks to $c_\varepsilon > 0$.

Secondly, we suppose that $\gamma = 1$. By (f_1) and (f'_2) , it follows that

$$\int_{\mathbb{R}^N} F(u_n) dx = \int_{\mathbb{R}^N} f(u_n) u_n dx = o_n(1).$$

This implies that

$$\int_{\mathbb{R}^N} G(\varepsilon x, u_n) dx \leq \frac{1}{p_s^*} \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{p_s^*} dx + \frac{V_1}{pK} \int_{\Lambda_\varepsilon^c \cap \{u_n > a\}} u_n^p dx + o_n(1) \quad (3.7)$$

and

$$\int_{\mathbb{R}^N} g(\varepsilon x, u_n) u_n dx = \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{p_s^*} dx + \frac{V_1}{K} \int_{\Lambda_\varepsilon^c \cap \{u_n > a\}} u_n^p dx + o_n(1), \quad (3.8)$$

where we used the notation $\Lambda_\varepsilon^c = \mathbb{R}^N \setminus \Lambda_\varepsilon$. Taking into account $\langle \mathcal{J}'_\varepsilon(u_n), u_n \rangle = o_n(1)$ and (3.8), we can deduce that

$$\|u_n\|_\varepsilon^p - \frac{V_1}{K} \int_{\Lambda_\varepsilon^c \cap \{u_n > a\}} u_n^p dx = \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{p_s^*} dx + o_n(1). \quad (3.9)$$

Let $\ell \geq 0$ be such that

$$\|u_n\|_\varepsilon^p - \frac{V_1}{K} \int_{\Lambda_\varepsilon^c \cap \{u_n > a\}} u_n^p dx \rightarrow \ell.$$

It is easy to see that $\ell > 0$, otherwise $u_n \rightarrow 0$ in $\mathcal{W}_\varepsilon$, and this implies that $\mathcal{J}_\varepsilon(u_n) \rightarrow 0$ which gives a contradiction since $c_\varepsilon > 0$. It follows from (3.9) that

$$\int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{p_s^*} dx \rightarrow \ell.$$

Using $\mathcal{J}_\varepsilon(u_n) - \frac{1}{p_s^*} \langle \mathcal{J}'_\varepsilon(u_n), u_n \rangle = c_\varepsilon + o_n(1)$, (3.7) and (3.8) we can see that

$$\frac{s}{N} \ell \leq c_\varepsilon. \quad (3.10)$$

On the other hand, by the definition of S_* , we can see that

$$\|u_n\|_\varepsilon^p - \frac{V_1}{K} \int_{\Lambda_\varepsilon^c \cap \{u_n > a\}} u_n^p dx \geq S_* \left(\int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{p_s^*} dx \right)^{\frac{p}{p_s^*}},$$

and taking the limit as $n \rightarrow \infty$ we can infer that

$$\ell \geq S_* \ell^{\frac{p}{p_s^*}}. \quad (3.11)$$

Then, by $\ell > 0$, (3.10) and (3.11), we can deduce that $c_\varepsilon \geq \frac{s}{N} S_*^{\frac{N}{s}}$ which contradicts Lemma 3.2. Finally, we prove the boundedness of $\{x_n\}_{n \in \mathbb{N}}$. Let us consider the function $\eta_R \in C^\infty(\mathbb{R}^N)$ defined as

$$\eta_R(x) = \begin{cases} 0 & \text{if } x \in \mathcal{B}_R \\ 1 & \text{if } x \notin \mathcal{B}_{2R} \end{cases}$$

and $|\nabla \eta_R|_\infty \leq C/R$. Since $\langle \mathcal{J}'_\varepsilon(u_n), \eta_R u_n \rangle = o_n(1)$, we get

$$\begin{aligned} & V_1 \left(1 - \frac{1}{K} \right) \int_{\mathbb{R}^N} |u_n|^p \eta_R dx \\ & \leq \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \eta_R(x) dx dy \\ & \quad + \int_{\mathbb{R}^N} \left(V(\varepsilon x) - \frac{V_1}{K} \right) |u_n|^p \eta_R dx \\ & = - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y))}{|x - y|^{N+sp}} u_n(y) dx dy \\ & \quad + \int_{\mathbb{R}^N} \left[g(\varepsilon x, u_n) u_n - \frac{V_1}{K} |u_n|^p \right] \eta_R dx + o_n(1). \end{aligned}$$

Take $R > 0$ such that $\Lambda_\varepsilon \subset \mathcal{B}_R$. Then, by (g₃) we deduce that

$$\begin{aligned} & V_1 \left(1 - \frac{1}{K} \right) \int_{\mathbb{R}^N} |u_n|^p \eta_R dx \\ & \leq - \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y))}{|x - y|^{N+sp}} u_n(y) dx dy + o_n(1). \end{aligned} \quad (3.12)$$

Exploiting the Hölder inequality and the boundedness of $\{u_n\}_{n \in \mathbb{N}}$ in $\mathcal{W}_\varepsilon$ we have that

$$\begin{aligned} & \left| \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y))}{|x - y|^{N+sp}} u_n(y) dx dy \right| \\ & \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(y)|^p dx dy \right)^{\frac{1}{p}}. \end{aligned}$$

Now, we show that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p dx dy = 0. \quad (3.13)$$

