Annali di Matematica Pura ed Applicata (1923 -) (2020) 199:23–63
https://doi.org/10.1007/s10231-019-00865-6

Groundstate asymptotics for a class of singularly perturbed p-Laplacian problems in $\mathbb{R}^N$

Wedad Albalawi[1] · **Carlo Mercuri**[2] · **Vitaly Moroz**[2]

Received: 27 July 2018 / Accepted: 3 May 2019 / Published online: 17 May 2019

Abstract

We study the asymptotic behaviour of positive groundstate solutions to the quasilinear elliptic equation

$$-\Delta_p u + \varepsilon u^{p-1} - u^{q-1} + u^{l-1} = 0 \quad \text{in} \quad \mathbb{R}^N \qquad (P_\varepsilon)$$

where $1 < p < N$, $p < q < l < +\infty$ and $\varepsilon > 0$ is a small parameter. For $\varepsilon \to 0$, we give a characterization of asymptotic regimes as a function of the parameters q, l and N. In particular, we show that the behaviour of the groundstates is sensitive to whether q is less than, equal to, or greater than the critical Sobolev exponent $p^* := \frac{pN}{N-p}$.

Keywords Groundstates · Liouville-type theorems · Quasilinear equations · Singular perturbation

Mathematics Subject Classification 35J92 (35B33 · 35B53 · 35B38)

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✉ Carlo Mercuri
C.Mercuri@swansea.ac.uk

Wedad Albalawi
wsalbalawi@pnu.edu.sa

Vitaly Moroz
V.Moroz@swansea.ac.uk

[1] College of Sciences, Princess Nourah Bint Abdulrahman University, PO Box 84428, Riyadh, Kingdom of Saudi Arabia

[2] Department of Mathematics, Computational Foundry, Swansea University, Fabian Way, Swansea SA1 8EN, UK

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1 Introduction

The present paper is devoted to the study of positive solutions to the quasilinear elliptic equation

$$-\Delta_p u + \varepsilon u^{p-1} - u^{q-1} + u^{l-1} = 0 \qquad \text{in} \quad \mathbb{R}^N, \tag{P_ε}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

is the p-Laplacian operator, $1 < p < N$, $p < q < l$ and $\varepsilon > 0$ is a small parameter. Our main aim is to understand the behaviour of positive groundstate solutions to (P_ε) as $\varepsilon \to 0$. By a solution to (P_ε), we mean a weak solution $u_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N)$. These solutions are constructed as critical points of the energy

$$\mathcal{E}_\varepsilon(u) := \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \int_{\mathbb{R}^N} F_\varepsilon(u)\mathrm{d}x, \tag{$\mathcal{E}_\varepsilon$}$$

where

$$F_\varepsilon(u) = \int_0^u \tilde{f}_\varepsilon(s)\mathrm{d}s,$$

and the expression $\tilde{f}_\varepsilon$ is a suitable bounded truncation of

$$f_\varepsilon(s) := -\varepsilon|s|^{p-2}s + |s|^{q-2}s - |s|^{l-2}s. \tag{1.1}$$

Throughout the paper by groundstate solution to (P_ε), we mean a positive weak solution which has the least energy $\mathcal{E}_\varepsilon$ amongst all the other non-trivial solutions.

In the first part of the paper, for all $1 < p < N$ and $p < q < l$, we prove the existence of a radial groundstate solution u_ε of (P_ε) for all sufficiently small $\varepsilon > 0$, see Theorem 2.1, extending classical results of Berestycki and Lions [3] from the Laplacian ($p = 2$) to the p-Laplacian setting, for any $1 < p < N$. As a by-product of the method [3] which is adapted to the present quasilinear context, the weak solutions to (P_ε) which are found, are essentially bounded and decay uniformly to zero as $|x| \to \infty$. We recall that, as in the known case $p = 2$ treated in [3], the symmetry of the solutions is achieved as a limit of a suitable (minimizing) sequence of radially decreasing rearrangements constructed from a possibly non-radial minimizing sequence. Theorem 2.1 in Sect. 3.2 summarizes all the above results about the existence and basic properties of these groundstates to (P_ε).

We point out that for large $\varepsilon > 0$ equation (P_ε) has no finite energy solutions, so the restriction on the size of ε is essential for the existence of the groundstates. The uniqueness (up to translations) of a spherically symmetric groundstate of (P_ε) is rather delicate. For $p \leq 2$, Serrin and Tang [33, Theorem 4] proved that equation (P_ε) admits at most one positive groundstate solution. For $p > 2$ the uniqueness could be also expected but to the best of our knowledge this remains an open question. We do not study the question of uniqueness in this paper and none of our result rely on the information about the uniqueness of the groundstate to (P_ε).

The question of understanding the asymptotic behaviour of the groundstates u_ε of (P_ε) as $\varepsilon \to 0$, naturally arises in the study of various bifurcation problems, for which (P_ε) at least in the case $p = 2$ can be considered as a canonical normal form (see, e.g. [8,40]). This problem may also be regarded as a bifurcation problem for quasilinear elliptic equations

$$-\Delta_p u = f_\varepsilon(u) \quad \text{in} \quad \mathbb{R}^N,$$

whose nonlinearity f_ε has the leading term in the expansion around zero which coincides with the ones in (P_ε). Let us also mention that problem (P_ε) in the case $p = 2$ appears in the study of phase transitions [6,25,44], as well as in the study of the decay of false vacuum in quantum field theories [7].

Loosely speaking, to understand the asymptotic behaviour of the groundstates u_ε as $\varepsilon \to 0$, one notes that elliptic regularity implies that locally the solution u_ε converges as $\varepsilon \to 0$ to a radial solution of the limit equation (see Theorem 6.4)

$$-\Delta_p u - u^{q-1} + u^{l-1} = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{P_0}$$

It is known that (here and in the rest of the paper $p^* := \frac{pN}{N-p}$ is the critical Sobolev exponent): when $q \leq p^*$ equation (P_0) has no non-trivial finite energy solutions, by Pohožaev's identity (3.1), whereas for $q > p^*$ equation (P_0) admits a radial groundstate solution. Existence goes back to Berestycki–Lions [3] and Merle–Peletier [23] in the case $p = 2$ and, in the context of the present paper, it is proved in the general p-Laplacian case (see Theorem 4.3); uniqueness questions have been studied by Tang [38, Theorem 4.1], see also Remark 4.4 .

In Theorem 2.8 we prove using direct variational arguments that, as expected, for $q > p^*$ solutions u_ε converge as $\varepsilon \to 0$ to a non-trivial radial groundstate solution to the formal limit equation (P_0). The fact that for $q \leq p^*$ equation (P_0) has no non-trivial positive solutions, suggests that for $q \leq p^*$ the solutions u_ε should converge almost everywhere, as $\varepsilon \to 0$, to the trivial zero solution of equation (P_0) (see estimate 2.2). This however does not reveal any information about the limiting profile of u_ε. Therefore, instead of looking at the formally obtained limit equation (P_0), we are going to show that for $q \leq p^*$ solutions u_ε converge to a non-trivial limit *after a rescaling*. The limiting profile of u_ε will be obtained from the groundstate solutions of the *limit equations* associated with the rescaled equation (P_ε), where the choice of the associated rescaling and limit equation depends on the value of p and on the space dimension N in a highly non-trivial way.

The convergence of rescaled solutions u_ε to their limiting profiles will be proved using a variational analysis similar to the techniques developed in [24] in the case $p = 2$. Note that the natural energy space for equation (P_ε) is the usual Sobolev space

$$W^{1,p}(\mathbb{R}^N) := \Big\{u : u \in L^p(\mathbb{R}^N) \quad \text{and} \quad \nabla u \in L^p(\mathbb{R}^N)\Big\},$$

with the norm

$$||u||_{1,p} = ||u||_p + ||\nabla u||_p,$$

while for $q > p^*$ the natural functional setting associated with the limit equation (P_0) is the homogeneous Sobolev space $D^{1,p}(\mathbb{R}^N)$ defined for $1 < p < N$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $||\nabla u||_{L^p}$. Since $W^{1,p}(\mathbb{R}^N) \subsetneq D^{1,p}(\mathbb{R}^N)$, it follows that no natural perturbation setting (suitable to apply the implicit function theorem or Lyapunov–Schmidt-type reduction methods) is available to analyse the family of equations (P_ε) as $\varepsilon \to 0$. In fact, even for $p = 2$ a linearization of (P_0) around the groundstate solution is not a Fredholm operator and has zero as the bottom of the essential spectrum in $L^2(\mathbb{R}^N)$. In the case of the p–Laplace equations, the difficulty in applying classical perturbation methods is even more striking, as for $1 < p < 2$ the energy associated with the p-Laplacian is not twice Fréchet differentiable.

In order to understand the limiting profile of u_ε in the case $q \le p^*$, we introduce the *canonical* rescaling associated with the lowest order nonlinear term in (P_ε):

$$v_\varepsilon(x) = \varepsilon^{-\frac{1}{q-p}} u_\varepsilon\left(\frac{x}{\sqrt[p]{\varepsilon}}\right). \tag{1.2}$$

Then (P_ε) reads as

$$-\Delta_p v + v^{p-1} = v^{q-1} - \varepsilon^{\frac{l-q}{q-p}} v^{l-1} \qquad \text{in} \quad \mathbb{R}^N, \tag{R_ε}$$

from which we formally get, as $\varepsilon \to 0$, the limit problem

$$-\Delta_p v + v^{p-1} = v^{q-1} \qquad \text{in} \quad \mathbb{R}^N. \tag{R_0}$$

We recall that for $q \ge p^*$ equation (R_0) has no non-trivial finite energy solutions, as a consequence of Pohožaev's identity (3.1), whereas for $p < q < p^*$ equation (R_0) possesses a unique radial groundstate solution. Existence was proved by Gazzola, Serrin and Tang [15] and uniqueness by Pucci–Serrin [29, Theorem 2]. The particular rescaling (1.2) allows to have, when $p < q < p^*$, for both (R_ε) and the limit problem (R_0), a variational formulation on the same Sobolev space $W^{1,p}(\mathbb{R}^N)$. This indicates that problem (R_ε) could be considered as a small perturbation of the limit problem (R_0). In particular in the case $p = 2$ the family of the groundstates (v_ε) of problem (R_ε) could be rigorously interpreted as a perturbation of the groundstate solution of the limit problem (R_0) using the perturbation techniques and framework developed by Ambrosetti, Malchiodi et al., see [2] and references. However, for $p \neq 2$ the Lyapunov–Schmidt reduction technique, in the spirit of [2] is not directly applicable. Instead, in this work, using a direct variational argument inspired by [24, Theorem 2.1] we prove (see Theorem 2.2) that for $p < q < p^*$ groundstate solutions (v_ε) of the rescaled problem (R_ε) converge to the (unique) radial groundstate of the limit problem (R_0).

In the critical case $q = p^*$, the limit problem (R_0) has no non-trivial positive solutions. This means that in this case the *canonical* rescaling (1.2) does not accurately capture the behaviour of (u_ε). In the present paper, extending the results obtained in [24] for $p = 2$, we show that for $q = p^*$ the asymptotic behaviour of the groundstate solutions to (P_ε) after a rescaling is given by a particular solution of the critical Emden–Fowler equation

$$-\Delta_p U = U^{p^*-1} \quad \text{in } \mathbb{R}^N. \tag{R_*}$$

It is well known that equation (R_*) admits a continuum of radial groundstate solutions. We will prove that the choice of the rescaling (and a particular solution of (R_*)) which provides the limit asymptotic profile for groundstate solutions to equation (P_ε) depends on the dimension N in a non-trivial way (see Theorem 2.3).

