An analogue of the Wielandt subgroup in infinite groups

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Abstract

In this paper we define analogues of the Wielandt subgroup of a group. We say that a subgroup H of a group G is f -subnormal in G if there is a finite chain of subgroups $H = H_0 \leq H_1 \leq \dots \leq H_n = G$ such that either $|H_{i+1} : H_i|$ is finite or H_i is normal in H_{i+1} , for $0 \leq i \leq n-1$. We study in a group G the connection between the subgroups $\overline{w}(G)$ and $\overline{w}_i(G)$ which are, respectively, the sets of elements of G normalizing all f -subnormal subgroups of G and those normalizing all infinite f -subnormal subgroups of G . In particular we show that $\overline{w}_i(G)/\overline{w}(G)$ is always Dedekind and often abelian.

Keywords f -subnormal subgroup · Generalized f -Wielandt subgroup · f -Wielandt subgroup · Dedekind group · Subsoluble group

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1 Introduction

A subgroup H of a group G is called f -subnormal in G if there is a finite chain of subgroups

$$H = H_0 \leq H_1 \leq \dots \leq H_n = G \quad (1)$$

such that either $|H_{i+1} : H_i|$ is finite or H_i is normal in H_{i+1} , for $0 \leq i \leq n-1$. In this case we call the chain of subgroups (1) an f -subnormal series of length n for H in G . Clearly

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subnormal subgroups and subgroups of finite index in G are f -subnormal in G . In particular all subgroups of a finite group are f -subnormal in the group, so f -subnormal subgroups need not be subnormal, although it was shown in [4] that for locally nilpotent groups f -subnormality is equivalent to subnormality.

This generalization of subnormality was introduced by Phillips in [16], and groups all of whose subgroups are f -subnormal were shown to be finite-by-soluble in [4], while further results have been obtained in the recent papers [5, 8–10, 14].

In this paper we define and study the relationship between two subgroups of a group G . The first of these is the f -Wielandt subgroup of G , which we denote by $\overline{\omega}(G)$ and define by

$$\overline{\omega}(G) = \bigcap \{N_G(H) \mid H \text{ is an } f\text{-subnormal subgroup of } G\}.$$

This is an analogue of the *Wielandt subgroup* of a group G ,

$$\omega(G) = \bigcap \{N_G(H) \mid H \text{ is a subnormal subgroup of } G\},$$

first introduced by Wielandt in [23] and studied in various papers including [3] and [6]. It is always the case that $\overline{\omega}(G) \leq \omega(G)$, but easy examples show that in general the two subgroups are different. The Wielandt subgroup is itself analogous to the *kern* or *norm* $N(G)$ of a group G , defined to be the intersection of the normalizers of all the subgroups of G , originally introduced by Baer in [1]. It was shown in [21] that $N(G) \leq \zeta_2(G)$, the second centre of G , and that $G/C_G(N(G))$ is abelian. Some information concerning $\overline{\omega}(G)$ was obtained in [9]. We note that the intersection of the normalizers of the finite f -subnormal subgroups coincides with the intersection of the normalizers of the finite cyclic f -subnormal subgroups of prime power order, so we usually focus on these latter subgroups.

In [2] the *generalized Wielandt subgroup* of a group G , here denoted by $\omega_i(G)$, was introduced and defined to be

$$\omega_i(G) = \bigcap \{N_G(H) \mid H \text{ is an infinite subnormal subgroup of } G\}.$$

We remark that of course $\omega(G) \leq \omega_i(G)$ and that $\omega_i(G) = G$ if there are no infinite subnormal subgroups. Here we introduce the *generalized f -Wielandt subgroup*

$$\overline{\omega}_i(G) = \bigcap \{N_G(H) \mid H \text{ is an infinite } f\text{-subnormal subgroup of } G\},$$

as a natural generalization of $\omega_i(G)$ for f -subnormal subgroups. Clearly $\overline{\omega}_i(G) \leq \omega_i(G)$ and $\overline{\omega}_i(G) = G$, if G has no infinite f -subnormal subgroups. Of course $\overline{\omega}(G) \leq \overline{\omega}_i(G)$ in general.

One purpose of the article [2] was to discuss the situation when $\omega_i(G) \neq \omega(G)$; a further goal was to discuss the structure of the quotient group $\omega_i(G)/\omega(G)$. In this paper we discuss the situation when $\overline{\omega}_i(G) \neq \overline{\omega}(G)$ and also obtain structural results for $\overline{\omega}_i(G)/\overline{\omega}(G)$.

In Sect. 2 we obtain some interesting useful preliminary results. In Sect. 3 we obtain some of the main structural results concerning the quotient $\overline{\omega}_i(G)/\overline{\omega}(G)$. In particular Theorem 3.3 and Theorem 3.4 show that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is always residually finite and even a Dedekind group when $\overline{\omega}_i(G)$ is infinite. Section 4 discusses the case in which $\overline{\omega}_i(G)$ is finite and shows that in such circumstances $\overline{\omega}_i(G)/\overline{\omega}(G)$ is actually abelian. The results of Sects. 3 and 4 are combined in Theorem 4.7 to show that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is always Dedekind whenever G is infinite. In Sect. 5 we give some applications of our results in Sect. 3 to the class of soluble groups with finitely many normalizers of f -subnormal subgroups. Finally in Sect. 6 we discuss several examples and prove that every finite Dedekind group occurs as the quotient $\overline{\omega}_i(G)/\overline{\omega}(G)$.

2 Some preliminary results

In this section we collect some preliminary results which are of general interest and finish with specific structural results concerned with the situation when $\overline{\omega}_i(G) \neq \overline{\omega}(G)$.

Lemma 2.1 *Let H be a finite f -subnormal subgroup of a group G and suppose that H has infinitely many conjugates in G . Then there are f -subnormal subgroups H_1, H_2, H_3 of G such that $H \leq H_1 \triangleleft H_2 \triangleleft H_3$ satisfying*

- (i) H_1 is finite;
- (ii) $H_2/H_1, H_3/H_2$ are infinite and H has infinitely many conjugates in H_3 .

Proof We consider an f -subnormal series

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

for H in G of minimal length n . Clearly, there are consecutive terms of this series $H_1 \triangleleft H_2$ such that H_1 is finite and H_2/H_1 is infinite. Since H has infinitely many conjugates in G it is also clear that $H_2 \neq G$, so $H_2 \not\leq H_3$. If H_2 has finite index in H_3 , then Dietzmann's lemma (see [18, Corollary 3 to Lemma 2.14]) implies that there is a finite normal subgroup L of H_3 containing H_1 and we then obtain a shorter f -subnormal series for H in G . Therefore, H_2 has infinite index in H_3 , so $H_2 \triangleleft H_3$. If H has finitely many conjugates in H_3 , then Dietzmann's lemma shows there is a finite normal subgroup of H_3 containing H again contradicting the minimality of n . Hence H has infinitely many conjugates in H_3 . $\square$

The following corollary will be useful later.

Corollary 2.2 *Let G be a group with the property that all finite subnormal subgroups of infinite f -subnormal subgroups have finitely many conjugates in them. Then all finite f -subnormal subgroups of G have finitely many conjugates.*

Proof If X is a finite f -subnormal subgroup with infinitely many conjugates, then we may apply Lemma 2.1 to obtain a contradiction. $\square$

Lemma 2.3 *Let G be a group and let H be an f -subnormal subgroup of G .*

- (i) *If K is a subnormal subgroup of G and $K^H = K$, then HK is f -subnormal in G .*
- (ii) *If H is finite and G is infinite, then there is an infinite f -subnormal subgroup K of G containing H which normalizes H .*

Proof (i) Let $K = K_n \triangleleft K_{n-1} \triangleleft \cdots \triangleleft K_1 = K^G \triangleleft K_0 = G$ be the normal closure series for K in G . An easy induction shows that H normalizes K_i for each i , so $K_{i+1} \triangleleft HK_i$ and HK_{i+1} is an f -subnormal subgroup of HK_i . Hence HK is f -subnormal in G as required.

(ii) There are f -subnormal subgroups H_1, H_2 with $H_1 \triangleleft H_2$ such that $H \leq H_1$, H_1 is finite and $|H_2 : H_1|$ is infinite. Clearly H has finitely many conjugates in H_2 so we may take $K = N_{H_2}(H)$. $\square$

If X, Y are subgroups of a group G and i is a natural number, then as usual we write

$$[X, \underbrace{Y, Y, \dots, Y}_i]$$

as $[X, {}_i Y]$. The following result will often be useful.

Lemma 2.4 *Let G be a group and let H be a finite f -subnormal subgroup of G .*

- (i) *If D is a divisible abelian normal subgroup of G , then $[D, H] = 1$.*
- (ii) *If $G = PH$, where P is a Prüfer group, then H centralizes P .*
- (iii) *If K is an f -subnormal subgroup of G and P is a normal Prüfer subgroup of G , then $P \leq K$ or $[P, K] = 1$.*

Proof (i) Note that $[D, {}_m H]$ is a divisible abelian subgroup for each natural number m . There is an f -subnormal series

$$H = H_0 \leq H_1 \leq \cdots \leq H_m = G$$

for H in G of length $m > 0$. If H_{m-1} has finite index in H_m , then $D \leq H_{m-1}$, whereas if H_{m-1} has infinite index in H_m , then $H_{m-1} \triangleleft H_m$, so $[D, H] \leq [D, H_{m-1}] \leq H_{m-1}$. Thus in any case $[D, H] \leq H_{m-1}$. Repeating this argument we see that $[D, H, H] \leq H_{m-2}$ and inductively we have $[D, {}_m H] \leq H$. Since H is finite and $[D, {}_m H]$ is divisible we have $[D, {}_m H] = 1$ and [18, Lemma 3.13] implies that $[D, H] = 1$.