Indeed, recalling that $0 \leq \eta_R \leq 1$, $|\nabla \eta_R|_\infty \leq C/R$ and using polar coordinates, we obtain

$$\begin{aligned}
 & \iint_{\mathbb{R}^{2N}} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p dx dy \\
 &= \int_{\mathbb{R}^N} \int_{|y-x|>R} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p dx dy \\
 &\quad + \int_{\mathbb{R}^N} \int_{|y-x|\leq R} \frac{|\eta_R(x) - \eta_R(y)|^p}{|x - y|^{N+sp}} |u_n(x)|^p dx dy \\
 &\leq C \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|y-x|>R} \frac{dy}{|x - y|^{N+sp}} \right) dx \\
 &\quad + \frac{C}{R^p} \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|y-x|\leq R} \frac{dy}{|x - y|^{N+sp-p}} \right) dx \\
 &\leq C \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|z|>R} \frac{dz}{|z|^{N+sp}} \right) dx + \frac{C}{R^p} \int_{\mathbb{R}^N} |u_n(x)|^p \left(\int_{|z|\leq R} \frac{dz}{|z|^{N+sp-p}} \right) dx \\
 &\leq C \int_{\mathbb{R}^N} |u_n(x)|^p dx \left(\int_R^\infty \frac{d\rho}{\rho^{sp+1}} \right) + \frac{C}{R^p} \int_{\mathbb{R}^N} |u_n(x)|^p dx \left(\int_0^R \frac{d\rho}{\rho^{sp-p+1}} \right) \\
 &\leq \frac{C}{R^{sp}} \int_{\mathbb{R}^N} |u_n(x)|^p dx + \frac{C}{R^p} R^{-sp+p} \int_{\mathbb{R}^N} |u_n(x)|^p dx \\
 &\leq \frac{C}{R^{sp}} \int_{\mathbb{R}^N} |u_n(x)|^p dx \leq \frac{C}{R^{sp}}
 \end{aligned}$$

where in the last passage we used the boundedness of $\{u_n\}_{n \in \mathbb{N}}$ in $\mathcal{W}_\varepsilon$.

Taking the limit as $n \rightarrow \infty$ and $R \rightarrow \infty$, we can see that (3.13) holds true. Then, if $\{x_n\}_{n \in \mathbb{N}}$ is unbounded, by (3.6), (3.12) and (3.13), we can infer that

$$0 < V_1 \left(1 - \frac{1}{K} \right) \beta \leq 0$$

that is a contradiction. In conclusion, the sequence $\{x_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}^N$. $\square$

Now, we are ready to provide an existence result for (3.2).

Theorem 3.1 *Assume that (V_1) – (V_2) hold, and suppose that (f_1) – (f_4) hold if $\gamma = 0$, and (f_1) , (f'_2) , (f_3) , (f_4) hold if $\gamma = 1$. Then, problem (3.2) admits a positive ground state for all $\varepsilon > 0$.*

Proof Taking into account Lemmas 3.1–3.4, we can see that $\mathcal{J}_\varepsilon$ admits a nontrivial critical point $u \in \mathcal{W}_\varepsilon$. Since $\langle \mathcal{J}'_\varepsilon(u), u^- \rangle = 0$, where $u^- = \min\{u, 0\}$, it is easy to check that $u \geq 0$ in $\mathbb{R}^N$. Indeed, using $g(x, t) = 0$ for $t \leq 0$ and $|x - y|^{p-2}(x - y)(x^- - y^-) \geq |x^- - y^-|^p$, where $x^- = \min\{x, 0\}$, we get

$$\begin{aligned}
 \|u^-\|_\varepsilon^p &\leq \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} (u^-(x) - u^-(y)) dx dy \\
 &\quad + \int_{\mathbb{R}^N} V(\varepsilon x) |u|^{p-2} u u^- dx \\
 &= \int_{\mathbb{R}^N} g(\varepsilon x, u) u^- dx = 0,
 \end{aligned}$$

which implies that $u^- = 0$, that is $u \geq 0$. Moreover, proceeding as in the proof of Lemma 4.1, we can see that $u \in L^\infty(\mathbb{R}^N)$. By Corollary 5.5 in [27], we deduce that $u \in C^{0,\alpha}(\mathbb{R}^N)$, and applying maximum principle [14] we can conclude that $u > 0$ in $\mathbb{R}^N$.

Finally, we show that u is a ground state solution to (3.2). Indeed, in view of (g_3) and applying Fatou's Lemma we obtain

$$\begin{aligned} c_\varepsilon &\leq \mathcal{J}_\varepsilon(u) - \frac{1}{\vartheta} \langle \mathcal{J}'_\varepsilon(u), u \rangle \\ &= \left(\frac{1}{p} - \frac{1}{\vartheta} \right) \|u\|_\varepsilon^p + \int_{\mathbb{R}^N} \frac{1}{\vartheta} g(\varepsilon x, u) u - G(\varepsilon x, u) \, dx \\ &\leq \liminf_{n \rightarrow \infty} \left[\left(\frac{1}{p} - \frac{1}{\vartheta} \right) \|u_n\|_\varepsilon^p + \int_{\mathbb{R}^N} \frac{1}{\vartheta} g(\varepsilon x, u_n) u_n - G(\varepsilon x, u_n) \, dx \right] \\ &= \liminf_{n \rightarrow \infty} \left[\mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta} \langle \mathcal{J}'_\varepsilon(u_n), u_n \rangle \right] = c_\varepsilon \end{aligned}$$

which implies that $\mathcal{J}_\varepsilon(u) = c_\varepsilon$. $\square$

Now, we deal with the following family of autonomous problems, with $\mu > 0$

$$\begin{cases} (-\Delta)_p^s u + \mu u = f(u) + \gamma |u|^{p_s^*-2} u & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (3.14)$$

It is clear that the Euler–Lagrange functional associated with (3.14) is given by

$$\begin{aligned} \mathcal{I}_\mu(u) &= \frac{1}{p} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx dy + \mu \int_{\mathbb{R}^N} |u|^p \, dx \right) \\ &\quad - \int_{\mathbb{R}^N} F(u) \, dx - \frac{\gamma}{p_s^*} \int_{\mathbb{R}^N} (u^+)^{p_s^*} \, dx. \end{aligned}$$