Wrapping up, we provide a characterization of the three asymptotic regimes occurring as $\varepsilon \to 0$, i.e. the subcritical case $q < p^*$, the supercritical case $q > p^*$ and the critical case $q = p^*$, extending the results of [24,25], to both a singular ($p < 2$) and degenerate ($p > 2$) quasilinear setting.

Asymptotic notation

Throughout the paper we will extensively use the following asymptotic notation. For $\varepsilon \ll 1$ and $f(\varepsilon), g(\varepsilon) \geq 0$, we write $f(\varepsilon) \lesssim g(\varepsilon)$, $f(\varepsilon) \sim g(\varepsilon)$ and $f(\varepsilon) \simeq g(\varepsilon)$, implying that there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$:
$f(\varepsilon) \lesssim g(\varepsilon)$ if there exists $C > 0$ independent of ε such that $f(\varepsilon) \leq Cg(\varepsilon)$;
$f(\varepsilon) \sim g(\varepsilon)$ if $f(\varepsilon) \lesssim g(\varepsilon)$ and $g(\varepsilon) \lesssim f(\varepsilon)$;
$f(\varepsilon) \simeq g(\varepsilon)$ if $f(\varepsilon) \sim g(\varepsilon)$ and $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 1$.

We also use the standard Landau symbols $f = O(g)$ and $f = o(g)$, with the understanding that $f \geq 0$ and $g \geq 0$. As usual, C, c, c_1, etc., denote generic positive constants independent of ε.

2 Main results

The following theorem summarizes the existence results for the equation (P_ε). The proof is a standard adaptation of the Berestycki and Lions method [3]. For completeness, we sketch the arguments in Sect. 3.2.

Theorem 2.1 *Let $N \geq 2$, $1 < p < N$ and $p < q < l$. Then there exists $\varepsilon_* = \varepsilon_*(p, q, l) > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$, equation ($P_\varepsilon$) admits a groundstate $u_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, $u_\varepsilon(x)$ is, by construction, a positive monotone decreasing function of $|x|$ and*

$$u_\varepsilon(|x|) \leq Ce^{-\delta|x|}, \qquad x \in \mathbb{R}^N,$$

for some $C, \delta > 0$.

For $p \leq 2$, Serrin and Tang proved [33, Theorem 4] that equation (P_ε) admits at most one positive groundstate solution. For $p > 2$ the uniqueness to the best of our knowledge remains an open question. As anticipated earlier, none of our subsequent results rely on the uniqueness of groundstates of (P_ε). In what follows, u_ε always denotes *a* groundstate solution to (P_ε) constructed in Theorem 2.1 for an $\varepsilon \in (0, \varepsilon_*)$. When we say that groundstates u_ε converge to a certain limit (in some topology) as $\varepsilon \to 0$, we understand that for every $\varepsilon > 0$ a groundstate of (P_ε) is selected, so that $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_*)}$ is a branch of groundstates of (P_ε), which is not necessarily continuous in ε. In the present work, we study the limit behaviour of such a branch of groundstates when $\varepsilon \to 0$.

2.1 Subcritical case $p < q < p^*$

As anticipated earlier, since in the subcritical case the formal limit equation (P_0) has no groundstate solutions, the family of groundstates u_ε must converge to zero, uniformly on compact subsets. We describe the asymptotic behaviour of u_ε performing the rescaling (1.2) which transforms (P_ε) into equation (R_ε). In Sect. 7, using the variational approach developed in the main part of this work we prove the following result, which extends [24, Theorem 2.1] to the case $p \neq 2$.

Theorem 2.2 *Let $N \geq 2$, $1 < p < N$, $p < q < p^*$ and (u_ε) be a family of groundstates of (P_ε). As $\varepsilon \to 0$, the rescaled family*

$$v_\varepsilon(x) := \varepsilon^{-\frac{1}{q-p}} u_\varepsilon\left(\frac{x}{\sqrt[p]{\varepsilon}}\right) \tag{2.1}$$

converges in $W^{1,p}(\mathbb{R}^N)$, $L^l(\mathbb{R}^N)$ and $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ to the unique radial groundstate solution $v_0(x)$ of the limit equation (R_0). In particular,

$$u_\varepsilon(0) \simeq \varepsilon^{\frac{1}{q-p}} v_0(0). \tag{2.2}$$

2.2 Critical case $q = p^*$

In this case we show that after a suitable rescaling the correct limit equation for (P_ε) is given by the critical Emden–Fowler equation

$$-\Delta_p U = U^{p^*-1} \quad \text{in } \mathbb{R}^N. \tag{R_*}$$

It is well known by Guedda–Veron [16] that the only radial solution to (R_*) is given, by the family of rescalings

$$U_\lambda(|x|) := U_1(|x|/\lambda) \qquad (\lambda > 0), \tag{2.3}$$

where

$$U_1(|x|) := \left[\frac{\kappa^{1/p'} N^{1/p}}{1 + |x|^{p'}}\right]^{\kappa/p'}, \tag{2.4}$$

and where $p' := \frac{p}{p-1}$ and $\kappa := \frac{N-p}{p-1}$. Recently in [12] it has been observed that $\pm U_\lambda$ are the only nontrivial radial solutions to $\Delta_p u + |u|^{p^*-2}u = 0$ in $D^{1,p}(\mathbb{R}^N)$. Sciunzi [32] and Vétois [42], respectively in the ranges $p > 2$ and $p < 2$, proved that any positive solution to (R_*) in $D^{1,p}(\mathbb{R}^N)$ is necessarily radial about some point; this combined with [16] gives a complete classification of the positive finite energy solutions to (R_*).

Our main result in this work is the following theorem, which extends [24, Theorem 2.5] to the case $p \neq 2$.

Theorem 2.3 *Let $N \geq 2$, $1 < p < N$, $p^* = q < l$ and (u_ε) be a family of groundstates of (P_ε). There exists a rescaling*

$$\lambda_\varepsilon : (0, \varepsilon_*) \to (0, \infty) \tag{2.5}$$

such that as $\varepsilon \to 0$, the rescaled family

$$v_\varepsilon(x) := \lambda_\varepsilon^{\frac{N-p}{p}} u_\varepsilon(\lambda_\varepsilon x)$$

converges in $D^{1,p}(\mathbb{R}^N)$ *to the radial groundstate solution* $U_1(x)$ *of the Emden–Fowler equation* (R_*). *Moreover,*

$$\lambda_\varepsilon \gtrsim \begin{cases} \varepsilon^{-\frac{p^*-p}{p(l-p)}} & 1 < p < \sqrt{N}, \\ \left(\varepsilon(\log \frac{1}{\varepsilon})\right)^{-\frac{(p^*-p)}{p(l-p)}} & p = \sqrt{N}, \\ \varepsilon^{-\frac{1}{[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N, \end{cases} \tag{2.6}$$

and

$$\lambda_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{p^*-p}{p(l-p)}} & 1 < p < \sqrt{N}, \\ \varepsilon^{-\frac{(p^*-p)}{p(l-p)}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{(l-p^*)}{p(l-p)}} & p = \sqrt{N}, \\ \varepsilon^{-\frac{(p^2-N)(l-p^*)+p^2}{p^2[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N. \end{cases} \tag{2.7}$$

Remark 2.4 The lower bound (2.6) on λ_ε can be converted into an upper bound on the maximum of u_ε,

$$u_\varepsilon(0) \lesssim \begin{cases} \varepsilon^{\frac{l}{(l-p)}} & 1 < p < \sqrt{N}, \\ \left(\varepsilon(\log \frac{1}{\varepsilon})\right)^{\frac{l}{(l-p)}} & p = \sqrt{N}, \\ \varepsilon^{\frac{N-p}{p[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N, \end{cases} \tag{2.8}$$

see Corollary 5.20.

For $1 < p < \sqrt{N}$ lower bound (2.6) and upper bound (2.7) are equivalent and hence optimal. For $\sqrt{N} \le p < N$, the upper bounds in (2.7) do not match the lower bounds (2.6). However, under some additional restrictions, we could obtain optimal two-sided estimates.

Theorem 2.5 *Under the assumptions of Theorem* 2.3, *we additionally have*

$$\lambda_\varepsilon \sim \begin{cases} \varepsilon^{-\frac{p^*-p}{p(l-p)}} & 1 < p < \sqrt{N} \text{and} N \ge 2, \\ \left(\varepsilon(\log \frac{1}{\varepsilon})\right)^{-\frac{(p^*-p)}{p(l-p)}} & p = \sqrt{N} \text{ and } N \ge 4, \\ \varepsilon^{-\frac{1}{[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < \frac{N+1}{2} \text{ and } N \ge 4, \end{cases} \tag{2.9}$$

and

$$u_\varepsilon(0) \sim \begin{cases} \varepsilon^{\frac{l}{(l-p)}} & 1 < p < \sqrt{N} \text{ and } N \ge 2, \\ \left(\varepsilon(\log \frac{1}{\varepsilon})\right)^{\frac{l}{(l-p)}} & p = \sqrt{N} \text{ and } N \ge 4, \\ \varepsilon^{\frac{N-p}{p[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < \frac{N+1}{2} \text{ and } N \ge 4. \end{cases} \tag{2.10}$$

In the above cases v_ε *converges to* $U_1(x)$ *in* $L^l(\mathbb{R}^N)$ *and* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.

Remark 2.6 In the case $p = 2$ and $N \ge 3$, *two–sided* asymptotics of the form (2.9) were derived in [25] using methods of formal asymptotic expansions. Later, two sided bounds of the form (2.9) were rigorously established for $p = 2$ in [24, Theorem 2.5]. The barrier approach developed in [24, Lemma 4.8] in order to refine upper bounds on λ_ε in the difficult case $\sqrt{N} \le p < N$ cannot be fully extended to $p \ne 2$, see Lemma 5.11. In this difficult case, the matching upper bounds of the form (2.6) are valid for $\sqrt{N} \le p < \frac{N+1}{2}$ and $N \ge 4$.

Remark 2.7 Theorem 2.5 leaves open the following cases, where matching lower and upper bounds are not available:

- $N \geq 4$ and $\frac{N+1}{2} \leq p < N$
- $N = 3$ and $\sqrt{3} \leq p < 3$
- $N = 2$ and $\sqrt{2} \leq p < 2$

Note that the case $N = 3$ and $p = 2$ is not included in Theorem 2.5. However, matching bounds (2.9) and (2.10) remain valid in this case. This is one of the results in [24, Theorem 2.5]. We conjecture that the restriction $p < \frac{N+1}{2}$ is merely technical and is due to the method we use.

2.3 Supercritical case $q > p^*$

Unlike the subcritical and critical cases, for $q > p^*$ the formal limit equation (P_0) admits a nontrivial solution. Using a direct analysis of the family of constrained minimization problems associated with (P_ε), we prove the following result, which extends [24, Theorem 2.3] to the case $p \neq 2$.

Theorem 2.8 *Let $N \geq 2$, $1 < p < N$, $p^* < q < l$ and (u_ε) be a family of groundstates of (P_ε). As $\varepsilon \to 0$, the family u_ε converges in $D^{1,p}(\mathbb{R}^N)$, $L^l(\mathbb{R}^N)$ and $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ to a groundstate solution $u_0(x)$ of the limit equation (P_0), with*

$$u_0(x) \sim |x|^{-\frac{N-p}{p-1}} \quad \text{as} \quad |x| \to \infty.$$

Moreover, it holds that

$$u_\varepsilon(0) \simeq u_0(0),$$

and that $\varepsilon \| u_\varepsilon|_p^p \to 0$.