(ii) Since P has finite index in G and P has no subgroups of finite index, it follows that $P \triangleleft G$ and the result follows by (i).

(iii) The group $[P, K]$ is divisible and is a subgroup of P , so we may assume that $[P, K] = P$. By induction on the length of an f -subnormal series we may suppose that $K \triangleleft G$ or $|G : K|$ is finite. In both cases $P \leq K$. $\square$

This result enables us to immediately deduce the following corollary.

Corollary 2.5 *Let G be a group containing a normal Prüfer subgroup P . Then $\overline{\omega}(G)$ contains P .*

The next result is also very useful.

Lemma 2.6 *Let G be a group and suppose that H is a finite f -subnormal subgroup of G . Suppose that G contains an infinite residually finite H -invariant subnormal subgroup K . If $g \in \overline{\omega}_i(G)$, then $g \in N_G(H)$.*

Proof By Lemma 2.3 KH is an f -subnormal subgroup of G which is clearly also residually finite. Then K has a finite residual system $\{K_\lambda\}_{\lambda \in \Lambda}$ consisting of H -invariant subgroups with the property that if $\lambda, \mu \in \Lambda$ there is an element $\alpha \in \Lambda$ such that $K_\alpha \leq K_\lambda \cap K_\mu$. It is well known that

$$H^g \leq \bigcap_{\lambda \in \Lambda} (K_\lambda H)^g = \bigcap_{\lambda \in \Lambda} (K_\lambda H) = H$$

and the result follows. $\square$

Our next result and proof is based on the corresponding result [2, Lemma 1], but there are some differences in the proof, so we give it for completeness. In this paper, as in [2], we use the term H is Prüfer-by-finite to include the possibility that the group H is finite. If G is an arbitrary group, then $\pi(G)$ denotes the set of primes dividing the orders of elements in G of finite order.

We make the (obvious) remark that if H is an f -subnormal subgroup of a group G and if K is a subgroup of H of finite index in H , then K is also an f -subnormal subgroup of G .

Proposition 2.7 *Let G be a group with $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then every abelian normal subgroup of G is Prüfer-by-finite.*

Proof Let A be an abelian normal subgroup of G and note that by hypothesis there is a finite f -subnormal subgroup H of G that is cyclic of prime power order and an element $g \in \overline{\omega}_i(G)$ such that $H^g \neq H$.

We first show that A is periodic and to this end let $a \in A$ be an element of infinite order. Let $B_0 = \langle a^H \rangle$, a finitely generated abelian group. Then B_0 is a residually finite H -invariant subgroup of G , so Lemma 2.6 implies that $g \in N_G(H)$, giving a contradiction. Hence A is periodic.

Next we prove that $\pi(A)$ is finite, so suppose for a contradiction that $\pi(A)$ is infinite and let B denote the $\pi(H)'$ -component of A . Note that B is infinite and a normal subgroup of G . As H is f -subnormal in HB , there is an infinite f -subnormal subgroup K of HB such that $H \triangleleft K$ by Lemma 2.3. Clearly $K = H \times (K \cap B)$, so H is the Sylow $\pi(H)$ -subgroup of K . But $H^g(K \cap B) = H(K \cap B)$, so $H^g = H$, giving the required contradiction.

Next note that if $A[p] = \{a \in A \mid a^p = 1\}$ is infinite for some prime $p \in \pi(A)$, then Lemma 2.6 implies that $H^g = H$, contrary to the choice of H and g . Hence A is a Chernikov group.

If $R_p \neq 1$ and $R_q \neq 1$ are divisible components of A with p, q distinct primes then HR_p, HR_q are f -subnormal in G . Then

$$H^g \leq HR_p \cap HR_q = H(R_p \cap R_q).$$

By Lemma 2.4 $R_p \cap HR_q = H \cap R_p$ so $H^g \leq H$ and we obtain the usual contradiction. Hence the divisible component D of A is trivial or $\pi(D) = \{p\}$. Clearly we may assume that D is nontrivial and Lemma 2.4 implies that $[D, H] = 1$.

Suppose that $D = P_1 \times P_2$, where P_1, P_2 are nontrivial and $H \cap D \leq P_1$. Then HP_i is f -subnormal in G by Lemma 2.3 and

$$H^g \leq HP_1 \cap HP_2 = H(HP_1 \cap P_2) = H((H \cap D)P_1 \cap P_2) = H.$$

Hence the minimal divisible subgroup containing $H \cap D$ is D itself and thus $D[p] \leq H$, which means that D is locally cyclic and so Prüfer. The result follows. $\square$

We have the following useful observation.

Corollary 2.8 *Let G be a group such that $\overline{\omega}_i(G) \neq \overline{\omega}(G)$ and suppose that H is a Chernikov f -subnormal subgroup of G . Then H is Prüfer-by-finite.*

Proof We may assume that H is infinite, so H contains a normal subgroup K of finite index such that K is subnormal in G by [4, Proposition 3.1]. The divisible part D of K is nontrivial and subnormal in G . It follows by [18, Lemma 4.46] that G has a nontrivial divisible normal periodic abelian subgroup, which is a Prüfer group by Proposition 2.7. The result follows. $\square$

Finally in this section we give an analogue of [2, Theorem 1] and parts of the proof are based on the proof of the corresponding result.

Theorem 2.9 *Let G be a group satisfying $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then the Baer radical of G is Prüfer-by-finite and hence is nilpotent.*

Proof Let B be the Baer radical of G . By assumption, there is a finite f -subnormal subgroup H and an element $g \in \overline{\omega}_i(G)$ such that $H^g \neq H$. Clearly we may assume that H is a cyclic q -group for some fixed prime q . We suppose for a contradiction that B is not Prüfer-by-finite.

First we note that B is periodic, for, if not, there is an element $b \in B$ of infinite order and clearly $L = \langle b^H \rangle$ is a finitely generated nilpotent subnormal subgroup of G . Then L

is residually finite and H -invariant, so Lemma 2.6 implies that $g \in N_G(H)$ which is a contradiction. Therefore, B is periodic so it is the direct product of its p -components.

Suppose that $B = C \times D$, where C and D are both infinite, G -invariant subgroups, and $H \cap B \leq D$. Then it is clear that HC, HD are infinite f -subnormal subgroups of G and

$$H^g \leq HC \cap HD = H(C \cap B \cap HD) = H(C \cap D(H \cap B)) = H,$$

a contradiction. It follows that B is the direct product of only finitely many p -components at most one of which is infinite. If B has finite rank, then Proposition 2.7 implies that B is Prüfer-by-finite. Hence B has infinite rank.

Let B_p be the infinite p -component of B for some prime p . Of course H is f -subnormal in HB_p so there exist f -subnormal subgroups $H_1 \triangleleft H_2 \leq HB_p$ such that H_1 is Chernikov, H_2 has infinite rank and $H \leq H_1$. Let $H_2 \cap B_p = X$. Then HX is f -subnormal in G .

Certainly $[X, H] \leq H_1$, so $[X, H, H]$ is finite by Lemma 2.4 and if $d = |H|$, then

$$[X, H]^d \leq [X, H^d][X, H, H] = [X, H, H],$$

so the Chernikov group $[X, H]$ is finite. Then $M = C_X(H^X)$ is of infinite rank and f -subnormal in B , so subnormal in G . Since M is a Baer group it does not satisfy the minimal condition on subnormal abelian subgroups by [2, Lemma 2]. Thus M contains an infinite elementary abelian subnormal subgroup A and Lemma 2.6 gives the contradiction sought. Finally, B is nilpotent by [2, Lemma 2]. $\square$

3 Some structural results concerning $\overline{\omega}_i(G)/\overline{\omega}(G)$

In this section we discuss the structure of $\overline{\omega}_i(G)/\overline{\omega}(G)$ which is very tightly controlled, much more so than with the corresponding factor $\omega_i(G)/\omega(G)$.

Let $V_f(G)$ denote the subgroup generated by all the finite f -subnormal subgroups of G .

We begin with the following analogue of [2, Theorem 6]. Example 3 shows that Lemma 3.1 cannot really be improved further.

Lemma 3.1 *Let G be a group in which $V_f(G)$ is Baer-by-finite. Then*

- (i) $\overline{\omega}_i(G)/\overline{\omega}(G)$ is finite;
- (ii) *There exists a finite normal subgroup N of G such that every f -subnormal subgroup of $\overline{\omega}_i(G)/N$ is a normal subgroup.*

Proof We may suppose that $\overline{\omega}_i(G) \neq \overline{\omega}(G)$; otherwise, (i) and (ii) hold trivially since $\overline{\omega}(G)$ is a T-group. Let $V = V_f(G)$.

(i) Since V is Baer-by-finite it contains a characteristic Baer subgroup of finite index and Theorem 2.9 implies that V is Prüfer-by-finite.

If V is finite, then G has only finitely many finite f -subnormal subgroups. If H is such a finite f -subnormal subgroup, then $|G : N_G(H)|$ is finite and as

$$\overline{\omega}(G) = \overline{\omega}_i(G) \cap \bigcap \{N_G(H) \mid H \text{ is finite } f\text{-subnormal in } G\}$$

(i) follows in this case.