Let us denote by $\mathcal{W}_\mu$ the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\mu^p = [u]_{s,p}^p + \mu \|u\|_p^p.$$

The Nehari manifold associated with $\mathcal{I}_\mu$ is given by

$$\mathcal{M}_\mu = \{u \in \mathcal{W}_\mu \setminus \{0\} : \langle \mathcal{I}'_\mu(u), u \rangle = 0\}.$$

It is easy to check that $\mathcal{I}_\mu$ has a mountain pass geometry, and we denote by m_μ its mountain pass level. Moreover, standard arguments (see [42]) show that

$$m_\mu = \inf_{\mathcal{M}_\mu} \mathcal{I}_\mu = \inf_{u \in \mathcal{W}_\mu \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\mu(tu).$$

As proved in Lemma 3.9 and Lemma 4.14 in [8], we know that

Theorem 3.2 *For all $\mu > 0$, problem (3.14) admits a positive ground state solution.*

In what follows, we establish the following useful relation between c_ε and m_{V_0} :

Lemma 3.5 *It holds $\limsup_{\varepsilon \rightarrow 0} c_\varepsilon \leq m_{V_0}$.*

Proof Let $\omega_\varepsilon(x) = \psi_\varepsilon(x)\omega(x)$, where ω is a positive ground state of (3.14) which is given by Theorem 3.2 with $\mu = V_0$, and $\psi_\varepsilon(x) = \psi(\varepsilon x)$ with $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$,

$\psi(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\psi(x) = 0$ if $|x| \geq 1$. For simplicity, we assume that $\text{supp}(\psi) \subset \mathcal{B}_1 \subset \Lambda$. Using Lemma 2.2 and the dominated convergence theorem, we can see that

$$\omega_\varepsilon \rightarrow \omega \quad \text{in } W^{s,p}(\mathbb{R}^N) \quad \text{and} \quad \mathcal{I}_{V_0}(\omega_\varepsilon) \rightarrow \mathcal{I}_{V_0}(\omega) = m_{V_0} \quad (3.15)$$

as $\varepsilon \rightarrow 0$. Now, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\mathcal{J}_\varepsilon(t_\varepsilon \omega_\varepsilon) = \max_{t \geq 0} \mathcal{J}_\varepsilon(t \omega_\varepsilon).$$

Then, $\langle \mathcal{J}'_\varepsilon(t_\varepsilon \omega_\varepsilon), \omega_\varepsilon \rangle = 0$ and this implies that

$$[\omega_\varepsilon]_{s,p}^p + \int_{\mathbb{R}^N} V(\varepsilon x) |\omega_\varepsilon|^p dx = \int_{\mathbb{R}^N} \frac{f(t_\varepsilon \omega_\varepsilon)}{(t_\varepsilon \omega_\varepsilon)^{p-1}} \omega_\varepsilon^p dx + \gamma t_\varepsilon^{p_s^*-p} \int_{\mathbb{R}^N} |\omega_\varepsilon|^{p_s^*} dx. \quad (3.16)$$

From (3.15), (3.16) and the growth assumptions on f , we obtain that $t_\varepsilon \rightarrow t_0 > 0$.

Taking the limit as $\varepsilon \rightarrow 0$ in (3.16) we get

$$[\omega]_{s,p}^p + \int_{\mathbb{R}^N} V_0 |\omega|^p dx = \int_{\mathbb{R}^N} \frac{f(t_0 \omega)}{(t_0 \omega)^{p-1}} \omega^p dx + \gamma t_0^{p_s^*-p} \int_{\mathbb{R}^N} |\omega|^{p_s^*} dx \quad (3.17)$$

which together with (f_4) , $\omega \in \mathcal{M}_{V_0}$ and (3.17) implies that $t_0 = 1$.

On the other hand, we can note that

$$c_\varepsilon \leq \max_{t \geq 0} \mathcal{J}_\varepsilon(t \omega_\varepsilon) = \mathcal{J}_\varepsilon(t_\varepsilon \omega_\varepsilon) = \mathcal{I}_{V_0}(t_\varepsilon \omega_\varepsilon) + \frac{t_\varepsilon^p}{p} \int_{\mathbb{R}^N} (V(\varepsilon x) - V_0) \omega_\varepsilon^p dx.$$

Since $V(\varepsilon x)$ is bounded on the support of ω_ε , by the dominated convergence theorem, (3.15) and the above inequality, we can conclude the proof. $\square$

We conclude this section by proving the following compactness result which will be fundamental for showing that, for $\varepsilon > 0$ small enough the solutions of the modified problem are also solutions of the original one.

Lemma 3.6 *Let $\varepsilon_n \rightarrow 0$ and $\{u_n\}_{n \in \mathbb{N}} := \{u_{\varepsilon_n}\}_{n \in \mathbb{N}} \subset \mathcal{W}_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$ and $\mathcal{J}'_{\varepsilon_n}(u_n) = 0$. Then, there exists $\{\tilde{y}_n\}_{n \in \mathbb{N}} := \{\tilde{y}_{\varepsilon_n}\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ such that the translated sequence*

$$\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$$

has a subsequence which strongly converges in $W^{s,p}(\mathbb{R}^N)$. Moreover, up to a subsequence, $\{y_n\}_{n \in \mathbb{N}} := \{\varepsilon_n \tilde{y}_n\}_{n \in \mathbb{N}}$ is such that $y_n \rightarrow y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$.