2.4 Organisation of the paper

This paper is organized as follows. Section 3 is devoted to the existence and qualitative properties of groundstates u_ε to (P_ε); in Sect. 4 we deal with existence and qualitative properties of groundstates to the limiting PDEs (P_0), (R_0), (R_*). Both sections contain various facts about the equation (P_ε) and limiting equations which are involved in our analysis. In the rest of the paper we study the asymptotic behaviour of the groundstates u_ε. In Sect. 5 we study the most delicate critical case $q = p^*$ and prove Theorems 2.3 and 2.5. In Sect. 6 we consider the supercritical case $q > p^*$ and prove Theorem 2.8. In Sect. 7 we consider the subcritical case $q < p^*$ and prove Theorem 2.2. For the reader convenience we have collected in the sections A and B of Appendix some auxiliary results which have been used in the main body of the paper.

3 Groundstate solutions to (P_ε)

3.1 Necessary conditions and Pohožaev's identity

According to Pohožaev's classical identity [26] for p-Laplacian equations, a solution to (P_ε) which is smooth enough, necessarily satisfies the identity

$$\int\limits_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x = p^* \int\limits_{\mathbb{R}^N} F(u) \mathrm{d}x, \tag{3.1}$$

for $1 < p < N$. Identities of this type are classical, see for instance [28] for C^2 solutions and [9] for bounded domains. In the present paper the following version of Pohožaev's identity has been extensively used.

Proposition 3.1 *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that* $f(0) = 0$, *and set* $F(t) = \int_0^t f(s)\mathrm{d}s$. *Let*

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N), \quad \textit{and} \quad |\nabla u|^p,\ F(u) \in L^1(\mathbb{R}^N)$$

with u such that

$$-\Delta_p u = f(u)$$

holds in the sense of distributions. Then u satisfies (3.1).

Proof We first assume that $p \le 2$. By the classical regularity result of Tolksdorf [39], see also Theorem 2.5 in [31], we have

$$u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N), \quad p \le 2.$$

Having checked the existence and local summability of the second weak derivatives in this case we argue as follows. Multiply the equation by $x_i \partial_i u(x)$ and integrate over $B_R = B(0, R)$ and denote by $n(\cdot)$ the outer normal unit vector. Observe that the vector field

$$v = x_i \partial_i u |\nabla u|^{p-2} \nabla u$$

is such that $v \in C(\mathbb{R}^N, \mathbb{R}^N)$ and $\operatorname{div} v \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. By the divergence theorem (see, e.g. Lemma 2.1 in [22]), we have

$$\begin{aligned}\int_{B_R} \Delta_p u\, x_i \partial_i u(x)\mathrm{d}x &= \int_{\partial B_R} |\nabla u(\sigma)|^{p-2} \partial_i u(\sigma) \sigma_i \nabla u \cdot n \,\mathrm{d}\sigma \\ &\quad - \int_{B_R} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla [x_i \partial_i u(x)]\mathrm{d}x.\end{aligned}$$

Write the last integral as $A_i + B_i$, where

$$\begin{aligned}A_i &:= \int_{B_R} |\nabla u(x)|^{p-2} |\partial_i u(x)|^2 \mathrm{d}x, \\ B_i &:= \frac{1}{p} \int_{B_R} \partial_i (|\nabla u(x)|^p) x_i \mathrm{d}x.\end{aligned}$$

An integration by parts in B_i yields

$$B_i = \frac{1}{p} \int_{\partial B_R} |\nabla u(\sigma)|^p \sigma_i n_i \mathrm{d}\sigma - \frac{1}{p} \int_{B_R} |\nabla u(x)|^p \mathrm{d}x.$$

On the other hand, we have also

$$\begin{aligned}&\int_{B_R} f(u(x)) x_i \partial_i u(x)\mathrm{d}x \\ &\quad = -\int_{B_R} F(u(x))\mathrm{d}x + \int_{\partial B_R} F(u(x)) \sigma_i n_i \mathrm{d}\sigma.\end{aligned}$$

Summing up on i we have

$$(*)\qquad N\int_{B_R} F(u(x))\mathrm{d}x + \left(1-\frac{N}{p}\right)\int_{B_R} |\nabla u(x)|^p \mathrm{d}x = \int_{\partial B_R} |\nabla u(\sigma)|^{p-2}\nabla u\cdot\sigma\,\nabla u\cdot n\,\mathrm{d}\sigma$$
$$-\frac{1}{p}\int_{\partial B_R} |\nabla u(\sigma)|^p \sigma\cdot n\,\mathrm{d}\sigma + \int_{\partial B_R} F(u(x))\sigma\cdot n\,\mathrm{d}\sigma.$$

The right hand side is bounded by

$$M(R) = \left(1+\frac{1}{p}\right) R\int_{\partial B_R} |\nabla u(\sigma)|^p\,\mathrm{d}\sigma + R\int_{\partial B_R} |F(u(x))|\mathrm{d}\sigma.$$

Similarly as in Lemma 2.3 from [22], since $F(u)$, $|\nabla u|^p \in L^1(\mathbb{R}^N)$, there exists a sequence $R_k \to \infty$ such that $M(R_k) \to 0$. By using the monotone convergence theorem in $(*)$ we obtain the conclusion in the case $p \le 2$.

For $p > 2$ a regularisation argument similar to [11, p. 833] (see also [12,17,20]) allows to work with a $C^{1,\alpha}_{\mathrm{loc}}$-approximation $u_\varepsilon \in C^2$ which classically solves

$$\begin{aligned} -\mathrm{div}\left(\left(\varepsilon + |\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right) &= f(u) \qquad \text{in } B_{2R},\\ u_\varepsilon &= u \qquad \text{on } \partial B_{2R}. \end{aligned}$$

The proof can be then carried out with obvious modifications of the proof given in the case $p \le 2$, performing the ε-limit before letting $R \to +\infty$ along a suitable sequence $(R_k)_{k\in\mathbb{N}}$, and this concludes the proof. □

3.2 Existence and variational characterization of the groundstates

To prove the existence of groundstates, we first observe that the method of Berestycki–Lions [3] although focused on the case $p = 2$ is applicable in the present quasilinear context, we sketch the proof referring to [3] for the details. In fact, observe that $f_\varepsilon(s) = |s|^{q-2}s - |s|^{l-2}s - \varepsilon|s|^{p-2}s$ satisfies

(f_1) $-\infty < \liminf_{s\to 0^+} \dfrac{f_\varepsilon(s)}{s^{p-1}} \leqq \limsup_{s\to 0^+} \dfrac{f_\varepsilon(s)}{s^{p-1}} = -\varepsilon < 0.$

(f_2) $-\infty \leqq \limsup_{s\to+\infty} \dfrac{f_\varepsilon(s)}{s^{p^*-1}} \leqq 0,$ where $p^* = \frac{pN}{N-p}$.

(f_3) There exists $\varepsilon_* > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$ the following property holds: there exists $\zeta > 0$ such that $F_\varepsilon(\zeta) = \int_0^\zeta f_\varepsilon(s)\mathrm{d}s > 0$.

To prove the existence of an optimizer, one carries on with the constrained minimization argument as in [3], based on the truncation of the nonlinearity f_ε, which allows to use $W^{1,p}(\mathbb{R}^N)$ for the functional setting. For all $\varepsilon \in (0, \varepsilon_*)$ in the present context $p \neq 2$ a suitable truncated function $\tilde{f}_\varepsilon : \mathbb{R} \to \mathbb{R}$ is provided by:

$$\tilde{f}_\varepsilon(u) = \begin{cases} 0, & u < 0,\\ u^{q-1} - u^{l-1} - \varepsilon u^{p-1}, & u \in [0,1],\\ -\varepsilon, & u > 1. \end{cases} \qquad \tilde{F}_\varepsilon(u) := \int\limits_0^u \tilde{f}_\varepsilon(s)\mathrm{d}s, \tag{3.2}$$

Replacing in (P_ε) the nonlinearity with the above bounded truncation $\tilde{f}_\varepsilon(u)$ makes the minimization problem

$$S_\varepsilon = \inf\left\{\int\limits_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x; \quad w \in W^{1,p}(\mathbb{R}^N), \quad p^* \int\limits_{\mathbb{R}^N} \tilde{F}_\varepsilon(w)\mathrm{d}x = 1\right\} \qquad (S_\varepsilon)$$

well-posed in $W^{1,p}(\mathbb{R}^N)$ even for supercritical $l > p^*$. Standard compactness arguments using radially symmetric rearrangements of minimizing sequences allows to obtain a radially decreasing optimizer w_ε, see also “Appendix A”. If w_ε is an optimizer for (S_ε) then a Lagrange multiplier θ_ε exists such that

$$-\Delta_p w_\varepsilon = \theta_\varepsilon \tilde{f}_\varepsilon(w_\varepsilon) \quad \text{in}\, \mathbb{R}^N. \tag{3.3}$$

Note that by construction $\tilde{f}_\varepsilon(u) \in L^\infty(\mathbb{R}^N)$ and then by a classical result of DiBenedetto, see, e.g. Corollary p. 830 in [11], any solution $u \in W^{1,p}(\mathbb{R}^N)$ to the truncated problem with $\tilde{f}_\varepsilon$ is regular, i.e. $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Then the maximum principle implies that any solution for the truncated problem is strictly positive and solves the problem

$$-\Delta_p w_\varepsilon = \theta_\varepsilon f_\varepsilon(w_\varepsilon) \quad \text{in}\, \mathbb{R}^N, \tag{3.4}$$

involving the original nonlinearity. The exponential decay estimate (3.10) on w_ε follows by Gazzola–Serrin ([14, Theorem 8]). As a consequence of the regularity and summability, w_ε satisfies both Nehari’s identity

$$\int\limits_{\mathbb{R}^N} |\nabla w_\varepsilon|^p \mathrm{d}x = \theta_\varepsilon \int\limits_{\mathbb{R}^N} f_\varepsilon(w_\varepsilon) w_\varepsilon \mathrm{d}x, \tag{3.5}$$

and Pohožaev’s identity (3.1)

$$\int\limits_{\mathbb{R}^N} |\nabla w_\varepsilon|^p \mathrm{d}x = \theta_\varepsilon p^* \int\limits_{\mathbb{R}^N} F_\varepsilon(w_\varepsilon) \mathrm{d}x. \tag{3.6}$$

The latter immediately implies that

$$\theta_\varepsilon = S_\varepsilon. \tag{3.7}$$

Then a direct calculation involving (3.7) shows that the rescaled function

$$u_\varepsilon(x) := w_\varepsilon(x/\sqrt[p]{S_\varepsilon}) \tag{3.8}$$

is the radial groundstate of (P_ε), described in Theorem 3.2 below.

One more consequence of Pohožaev’s identity (3.6) is an expression for the total energy of the solution

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \left(\frac{1}{p} - \frac{1}{p^*}\right) S_\varepsilon^{N/p},$$

(see [3, Corollary 2]), which shows that u_ε is indeed a groundstate, i.e. a nontrivial solution with the least energy. Another simple consequence of (3.6) is that (P_ε) has no nontrivial finite energy solutions for $\varepsilon \geq \varepsilon_*$. The threshold value ε_* is simply the smallest value of $\varepsilon > 0$ for which the energy $\mathcal{E}_\varepsilon$ is nonnegative and can be computed explicitly.