Hence we may suppose that V is infinite and let P be the Prüfer p -subgroup of V , for some prime p . Of course, $P \triangleleft G$. It follows by Lemma 2.4 that $V \leq C_G(P)$. Furthermore, there is a finite subgroup F of G such that $V = FP$. Let p^d denote the highest power of p dividing $|F|$ and let Q denote the set of elements of P of order at most p^d . Then $K = QF$

is a finite characteristic subgroup of G , so $G/C_G(K)$ is finite. Let $C = C_G(K)$ and note that C acts as a group of power automorphisms on V . Hence C normalizes the finite f -subnormal subgroups of G , so $C \cap \bar{\omega}_i(G) \leq \bar{\omega}(G)$. Then $|\bar{\omega}_i(G) : C \cap \bar{\omega}_i(G)|$ is also finite, as is $|\bar{\omega}_i(G) : \bar{\omega}(G)|$.

(ii) If V is infinite, then we choose K and P as above. In this case, if H/K is f -subnormal in $\bar{\omega}_i(G)K/K$ then of course H is normalized by $\bar{\omega}_i(G)$, if H is infinite. If H is finite, then $H \leq V = KP$, so H/K is a characteristic subgroup of V/K and hence is normalized by G/K . It follows that $\bar{\omega}_i(G)K/K$ has the property stated. Hence we take $N = \bar{\omega}_i(G) \cap K$.

If V is finite, then all nontrivial f -subnormal subgroups of $\bar{\omega}_i(G)V/V$ are infinite and hence are normalized by $\bar{\omega}_i(G)V/V$. Thus every f -subnormal subgroup of $\bar{\omega}_i(G)V/V$ is a normal subgroup and we take $N = \bar{\omega}_i(G) \cap V$ in this case. This completes the proof. $\square$

It was shown in [2] that $\omega_i(G)/\omega(G)$ is always residually finite and we obtain the same result concerning $\bar{\omega}_i(G)/\bar{\omega}(G)$. However, its structure is even more restricted. First we prove the following preliminary result.

Lemma 3.2 *Let G be a group and let F be a finite f -subnormal subgroup of G . Then $|\bar{\omega}_i(G) : N_G(F) \cap \bar{\omega}_i(G)|$ is finite.*

Proof We may assume that $\bar{\omega}_i(G)$ is infinite. It is easy to see that we only need prove the result when F is abelian, so we assume this from now on. Let

$$\bar{F} = \bigcap \{K | F \leq K, K \text{ is an infinite } f\text{-subnormal of } G\}.$$

Clearly $F \leq \bar{F}$, whereas if $x \in \bar{\omega}_i(G)$, then $x \in N_G(K)$ for all infinite f -subnormal subgroups K of G , so $x \in N_G(\bar{F})$. Hence $\bar{F}$ is normalized by $\bar{\omega}_i(G)F$. Since $\bar{\omega}_i(G)F$ is an infinite f -subnormal subgroup of G we have $\bar{F} \leq \bar{\omega}_i(G)F$, so $\bar{F} \triangleleft \bar{\omega}_i(G)F$.

If $\bar{F}$ is finite, then F has only finitely many conjugates in $\bar{\omega}_i(G)F$, so that $|\bar{\omega}_i(G) : N_G(F) \cap \bar{\omega}_i(G)|$ is finite in this case.

On the other hand, suppose that $\bar{F}$ is infinite. Since F is f -subnormal in G there is a finite subgroup L such that $F \leq L \triangleleft \bar{F}$. Clearly $C_{\bar{F}}(L)F$ is an infinite f -subnormal subgroup of G containing F , so $\bar{F} = C_{\bar{F}}(L)F$. Since F is abelian we have $F \leq Z(\bar{F})$; also, $Z(\bar{F})$ is a normal subgroup of $\bar{\omega}_i(G)F$, which is f -subnormal in G .

If $Z(\bar{F})$ is finite, then F has finitely many conjugates in $\bar{\omega}_i(G)F$, and hence, $|\bar{\omega}_i(G) : N_G(F) \cap \bar{\omega}_i(G)|$ is finite. If $Z(\bar{F})$ is infinite, then it coincides with $\bar{F}$. Since $\bar{F}/F$ contains no proper infinite subgroups it is Prüfer. If $|F| = d$, then the subgroup of elements of $\bar{F}$ whose order divides d is a finite characteristic subgroup, M say, of $\bar{F}$ containing F and of course $M \triangleleft \bar{\omega}_i(G)F$. Hence, as before, $|\bar{\omega}_i(G) : N_G(F) \cap \bar{\omega}_i(G)|$ is finite. $\square$

Theorem 3.3 *For all groups G the quotient group $\bar{\omega}_i(G)/\bar{\omega}(G)$ is residually finite. Furthermore, $\bar{\omega}_i(G)$ is either finite or $\bar{\omega}_i(G)/\bar{\omega}(G)$ is Dedekind.*

Proof We may assume that $\bar{\omega}_i(G)/\bar{\omega}(G)$ is infinite. Note that

$$\bar{\omega}(G) = \bar{\omega}_i(G) \cap X \text{ where}$$

$$X = \bigcap \{N_G(F) | F \text{ is a finite } f\text{-subnormal subgroup of } G\}.$$

Let F be a finite f -subnormal subgroup of G . By Lemma 3.2 it follows that $|\bar{\omega}_i(G) : N_G(F) \cap \bar{\omega}_i(G)|$ is finite and it is immediate that $\bar{\omega}_i(G)/\bar{\omega}(G)$ is residually finite.

Next note that if R is a normal subgroup of $\bar{\omega}_i(G)$ of finite index and if $R \leq H \leq \bar{\omega}_i(G)$, then H is f -subnormal in G so H is normalized by $\bar{\omega}_i(G)$. Hence $\bar{\omega}_i(G)/\bar{\omega}(G)$ is actually residually (finite Dedekind).

Factoring by $\overline{\omega}(G)$ we may assume that $\overline{\omega}_i(G)$ is an infinite residually (finite Dedekind) subgroup of G and suppose that $\{R_\lambda\}_{\lambda \in \Lambda}$ is a finite residual system for $\overline{\omega}_i(G)$ such that $\overline{\omega}_i(G)/R_\lambda$ is a finite Dedekind group and with the property that if $\lambda, \mu \in \Lambda$ there is an element $\alpha \in \Lambda$ such that $R_\alpha \leq R_\lambda \cap R_\mu$. Then $\overline{\omega}_i(G)'R_\lambda/R_\lambda$ has order at most 2 and it is easy to see that this means that $(\overline{\omega}_i(G)/\overline{\omega}(G))'$ itself has order at most 2. Lemma 2.6 shows immediately that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is Dedekind, as required. $\square$

If we add a generalized solubility condition, then we can further strengthen our results. We recall that a group G is subsoluble if it has an ascending subnormal series whose factors are abelian. We note that this class of groups is $\mathbf{N}$ -closed and that if G is subsoluble then $C_G(B) \leq B$, where B is the Baer radical of G . These results appear in [17].

Theorem 3.4 *Let G be an infinite subsoluble group such that $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then*

- (i) *G is a soluble group with a normal Prüfer p -subgroup P such that G/P is finite-by-(torsion-free abelian);*
- (ii) *$G/\overline{\omega}(G)$ has finite exponent;*
- (iii) *$\overline{\omega}_i(G)/\overline{\omega}(G)$ is a finite abelian $\{p, p-1\}$ -group.*

Proof (i) Let B be the Baer radical of G and note that $C_G(B) \leq B$. By Theorem 2.9 it follows that B is Prüfer-by-finite and nilpotent. Since G is infinite it is clear that we may write $B = PF$, where P is a Prüfer p -group and F is a finite characteristic subgroup of B .

Clearly $G/C_G(P)$ is abelian and has finite torsion subgroup. Also $G/C_G(F)$ is finite, so if $x \in C_G(P)$, then there exists a natural number n such that $x^n \in C_G(B) \leq B$, and hence, x has finite order. Thus $C_G(P)$ is periodic. Also $C_G(P)/C_G(B) \cong C_G(P)C_G(F)/C_G(F)$ is finite, so $C_G(P)$ is Prüfer-by-finite. Thus (i) follows. Furthermore, $V_f(G) \leq C_G(P)$ by Lemma 2.4.

(ii) By (i) G/P is centre-by-(finite exponent). Let Y be an infinite f -subnormal subgroup of G . If T/P is the torsion subgroup of G/P , then T is Prüfer-by-finite and $P \leq Y \leq T$ if Y is periodic. If Y contains an element of infinite order, then $Y \not\leq C_G(P)$, so Lemma 2.4 implies that $P \leq Y$. In any case, if Z/P is the centre of G/P it follows that $[Z, Y] \leq P \leq Y$, so $Z \leq \overline{\omega}_i(G)$. In particular (ii) follows since Lemma 3.1 implies that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is finite.

(iii) It remains to prove that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is an abelian $\{p, p-1\}$ -group. Let q be a prime not dividing p or $p-1$ and suppose that there is a nontrivial element $a\overline{\omega}(G)$ of $\overline{\omega}_i(G)/\overline{\omega}(G)$ of order q . There is a finite cyclic f -subnormal subgroup $E = \langle e \rangle$ of G of prime power order which is not normalized by a . Since EP is infinite, Lemma 2.4 implies that $|e|$ is a power of p .

If aP has finite order, then we may choose a so that aP has order a power of q . Then the q -component of $\overline{\omega}_i(G)/P$ commutes with the subgroup EP/P , so $e^a = ec$ for some element $1 \neq c \in P$.