Proof Using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and (g_1) , (g_2) , it is easy to see that there is $\kappa > 0$ such that

$$\|u_n\|_{\varepsilon_n} \geq \kappa > 0 \quad \text{for any } n \in \mathbb{N}.$$

Taking into account $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$, $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and Lemma 3.5, we can argue as in the proof of Lemma 3.3 to deduce that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{W}_{\varepsilon_n}$. Therefore, proceeding as in Lemma 3.4, we can find a sequence $\{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \alpha > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{\mathcal{B}_R(\tilde{y}_n)} |u_n|^p dx \geq \alpha.$$

Set $\tilde{u}_n(x) := u_n(x + \tilde{y}_n)$. Then, $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ is bounded in $W^{s,p}(\mathbb{R}^N)$, and we may assume that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{weakly in } W^{s,p}(\mathbb{R}^N). \quad (3.18)$$

Moreover, $\tilde{u} \neq 0$ in view of

$$\int_{\mathcal{B}_R} |\tilde{u}|^p dx \geq \alpha. \quad (3.19)$$

Now, we set $y_n := \varepsilon_n \tilde{y}_n$. Let us begin by proving that $\{y_n\}_{n \in \mathbb{N}}$ is bounded. To this end, it is enough to show the following claim:

Claim 1 $\lim_{n \rightarrow \infty} \text{dist}(y_n, \overline{\Lambda}) = 0$.

Indeed, if the claim does not hold, there is $\delta > 0$ and a subsequence of $\{y_n\}_{n \in \mathbb{N}}$, still denoted by itself, such that

$$\text{dist}(y_n, \overline{\Lambda}) \geq \delta \quad \forall n \in \mathbb{N}.$$

Then, we can find $r > 0$ such that $\mathcal{B}_r(y_n) \subset \Lambda^c$ for all $n \in \mathbb{N}$. Since $\tilde{u} \geq 0$ and $\mathcal{C}_c^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, we can find a sequence $\{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\psi_j \geq 0$ and $\psi_j \rightarrow \tilde{u}$ in $W^{s,p}(\mathbb{R}^N)$. Fixed $j \in \mathbb{N}$ and using $\psi = \psi_j$ as test function in $\langle \mathcal{J}'_{\varepsilon_n}(u_n), \psi \rangle = 0$ we get

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2} (\tilde{u}_n(x) - \tilde{u}_n(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy \\ & + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) \tilde{u}_n^{p-1} \psi_j dx \\ & = \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) \psi_j dx. \end{aligned} \quad (3.20)$$

Recalling that $u_{\varepsilon_n}, \psi_j \geq 0$ and the definition of g , we have

$$\begin{aligned} \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) \psi_j dx &= \int_{\mathcal{B}_{r/\varepsilon_n}} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) \psi_j dx \\ &+ \int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) \psi_j dx \\ &\leq \frac{V_1}{K} \int_{\mathcal{B}_{r/\varepsilon_n}} \tilde{u}_n^{p-1} \psi_j dx \\ &+ \int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} \left(f(\tilde{u}_n) \psi_j + \gamma \tilde{u}_n^{p_s^*-1} \psi_j \right) dx \end{aligned}$$

which together with (3.20) implies that

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2} (\tilde{u}_n(x) - \tilde{u}_n(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy + A \int_{\mathbb{R}^N} \tilde{u}_n^{p-1} \psi_j dx \\ & \leq \int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} \left(f(\tilde{u}_n) \psi_j + \gamma \tilde{u}_n^{p_s^*-1} \psi_j \right) dx, \end{aligned} \quad (3.21)$$

where $A = V_1(1 - \frac{1}{K})$. By (3.18), ψ_j has compact support in $\mathbb{R}^N$ and $\varepsilon_n \rightarrow 0$, we can deduce that as $n \rightarrow \infty$

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2} (\tilde{u}_n(x) - \tilde{u}_n(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy \\ & \rightarrow \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2} (\tilde{u}(x) - \tilde{u}(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy \end{aligned}$$

and

$$\int_{\mathbb{R}^N \setminus \mathcal{B}_{r/\varepsilon_n}} \left(f(\tilde{u}_n) \psi_j + \gamma \tilde{u}_n^{p_s^*-1} \psi_j \right) dx \rightarrow 0.$$

The above limits and (3.21) give

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2} (\tilde{u}(x) - \tilde{u}(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy + A \int_{\mathbb{R}^N} \tilde{u}^{p-1} \psi_j dx \leq 0,$$

and taking the limit as $j \rightarrow \infty$ we obtain

$$\|\tilde{u}\|_A^p = [\tilde{u}]_{s,p}^p + A \|\tilde{u}\|_p^p \leq 0$$

which contradicts (3.19). Hence, there exists a subsequence of $\{y_n\}_{n \in \mathbb{N}}$ such that $y_n \rightarrow y_0 \in \bar{\Lambda}$.

Claim 2 $y_0 \in \Lambda$. From (g₂) and (3.20), we can see that

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|^{p-2} (\tilde{u}_n(x) - \tilde{u}_n(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy \\ & \quad + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) \tilde{u}_n^{p-1} \psi_j dx \\ & \leq \int_{\mathbb{R}^N} (f(\tilde{u}_n) + \gamma \tilde{u}_n^{p_s^*-1}) \psi_j dx. \end{aligned}$$

Letting $n \rightarrow \infty$, we find

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2} (\tilde{u}(x) - \tilde{u}(y)) (\psi_j(x) - \psi_j(y))}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} V(y_0) \tilde{u}^{p-1} \psi_j dx \\ & \leq \int_{\mathbb{R}^N} (f(\tilde{u}) + \gamma \tilde{u}^{p_s^*-1}) \psi_j dx, \end{aligned}$$

and passing to the limit as $j \rightarrow \infty$ we have

$$\iint_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} V(y_0) \tilde{u}^p dx \leq \int_{\mathbb{R}^N} (f(\tilde{u}) + \gamma \tilde{u}^{p_s^*-1}) \tilde{u} dx.$$

Then, there exists $\tau \in (0, 1)$ such that $\tau \tilde{u} \in \mathcal{M}_{V(y_0)}$. Therefore, denoting by $m_{V(y_0)}$ the mountain pass level associated with $\mathcal{I}_{V(y_0)}$, we have

$$m_{V(y_0)} \leq \mathcal{I}_{V(y_0)}(\tau \tilde{u}) \leq \liminf_{n \rightarrow \infty} \mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) = \liminf_{n \rightarrow \infty} c_{\varepsilon_n} \leq m_{V_0},$$

from which we deduce that $V(y_0) \leq V(0) = V_0$. Since $V_0 = \inf_{\bar{\Lambda}} V$, we can infer that $V(y_0) = V_0$. Using (V₂), we can see that $y_0 \notin \partial \Lambda$, that is $y_0 \in \Lambda$.