To summarize, in the spirit of [3, Theorem 2] we have the following

Theorem 3.2 *Let $N \geq 2$, $1 < p < N$ and $p < q < l$. Then there exists $\varepsilon_* = \varepsilon_*(p, q, l) > 0$ such that for all $\varepsilon \in (0, \varepsilon_*)$, the minimization problem (S_ε) has a minimizer $w_\varepsilon \in W^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$. The minimizer w_ε satisfies*

$$-\Delta_p w_\varepsilon = S_\varepsilon f_\varepsilon(w_\varepsilon) \quad \text{in } \mathbb{R}^N. \tag{3.9}$$

Moreover, $w_\varepsilon(x)$ is a positive monotone decreasing function of $|x|$ and

$$w_\varepsilon(|x|) \leq Ce^{-\delta|x|}, \qquad x \in \mathbb{R}^N, \tag{3.10}$$

for some $C, \delta > 0$. The rescaled function

$$u_\varepsilon(x) := w_\varepsilon(x/\sqrt[p]{S_\varepsilon})$$

is a groundstate solution to (P_ε).

In view of (3.2) and since we are interested only in positive solutions of (P_ε), in what follows, we always assume that the nonlinearity $f_\varepsilon(u)$ in (P_ε) is replaced by its bounded truncation $\tilde{f}_\varepsilon(u)$ from (3.2), without mentioning this explicitely.

Remark 3.3 Equivalently to (S_ε), we can consider minimizing the quotient

$$\mathcal{S}_\varepsilon(w) := \frac{||\nabla w||_p^p}{\left(p^* \int\limits_{\mathbb{R}^N} F_\varepsilon(w)\mathrm{d}x\right)^{(N-p)/N}}, \quad w \in \mathcal{M}_\varepsilon,$$

where

$$\mathcal{M}_\varepsilon := \left\{0 \leq w \in W^{1,p}(\mathbb{R}^N), \int\limits_{\mathbb{R}^N} F_\varepsilon(w)\mathrm{d}x > 0\right\}.$$

Setting $w_\lambda(x) := w(\lambda x)$, it is easy to check that $\mathcal{S}_\varepsilon(w_\lambda) = \mathcal{S}_\varepsilon(w)$ for all $\lambda > 0$. Therefore it holds that

$$S_\varepsilon = \inf_{w \in \mathcal{M}_\varepsilon} \mathcal{S}_\varepsilon(w). \tag{3.11}$$

Moreover, the inclusion $\mathcal{M}_{\varepsilon_2} \subset \mathcal{M}_{\varepsilon_1}$ for $\varepsilon_2 > \varepsilon_1 > 0$, (3.11) implies that S_ε is a nondecreasing function of $\varepsilon \in (0, \varepsilon_*)$.

4 Limiting PDEs

4.1 Critical Emden–Fowler equation

In this section, we recall some old and new results for the critical Emden–Fowler equation

$$-\Delta_p u = u^{p^*-1}, \quad u \in D^{1,p}(\mathbb{R}^N), \quad u > 0, \tag{R_*}$$

where $1 < p < N$, $p^* = pN/(N-p)$ is the critical exponent for the Sobolev embedding. We observe that any nontrivial nonnegative solution to (R_*) is necessarily positive as a consequence of strong maximum principle (see [41]). Solutions of (R_*) are critical points of the functional

$$\mathcal{J}(u) := \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x. \tag{4.1}$$

By the Sobolev embedding $D^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$, $\mathcal{J}$ is defined in $D^{1,p}(\mathbb{R}^N)$. Since by [18] all the minimizing sequences for

$$S_* := \inf \left\{ \int_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x; \quad w \in D^{1,p}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |w|^{p^*} \mathrm{d}x = 1 \right\}, \tag{S_*}$$

are relatively compact modulo translations and dilations, critical points for $\mathcal{J}$ are provided by direct minimization, after suitable rescaling of positive solutions W to the Euler–Lagrange equation for S_*

$$-\Delta_p W = \theta W^{p^*-1} \quad \text{in } \mathbb{R}^N. \tag{4.2}$$

Here since

$$\int_{\mathbb{R}^N} |\nabla W|^p \mathrm{d}x = \theta \int_{\mathbb{R}^N} |W|^{p^*} \mathrm{d}x = \theta,$$

it follows that $S_* = \theta$. Positive finite energy solutions to this equation are classified after the works of Guedda–Veron [16] and of Sciunzi [32] and Vétois [42] mentioned in the Introduction, which we recall in the following

Theorem 4.1 *Let $1 < p < N$. Then every radial solution U to (R_*) is represented as*

$$U(|x|) = U_{\lambda,0}(|x|) := \left[\frac{\lambda^{p'/p} k^{1/p'} N^{1/p}}{\lambda^{p'} + |x|^{p'}} \right]^{k/p'}, \tag{4.3}$$

for some $\lambda > 0$, where $p' := \frac{p}{p-1}$ and $k := \frac{N-p}{p-1}$, [16].
In fact, every solution U to (R_*) is radially symmetric about some points $y \in \mathbb{R}^N$ and therefore it holds that

$$U(x) = U_{\lambda,y}(x) := \left[\frac{\lambda^{p'/p} k^{1/p'} N^{1/p}}{\lambda^{p'} + |x-y|^{p'}} \right]^{k/p'}, \tag{4.4}$$

for some $\lambda > 0$ and $y \in \mathbb{R}^N$, [32,42].

In the case $p = 2$ and $N \geq 3$ this result is classical, see [5]. Hence, the radial groundstate of (R_*) is given by rescaling the function

$$U_{1,0}(x) := \left[\frac{k^{1/p'} N^{1/p}}{1 + |x|^{p'}} \right]^{k/p'}, \tag{4.5}$$

and moreover it holds that

$$||\nabla U_{\lambda,0}||_p^p = ||U_{\lambda,0}||_{p^*}^{p^*} = S_*^{N/p}, \tag{4.6}$$

see, e.g. [37]. In conclusion all the positive minimisers for (S_*) are translations of the radial family

$$W_\lambda(x) := U_{\lambda,0}(\sqrt[p]{S_*} x). \tag{4.7}$$

4.2 Supercritical zero mass equation

This section is devoted to the supercritical equation

$$-\Delta_p u - |u|^{q-2}u + |u|^{l-2}u = 0 \quad \text{in } \mathbb{R}^N, \tag{P_0}$$

where $1 < p < N$ and $p^* < q < l$.

Remark 4.2 Note that by Pohožaev's identity (3.1), equation (P_0) has no solution in $D^{1,p}(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ $q \leq p^*$.

We prove the following existence result in the spirit of Merle-Peletier [23] to the case $p \neq 2$.

Theorem 4.3 *Let $N \geq 2$, $1 < p < N$ and $p^* < q < l$. Equation (P_0) admits a groundstate solution $u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, such that $u_0(x)$ is a positive monotone decreasing function of $|x|$ and*

$$u_0(x) \sim |x|^{-\frac{N-p}{p-1}} \quad as \quad |x| \to \infty. \tag{4.8}$$

Remark 4.4 The uniqueness result of [38] is applicable to fast decay solutions to (P_0). However the regularity hypothesis H1 as stated at p. 155 in [38] would require $p^* \geq 2$, namely $p \geq \frac{2N}{N+2}$.

Proof Following Berestycki–Lions [3] in the present zero-mass case context, we solve the variational problem in $D^{1,p}(\mathbb{R}^N)$ namely

$$S_0 := \inf \left\{ \int_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x | w \in D^{1,p}(\mathbb{R}^N), \quad p^* \int_{\mathbb{R}^N} \tilde{F}_0(w) \mathrm{d}x = 1 \right\}, \tag{S_0}$$

where

$$\tilde{F}_0(w) = \int_0^w \tilde{f}_0(s) \mathrm{d}s,$$

and $\tilde{f}_0(s)$ is a bounded truncation of the nonlinearity

$$f_0(s) = |s|^{q-2}s - |s|^{l-2}s,$$

e. g.

$$\tilde{f}_0(u) = \begin{cases} 0, & u < 0, \\ u^{q-1} - u^{l-1}, & u \in [0,1], \\ 0, & u > 1. \end{cases} \tag{4.9}$$

The above bounded truncation makes the minimization problem well-posed in $D^{1,p}(\mathbb{R}^N)$. Arguing as for the positive mass case the existence of a radially decreasing optimizer u is standard.

The global boundedness of the truncation allows to use the classical result of DiBenedetto, see, e.g. Corollary p. 830 in [11], to show that $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$.
Then the maximum principle implies that any solution for the truncated problem solves in fact (P_0) and is strictly positive.

Note that by Ni's inequality A.3 and the $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ regularity it follows that $u \in L^\infty(\mathbb{R}^N)$. By interpolation with Sobolev's inequality, this implies that $u \in L^l(\mathbb{R}^N)$ for all $l > p^*$.
With the lemmas below on the asymptotic decay, we conclude the proof. □

The following lemma about asymptotic properties of solutions is taken from [13].

Lemma 4.5 ([13, Corollary 8.3.]) *Let* $1 < p < N$. *Assume that*

$$|V(x)| \leq \frac{g(|x|)}{1+|x|^p},$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ *is bounded and continuous and satisfies the following conditions:*

(C1) $\left|\int\limits_1^\infty \left| t^{1-N} \int\limits_1^t \frac{g(|x|)}{|x|^p} |x|^{N-1} \mathrm{d}|x| \right|^{\frac{1}{p-1}} \mathrm{d}t\right| < \infty.$

(C2) $\left|\int\limits_1^\infty \frac{g(|x|)}{|x|} \mathrm{d}|x|\right| < \infty.$

Assume that

$$-\Delta_p u + V(x)u^{p-1} = 0, \qquad in\ \mathbb{R}^N \backslash B_1(0), \tag{4.10}$$

admits a positive supersolution. Then (4.10) *admits a solution which satisfies*

$$U_0(x) \sim |x|^{-\frac{N-p}{p-1}} \qquad as\ |x| \to \infty. \tag{4.11}$$

Corollary 4.6 *If*

$$V(x) = \frac{c}{(1+|x|)^{p+\delta}},$$

and c is sufficiently small then (4.10) *admits a positive solution that satisfies* (4.11)

Proof We can take

$$g(|x|) = |x|^{-\delta}.$$

Then $(C1)$, $(C2)$ are elementary to check. □

The decay estimate (4.8) is proved in the following lemma.