On the other hand, if aP has infinite order we have $e^a = e^n c$ for some nontrivial element $c \in P$ and some integer n . Thus

$$(aeP)^{aP} = ae^n P.$$

However, G/P is finite-by-abelian, so every subgroup of G/P is f -subnormal and in particular this is true of $\langle aeP \rangle$. But $\langle ae \rangle P$ is normalized by a , so it follows that

$$(aeP)^{aP} = aeP.$$

Hence $e^a = ec'$ for some $c' \in P$.

Thus in either case, there is a natural number m such that $a^{p^m(p-1)}$ centralizes E . However, a^q also normalizes E , and hence, a normalizes E , which is a contradiction. Thus $\bar{\omega}_i(G)/\bar{\omega}(G)$ is a $\{p, p-1\}$ -group.

Clearly $\bar{\omega}_i(G)/P$ is a Dedekind group. Thus if G/P is not periodic, then $\bar{\omega}_i(G)/P$ is abelian as is $\bar{\omega}_i(G)/\bar{\omega}(G)$ in this case.

Hence we may assume by (ii) that G is periodic, so it is Prüfer-by-finite. We suppose $\bar{\omega}_i(G)/\bar{\omega}(G)$ is nonabelian. Let Q/P be the 2-component of $\bar{\omega}_i(G)/P$. Let $Q'P/P = \langle aP \rangle$ and note that $Q'P/P \leq Z(G/P)$. Clearly there is an f -subnormal subgroup $X = \langle x \rangle$ of prime power order that is not normalized by a . Hence $x^a = xc$ for some nontrivial $c \in P$. Of course $G/C_G(P)$ is cyclic, so $a \in C_G(P)$ and Lemma 2.4 shows that $x = x^{a^2} = xc^2$. This implies that P is a 2-group. By Lemma 2.4 $[x, c] = 1$, so $\pi(\langle x \rangle) = \{2\}$. Thus $H = QX$ is locally nilpotent and f -subnormal in G . In particular $Q \leq \bar{\omega}_i(G) \leq \bar{\omega}_i(H) = \omega_i(H)$, so $Q' \leq \omega_i(H) = \bar{\omega}_i(H)$ by [2, Theorem 8], which means that a normalizes X , a contradiction. The result follows. $\square$

We note that, in contrast, the authors of [2] exhibit an example of a periodic soluble group G in which $\omega_i(G)/\omega(G)$ is nonabelian.

Following [4] we shall call a group in which every subgroup is f -subnormal an S -group.

Corollary 3.5 *Let G be a group.*

- (i) *Suppose that $\bar{\omega}_i(G)$ is contained in an infinite subsoluble normal subgroup of G . Then there is a prime p such that $\bar{\omega}_i(G)/\bar{\omega}(G)$ is an abelian residually finite $\{p, p-1\}$ -group.*
- (ii) *If G is an (infinite subsoluble)-by- S -group, then there is a prime p such that $\bar{\omega}_i(G)/\bar{\omega}(G)$ is an abelian residually finite $\{p, p-1\}$ -group.*
- (iii) *If G is an infinite subsoluble-by-finite group, then there is a prime p such that $\bar{\omega}_i(G)/\bar{\omega}(G)$ is a finite abelian $\{p, p-1\}$ -group.*

Proof We may suppose throughout that $\bar{\omega}_i(G) \neq \bar{\omega}(G)$.

(i) Let H be an infinite subsoluble normal subgroup of G containing $\bar{\omega}_i(G)$ and let X be a finite cyclic f -subnormal subgroup of G which is not normalized by some element of $\bar{\omega}_i(G)$. Then XH is an infinite subsoluble f -subnormal subgroup of G and it is easy to see that $\bar{\omega}_i(XH) \neq \bar{\omega}(XH)$. Since $\bar{\omega}_i(G) \leq \bar{\omega}_i(HX)$, it follows that $\bar{\omega}_i(G)/(\bar{\omega}_i(G) \cap N_G(X))$ is a finite abelian $\{p, p-1\}$ -group, applying Theorem 3.4 to XH . This implies that $\bar{\omega}_i(G)/\bar{\omega}(G)$ is abelian and a residually finite $\{p, p-1\}$ -group.

(ii) Let H be the subsoluble radical of G . Since $\bar{\omega}_i(G)H/H$ is Dedekind we have $\bar{\omega}_i(G) \leq H$ and the result follows by (i).

(iii) Let H be the subsoluble radical of G . Then $\bar{\omega}_i(G) \leq H$, so by (ii) $\bar{\omega}_i(G)/\bar{\omega}(G)$ is an abelian residually finite $\{p, p-1\}$ -group. Suppose that this quotient is infinite. Then there is an infinite family of finite cyclic f -subnormal subgroups $\{\langle x_i \rangle | i \in \mathbb{N}\}$ of G such that

$$\bar{\omega}_i(G) \cap H_{n+1} \not\leq \bar{\omega}_i(G) \cap H_n,$$

where $H_n = \bigcap_{j \leq n} N_G(\langle x_j \rangle)$. If T is a transversal to H in G , then we may write $x_i = t_i h_i$, where $t_i \in T$ and $h_i \in H$. It follows that there exists $t \in T$ and an infinite subset I of $\mathbb{N}$ such that $x_i = th_i$ for all $i \in I$. Then $H\langle t \rangle$ is subsoluble and f -subnormal in G and contains all these particular $\langle x_i \rangle$. By Theorem 3.4 applied to $H\langle t \rangle$ there exists $k \in I$ such that $\bar{\omega}_i(H\langle t \rangle) \cap H_k = \bar{\omega}_i(H\langle t \rangle) \cap H_n$ for all $n \in I$ such that $n \geq k$. Then $\bar{\omega}_i(G) \cap H_k = \bar{\omega}_i(G) \cap H_n$ for all such n , which is a contradiction. Thus $\bar{\omega}_i(G)/\bar{\omega}(G)$ is a finite abelian $\{p, p-1\}$ -group as required. $\square$

There are many classes of soluble groups where $\overline{\omega}_i(G) = \overline{\omega}(G)$; the following corollary exhibits two such.

Corollary 3.6 *Let G be an infinite subsoluble group.*

- (i) *If G is finitely generated, then $\overline{\omega}_i(G) = \overline{\omega}(G)$.*
- (ii) *If G contains no Prüfer subgroups, then $\overline{\omega}_i(G) = \overline{\omega}(G)$.*

Proof (i) If we assume the contrary, then Theorem 3.4 implies that there is a Prüfer subgroup P such that G/P is finitely presented. Then P is finitely generated and this is a contradiction.

(ii) We assume the contrary, and in this case, the Baer radical is finite, and hence, G is finite. $\square$

We next consider the situation when $\overline{\omega}_i(G)/\overline{\omega}(G)$ is infinite and collect a certain amount of information concerning this situation. We note that Example 3 shows that every finite Dedekind group appears as the quotient $\overline{\omega}_i(G)/\overline{\omega}(G)$ for some group G .

Lemma 3.7 *Let G be a group such that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is infinite. Then $\gamma_3(\overline{\omega}_i(G))$ is an infinite $\mathfrak{F}$ -perfect subgroup, centralizing all finite f -subnormal subgroups of G . Moreover, $\gamma_3(\overline{\omega}_i(G))$ is the nilpotent residual of $\overline{\omega}_i(G)$.*

Proof Let

$$W = \overline{\omega}_i(G)/\overline{\omega}(G)$$

and note that W is nilpotent of class 2, so $\gamma_3(\overline{\omega}_i(G)) \leq \overline{\omega}(G)$. If $\gamma_k(\overline{\omega}_i(G))$ is finite for some k , then $\overline{\omega}_i(G)$ is finite-by-nilpotent and hence nilpotent-by-finite. However, Theorem 2.9 shows that the Baer radical of G is Prüfer-by-finite, so $\overline{\omega}_i(G)/\overline{\omega}(G)$ would be finite by Corollary 2.5. Hence $\gamma_k(\overline{\omega}_i(G))$ is infinite for all k , and in particular, the nilpotent group $\overline{\omega}_i(G)/\gamma_k(\overline{\omega}_i(G))$ is always Dedekind. Hence $\gamma_3(\overline{\omega}_i(G))$ is infinite and also the nilpotent residual of $\overline{\omega}_i(G)$. Furthermore, if M is a subgroup of finite index in $\gamma_3(\overline{\omega}_i(G))$, then M is an infinite f -subnormal subgroup of G , so M is normal in $\overline{\omega}_i(G)$. Then $\overline{\omega}_i(G)/M$ is also Dedekind and hence nilpotent of class at most 2. It follows that $M = \gamma_3(\overline{\omega}_i(G))$, so $\gamma_3(\overline{\omega}_i(G))$ has no proper subgroups of finite index. Furthermore, let $H = \langle h \rangle$ be a finite cyclic f -subnormal subgroup of G of prime power order. By Lemma 3.2 we know that $|\overline{\omega}_i(G) : N_H|$ is finite, where $N_H = N_G(H) \cap \overline{\omega}_i(G)$. Since $N_H/C_{N_H}(H)$ is finite it follows that $\gamma_3(\overline{\omega}_i(G)) \leq C_{\overline{\omega}_i(G)}(H)$. $\square$

Lemma 3.8 *Let G be a group such that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is periodic. Then $\pi(\overline{\omega}_i(G)/\overline{\omega}(G))$ is finite.*

Proof We may assume that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is infinite. We let

$$Z = Z(\gamma_3(\overline{\omega}_i(G)))$$

and note that Z is Prüfer-by-finite by Theorem 2.9.