Claim 3 $\tilde{u}_n \rightarrow \tilde{u}$ in $W^{s,p}(\mathbb{R}^N)$ as $n \rightarrow \infty$.

Let us define

$$\tilde{\Lambda}_n = \frac{\Lambda - \varepsilon_n \tilde{y}_n}{\varepsilon_n}$$

and consider

$$\begin{aligned}\tilde{\chi}_n^1(x) &= \begin{cases} 1 & \text{if } x \in \tilde{\Lambda}_n \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \tilde{\Lambda}_n \end{cases} \\ \tilde{\chi}_n^2(x) &= 1 - \tilde{\chi}_n^1(x).\end{aligned}$$

Now, we introduce the following functions for all $x \in \mathbb{R}^N$

$$\begin{aligned}h_n^1(x) &= \left(\frac{1}{p} - \frac{1}{\vartheta}\right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) |\tilde{u}_n(x)|^p \tilde{\chi}_n^1(x) \\ h^1(x) &= \left(\frac{1}{p} - \frac{1}{\vartheta}\right) V(y_0) |\tilde{u}(x)|^p \\ h_n^2(x) &= \left[\left(\frac{1}{p} - \frac{1}{\vartheta}\right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) |\tilde{u}_n(x)|^p \right. \\ &\quad \left. + \frac{1}{\vartheta} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n(x)) \tilde{u}_n(x) - G(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n(x))\right] \tilde{\chi}_n^2(x) \\ &\geq \left(\left(\frac{1}{p} - \frac{1}{\vartheta}\right) - \frac{1}{K}\right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) |\tilde{u}_n(x)|^p \tilde{\chi}_n^2(x) \\ h_n^3(x) &= \left(\frac{1}{\vartheta} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n(x)) \tilde{u}_n(x) - G(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n(x))\right) \tilde{\chi}_n^1(x) \\ &= \left[\frac{1}{\vartheta} \left(f(\tilde{u}_n(x)) \tilde{u}_n(x) + \gamma |\tilde{u}_n(x)|^{p_s^*}\right) - \left(F(\tilde{u}_n(x)) + \frac{\gamma}{p_s^*} |\tilde{u}_n(x)|^{p_s^*}\right)\right] \tilde{\chi}_n^1(x) \\ h^3(x) &= \frac{1}{\vartheta} \left(f(\tilde{u}(x)) \tilde{u}(x) + \gamma |\tilde{u}(x)|^{p_s^*}\right) - \left(F(\tilde{u}(x)) + \frac{\gamma}{p_s^*} |\tilde{u}(x)|^{p_s^*}\right).\end{aligned}$$

In view of (f_3) and (g_3) , we can observe that the above functions are nonnegative. Moreover, by (3.18) and Claim 2, we know that

$$\begin{aligned}\tilde{u}_n(x) &\rightarrow \tilde{u}(x) \quad \text{a.e. } x \in \mathbb{R}^N, \\ \varepsilon_n \tilde{y}_n &\rightarrow y_0 \in \Lambda,\end{aligned}$$

which imply that

$$\tilde{\chi}_n^1(x) \rightarrow 1, \quad h_n^1(x) \rightarrow h^1(x), \quad h_n^2(x) \rightarrow 0 \quad \text{and} \quad h_n^3(x) \rightarrow h^3(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

Then, applying Lemma 3.5, Fatou's Lemma and using a change of variable we can see that

$$\begin{aligned}m_{V_0} &\geq \limsup_{n \rightarrow \infty} c_{\varepsilon_n} = \limsup_{n \rightarrow \infty} \left(\mathcal{J}_{\varepsilon_n}(u_n) - \frac{1}{\vartheta} \langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle \right) \\ &\geq \limsup_{n \rightarrow \infty} \left[\left(\frac{1}{p} - \frac{1}{\vartheta}\right) [\tilde{u}_n]_{s,p}^p + \int_{\mathbb{R}^N} (h_n^1 + h_n^2 + h_n^3) dx \right] \\ &\geq \liminf_{n \rightarrow \infty} \left[\left(\frac{1}{p} - \frac{1}{\vartheta}\right) [\tilde{u}_n]_{s,p}^p + \int_{\mathbb{R}^N} (h_n^1 + h_n^2 + h_n^3) dx \right] \\ &\geq \left(\frac{1}{p} - \frac{1}{\vartheta}\right) [\tilde{u}]_{s,p}^p + \int_{\mathbb{R}^N} (h^1 + h^3) dx \geq m_{V_0}.\end{aligned}$$

Accordingly,

$$\lim_{n \rightarrow \infty} [\tilde{u}_n]_{s,p}^p = [\tilde{u}]_{s,p}^p \quad (3.22)$$

and

$$h_n^1 \rightarrow h^1, h_n^2 \rightarrow 0 \text{ and } h_n^3 \rightarrow h^3 \text{ in } L^1(\mathbb{R}^N).$$

Hence,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) |\tilde{u}_n|^p dx = \int_{\mathbb{R}^N} V(y_0) |\tilde{u}|^p dx,$$

from which we deduce that

$$\lim_{n \rightarrow \infty} |\tilde{u}_n|_p^p = |\tilde{u}|_p^p. \quad (3.23)$$

Putting together (3.22) and (3.23) and using Brezis–Lieb Lemma [11], we obtain

$$\|\tilde{u}_n - \tilde{u}\|_{s,p}^p = \|\tilde{u}_n\|_{s,p}^p - \|\tilde{u}\|_{s,p}^p + o_n(1) = o_n(1).$$

This fact ends the proof of Lemma. $\square$

4 Proof of Theorem 1.1

This last section is devoted to the proof of Theorem 1.1. Firstly, we use a Moser iteration argument [35] to prove the following useful L^∞ -estimate for the solutions of the modified problem (3.2).