Lemma 4.7 *Let* $u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N)$ *be a positive radial solution of* (P_0). *Then*

$$u_0(x) \sim |x|^{-\frac{N-p}{p-1}} \quad as \quad |x| \to \infty. \tag{4.12}$$

Proof Since $u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N)$ is radial then by the Ni type inequality A.3, we have

$$u_0 \leq c|x|^{-\frac{N-p}{p}}, \quad \text{in } \mathbb{R}^N \backslash B_1(0),$$

and since $l > p^*$ then we have for some $\delta_1 > 0$

$$u_0^{l-p} \leq c|x|^{-\frac{N-p}{p}(l-p)} = \frac{c}{|x|^{p+\delta_1}}, \qquad \text{in } \mathbb{R}^N \backslash B_1(0), \tag{4.13}$$

implying

$$u_0^{l-p} \leq \frac{C}{(1+|x|)^{p+\delta_1}}, \qquad \text{in } \mathbb{R}^N,$$

for sufficiently large constant C independent of x. Now set

$$-\Delta_p u_0 + (u_0^{l-p})u_0^{p-1} = u_0^{q-1} \geq 0, \quad \text{in } \mathbb{R}^N,$$

and then we have

$$-\Delta_p u_0 + \frac{C}{(1+|x|)^{p+\delta_1}} u_0^{p-1} \geq 0, \qquad \text{in } \mathbb{R}^N.$$

As a consequence, u_0 is a supersolution of (4.10) and then by comparison principle (see Theorem B.1 in Appendix), we obtain

$$u_0 \geq c|x|^{-\frac{N-p}{p-1}} \quad \text{in} \quad |x| > 1. \tag{4.14}$$

Similarly, we can set

$$-\Delta_p u_0 - (u_0^{q-p}) u_0^{p-1} = -u_0^{l-1} \leq 0, \qquad \text{in } \mathbb{R}^N,$$

and since $q > p^*$ we have for some $\delta_2 > 0$,

$$u_0^{q-p} \leq c'|x|^{-\frac{N-p}{p}(q-p)} \leq \frac{c'}{|x|^{p+\delta_2}}, \qquad \text{in } |x| > 1,$$

implying

$$u_0^{q-p} \leq \frac{C'}{(1+|x|)^{p+\delta_2}}, \qquad \text{in } \mathbb{R}^N,$$

and hence

$$-\Delta_p u_0 - \frac{C'}{(1+|x|)^{p+\delta_2}} u_0^{p-1} \leq 0 \qquad \text{in } \mathbb{R}^N.$$

Now since $u_0 \in D^{1,p}_{rad}(\mathbb{R}^N)$ is a subsolution of (4.10), then by Lemma B.2 u_0 satisfies condition (S) and hence by comparison principle Theorem B.1, we have

$$u_0 \leq c'|x|^{-\frac{N-p}{p-1}} \qquad \text{in} \quad |x| > 1, \tag{4.15}$$

and hence from (4.14) and(4.15) the conclusion follows. □

5 Proof of Theorems 2.3 and 2.5: critical case $q = p^*$

In this section we analyse the behaviour of the groundstates u_ε of equation (P_ε) as $\varepsilon \to 0$ in the critical case $q = p^*$ and prove Theorem 2.3. Although our approach follows the ideas of [24], the present p-Laplacian setting requires substantial modifications.

5.1 Variational estimates for S_ε

Equivalently to the Sobolev constant (S_*), we consider the Rayleigh type Sobolev quotient

$$\mathcal{S}_*(w) := \frac{\int_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |w|^{p^*} \mathrm{d}x\right)^{(N-p)/N}}, \quad w \in D^{1,p}(\mathbb{R}^N), \quad w \neq 0,$$

which is invariant with respect to the dilations $w_\lambda(x) := w(x/\lambda)$, so that

$$S_* = \inf_{0 \neq w \in D^{1,p}(\mathbb{R}^N)} \mathcal{S}_*(w).$$

We define the gap

$$\sigma_\varepsilon := S_\varepsilon - S_*. \tag{5.1}$$

To estimate σ_ε in terms of ε, we shall use the Sobolev minimisers W_μ from (4.7) as test functions for ($\mathcal{S}_\varepsilon$). Since $W_\lambda \in L^p(\mathbb{R}^N)$ only if $1 < p < \sqrt{N}$, we analyse the higher and lower dimensions separately. It is easy to check that $W_\lambda \in L^s(\mathbb{R}^N)$ for all $s > \frac{N(p-1)}{N-p}$, with

$$||W_\lambda||_s^s = \lambda^{-\frac{N-p}{p}s+N}||W_1||_s^s = \lambda^{-\frac{N-p}{p}(s-p^*)}||W_1||_s^s,$$

and that, if $1 < p < \sqrt{N}$ then $W_\lambda \in L^p(\mathbb{R}^N)$ it holds that

$$||W_\lambda||_p^p = \lambda^p||W_1||_p^p.$$

In the case of dimensions $p = \sqrt{N}$ and $\sqrt{N} < p < N$, given $R \gg \mu$, we introduce a cutoff function $\eta_R \in C_0^\infty(\mathbb{R})$ such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta_R < 1$ for $R < |r| < 2R$, $\eta_R(r) = 0$ for $|r| > 2R$ and $|\eta_R^{'}(r)| \leq 2/R$. We then compute as in, e.g. [35, Chapter III, proof of Theorem 2.1]

$$||\nabla\left(\eta_R W_\mu(x)\right)||_p^p = S_* + O\left(\left(\frac{R}{\mu}\right)^{-\frac{N-p}{p-1}}\right), \tag{5.2}$$

$$||\eta_R W_\mu||_{p^*}^{p^*} = 1 - O\left(\left(\frac{R}{\mu}\right)^{-\frac{N}{p-1}}\right), \tag{5.3}$$

$$||\eta_R W_\mu||_l^l = \mu^{-\frac{N-p}{p}(l-p^*)}||W_1||_l^l\Big(1 - O\Big(\Big(\frac{R}{\mu}\Big)^{-\frac{(N-p)l}{p-1}+N}\Big), \tag{5.4}$$

and

$$||\eta_R W_\mu||_p^p = \begin{cases} O\left(\mu^p \log R\right), & p = \sqrt{N}, \\ O\left(\mu^{\frac{N-p}{p-1}} R^{\frac{p^2-N}{p-1}}\right), & \sqrt{N} < p < N. \end{cases} \tag{5.5}$$

As a consequence of these expansions we get an upper estimate for σ_ε which plays a key role in what follows.

Lemma 5.1 *It holds that*

$$0 < \sigma_\varepsilon \lesssim \begin{cases} \varepsilon^{\frac{l-p^*}{l-p}} & 1 < p < \sqrt{N}, \\ \varepsilon^{\frac{(N-p)(l-p^*)}{p[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N, \\ \left(\varepsilon(\log\frac{1}{\varepsilon})\right)^{\frac{(l-p^*)}{(l-p)}} & p = \sqrt{N}. \end{cases} \tag{5.6}$$

Hence, $\sigma_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof We first observe that since

$$S_* \leq \mathcal{S}_*(w_\varepsilon) < \mathcal{S}_\varepsilon(w_\varepsilon) = S_\varepsilon,$$

it follows that $\sigma_\varepsilon > 0$. We now obtain the upper bounds on σ_ε.
Case $1 < p < \sqrt{N}$. Note that $W_\mu \in \mathcal{M}_\varepsilon$ for all sufficiently small ε and sufficiently large μ, and we have

$$\mathcal{S}_\varepsilon(W_\mu) \leq \frac{S_*}{\left(1 - \varepsilon\mu^p\beta_p - \mu^{\frac{-(N-p)}{p}(l-p^*)}\beta_l\right)^{(N-p)/N}}, \tag{5.7}$$

where

$$\beta_p := \frac{p^*}{p}||W_1||_p^p, \qquad \beta_l := \frac{p^*}{l}||W_1||_l^l.$$

We now optimize the right hand side of the estimate (5.7) picking μ such that the function

$$\psi_\varepsilon(\mu) := \beta_p \varepsilon \mu^p + \beta_l \mu^{-\frac{N-p}{p}(l-p^*)}.$$

achieves its minimum. This occurs at

$$\mu_\varepsilon \sim \varepsilon^{-\frac{p}{(N-p)(l-p)}} \tag{5.8}$$

and we have

$$\min_{\mu>0} \psi_\varepsilon \sim \psi_\varepsilon(\mu_\varepsilon) \sim \varepsilon^{\frac{l-p^*}{l-p}}.$$

In the present case $1 < p < \sqrt{N}$, we may conclude that

$$\mathcal{S}_\varepsilon(W_\mu) \lesssim \frac{S_*}{(1-\psi_\varepsilon(\mu_\varepsilon))^{(N-p)/N}} = S_*\left(1 + O(\psi_\varepsilon(\mu_\varepsilon))\right) = S_* + O\left(\varepsilon^{\frac{l-p^*}{l-p}}\right), \tag{5.9}$$

and (5.8) is the value of μ_ε such that the bound (5.6) is achieved on the function W_{μ_ε}.
Case $p > \sqrt{N}$. We assume here that $R \gg \mu$. Using $\eta_R W_\mu$ as test function and using the calculation in (5.2)–(5.5), we get

$$\begin{aligned}\mathcal{S}_\varepsilon(\eta_R W_\mu) \le &\left(S_* + O\left(\left(\frac{R}{\mu}\right)^{-\frac{N-p}{p-1}}\right)\right)\\ &\times\Bigg(1 - \Bigg\{O\Big(\Big(\frac{R}{\mu}\Big)^{-\frac{N}{p-1}}\Big) + \varepsilon O\Big(\mu^{\frac{N-p}{p-1}} R^{\frac{p^2-N}{p-1}}\Big)\\ &+\mu^{-\frac{N-p}{p}(l-p^*)}||W_1||_l^l\Big[1 - O\Big(\Big(\frac{R}{\mu}\Big)^{-\frac{(N-p)l}{p-1}+N}\Big)\Big]\Bigg\}\Bigg)^{-\frac{N-p}{N}},\end{aligned}$$

and hence as $\frac{R}{\mu} \to \infty$, we have

$$\mathcal{S}_\varepsilon(\eta_R W_\mu) \le S_*\Big(1 + \psi_\varepsilon(\mu, R)\Big),$$

and hence as $\frac{R}{\mu} \to \infty$, we have

$$\mathcal{S}_\varepsilon(\eta_R W_\mu) \le S_*\Big(1 + \psi_\varepsilon(\mu, R)\Big),$$

where

$$\psi_\varepsilon(\mu, R) := \left(\frac{R}{\mu}\right)^{-\frac{N-p}{p-1}} + \varepsilon \mu^{\frac{N-p}{p-1}} R^{\frac{p^2-N}{p-1}} + \mu^{-\frac{N-p}{p}(l-p^*)}. \tag{5.10}$$

If in particular we choose

$$\mu_\varepsilon = \varepsilon^{-\frac{1}{(l-p^*)(p-1)+p}}, \qquad R_\varepsilon = \varepsilon^{-\frac{1}{p}}. \tag{5.11}$$

we then find that

$$\psi_\varepsilon(\mu_\varepsilon, R_\varepsilon) \sim \varepsilon^{\frac{(N-p)(l-p^*)}{p[(l-p^*)(p-1)+p]}},$$

and, similarly to the above case, the bound (5.6) is achieved on the test function $\eta_{R_\varepsilon} W_{\mu_\varepsilon}$ provided μ_ε and R_ε are as in (5.11).