Let $H = \langle h \rangle$ be a finite cyclic f -subnormal subgroup of G of prime power order. Clearly $h \in Z(\gamma_3(\overline{\omega}_i(G))H)$ and also $\gamma_3(\overline{\omega}_i(G))H$ is normalized by $\overline{\omega}_i(G)$ by Lemma 3.7. Hence for each $g \in \overline{\omega}_i(G)$ we have

$$h^g = h^n l \in Z(\gamma_3(\overline{\omega}_i(G))H), \quad (2)$$

for some element $l \in Z$ and some natural number n . In particular the order of l must be a power of the only prime belonging to $\pi(H)$. We note that if H is not normalized by g , then $l \neq 1$.

Let $\mathcal{L}$ be the set of all finite cyclic f -subnormal subgroups of prime power order which are not normalized by $\overline{\omega}_i(G)$ and let $\pi(\mathcal{L})$ denote the set of all primes p such that there is a subgroup in $\mathcal{L}$ of order a power of the prime p . Our work above shows that $\pi(\mathcal{L})$ is finite since $\pi(Z)$ is finite.

Since Z is Prüfer-by-finite it follows that $|\overline{\omega}_i(G) : C_{\overline{\omega}_i(G)}(Z)|$ is finite. Let

$$g\gamma_3(\overline{\omega}_i(G)) \in C_{\overline{\omega}_i(G)}(Z)/\gamma_3(\overline{\omega}_i(G))$$

be an element of prime power order p_1^c which does not normalize some subgroup $H = \langle h \rangle \in \mathcal{L}$ of prime power order p_2^d and suppose that $p_1 \neq p_2$. Then $H\gamma_3(\overline{\omega}_i(G))/\gamma_3(\overline{\omega}_i(G))$ is a p_2 -group and it is centralized by the p_2' -component of $\overline{\omega}_i(G)/\gamma_3(\overline{\omega}_i(G))$. In particular we have

$$h^g = hl, \text{ where } l \in Z \text{ is a } p_2\text{-element.}$$

On the other hand,

$$h = h^{g^{p_1^c}} = hl^{p_1^c}$$

since $g \in C_{\overline{\omega}_i(G)}(Z)$, so that l has order a power of p_1 . Hence $p_1 = p_2$. In particular we see that

$$\pi(\overline{\omega}_i(G)/\overline{\omega}(G)) \text{ is finite,}$$

since $\pi(\mathcal{L})$ is finite and the result follows. $\square$

In particular we have the following.

Corollary 3.9 *Let G be a group such that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is an infinite nonabelian group. Then $\pi(\overline{\omega}_i(G)/\overline{\omega}(G))$ is finite.*

Proposition 3.10 *Let G be a group such that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is an infinite nonabelian group. Let $Z = Z(\gamma_3(\overline{\omega}_i(G)))$.*

- (i) *If Z contains a Prüfer p -subgroup P , then $p = 2$ and $\overline{\omega}_i(G)/\overline{\omega}(G)$ is a 2-group.*
- (ii) *If Z is finite, then $\overline{\omega}_i(G)/\overline{\omega}(G)$ has finite exponent.*

Proof (i) Suppose that Z is infinite and contains the Prüfer p -subgroup P . Let $g\gamma_3(\overline{\omega}_i(G)) \in C_{\overline{\omega}_i(G)}(Z)/\gamma_3(\overline{\omega}_i(G))$ be an element of order q^m for some prime $q \neq p$. Then g does not normalize some finite cyclic f -subnormal subgroup H of prime power order and the proof of Lemma 3.7 shows that H also has order a power of the prime q . However, by Lemma 2.4, H is then a characteristic subgroup of $HP = H \times P$ and since HP is infinite, we obtain that g normalizes H , a contradiction. Hence $C_{\overline{\omega}_i(G)}(Z)/\gamma_3(\overline{\omega}_i(G))$ is a p -group. Since $|\overline{\omega}_i(G) : C_{\overline{\omega}_i(G)}(Z)|$ is finite the p' -component of $\overline{\omega}_i(G)/\overline{\omega}(G)$ is finite.

Suppose that $p > 2$. Since $C_{\overline{\omega}_i(G)}(P) \geq \gamma_3(\overline{\omega}_i(G))$ it follows that $\overline{\omega}_i(G)/C_{\overline{\omega}_i(G)}(P)$ is cyclic with 2-component of order at most 2. Let x be any element of $\overline{\omega}_i(G)$ which has order 4 modulo $\overline{\omega}(G)$. Then there is a cyclic subgroup E of prime power order which is not normalized by x^2 . Now PE is an abelian f -subnormal p -subgroup (as above) and we may write $PE = P \times \langle a \rangle$ for some element a of order a power of p . Clearly,

$$a^x = a^n c_x$$

for some element $c_x \in P$ and n coprime to p . Also $x^4 \in C_{\overline{\omega}_i(G)}(P) \cap \overline{\omega}(G)$, and hence, x^4 acts as a group of power automorphisms on the abelian p -group PE . Then [22, Lemma 1.5.4]

shows that this automorphism is universal, so x^4 centralizes PE . Therefore, x acts on $\langle a \rangle P/P$ as an automorphism of order precisely 4; if not $a^{x^2} = ac$, with $1 \neq c \in P$, so $p = 2$, or $a^{x^2} = a$ in which case E would be normalized by x^2 . In particular we have $a^{x^2} = a^{-1}c_{x^2}$ for some $c_{x^2} \in P$.

If y is another element of $\overline{\omega}_i(G)$ which has order 4 modulo $\overline{\omega}(G)$, then $y^2 = x^2c$ for some $c \in \overline{\omega}(G)$, but $c = x^{-2}y^2 \in C_{\overline{\omega}_i(G)}(P)$, so c centralizes PE as above. It follows that $a^{x^2} = a^{y^2}$. We choose $x, y, w = xy$ to be distinct elements of $\overline{\omega}_i(G)$ having order 4 modulo $\overline{\omega}(G)$ and such that x, y act in the opposite way on $\langle a \rangle P/P$. Then we have

$$a^w = ac_w$$

for some $c_w \in P$. But then

$$a^{w^2} = ac_w c_w^w = a^{-1}c_{w^2},$$

a contradiction.

Consequently $p = 2$.

Suppose next that $g\gamma_3(\overline{\omega}(G)) \in \overline{\omega}_i(G)/\gamma_3(\overline{\omega}(G))$ is a nontrivial element of order some prime $q \neq 2$. Then there is a cyclic f -subnormal 2-subgroup E which is not normalized by g , so $PE = P \times \langle a \rangle$ for some element a of order a power of 2. Now $a^g = a^n c$ for some $c \in P$ and integer n coprime to 2. Hence there is a power of 2, say l such that $g_1 = g^l$ centralizes P and $\langle a \rangle P/P$. Thus $a^{g_1} = ac$ for some $c \neq 1$ and this implies the contradiction $c^q = 1$. It follows that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is actually a 2-group.

(ii) Suppose now that Z is finite of order s say and that, for a contradiction, there is a prime q such that the exponent of the q -component of $\overline{\omega}_i(G)/\overline{\omega}(G)$ is infinite. For each $m \in \mathbb{N}$ there is some $g_m \gamma_3(\overline{\omega}_i(G)) \in C_{\overline{\omega}_i(G)}(Z)/\gamma_3(\overline{\omega}_i(G))$ for which $g_m^{q^m}$ does not normalize some cyclic f -subnormal subgroup $H_m = \langle h_m \rangle$ of order some power of q as in the proof of Lemma 3.8; we set

$$h_m^{g_m} = h_m^{n_m} l_m \quad (3)$$

for some $n_m \in \mathbb{N}$ and $1 \neq l_m \in Z$. Let $\mathcal{L}_q$ be the set of all these H_m .

Suppose that the subgroups in $\mathcal{L}_q$ have orders at most k and set $g = g_t$ where $t > ks$. Then (3) implies

$$h_t^{g^r} = h_t^{n_t^r} l_t^{1+n_t+\dots+n_t^{r-1}}$$

for $r \leq q^t$. Hence there exist $u, v \in \mathbb{N}$ such that $u < v \leq q^t$ and $h_t^{g^u} = h_t^{g^v}$, so g^{v-u} normalizes h_t , a contradiction to the choice of g .

Thus the orders of the elements of $\mathcal{L}_q$ are unbounded and we take a subset $\mathcal{L}'_q$ of $\mathcal{L}_q$ having infinitely many, and all distinct, orders. Let $u \in \mathbb{N}$. If q divides n_u , then (3) shows that the order of l_u is the same as that of h_u . Since Z is finite there are only finitely many $m \in \mathbb{N}$ for which $H_m \in \mathcal{L}'_q$ and q divides n_m . We remove the corresponding H_m from $\mathcal{L}'_q$ and denote the resulting subset by $\mathcal{L}''_q$. Let $H_m \in \mathcal{L}''_q$. Then $1 \neq l_m^{1+n_m+\dots+n_m^{r-1}} \in Z$ for all $r \leq q^m$. Moreover, if $i < j \leq q^m$ and

$$l_m^{1+n_m+\dots+n_m^{i-1}} = l_m^{1+n_m+\dots+n_m^{j-1}}$$

then

$$l_m^{n^i(1+n_m+\dots+n_m^{j-i-1})} = 1$$

and so also

$$l_m^{1+n_m+\dots+n_m^{j-i-1}} = 1,$$

which is a contradiction. Thus there are infinitely many elements in Z , which is the final contradiction. $\square$

A Dedekind 2-group of course has exponent at most 4, so we state the following immediate corollary.