Lemma 4.1 *Let $\varepsilon_n \rightarrow 0$ and $u_n \in \mathcal{W}_{\varepsilon_n}$ be a mountain pass solution to (3.2). Then, up to a subsequence, $v_n := u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N)$, where $\{\tilde{y}_n\}_{n \in \mathbb{N}}$ is defined as in Lemma 3.6, there exists $C > 0$ such that*

$$|v_n|_\infty \leq C \quad \text{for all } n \in \mathbb{N}.$$

Moreover,

$$|v_n(x)| \rightarrow 0 \text{ as } |x| \rightarrow \infty, \text{ uniformly in } n \in \mathbb{N}.$$

Proof For any $L > 0$ and $\beta > 1$, let us define the function

$$\gamma(v_n) = \gamma_{L,\beta}(v_n) = v_n v_{L,n}^{p(\beta-1)} \in \mathcal{W}_\varepsilon$$

where $v_{L,n} = \min\{v_n, L\}$. Since γ is an increasing function, we have

$$(a - b)(\gamma(a) - \gamma(b)) \geq 0 \quad \text{for any } a, b \in \mathbb{R}.$$

Let us introduce the following functions

$$\mathcal{E}(t) = \frac{|t|^p}{p} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(\tau))^{\frac{1}{p}} d\tau.$$

Then, applying Jensen's inequality, we get for all $a, b \in \mathbb{R}$ such that $a > b$,

$$\begin{aligned} \mathcal{E}'(a - b)(\gamma(a) - \gamma(b)) &= (a - b)^{p-1}(\gamma(a) - \gamma(b)) = (a - b)^{p-1} \int_b^a \gamma'(t) dt \\ &= (a - b)^{p-1} \int_b^a (\Gamma'(t))^p dt \geq \left(\int_b^a (\Gamma'(t)) dt \right)^p. \end{aligned}$$

Since the same argument works when $a \leq b$, we can deduce that

$$\mathcal{E}'(a - b)(\gamma(a) - \gamma(b)) \geq |\Gamma(a) - \Gamma(b)|^p \text{ for any } a, b \in \mathbb{R}.$$

The above inequality implies that

$$|\Gamma(v_n)(x) - \Gamma(v_n)(y)|^p \leq |v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y)) ((v_n v_{L,n}^{p(\beta-1)})(x) - (v_n v_{L,n}^{p(\beta-1)})(y)). \quad (4.1)$$

Choosing $\gamma(v_n) = v_n v_{L,n}^{p(\beta-1)}$ as test function in (3.2) and using (4.1) we obtain

$$\begin{aligned} & [\Gamma(v_n)]_{s,p}^p + \int_{\mathbb{R}^N} V_n(x) |v_n|^p v_{L,n}^{p(\beta-1)} dx \\ & \leq \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2} (v_n(x) - v_n(y))}{|x - y|^{N+sp}} ((v_n v_{L,n}^{p(\beta-1)})(x) - (v_n v_{L,n}^{p(\beta-1)})(y)) dx dy \\ & \quad + \int_{\mathbb{R}^N} V_n(x) |v_n|^p v_{L,n}^{p(\beta-1)} dx \\ & = \int_{\mathbb{R}^N} g_n(v_n) v_n v_{L,n}^{p(\beta-1)} dx, \end{aligned} \quad (4.2)$$

where we used the notations $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $g_n(v_n) = g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n)$. Observing that

$$\Gamma(v_n) \geq \frac{1}{\beta} v_n v_{L,n}^{\beta-1},$$

and applying Theorem 2.1, we have

$$[\Gamma(v_n)]_{s,p}^p \geq C_*^{-1} |\Gamma(v_n)|_{p_s^*}^p \geq \left(\frac{1}{\beta}\right)^p C_*^{-1} |v_n v_{L,n}^{\beta-1}|_{p_s^*}^p. \quad (4.3)$$

From assumptions (g_1) and (g_2) , for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$|g_n(v_n)| \leq \xi |v_n|^{p-1} + C_\xi |v_n|^{p_s^*-1}. \quad (4.4)$$

Taking $\xi \in (0, V_1)$, and using (4.3) and (4.4), we can see that (4.2) yields

$$|w_{L,n}|_{p_s^*}^p \leq C \beta^p \int_{\mathbb{R}^N} |v_n|^{p_s^*} v_{L,n}^{p(\beta-1)} dx, \quad (4.5)$$

where $w_{L,n} := v_n v_{L,n}^{\beta-1}$. Now, we take $\beta = \frac{p_s^*}{p}$ and fix $R > 0$. Noting that $0 \leq v_{L,n} \leq v_n$, we can infer that