Case $p = \sqrt{N}$. Again we assume that $R \gg \mu$. Testing again against $\eta_R W_\mu$ and by(5.2)–(5.5) with $p = \sqrt{N}$, we get

$$\begin{aligned}\mathcal{S}_\varepsilon(\eta_R W_\mu) \leq \Big(S_* &+ O\Big(\Big(\frac{R}{\mu}\Big)^{-\frac{N-p}{p-1}}\Big)\Big)\\ &\times\Big(1 - \Big(O\Big(\Big(\frac{R}{\mu}\Big)^{\frac{-N}{p-1}}\Big) + \varepsilon O\Big(\mu^p \log R\Big)\\ &+ \mu^{-\frac{N-p}{p}(l-p^*)} ||W_1||_l^l \Big[1 - O\Big(\Big(\frac{R}{\mu}\Big)^{-\frac{(N-p)l}{p-1}+N}\Big)\Big]\Big)\Big)^{-(N-p)/N},\end{aligned}$$

and then as $\frac{R}{\mu} \to \infty$, we have

$$\mathcal{S}_\varepsilon(\eta_R W_\mu) \leq S_*\Big(1 + \psi_\varepsilon(\mu, R)\Big),$$

where

$$\psi_\varepsilon(\mu, R) := \Big(\frac{R}{\mu}\Big)^{-\frac{N-p}{p-1}} + \varepsilon\mu^p \log R + \mu^{-\frac{N-p}{p}(l-p^*)}. \tag{5.12}$$

Choose

$$R_\varepsilon := \varepsilon^{-\frac{1}{p}}, \qquad \mu_\varepsilon := \Big(\varepsilon \log \frac{1}{\varepsilon}\Big)^{\frac{-p}{(N-p)(l-p)}}, \tag{5.13}$$

and hence

$$\psi_\varepsilon(\mu_\varepsilon, R_\varepsilon) \sim \Big(\varepsilon \log \frac{1}{\varepsilon}\Big)^{\frac{(l-p^*)}{(l-p)}}.$$

Thus the bound (5.6) is achieved by the test function $\eta_{R_\varepsilon} W_{\mu_\varepsilon}$, where μ_ε and R_ε are defined in (5.13). □

5.2 Pohožaev estimates

For $\varepsilon \in (0, \varepsilon_*)$, let $w_\varepsilon > 0$ be a family of the minimisers for $(\mathcal{S}_\varepsilon)$ (or equivalently 3.11). This minimisers w_ε solve the Euler–Lagrange equation

$$-\Delta_p w_\varepsilon = \mathcal{S}_\varepsilon\big(-\varepsilon w_\varepsilon^{p-1} + w_\varepsilon^{p^*-1} - w_\varepsilon^{l-1}\big) \quad \text{in } \mathbb{R}^N \tag{5.14}$$

with the original (untruncated) nonlinearity.

Our next step is to use Nehari's identity combined with Pohožaev's identity for (5.14) in order to obtain the following useful relations between the norms of w_ε.

Lemma 5.2 *For all* $1 < p < N$, *set* $k := \frac{l(p^*-p)}{p(l-p^*)} > 0$. *Then, it holds that*

$$\begin{aligned}||w_\varepsilon||_l^l &= k\varepsilon ||w_\varepsilon||_p^p,\\ ||w_\varepsilon||_{p^*}^{p^*} &= 1 + (k+1)\varepsilon ||w_\varepsilon||_p^p.\end{aligned}$$

Proof Since w_ε is a minimizer of $(\mathcal{S}_\varepsilon)$, identities (3.5)–(3.6) read

$$1 = ||w_\varepsilon||_{p^*}^{p^*} - \varepsilon ||w_\varepsilon||_p^p - ||w_\varepsilon||_l^l, \quad 1 = ||w_\varepsilon||_{p^*}^{p^*} - \frac{p^*\varepsilon}{p}||w_\varepsilon||_p^p - \frac{p^*}{l}||w_\varepsilon||_l^l. \tag{5.15}$$

An easy calculation yields the conclusion. □

Lemma 5.3 *For all* $1 < p < N$, *we have*

$$\varepsilon(k+1)||w_\varepsilon||_p^p \leq \frac{N}{N-p} S_*^{-1} \sigma_\varepsilon \big(1 + o(1)\big).$$

Proof Using that w_ε is a minimizer for (S_ε), by Lemma 5.2 it follows that

$$S_* \leq \mathcal{S}_*(w_\varepsilon) = \frac{||\nabla w_\varepsilon||_p^p}{||w_\varepsilon||_{p^*}^p} = \frac{S_\varepsilon}{\Big(1 + (k+1)\varepsilon||w_\varepsilon||_p^p\Big)^{(N-p)/N}},$$

namely,

$$S_*^{N/(N-p)}\Big(1 + (k+1)\varepsilon||w_\varepsilon||_p^p\Big) \leq S_\varepsilon^{N/(N-p)}.$$

Setting $\sigma_\varepsilon := S_\varepsilon - S_*$, as $\varepsilon \to 0$ we obtain

$$S_*^{N/(N-p)}(k+1)\varepsilon||w_\varepsilon||_p^p \leq \sigma_\varepsilon \frac{N}{N-p} S_*^{\frac{N}{N-p}-1} + o(\sigma_\varepsilon),$$

and this concludes the proof. □

We note that the above results allow us to understand the behaviour of the norms associated with the minimizer w_ε to (S_ε). In fact we have the following

Corollary 5.4 *As* $\varepsilon \to 0$, *we have*

$$\varepsilon||w_\varepsilon||_p^p \to 0, \quad ||w_\varepsilon||_l^l \to 0, \quad ||w_\varepsilon||_{p^*}^{p^*} \to 1.$$

5.3 Optimal rescaling

We are now in a position to introduce an optimal rescaling which captures the convergence of the minimisers w_ε to the limit Emden–Fowler optimizer W_1.
Following [35, pp.38 and 44], consider the concentration function

$$Q_\varepsilon(\lambda) = \int\limits_{B_\lambda} |w_\varepsilon|^{p^*} \mathrm{d}x,$$

where B_λ is the ball of radius λ centred at the origin. Note that $Q_\varepsilon(\cdot)$ is strictly increasing, with

$$\lim_{\lambda \to 0} Q_\varepsilon(\lambda) = 0,$$

and

$$\lim_{\lambda \to \infty} Q_\varepsilon(\lambda) = ||w_\varepsilon||_{p^*}^{p^*} \to 1, \qquad \text{as } \varepsilon \to 0,$$

by Corollary 5.4. It follows that the equation $Q_\varepsilon(\lambda) = Q_*$ with

$$Q_* := \int\limits_{B_1} |W_1(x)|^{p^*} \mathrm{d}x < 1,$$

has a unique solution $\lambda = \lambda_\varepsilon > 0$ for $\varepsilon \ll 1$, namely

$$Q_\varepsilon(\lambda_\varepsilon) = Q_*. \tag{5.16}$$

By means of the value of λ_ε implicitly defined by (5.16), we set

$$v_\varepsilon(x) := \lambda_\varepsilon^{\frac{N-p}{p}} w_\varepsilon(\lambda_\varepsilon x), \tag{5.17}$$

and easily check that

$$||v_\varepsilon||_{p^*} = ||w_\varepsilon||_{p^*} = 1 + o(1), \quad ||\nabla v_\varepsilon||_p^p = ||\nabla w_\varepsilon||_p^p = S_* + o(1). \tag{5.18}$$

namely (v_ε) is a minimizing family for (S_*). Moreover

$$\int\limits_{B_1} |v_\varepsilon(x)|^{p^*} \mathrm{d}x = Q_*.$$

The following convergence lemma follows by the concentration–compactness principle of P.-L. Lions [35, Theorem 4.9].

Lemma 5.5 *For all* $1 < p < N$, *it holds that*

$$||\nabla(v_\varepsilon - W_1)||_p \to 0,$$

and

$$||v_\varepsilon - W_1||_{p^*} \to 0,$$

as $\varepsilon \to 0$.

Proof By (5.18), for any sequence $\varepsilon_n \to 0$, there exists a subsequence $(\varepsilon_{\acute{n}})$ such that $(v_{\varepsilon_{\acute{n}}})$ converges weakly in $D^{1,p}(\mathbb{R}^N)$ to some radial functions $w_0 \in D^{1,p}(\mathbb{R}^N)$. By the concentration–compactness Principle [35, Theorem 4.9] applied to $||v_\varepsilon||_{p^*}^{-1} v_\varepsilon$, we have in fact that $(v_{\varepsilon_{\acute{n}}})$ converges to w_0 strongly in $D^{1,p}(\mathbb{R}^N)$ and $L^{p^*}(\mathbb{R}^N)$. Hence, $||w_0||_{p^*} = 1$ and therefore w_0 is a radial minimizer of (S_*), that is necessarily $w_0 \in \{W_\lambda\}_{\lambda>0}$. Note that it also holds

$$\int\limits_{B_1} |w_0(x)|^{p^*} \mathrm{d}x = Q_*.$$

As a consequence $w_0 = W_1$. Since the sequence (ε_n) was arbitrary, the whole sequence (v_n) converges to W_1 strongly in $D^{1,p}(\mathbb{R}^N)$ and $L^{p^*}(\mathbb{R}^N)$, and this concludes the proof. □

5.4 Rescaled equation estimates

Our next step is to obtain upper and lower estimates on the rescaling function λ_ε, which is implicitly determined by (5.16). The rescaled function v_ε introduced in (5.17) is such that

$$-\Delta_p v_\varepsilon = S_\varepsilon\Big(-\varepsilon\lambda_\varepsilon^p v_\varepsilon^{p-1} + v_\varepsilon^{p^*-1} - \lambda_\varepsilon^{-(N-p)(\frac{l-p}{p})+p} v_\varepsilon^{l-1}\Big), \tag{R_ε^*}$$

as (S_ε) is achieved by w_ε. By construction, for v_ε we obtain

$$||v_\varepsilon||_l^l = \lambda_\varepsilon^{\frac{p(l-p^*)}{(p^*-p)}} ||w_\varepsilon||_l^l, \qquad ||v_\varepsilon||_p^p = \lambda_\varepsilon^{-p} ||w_\varepsilon||_p^p.$$

Putting Lemmas 5.2 and 5.3together we then achieve the relation

$$\lambda_\varepsilon^{-\frac{p(l-p^*)}{(p^*-p)}} ||v_\varepsilon||_l^l = \lambda_\varepsilon^p k\varepsilon ||v_\varepsilon||_p^p \lesssim \sigma_\varepsilon, \tag{5.19}$$

which yields the following

Lemma 5.6 *Let* $1 < p < N$. *Then*

$$\sigma_\varepsilon^{-\frac{(p^*-p)}{p(l-p^*)}} \lesssim \lambda_\varepsilon \lesssim \varepsilon^{-\frac{1}{p}} \sigma_\varepsilon^{\frac{1}{p}}.$$

Proof The statement will follow by (5.19) combined with the observation that

$$\liminf_{\varepsilon \to 0} ||v_\varepsilon||_l > 0, \quad \liminf_{\varepsilon \to 0} ||v_\varepsilon||_p > 0.$$

The former is a consequence of Lemma 5.5 and Hölder's inequality, which yields $L^l(B_1) \subset L^{p^*}(B_1)$ since $l > p^*$, hence

$$\begin{aligned} c||v_\varepsilon \mathcal{X}_{B_1}||_l \geq ||v_\varepsilon \mathcal{X}_{B_1}||_{p^*} &\geq ||W_1 \mathcal{X}_{B_1}||_{p^*} - ||(W_1 - v_\varepsilon)\mathcal{X}_{B_1}||_{p^*} \\ &= ||W_1 \mathcal{X}_{B_1}||_{p^*} - o(1). \end{aligned}$$

Here $\mathcal{X}_{B_R}$ is the characteristic function of B_R. To show the latter, by the embedding $L^{p^*}(B_1) \subset L^p(B_1)$ since $p^* > p$, we obtain

$$c||v_\varepsilon \mathcal{X}_{B_1}||_{p^*} \geq ||v_\varepsilon \mathcal{X}_{B_1}||_p \geq ||W_1 \mathcal{X}_{B_1}||_p - ||(W_1 - v_\varepsilon)\mathcal{X}_{B_1}||_p = ||W_1 \mathcal{X}_{B_1}||_p - o(1),$$

and this concludes the proof. □

By (5.6) and Lemma 5.6 we obtain both an estimate from below

$$\lambda_\varepsilon \gtrsim \sigma_\varepsilon^{-\frac{(p^*-p)}{p(l-p^*)}} \gtrsim \begin{cases} \varepsilon^{-\frac{(p^*-p)}{p(l-p)}} & 1 < p < \sqrt{N}, \\ \varepsilon^{-\frac{1}{[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N, \\ \left(\varepsilon(\log \frac{1}{\varepsilon})\right)^{-\frac{(p^*-p)}{p(l-p)}} & p = \sqrt{N}, \end{cases} \tag{5.20}$$

and from above

$$\lambda_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{p^*-p}{p(l-p)}} & 1 < p < \sqrt{N}, \\ \varepsilon^{-\frac{(p^2-N)(l-p^*)+p^2}{p^2[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N, \\ \varepsilon^{-\frac{(p^*-p)}{p(l-p)}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{(l-p^*)}{p(l-p)}} & p = \sqrt{N}. \end{cases} \tag{5.21}$$

We note that in the case $1 < p < \sqrt{N}$ the above lower and upper estimates are equivalent, therefore we have the following

Corollary 5.7 *Let* $1 < p < \sqrt{N}$. *Then* $||v_\varepsilon||_l$ *and* $||v_\varepsilon||_p$ *are bounded.*

Proof This is an immediate consequence of (5.19)–(5.21). □

In the case $p \geq \sqrt{N}$ we take into account the growth of $||v_\varepsilon||_p$ to obtain matching bounds. In this case instead of (5.21) we use the more explicit upper bound

$$\lambda_\varepsilon \lesssim \frac{\varepsilon^{-1/p} \sigma_\varepsilon^{1/p}}{||v_\varepsilon||_p} \lesssim ||v_\varepsilon||_p^{-1} \begin{cases} \varepsilon^{-\frac{(p^2-N)(l-p^*)+p^2}{p^2[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < N. \\ \varepsilon^{-\frac{(p^*-p)}{p(l-p)}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{(l-p^*)}{p(l-p)}} & p = \sqrt{N}, \end{cases} \tag{5.22}$$

which follows from (5.19) and (5.6).