Corollary 3.11 *Let G be a group such that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is an infinite nonabelian group. Then $\overline{\omega}_i(G)/\overline{\omega}(G)$ has finite exponent.*

4 Groups in which $\overline{\omega}_i(G)$ is finite

In this section we discuss the situation when $\overline{\omega}_i(G)$ is finite. It is easy to see that if G is the direct product of a quaternion group Q_8 of order 8 with infinitely many copies of the alternating group A_5 , then $\overline{\omega}_i(G) \cong Q_8 \cong \overline{\omega}(G)$. On the other hand, in this case we have $\omega_i(G) = G = \omega(G)$.

Lemma 4.1 *Let G be a group such that $C_G(\overline{\omega}_i(G))$ is infinite. Suppose that either*

- (i) *all cyclic subgroups of $G/C_G(\overline{\omega}_i(G))$ are f -subnormal or*
- (ii) *all cyclic subgroups of $\overline{\omega}_i(G)$ are f -subnormal.*

Then $\overline{\omega}_i(G)$ is nilpotent of class at most 2.

Proof Let F be a cyclic subgroup of prime power order in $\overline{\omega}_i(G)$ and let $X = C_G(\overline{\omega}_i(G))$. Since X is infinite, either of (i) or (ii) imply that FX is an infinite f -subnormal subgroup of G , so

$$\overline{\omega}_i(G) \leq N_G(FX) \leq N_G(Z(FX)) = N_G(FZ(X)).$$

It follows that $FZ(X)$ is an abelian normal subgroup of $\overline{\omega}_i(G)X$. This implies that $\overline{\omega}_i(G)Z(X)/Z(X) \cong \overline{\omega}_i(G)/Z(\overline{\omega}_i(G))$ is a Dedekind group, so $\overline{\omega}_i(G)$ is also nilpotent of class at most 3.

Suppose that $\overline{\omega}_i(G)/Z(\overline{\omega}_i(G))$ is nonabelian and hence periodic. Let p be an odd prime and let Y be the Sylow p -subgroup of $\overline{\omega}_i(G)$. Then $YZ(\overline{\omega}_i(G))/Z(\overline{\omega}_i(G))$ is abelian, so Y is nilpotent of class at most 2. Let L be the Sylow 2-subgroup of $\overline{\omega}_i(G)$. Then $L/Z(L)$ is Dedekind and nonabelian, so

$$L/Z(L) = \langle aZ(L), bZ(L) \rangle / Z(L) \times C/Z(L),$$

where

$$\langle aZ(L), bZ(L) \rangle / Z(L)$$

is a quaternion group of order 8 and $C/Z(L)$ is an elementary abelian 2-group generated by elements $c_i Z(L)$ for $i \in I$ an index set.

We shall prove that $a^2 \in Z(L)$, giving a contradiction. To see this, observe that a^2 commutes with a and $Z(L)$. Also $a^2 = b^2 z$ for some $z \in Z(L)$, so a^2 also commutes with b . However, $(ac_i)^2 = a^2 z_i$ for some $z_i \in Z(L)$ and $(ac_i)^2$ commutes with ac_i for all $i \in I$. Thus

$$ac_i = (ac_i)^{a^2 z_i} = a(c_i^{a^2})$$

which implies that $c_i = c_i^{a^2}$, so that a^2 commutes with each c_i and it follows that $a^2 \in Z(L)$, giving the contradiction sought. Hence $L/Z(L)$ is also abelian, so L is nilpotent of class at most 2, as is $\bar{\omega}_i(G)$, giving the result. $\square$

We may immediately deduce the following result.

Corollary 4.2 *Let G be an infinite group.*

- (i) *If $\bar{\omega}_i(G)$ is finite, then $\bar{\omega}_i(G)$ is nilpotent of class at most 2.*
- (ii) *If $\bar{\omega}_i(G)$ is a subgroup of the Baer radical of G , then $\bar{\omega}_i(G)$ is nilpotent of class at most 2.*

Proof (i) Clear from Lemma 4.1.

(ii) By (i) we may suppose that $\bar{\omega}_i(G)$ is infinite. If $\bar{\omega}_i(G) = \bar{\omega}(G)$, then $\bar{\omega}_i(G)$ is Dedekind and the result follows. Hence we may suppose that $\bar{\omega}_i(G) \neq \bar{\omega}(G)$, so Theorem 2.9 implies that $\bar{\omega}_i(G)$ is nilpotent and Prüfer-by-finite. Therefore, the Prüfer subgroup of $\bar{\omega}_i(G)$ is central in $\bar{\omega}_i(G)$, and hence, $C_G(\bar{\omega}_i(G))$ is infinite. We may now apply Lemma 4.1(ii) to deduce the result. $\square$

Lemma 4.3 *Let G be an infinite group such that $\bar{\omega}_i(G)$ is finite and let $\langle g \rangle$ be a finite f -subnormal subgroup of G of prime power order. Then $\bar{\omega}_i(G)$ contains a normal subgroup Z such that $Z \leq N_G(\langle g \rangle)$ and $\bar{\omega}_i(G)/Z$ is Dedekind. Furthermore,*

$$\bar{\omega}_i(G)/(\bar{\omega}_i(G) \cap N_G(\langle g \rangle))$$

is abelian.

Proof Let $W = \bar{\omega}_i(G)$. There are f -subnormal subgroups F and H such that

$$\langle g \rangle \leq F \triangleleft H$$

with F finite and H infinite. Let $H_1 = C_H(FW)$. Then $H_1W \triangleleft HW$, so $H_1W\langle g \rangle$ is f -subnormal in G . Now $T = H_1 \cap W\langle g \rangle$ is clearly central in $H_1W\langle g \rangle$, so $T \leq N_G(\langle g \rangle)$. Moreover, as H_1K is an infinite f -subnormal subgroup of G for each subgroup $K/T \leq W\langle g \rangle/T$ it follows that each subgroup of $W\langle g \rangle/T$ is normalized by W , so WT/T is Dedekind. Let $Z = T \cap W$.

We may clearly suppose that W/Z is nonabelian. Then W contains a subgroup Q such that Q/Z is quaternion. The structure of Q in this situation is given in [12, Theorem 1]. In this case $Q = ZR$, where R is a 2-group generated by elements a, b such that $Q/Z = \langle aZ, bZ \rangle$. We shall prove that $[a, b] \in N_g = N_G(\langle g \rangle)$ so that then $W/(N_g \cap W)$ is abelian.

If $QT/T \cap \langle g \rangle T/T \neq 1$, then $[a, b] = g^i x$ for some $x \in T$ and $i \in \mathbb{N}$. Then $[a, b] \in N_g$ since T centralizes g . Hence we may suppose that $QT/T \cap \langle g \rangle T/T = 1$. In this case, if the order of g is a power of an odd prime, then $\langle g \rangle$ is subnormal in $W\langle g \rangle$ and W is nilpotent by Corollary 4.2, it follows that $W\langle g \rangle$ is nilpotent and $R \leq N_g$ in this case. Thus we may assume that g has order a power of 2.

Every subgroup of $W\langle g \rangle/T$ is W -invariant, so if gT has order 2, then $g^a = gx$, $g^b = gy$, for some $x, y \in T$ and clearly $[a, b] \in N_g$. If gT has order at least 4 and if hT is an element of order precisely 4 in $\langle gT \rangle$, then it is easy to check that if aT, bT centralize hT , then $\langle ah \rangle T/T$ is not normalized by b . If say $h^a T = h^{-1}T$, then $(ah)^a T = ah^{-1}T \neq T, ahT, (ahT)^2 = a^2T, (ah)^3 T = a^{-1}hT$, so in this case we obtain a contradiction. The required result follows. $\square$

The following result is now immediate.

Theorem 4.4 *If G is an infinite group in which $\overline{\omega}_i(G)$ is finite, then $\overline{\omega}_i(G)/\overline{\omega}(G)$ is abelian.*

Lemma 4.5 *Let G be a group. Then $\overline{\omega}_i(G)$ is finite if and only if $\overline{\omega}(G)$ is finite.*

Proof Let $W = \overline{\omega}_i(G)$ and let $V = \overline{\omega}(G)$. Clearly, if W is finite, then so is V , so suppose that V is finite, but that W/V is infinite. Then Theorem 3.3 implies that W/V is nilpotent. Let S be a subgroup of W and note that $|SV : S|$ is finite. However, SV is subnormal in W , so S is f -subnormal in G . Hence all subgroups of W are f -subnormal in G . Thus if $X \leq W$, then $V \leq N_W(X)$ and by a result of Schenkman [21] $V \leq Z_2(W)$, the second centre of W . Thus W is itself nilpotent. Theorem 2.9 implies that W contains a G -invariant Prüfer subgroup P and Corollary 2.5 yields the contradiction that $P \leq V$. It follows that W must be finite. $\square$

We may therefore state Theorem 4.4 slightly differently.

Corollary 4.6 *If G is an infinite group in which $\overline{\omega}(G)$ is finite, then $\overline{\omega}_i(G)/\overline{\omega}(G)$ is a finite abelian group.*

We observe that in any case, using Theorems 3.3 and 4.4:

Theorem 4.7 *If G is an infinite group, then $\overline{\omega}_i(G)/\overline{\omega}(G)$ is Dedekind.*

We close this section by exhibiting a further class of groups in which $\overline{\omega}_i(G)/\overline{\omega}(G)$ is abelian for all groups G in the class.