$$\begin{aligned} \int_{\mathbb{R}^N} v_n^{p_s^*} v_{L,n}^{p(\beta-1)} dx &= \int_{\mathbb{R}^N} v_n^{p_s^*-p} v_n^p v_{L,n}^{p_s^*-p} dx \\ &= \int_{\mathbb{R}^N} v_n^{p_s^*-p} (v_n v_{L,n}^{\frac{p_s^*-p}{p}})^p dx \\ &\leq \int_{\{v_n < R\}} R^{p_s^*-p} v_n^{p_s^*} dx + \int_{\{v_n > R\}} v_n^{p_s^*-p} (v_n v_{L,n}^{\frac{p_s^*-p}{p}})^p dx \\ &\leq \int_{\{v_n < R\}} R^{p_s^*-p} v_n^{p_s^*} dx \\ &\quad + \left(\int_{\{v_n > R\}} v_n^{p_s^*} dx \right)^{\frac{p_s^*-p}{p_s^*}} \left(\int_{\mathbb{R}^N} (v_n v_{L,n}^{\frac{p_s^*-p}{p}})^{p_s^*} dx \right)^{\frac{p}{p_s^*}}. \end{aligned} \quad (4.6)$$

Since $\{v_n\}_{n \in \mathbb{N}}$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$, we can see that for any R sufficiently large

$$\left(\int_{\{v_n > R\}} v_n^{p_s^*} dx \right)^{\frac{p_s^* - p}{p_s^*}} \leq \frac{1}{2C\beta^p}. \quad (4.7)$$

In light of (4.5), (4.6) and (4.7), we get

$$\left(\int_{\mathbb{R}^N} (v_n v_{L,n}^{\frac{p_s^* - p}{p}})^{p_s^*} dx \right)^{\frac{p}{p_s^*}} \leq C\beta^p \int_{\mathbb{R}^N} R^{p_s^* - p} v_n^{p_s^*} dx < \infty$$

and taking the limit as $L \rightarrow \infty$, we obtain $v_n \in L^{\frac{(p_s^*)^2}{p}}(\mathbb{R}^N)$.

Now, using $0 \leq v_{L,n} \leq v_n$ and passing to the limit as $L \rightarrow \infty$ in (4.5), we have

$$|v_n|_{\beta p_s^*}^{\beta p} \leq C\beta^p \int_{\mathbb{R}^N} v_n^{p_s^* + p(\beta - 1)} dx,$$

from which we deduce that

$$\left(\int_{\mathbb{R}^N} v_n^{\beta p_s^*} dx \right)^{\frac{1}{(\beta - 1)p_s^*}} \leq (C\beta)^{\frac{1}{\beta - 1}} \left(\int_{\mathbb{R}^N} v_n^{p_s^* + p(\beta - 1)} dx \right)^{\frac{1}{p(\beta - 1)}}.$$

For $m \geq 1$, we define β_{m+1} inductively so that $p_s^* + p(\beta_{m+1} - 1) = p_s^* \beta_m$ and $\beta_1 = \frac{p_s^*}{p}$. Then, we have

$$\left(\int_{\mathbb{R}^N} v_n^{\beta_{m+1} p_s^*} dx \right)^{\frac{1}{(\beta_{m+1} - 1)p_s^*}} \leq (C\beta_{m+1})^{\frac{1}{\beta_{m+1} - 1}} \left(\int_{\mathbb{R}^N} v_n^{p_s^* \beta_m} dx \right)^{\frac{1}{p_s^* (\beta_m - 1)}}.$$

Let us define

$$D_m = \left(\int_{\mathbb{R}^N} v_n^{p_s^* \beta_m} dx \right)^{\frac{1}{p_s^* (\beta_m - 1)}}.$$

Using a standard iteration argument, we can find $C_0 > 0$ independent of m such that

$$D_{m+1} \leq \prod_{k=1}^m (C\beta_{k+1})^{\frac{1}{\beta_{k+1} - 1}} D_1 \leq C_0 D_1.$$

Taking the limit as $m \rightarrow \infty$, we get $|v_n|_\infty \leq K$ for all $n \in \mathbb{N}$. Moreover, from Corollary 5.5 in [27] (see also [17]), we can deduce that $v_n \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha > 0$ (independent of n) and $[v_n]_{C^{0,\alpha}(\mathbb{R}^N)} \leq C$, with C independent of n . Since $v_n \rightarrow v$ in $W^{s,p}(\mathbb{R}^N)$ (see Lemma 3.6), we can infer that $\lim_{|x| \rightarrow \infty} v_n(x) = 0$ uniformly in $n \in \mathbb{N}$. $\square$

Now, we are ready to give the proof of our main result.

Proof of Theorem 1.1 We begin by proving that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any mountain pass solution $u_\varepsilon \in \mathcal{W}_\varepsilon$ of (3.2), it holds

$$|u_\varepsilon|_{L^\infty(\mathbb{R}^N \setminus \Lambda_\varepsilon)} < a. \quad (4.8)$$

Assume by contradiction that for some subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $\varepsilon_n \rightarrow 0$, we can find $u_n := u_{\varepsilon_n} \in \mathcal{W}_{\varepsilon_n}$ such that $\mathcal{J}_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$, $\mathcal{J}'_{\varepsilon_n}(u_n) = 0$ and

$$|u_n|_{L^\infty(\mathbb{R}^N \setminus \Lambda_{\varepsilon_n})} \geq a. \quad (4.9)$$

In view of Lemma 3.6, we can find $\{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ such that $\tilde{u}_n = u_n(\cdot + \tilde{y}_n) \rightarrow \tilde{u}$ in $W^{s,p}(\mathbb{R}^N)$ and $\varepsilon_n \tilde{y}_n \rightarrow y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$.