5.5 A lower barrier for $p \geq 2$

To refine the upper bound (5.21) we shall construct a lower barrier for w_ε in the critical regimés $\sqrt{N} \leq p < N$. For $p \geq 2$ this will be done using the following uniform estimate.

Lemma 5.8 *Given $\mu > 0$ and $\gamma > 0$, set*

$$h(r) := r^{-\gamma} e^{-\mu r}.$$

Assume that $p \geq 2$ and that $N - 1 - 2\gamma(p-1) \leq 0$ and $\gamma(N - p - \gamma(p-1)) \leq 0$. Then for all $\mu > 0$ and $r > 0$,

$$\begin{aligned} &-\Delta_p h + \mu^p (p-1) h^{p-1} \\ &\leq \mu \frac{\gamma^{p-2}(N - 1 - 2\gamma(p-1))}{r^{p-1}} h^{p-1} + \frac{\gamma^{p-1}(N - p - \gamma(p-1))}{r^p} h^{p-1}. \end{aligned} \tag{5.23}$$

Remark 5.9 If $p = 2$ then (5.23) becomes an equality.

Proof By direct calculations, we have

$$\begin{aligned} -\Delta_p h + \mu^p (p-1) h^{p-1} &= (p-1)\mu^2 \left\{ \mu^{p-2} - \left(\mu + \frac{\gamma}{r}\right)^{p-2} \right\} h^{p-1} \\ &+ \left(\mu + \frac{\gamma}{r}\right)^{p-2} \left\{ \mu \frac{N - 1 - 2\gamma(p-1)}{r} + \frac{\gamma(N - p - \gamma(p-1))}{r^2} \right\} h^{p-1}. \end{aligned}$$

For all $\mu > 0$ and $r > 0$, by monotonicity we have

$$\left\{ \mu^{p-2} - \left(\mu + \frac{\gamma}{r}\right)^{p-2} \right\} \leq 0,$$

$$\left(\mu + \frac{\gamma}{r}\right)^{p-2} \geq \left(\frac{\gamma}{r}\right)^{p-2}.$$

Therefore, assuming that $N - 1 - 2\gamma(p-1) \leq 0$ and $\gamma(N - p - \gamma(p-1)) \leq 0$ we can estimate,

$$\begin{aligned} &-\Delta_p h + \mu^p (p-1) h^{p-1} \\ &\leq \left(\frac{\gamma}{r}\right)^{p-2} \left\{ \mu \frac{N - 1 - 2\gamma(p-1)}{r} + \frac{\gamma(N - p - \gamma(p-1))}{r^2} \right\} h^{p-1} \\ &\leq \mu \frac{\gamma^{p-2}(N - 1 - 2\gamma(p-1))}{r^{p-1}} h^{p-1} + \frac{\gamma^{p-1}(N - p - \gamma(p-1))}{r^p} h^{p-1}, \end{aligned}$$

uniformly for all $\mu > 0$ and $r > 0$. ☐

Remark 5.10 In the case $1 < p < 2$ by monotonicity, convexity and Taylor for all $\mu > 0$ and $r > 0$ we have

$$0 \leq \left\{ \mu^{p-2} - \left(\mu + \frac{\gamma}{r}\right)^{p-2} \right\} \leq (2-p)\mu^{p-3} \frac{\gamma}{r}.$$

Similarly, we can estimate

$$\left(\mu + \frac{\gamma}{r}\right)^{p-2} \geq \mu^{p-2} - (2-p)\mu^{p-3} \frac{\gamma}{r}, \tag{5.24}$$

or, alternatively,

$$\left(\mu + \frac{\gamma}{r}\right)^{p-2} \geq \left(\frac{\gamma}{r}\right)^{p-2} - (2-p)\left(\frac{\gamma}{r}\right)^{p-3} \mu. \tag{5.25}$$

Therefore, assuming that $N-1-2\gamma(p-1)\leq 0$ and $\gamma(N-p-\gamma(p-1))\leq 0$ we can estimate,

$$\begin{aligned} -\Delta_p h+\mu^p(p-1)h^{p-1} &\leq \mu^{p-1}\frac{(2-p)(p-1)\gamma}{r}h^{p-1} \\ &+\Big(\quad (5.24) \text{ or } (5.25) \quad\Big)\left\{\mu\frac{N-1-2\gamma(p-1)}{r}+\frac{\gamma(N-p-\gamma(p-1))}{r^2}\right\}h^{p-1}. \end{aligned} \tag{5.26}$$

Both (5.24) and (5.25) introduce a large positive term in (5.26) which we cannot control.

To estimate the norm $||v_\varepsilon||_p$, we note that

$$-\Delta_p v_\varepsilon+S_\varepsilon\varepsilon\lambda_\varepsilon^p|v_\varepsilon|^{p-1}=S_\varepsilon|v_\varepsilon|^{p^*-1}-S_\varepsilon\lambda_\varepsilon^{-(N-p)\frac{l-p}{p}+p}|v_\varepsilon|^{l-1}\geq -V_\varepsilon(x)v_\varepsilon^{p-1},$$

where we have set

$$V_\varepsilon(x):=S_\varepsilon\lambda_\varepsilon^{-(N-p)\frac{l-p}{p}+p}v_\varepsilon^{l-p}(x).$$

By the radial decay estimate (A.3) we have

$$v_\varepsilon(x)\leq C_{N,p^*}|x|^{-N/p^*}||v_\varepsilon||_{p^*}.$$

By (5.18) and since $\lambda_\varepsilon^{-\frac{p(l-p^*)}{(p^*-p)}}\lesssim\sigma_\varepsilon\to 0$ Lemmas 5.1 and 5.6 yield, for sufficiently small $\varepsilon>0$, the following decay estimate

$$\begin{aligned} V_\varepsilon(x):=S_\varepsilon\lambda_\varepsilon^{-(N-p)\frac{l-p}{p}+p}v_\varepsilon^{l-p}(x) &\leq S_\varepsilon\lambda_\varepsilon^{-(N-p)\frac{l-p}{p}+p}c_{p^*}^{l-p}||v_\varepsilon||_{p^*}^{l-p}|x|^{-\frac{N}{p^*}(l-p)} \\ &\leq C\lambda_\varepsilon^{-(N-p)\frac{l-p}{p}+p}|x|^{-(p+\delta)}, \end{aligned}$$

where $\delta:=\frac{N-p}{p}(l-p)-p>0$ and the constant $C>0$ does not depend on ε or x. Hence, for small $\varepsilon>0$ the rescaled functions $v_\varepsilon>0$ satisfy the homogeneous inequality

$$-\Delta_p v_\varepsilon+S_\varepsilon\varepsilon\lambda_\varepsilon^p v_\varepsilon^{p-1}+V_\varepsilon(x)v_\varepsilon^{p-1}\geq 0,\quad x\in\mathbb{R}^N. \tag{5.27}$$

The following result provides a suitable lower barrier to (5.29) below.

Lemma 5.11 *Assume $N\geq 4$ and $2\leq p<\frac{N+1}{2}$. Then there exists $R>0$, independent on $\varepsilon>0$, such that for all small $\varepsilon>0$,*

$$h_\varepsilon(x):=|x|^{-\frac{N-p}{p-1}}e^{-\sqrt[p]{\varepsilon S_\varepsilon}\lambda_\varepsilon|x|}$$

satisfies

$$-\Delta_p h_\varepsilon+(p-1)S_\varepsilon\varepsilon\lambda_\varepsilon^p h_\varepsilon^{p-1}+V_\varepsilon(x)h_\varepsilon^{p-1}\leq 0,\quad |x|>R. \tag{5.28}$$

Proof By Lemma 5.8 with $\gamma=\frac{N-p}{p-1}$ we conclude that there exists $R>1$, independent of $\varepsilon>0$, such that

$$\begin{aligned} &-\Delta_p h_\varepsilon+(p-1)S_\varepsilon\varepsilon\lambda_\varepsilon^p h_\varepsilon^{p-1}+V_\varepsilon(x)h_\varepsilon^{p-1} \\ &\quad\leq(\varepsilon S_\varepsilon)^{\frac{1}{p}}\lambda_\varepsilon\frac{\gamma^{p-2}(N-1-2\gamma(p-1))}{r^{p-1}}h_\varepsilon^{p-1}+\lambda_\varepsilon^{-\frac{N-p}{p}(l-p)+p}\frac{C}{r^{p+\delta}}h_\varepsilon^{p-1} \\ &\quad\leq\left\{-\gamma^{p-2}\big(N+1-2p\big)(\varepsilon S_\varepsilon)^{\frac{1}{p}}\lambda_\varepsilon+C\lambda_\varepsilon^{-\frac{N-p}{p}(l-p)+p}\right\}\frac{1}{r^{p-1}}h_\varepsilon^{p-1}\quad\text{for } |x|>R. \end{aligned}$$

It is convenient to denote $s := l - p^* > 0$. Taking into account that $-\frac{N-p}{p}(l-p) + p = s(1 - N/p) < 0$, we can use the lower bound (5.20) on λ_ε to estimate

$$\begin{aligned} &-\gamma^{p-2}\big(N+1-2p\big)(\varepsilon S_\varepsilon)^{\frac{1}{p}}\lambda_\varepsilon + C\lambda_\varepsilon^{-\frac{N-p}{p}(l-p)+p} \\ &\quad \le -\gamma^{p-2}(N+1-2p)S_\varepsilon^{\frac{1}{p}}\varepsilon^{\frac{1}{p}-\frac{1}{[s(p-1)+p]}} + C\varepsilon^{\frac{(N-p)(l-p)-p^2}{p[s(p-1)+p]}} \\ &\quad \le -\gamma^{p-2}(N+1-2p)S_\varepsilon^{\frac{1}{p}}\varepsilon^{\frac{s(p-1)}{p[s(p-1)+p]}} + C\varepsilon^{\frac{(N-p)(l-p)-p^2}{p[s(p-1)+p]}} \le 0, \end{aligned}$$

for all sufficiently small $\varepsilon > 0$, provided that $p < (N+1)/2$, which completes the proof. □