Corollary 4.8 *Let G be an infinite FC-group. Then $\overline{\omega}_i(G)/\overline{\omega}(G)$ is abelian.*

Proof It is easy to see that for an FC-group, $\overline{\omega}(G)$ coincides with the norm of G , so $\overline{\omega}(G)$ is Dedekind in this case. By Theorem 4.7 $\overline{\omega}_i(G)/\overline{\omega}(G)$ is also Dedekind, so $\overline{\omega}_i(G)$ is subsoluble. If $\overline{\omega}_i(G)$ is finite, then the result follows by Corollary 4.6; otherwise, $\overline{\omega}_i(G)$ is infinite and Corollary 3.5 yields the result in this case. $\square$

5 Some applications of the results

In this short section we give an application of the results obtained to groups with finitely many normalizers of f -subnormal subgroups. Note that groups with finitely many normalizers of subnormal subgroups were studied in [7]. Let FC_{sn} be the class of groups such that all subnormal subgroups have finitely many conjugates. Let $\overline{FC}_{sn}$ be the largest subclass of FC_{sn} closed with respect to taking subgroups of finite index, normal subgroups and factor groups.

Theorem 5.1 *Let G be an $\overline{FC}_{sn}$ -group having finitely many normalizers of infinite f -subnormal subgroups. Then every f -subnormal subgroup of G has finitely many conjugates.*

Proof We use induction on the number of proper normalizers, k , of infinite f -subnormal subgroups. If $k = 0$, then all infinite f -subnormal subgroups of G are normal in G . The result follows in this case using Lemma 3.2.

Suppose that $k \geq 1$. Let X be an infinite f -subnormal subgroup of G . By [4, Proposition 3.1] there is a subnormal subgroup S of G with finite index in X and $X \leq N_G(S)$. Then $|G : N_G(S)|$ is finite since $G \in \overline{FC}_{sn}$. Let $Z = N_G(S)$ and suppose that $Z \neq G$. Since $\overline{FC}_{sn}$ is closed under taking subgroups of finite index we have $Z \in \overline{FC}_{sn}$. Then Z has one fewer

proper normalizer of infinite f -subnormal subgroups than G so by the induction hypothesis $|Z : N_Z(X)|$ is finite. Hence $|G : N_G(X)|$ is likewise finite.

If $S \triangleleft G$, then X/S is a finite f -subnormal subgroup of G/S . However, $G/S \in \overline{FC}_{sn}$ so Corollary 2.2 implies that $|G/S : N_{G/S}(X/S)|$ is finite, and hence, $|G : N_G(X)|$ is likewise finite.

On the other hand, if X is a finite f -subnormal subgroup of G , then X has finitely many conjugates by Corollary 2.2. $\square$

There are some very easy consequences of this result which enable us to prove results concerning the quotient group $G/\overline{\omega}_i(G)$. The proof of the next corollary uses [7, Theorem 3.6] and Theorem 5.1.

Corollary 5.2 *Let G be a soluble group with finitely many normalizers of f -subnormal subgroups. If G is locally polycyclic, then $|G : \overline{\omega}(G)|$ is finite.*

By [7, Theorem 4.4] the class of periodic soluble groups with finitely many normalizers of infinite f -subnormal subgroups lies in the class $\overline{FC}_{sn}$. We note also that in [7, p.385] it is shown that if G is a periodic soluble IT-group, then $G/\omega(G)$ is finite.

The next result follows using Theorem 3.4 and Theorem 5.1.

Corollary 5.3 *Let G be a periodic soluble group with finitely many normalizers of infinite f -subnormal subgroups. Then $|G : \overline{\omega}(G)|$ is finite, and in particular, G has finitely many normalizers of f -subnormal subgroups.*

Corollary 5.4 *Let G be a soluble group with finitely many normalizers of infinite f -subnormal subgroups. If $G/Z(G)$ is periodic, then $|G : \overline{\omega}(G)|$ is finite.*

Proof Using Corollary 5.3 we may assume $Z(G)$ is not periodic and hence contains an infinite cyclic subgroup J . If F is a finite f -subnormal subgroup of G , then FJ is f -subnormal in G and F is characteristic in FJ which implies that $\overline{\omega}_i(G) \leq N_G(FJ) \leq N_G(F)$ so $\overline{\omega}_i(G) = \overline{\omega}(G)$. This completes the proof. $\square$

Corollary 5.5 *Let G be a finitely generated soluble-by-finite group with finitely many normalizers of infinite f -subnormal subgroups. Then $|G : \overline{\omega}(G)|$ is finite. Furthermore, if G is soluble, then G is nilpotent and centre-by-finite. In particular if G is torsion-free and soluble, then G is abelian.*

Proof It follows from [7, Theorem 3.2] that G is abelian-by-finite so is a polycyclic-by-finite group. This implies in particular that G is an $\overline{FC}_{sn}$ -group. From Theorem 3.4 and Theorem 5.1 we see that $|G : \overline{\omega}(G)|$ is finite. The last part of the statement follows from [7, Theorem 3.2]. $\square$

6 Examples and closing remarks

In this final section we collect together some examples which further exhibit the relationships between $\overline{\omega}(G)$, $\overline{\omega}_i(G)$, $\omega(G)$ and $\omega_i(G)$. If G is the dihedral group of order 6, then it is easy to see that $\omega(G) = \omega_i(G)$ but that $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. On the other hand, if G is finite and if $\overline{\omega}_i(G) = \overline{\omega}(G)$, then $\overline{\omega}(G) = G$, so G is a Dedekind group, and hence, $\omega(G) = \omega_i(G)$. If G is a finite non-Dedekind p -group then $\overline{\omega}_i(G) = \omega_i(G) = G \neq \overline{\omega}(G) = \omega(G)$.

For infinite groups the following example shows that if $\overline{\omega}(G) = \overline{\omega}_i(G)$ we need not have $\omega(G) = \omega_i(G)$.

Example 1 There is an infinite group G such that $\overline{\omega}_i(G) = \overline{\omega}(G)$, but $\omega(G) \neq \omega_i(G)$. Furthermore, $\omega_i(G)/\omega(G)$ is not Dedekind.

Proof Let $p > 2$ be prime and for $i \geq 1$ let X_i be a finite perfect group such that $X_i/Z(X_i)$ is a nonabelian simple group, where $Z(X_i) = \langle x_i \rangle$ has order p for all such i . Let $Y = \langle a, b \rangle$ be an elementary abelian p -group of order p^2 and define automorphism s, t of Y by $a^s = a^t = a, b^s = ab, b^t = b^{-1}$, so that $Z = \langle s, t \rangle$ is dihedral of order $2p$. Let $S = Y \rtimes Z$ be the semidirect product of Y and Z , so S has order $2p^3$ and $Z(S) = \langle a \rangle$. We let G be the central product of S and the groups X_i .

Then $Z(G) = \langle a \rangle$. If R is an infinite subnormal subgroup of G , then there is some i for which $X_i \cap R$ is nontrivial. Thus we have $Z(G) \leq X_i \leq X_i \cap R$ since X_i is perfect. However, using similar arguments it is easy to show that all the infinite subnormal subgroups of $G/Z(G)$ are normal and so $\omega_i(G) = G$. On the other hand, $\omega(G) = \langle X_1, X_2, \dots \rangle$. Since every finite subgroup of G is f -subnormal in G we have $\overline{\omega}_i(G) = \overline{\omega}(G) = \langle a \rangle$. $\square$

The next example provides in particular an infinite finitely generated group G in which $\omega_i(G) \neq \omega(G)$, answering a question posed in [2]. We know of no example of a finitely generated group G in which $\overline{\omega}_i(G)/\overline{\omega}(G)$ (or $\omega_i(G)/\omega(G)$) is nonabelian.

Example 2 Let A be a nontrivial finite abelian group. Then there is an infinite group G such that $\overline{\omega}_i(G)/\overline{\omega}(G) \cong A \times A$.

Proof Let p be a sufficiently large prime number and let $H = G(\infty)$ be the group constructed in [15, 27.2], where we take $n_0 = p^i$. Then H is an infinite 2-generator group whose maximal subgroups are cyclic of order p^i and any two distinct subgroups of order p^i have trivial intersection in H , by [15, Theorem 28.2]. The group H has a (graded) presentation F/N where F is the free group of rank 2 and N is a normal subgroup of F . Furthermore, this presentation satisfies Condition R by [15, Lemma 27.2]. It follows by [15, Corollary 31.1] that $N/[F, N]$ is a free abelian group of countable rank.

The Schur multiplier $(F' \cap N)/[F, N]$ is then a free abelian group of at most countable rank. However, $N/(N \cap F') \cong NF'/F' = F/F'$ is 2-generator, so the Schur multiplier must have countable rank. By the universal coefficient theorem (see [19, 11.4.18], for example) there is a nonsplitting central extension M of an arbitrary countable abelian group by H . Moreover, using [19, Exercise 10, p.354] and the fact that $M/Z(M)$ is a nonabelian simple group, we may arrange that M is perfect.