Now, if we choose $r > 0$ such that $\mathcal{B}_r(y_0) \subset \mathcal{B}_{2r}(y_0) \subset \Lambda$, we can see that $\mathcal{B}_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Then, for any $y \in \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, it holds

$$\left| y - \frac{y_0}{\varepsilon_n} \right| \leq |y - \tilde{y}_n| + \left| \tilde{y}_n - \frac{y_0}{\varepsilon_n} \right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \quad \text{for } n \text{ sufficiently large.}$$

Therefore,

$$\mathbb{R}^N \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \setminus \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \quad (4.10)$$

for any n big enough. Using Lemma 4.1, we can see that

$$\tilde{u}_n(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \quad (4.11)$$

uniformly in $n \in \mathbb{N}$. Therefore, there exists $R > 0$ such that

$$\tilde{u}_n(x) < a \quad \text{for } |x| \geq R, n \in \mathbb{N}.$$

Hence, $u_n(x) < a$ for any $x \in \mathbb{R}^N \setminus \mathcal{B}_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, by (4.10), there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ we have

$$\mathbb{R}^N \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \setminus \mathcal{B}_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \mathbb{R}^N \setminus \mathcal{B}_R(\tilde{y}_n),$$

which implies that $u_n(x) < a$ for any $x \in \mathbb{R}^N \setminus \Lambda_{\varepsilon_n}$ and $n \geq \nu$. This is impossible in view of (4.9). Now, since $u_\varepsilon \in \mathcal{W}_\varepsilon$ satisfies (4.8), by the definition of g , it follows that u_ε is a solution of (3.1). Consequently, $\hat{u}_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a solution to (1.1), and we can conclude that (1.1) has a nontrivial solution. Finally, we study the behavior of the maximum points of solutions to problem (1.1). Take $\varepsilon_n \rightarrow 0$ and consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{W}_{\varepsilon_n}$ of solutions to (3.1) as above. Let us observe that (g_1) implies that there exists $\omega \in (0, a)$ such that

$$g(\varepsilon_n x, t) = f(t)t + \gamma t^{p_s^*} \leq \frac{V_1}{K} t^p \quad \text{for any } x \in \mathbb{R}^N, t \leq \omega. \quad (4.12)$$

Arguing as before, we can find $R > 0$ such that

$$|u_n|_{L^\infty(\mathbb{R}^N \setminus \mathcal{B}_R(\tilde{y}_n))} < \omega. \quad (4.13)$$

Moreover, up to subsequences, we may assume that

$$|u_n|_{L^\infty(\mathcal{B}_R(\tilde{y}_n))} \geq \omega. \quad (4.14)$$

Indeed, if (4.14) does not hold, in view of (4.13) we can see that $|u_n|_\infty < \omega$. Then, using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and (4.12), we can infer

$$\|u_n\|_{\varepsilon_n}^p = \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) u_n \, dx \leq \frac{V_1}{K} \int_{\mathbb{R}^N} |u_n|^p \, dx$$

which yields $\|u_n\|_{\varepsilon_n} = 0$, and this is impossible. Hence, (4.14) holds true.

Taking into account (4.13) and (4.14), we can deduce that the maximum points $p_n \in \mathbb{R}^N$ of u_n belong to $\mathcal{B}_R(\tilde{y}_n)$. Therefore, $p_n = \tilde{y}_n + q_n$, for some $q_n \in \mathcal{B}_R$. Consequently, $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$ is the maximum point of $\hat{u}_n(x) = u_n(x/\varepsilon_n)$. Since $|q_n| < R$ for any $n \in \mathbb{N}$ and $\varepsilon_n \tilde{y}_n \rightarrow y_0$, from the continuity of V we can infer that

$$\lim_{n \rightarrow \infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0.$$

Next, we prove a decay estimate for $\hat{u}_n$. For this purpose, using Lemma 7.1 in [15], we can find a continuous positive function w and a constant $C > 0$ such that for large $|x| > R_0$ it holds that $w(x) \leq \frac{C}{1+|x|^{N+sp}}$ and $(-\Delta)_p^s w + \frac{V_0}{2} w^{p-1} \geq 0$. On the other hand, by (4.11) and (g_1) , it follows that for some large $R_1 > 0$

$$(-\Delta)_p^s \tilde{u}_n + \frac{V_0}{2} \tilde{u}_n^{p-1} \leq g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) - \frac{V_0}{2} \tilde{u}_n^{p-1} \leq 0 \quad \text{for } |x| > R_1.$$

In view of the continuity of $\tilde{u}_n$ and w , there exists a constant $C_1 > 0$ such that $z_n := \tilde{u}_n - C_1 w \leq 0$ on $|x| = R_2$, where $R_2 = \max\{R_0, R_1\}$. Then, we can argue as in Remark 3 in [8], replacing the function $\Gamma(x) = |x|^{-\frac{N-sp}{p-1}}$ by $w(x)$, to prove that $z_n \leq 0$ for $|x| \geq R_2$, that is $\tilde{u}_n \leq C_1 w$ for $|x| \geq R_2$. This fact together with $\hat{u}_n(x) = u_n(\frac{x}{\varepsilon_n}) = \tilde{u}_n(\frac{x}{\varepsilon_n} - \tilde{y}_n)$ yields

$$\begin{aligned} \hat{u}_n(x) &= u_n\left(\frac{x}{\varepsilon_n}\right) = \tilde{u}_n\left(\frac{x}{\varepsilon_n} - \tilde{y}_n\right) \\ &\leq \frac{\tilde{C}}{1 + \left|\frac{x}{\varepsilon_n} - \tilde{y}_n\right|^{N+sp}} \\ &= \frac{\tilde{C} \varepsilon_n^{N+sp}}{\varepsilon_n^{N+sp} + |x - \varepsilon_n \tilde{y}_n|^{N+sp}} \\ &\leq \frac{\tilde{C} \varepsilon_n^{N+sp}}{\varepsilon_n^{N+sp} + |x - \eta_{\varepsilon_n}|^{N+sp}}. \end{aligned}$$

This ends the proof of Theorem 1.1. $\square$

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