Write $A = \text{Dr } \langle a_i \rangle$ and define $B = \text{Dr } \langle b_i \rangle, C = \text{Dr } \langle c_i \rangle$ with $|a_i| = |b_i| = |c_i|$. Now use the above construction to form a group M with $Z(M) = A$ and set $G = (M \times B) \rtimes C$. We define an action of A on $G \times B$ by $b_i^{c_i} = b_i a_i, b_i^{c_j} = b_i$ for $j \neq i$ and $g^{c_i} = g$ for all $g \in M$, for all $i \geq 1$. Clearly G is a finitely generated group. If S is an infinite subnormal subgroup of G , then it is easy to see that, since M is perfect, $M \leq S$. Thus $\omega_i(G) = \overline{\omega}_i(G) = G$. It is also easy to see that $\omega(G) = \overline{\omega}(G) = M$ and this completes the proof. $\square$

We saw above that $\overline{\omega}_i(G)$ is finite if and only if $\overline{\omega}(G)$ is finite. We note that the results of [2] can be used to show that $\omega_i(G)$ is finite if and only if $\omega(G)$ is finite. Furthermore, if $\omega_i(G)$ is finite, then so is $\overline{\omega}_i(G)$. However, it is easy to construct an infinite periodic soluble group G in which $G/\omega(G)$ is finite, while $\overline{\omega}(G)$ is trivial. For example, let p_i, q_i for $i \geq 1$ be infinite sequences of distinct primes such that $p_i \equiv 1 \pmod{q_i}$ which can be constructed by Dirichlet's theorem on the number of primes in arithmetic progressions and for $i \geq 1$ let G_i be the natural semidirect product of the group of order p_i by the group of order q_i . In this case it is easy to see that $\overline{\omega}_i(G) = \overline{\omega}(G) = 1$, but that $\omega_i(G) = \omega(G) = G$.

In general the group $\bar{\omega}_i(G)$ need not be a Dedekind group. We know from [2, Example 1] that there is a finite perfect group X such that $X/Z(X)$ is a nonabelian simple group such that $Z(X)$ has order p . For each $i \geq 1$, let $X_i \cong X$ and let H be the central product of these groups X_i with amalgamated centres. Let $K = H \times \langle a \rangle$, where $\langle a \rangle$ is an element of order p . Let $\langle b \rangle$ denote the centre of H and let $G = K \rtimes \langle c \rangle$, where c is an element of order p , $h^c = h$ and $a^c = ab$. If Y is an infinite f -subnormal subgroup of G , then Y contains b and it follows easily that $\bar{\omega}_i(G) = \langle a, c \rangle$, which is nilpotent of class 2, but not Dedekind.

Next we adapt a construction due to Heineken (see [13, Example 3]) which shows that every finite Dedekind group can be realized as the quotient $\bar{\omega}_i(G)/\bar{\omega}(G)$ for some group G .

Example 3 If G is a finite Dedekind group, then there is a group R such that $\bar{\omega}_i(R)/\bar{\omega}(R) \cong G \cong \omega_i(R)/\omega(R)$.

Proof The group G is a subgroup of the Galois group of some finite Galois extension F of $\mathbb{Q}$, the field of rational numbers. Let $L = GL(V, F)$ be the group of invertible linear mappings of the F -vector space V of countably infinite dimension and let L^* denote the set of those mappings which fix elementwise some subspace of finite codimension. Rosenberg [20] showed that $L^+ = L/L^*$ is simple modulo its centre, that $L = L'$ and $Z(L^+) \cong F^*$, the multiplicative group of F . By [11, Theorem 127.2] F^* is isomorphic to the direct product of a finite cyclic group and a free abelian group of countably infinite rank. Let $K = L^+ \rtimes G$ where the action of G is as follows. We fix a basis for V , say $\{x_1, x_2, \dots\}$, let $\sigma \in G$ and define $(fx_i)^\sigma = f^\sigma x_i$; then the action of G on L is conjugation, namely if $\alpha \in L$ and $\sigma \in G$ then we define α^σ to be the composite map $\sigma^{-1}\alpha\sigma$ and under this action L^* is G -invariant, so there is an induced action of G on L^+ . We note that if $f \in F^* = Z(L^+)$, then $f^\sigma = \sigma(f) \in F^*$, for all $\sigma \in G$.

Let $Z(L^+) = S \times U$, where S is a finite cyclic group and U is freely generated by elements $u_1, u_2, \dots$. Let $1 \neq g \in G$. We shall define a subgroup H_g with finite index in $Z(L^+)$ and an element $f_g \in Z(L^+)$ such that $(f_g)^g H_g$ is not a power of $f_g H_g$.

First, suppose that, for some i , $u_i^g = u_i^n t u_{j_1}^{m_1} \dots u_{j_k}^{m_k}$, where $t \in S$, $i, j_1, \dots, j_k$ are pairwise distinct, $k \in \mathbb{N}$ and $m_1, \dots, m_k, n \in \mathbb{Z}$. If $t \neq 1$, take $H_g = U$. Then $u_i^g H_g = u_i^n t H_g \neq u_i^r H_g$, for any integer r .

If $t = 1$ and there is some index, j_1 say, such that $u_{j_1}^{m_1} \neq 1$, then we let

$$H_g = \langle u_j, u_{j_1}^p \mid j \neq i, j_1 \text{ and } p \text{ is a prime greater than } |m_1| \rangle.$$

Again we can easily see that $u_i^g H_g \neq u_i^r H_g$, for any integer r . In the case when u_i is mapped to some power $u_i^{m_i}$ for all i it follows that $m_i = \pm 1$, since g has finite order. If g fixes U , then there is some element $s \in S$ such that s has prime power order q^m , say, and $s^g \neq s$. Then $(su_1)^g \neq (su_1)^k$ for any integer k and we take

$$H_g = \langle u_j, u_1^{q^m} \mid j \geq 2 \rangle.$$

Suppose that there is an index l such that $u_l^g = u_l^{-1}$. If there is an index $m \neq l$ such that u_m is fixed by g , then we replace u_m by $u'_m = u_l u_m$. Then $(u'_m)^g = u'_m u_l^{-2}$ and we can use the argument above to define H_g . Hence we may assume that g acts by inversion on U . Now if $n \in \mathbb{Q}$, then n is fixed by g , but $n = su$ with $s \in S$, $u \in U$, so $su = n = n^g = s^g u^{-1}$ which gives a contradiction. This completes the construction of the group H_g and we let

$$J = \bigcap_{g \in G} H_g, T = J_G, R = K/T.$$

If there were an element $g \in G$ acting as a power automorphism on $Z(L^+)/T$, then this would also be the case for $Z(L^+)/H_g$, which is a contradiction to the choice we made. Suppose that H/T is an infinite f -subnormal subgroup of R , so that there is an infinite subnormal subgroup M/T of R such that $M \triangleleft H$ and H/M is finite. Since $L^+/Z(L^+)$ is simple and L^+ is perfect, it is easy to see that $L^+ \leq M \leq H$ and as G is Dedekind we have $H \triangleleft K$. Thus $\overline{\omega}_i(R) = R$.

However, every subgroup of $Z(L^+)/T$ is subnormal in K/T , so $\overline{\omega}(R) = \omega(R) = L^+/T$. Thus $\overline{\omega}_i(R)/\overline{\omega}(R) \cong G$ as required. $\square$

In fact we know no example of a group G for which $\overline{\omega}_i(G)/\overline{\omega}(G)$ is infinite.

Let $N(G)$ denote the norm of the group G . The final result shows in particular that, by contrast, there is no finite group G such that $\overline{\omega}_i(G)/\overline{\omega}(G)$ is isomorphic to the quaternion group of order 8.

Proposition 6.1 *There is no group G such that $G/N(G)$ is isomorphic to the quaternion group of order 8.*

Proof Let $N = N(G)$ and suppose, for a contradiction, that G is a group such that $G/N = \langle xN, yN \rangle$ is quaternion of order 8. Then there is a cyclic subgroup H of G which is not normalized by y^2 , and hence, x, y, xy do not normalize H either. By symmetry we may assume that H is one of the subgroups $\langle n \rangle, \langle nx \rangle, \langle ny \rangle, \langle ny^2 \rangle$, for some $n \in N$. Since $y^2 = n_1 x^2$ for some $n_1 \in N$ indeed we may assume that H is one of the three possibilities

$$\langle n \rangle, \langle ny \rangle, \langle ny^2 \rangle.$$

It is easy to see that the norm of $N\langle y \rangle$ is N and it follows from [1, Zusatz, p. 267] that N must be abelian. Let $H = \langle ny \rangle$. Since N normalizes $\langle y \rangle$ we have

$$z_y = [ny, y] = [n, y] \in N \cap \langle y \rangle \leq Z(N\langle y \rangle) \text{ and } y^n = yz_y^{-1}.$$

Furthermore, $z_y \notin \langle ny \rangle$ and since $(ny)^{y^2} = nyz_y^2 \notin \langle ny \rangle$, but $ny = (ny)^{y^4} = nyz_y^4$ we see that z_y has order exactly 4. Thus $\langle z_y \rangle \cap \langle ny \rangle = 1$. Then $(ny)^n = ny^n = nyz_y^{-1}$ and since $n \in N$, the norm of G , we obtain the contradiction that $z_y \in \langle ny \rangle$. We obtain a similar contradiction if $H = \langle ny^2 \rangle$.

On the other hand, let $H = \langle n \rangle$. Using the notation above, we see that $z_x^2 = z_y^2 = z_{xy}^2$, as $x^2N = y^2N = (xy)^2N$. Also $n^{xy} = nz_yz_x^y = nz_{xy}$ and z_{xy} commutes with xy so that

$$n^{(xy)(yx)^{-1}} = n^{(xy)^2} = n(z_yz_x^y)^2 = nz_y^2.$$

From this we obtain

$$z_yz_x^yz_yz_x^y = z_y^2, \text{ so } z_x^yz_yz_x^y = z_y \text{ and } z_x^2 = 1,$$

since N is abelian. This further contradiction gives the result. $\square$

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