Lemma 5.12 *Assume $N \ge 4$ and $2 \le p < \frac{N+1}{2}$. There exists $R > 0$ and $c > 0$, independent on $\varepsilon > 0$, such that for all small $\varepsilon > 0$,*

$$v_\varepsilon(x) \ge c|x|^{-\frac{N-p}{p-1}} e^{-\sqrt[p]{\varepsilon S_\varepsilon}\lambda_\varepsilon |x|} \quad (|x| > R).$$

Proof Define the barrier

$$h_\varepsilon(x) := |x|^{-\frac{N-p}{p-1}} e^{-\sqrt[p]{\varepsilon S_\varepsilon}\lambda_\varepsilon |x|},$$

which satisfies

$$-\Delta_p h_\varepsilon + \varepsilon S_\varepsilon \lambda_\varepsilon^p h_\varepsilon^{p-1} + V_\varepsilon(x) h_\varepsilon^{p-1} \le 0, \quad |x| > R. \tag{5.29}$$

by Lemma 5.11. Note that Lemma 5.5 and Lemma A.4 in Appendix imply

$$||(v_\varepsilon - W_1)_{B_R \setminus B_{R/2}}||_\infty \to 0,$$

and hence

$$v_\varepsilon(|x|) \to W_1(|x|), \qquad \text{for } |x| = R.$$

Hence for all sufficiently small $\varepsilon > 0$, we have

$$v_\varepsilon(R) \ge \frac{1}{2} W_1(R), \quad \text{for } |x| = R.$$

Since $h_\varepsilon(R)$ is a monotone decreasing function in ε, then by a suitable choice of a uniform small constant $c > 0$ we obtain

$$ch_\varepsilon(R) \le \frac{1}{2} W_1(R),$$

and hence

$$v_\varepsilon(R) \ge ch_\varepsilon(R), \quad \text{for all small } \varepsilon > 0.$$

Then the homogeneity of (5.29) implies

$$-\Delta_p(ch_\varepsilon) + \varepsilon S_\varepsilon \lambda_\varepsilon^p (ch_\varepsilon)^{p-1} + V_\varepsilon(x)(ch_\varepsilon)^{p-1} \le 0, \quad \text{in } |x| > R,$$

for all small $\varepsilon > 0$. Define a function $ch_{\varepsilon,k}$ by

$$ch_{\varepsilon,k} = ch_\varepsilon - k^{-1} < ch_\varepsilon, \qquad \text{for all } k > 0,$$

then

$$-\Delta_p(ch_{\varepsilon,k}) + V_\varepsilon(x)(ch_{\varepsilon,k})^{p-1} + \varepsilon S_\varepsilon \lambda_\varepsilon^p (ch_{\varepsilon,k})^{p-1} \le 0, \quad \text{in } |x| > R \tag{5.30}$$

and

$$v_\varepsilon \geq ch_\varepsilon > ch_{\varepsilon,k}, \qquad \text{for } |x| = R.$$

Now, since

$$ch_\varepsilon \to 0, \qquad \text{as } |x| \to +\infty,$$

then for k large enough there exists $R_k > R$ such that

$$ch_{\varepsilon,k} = 0, \qquad \text{for } |x| = R_k,$$

and since $v_\varepsilon > 0$, then

$$v_\varepsilon > ch_{\varepsilon,k}, \qquad \text{for } |x| = R_k.$$

As a consequence, from (5.29) and (5.30), using the comparison principle (see Theorem B.1 in Appendix) we obtain

$$v_\varepsilon \geq ch_{\varepsilon,k}, \qquad \text{for } R < |x| < R_k,$$

which can be achieved for every k. Since $R_k \to \infty$ as $k \to \infty$, the assertion follows. □

5.6 Critical dimensions $N \geq 4$ and $\sqrt{N} \leq p < \frac{N+1}{2}$ completed

We now apply Lemma 5.12 to obtain matching estimates for the blowup of $||v_\varepsilon||_p$ in dimensions $N \geq 4$ and $\sqrt{N} \leq p < \frac{N+1}{2}$.

Lemma 5.13 *If* $N \geq 4$ *and* $\sqrt{N} < p < \frac{N+1}{2}$, *then* $||v_\varepsilon||_p^p \gtrsim \left(\frac{1}{\sqrt[p]{\varepsilon}\lambda_\varepsilon}\right)^{\frac{p^2-N}{p-1}}$.

Proof Since $\sqrt{N} < p < \frac{N+1}{2}$, we directly calculate from Lemma 5.12:

$$||v_\varepsilon||_p^p \geq \int\limits_{\mathbb{R}^N \setminus B_R} |v_\varepsilon|^p \mathrm{d}x \geq \int\limits_R^\infty r^{N-1} \left| c r^{-\frac{N-p}{p-1}} e^{-\sqrt[p]{\varepsilon S_\varepsilon}\lambda_\varepsilon r} \right|^p \mathrm{d}r,$$

and as $\varepsilon \to 0$ (i.e. $\frac{1}{\sqrt[p]{\varepsilon}\lambda_\varepsilon} \to \infty$), we have

$$||v_\varepsilon||_p^p \geq c^p \int\limits_R^{\frac{1}{\sqrt[p]{\varepsilon S_\varepsilon}\lambda_\varepsilon}} r^{\frac{p^2-N}{p-1}-1} \mathrm{d}r \geq \left(\frac{C}{\sqrt[p]{\varepsilon}\lambda_\varepsilon}\right)^{\frac{p^2-N}{p-1}},$$

and this completes the proof. □

As an immediate consequence of the above result, by (5.22), we obtain an upper estimate of λ_ε which matches the lower bound of (5.20) in dimensions $N \geq 4$ and $\sqrt{N} < p < \frac{N+1}{2}$.

Corollary 5.14 *If* $N \geq 4$ *and* $\sqrt{N} < p < \frac{N+1}{2}$, *then* $\lambda_\varepsilon \lesssim \varepsilon^{-\frac{1}{[(l-p^*)(p-1)+p]}}$.

We now move to consider the case $p = \sqrt{N}$.

Lemma 5.15 *If* $N \geq 4$ *and* $p = \sqrt{N}$ *then it holds that* $||v_\varepsilon||_p^p \gtrsim \log(\frac{1}{\sqrt[p]{\varepsilon}\lambda_\varepsilon})$.

Proof Since $p = \sqrt{N}$, by Lemma 5.12 we immediately get

$$||v_\varepsilon||_p^p \geq \int\limits_{\mathbb{R}^N \setminus B_R} r^{N-1} |v_\varepsilon(r)|^p \mathrm{d}r \geq c^p \int\limits_R^{\frac{1}{\sqrt[p]{\varepsilon S_\varepsilon \lambda_\varepsilon}}} r^{\frac{p^2-N}{p-1}-1} \mathrm{d}r$$

$$= c^p \int\limits_R^{\frac{1}{\sqrt[p]{\varepsilon S_\varepsilon \lambda_\varepsilon}}} r^{-1} \mathrm{d}r \geq \log\left(\frac{C}{\sqrt[p]{\varepsilon}\lambda_\varepsilon}\right),$$

and this concludes the proof. □

Corollary 5.16 *If $N \geq 4$ and $p = \sqrt{N}$ then it holds that $\lambda_\varepsilon \lesssim \left(\varepsilon(\log \frac{1}{\varepsilon})\right)^{-\frac{(p^*-p)}{p(l-p)}}$.*

Proof By (5.19) and (5.6) we get

$$C\varepsilon\lambda_\varepsilon^p \log \frac{1}{\sqrt[p]{\varepsilon}\lambda_\varepsilon} \leq \left(\varepsilon\left(\log\frac{1}{\varepsilon}\right)\right)^{\frac{(l-p^*)}{(l-p)}}.$$

Clearly,

$$\varepsilon^{\delta_1} \leq \sqrt[p]{\varepsilon}\lambda_\varepsilon \leq \varepsilon^{\delta_2},$$

for some $\delta_{1,2} \geq 0$ and ε small enough, by (5.20) and (5.21). It follows that

$$\log\frac{1}{\sqrt[p]{\varepsilon}\lambda_\varepsilon} \sim \log\frac{1}{\varepsilon}.$$

Hence,

$$\lambda_\varepsilon^p \lesssim \left(\varepsilon(\log\frac{1}{\varepsilon})\right)^{\frac{(l-p^*)}{(l-p)}-1} = \left(\varepsilon\left(\log\frac{1}{\varepsilon}\right)\right)^{-\frac{(p^*-p)}{(l-p)}},$$

and

$$\lambda_\varepsilon \lesssim \left(\varepsilon\left(\log\frac{1}{\varepsilon}\right)\right)^{-\frac{(p^*-p)}{p(l-p)}},$$

and this concludes the proof. □

5.7 Proofs

The sharp upper estimates on λ_ε yield the following

Corollary 5.17 *Let either $1 < p < \sqrt{N}$, or $N \geq 4$ and $\sqrt{N} \leq p < \frac{N+1}{2}$. Then*

$$||v_\varepsilon||_l = O(1).$$

The boundedness of the L^l norm also allows one to reverse the estimates of $||v_\varepsilon||_p$ via (5.19).

Corollary 5.18 *It holds that*

$$||v_\varepsilon||_p^p = \begin{cases} O(1), & 1 < p < \sqrt{N}, \\ O(\log\frac{1}{\varepsilon}), & p = \sqrt{N},\ N \geq 4, \\ O(\varepsilon^{\frac{(l-p^*)[p-(p^*-p)(p-1)]}{(p^*-p)[(l-p^*)(p-1)+p]}}), & \sqrt{N} < p < \frac{N+1}{2},\ N \geq 4. \end{cases}$$

We now prove that the L^l bound implies an L^∞ bound.

Lemma 5.19 *Let either* $1 < p < \sqrt{N}$, *or* $N \geq 4$ *and* $\sqrt{N} \leq p < \frac{N+1}{2}$. *It holds that*

$$||v_\varepsilon||_\infty = O(1). \tag{5.31}$$

Proof We start observing that by (R_ε^*) v_ε is a positive solution to the inequality

$$-\Delta_p v_\varepsilon - V_\varepsilon(x) v_\varepsilon^{p-1} \leq 0, \quad x \in \mathbb{R}^N,$$

with

$$V_\varepsilon(x) := S_\varepsilon v_\varepsilon^{p^*-p}(x).$$

By Lemma A.5 in Appendix, we obtain

$$|v_\varepsilon(x)| \leq C_l ||v_\varepsilon||_l |x|^{-N/l} \quad x \neq 0, \tag{5.32}$$

which combined with Corollary 5.17 yields

$$V_\varepsilon(x) \leq S_\varepsilon C_l^{p^*-p} ||v_\varepsilon||_l^{p^*-p} |x|^{-N(p^*-p)/l} \leq C_* |x|^{-pp^*/l},$$

for some uniform constant $C_* > 0$ independent on ε or x. Hence, v_ε is a positive solution to the inequality

$$-\Delta_p v_\varepsilon - V_*(x) v_\varepsilon^{p-1} \leq 0, \quad x \in \mathbb{R}^N, \tag{5.33}$$

with $V_*(x) = C_*|x|^{-pp^*/l} \in L^s_{\text{loc}}(\mathbb{R}^N)$ for some $s > N/p$, since $l > p^*$. With these preliminaries in place, one can invoke here the result on local boundedness Theorem 7.1.1 in [30, p.154] for subsolutions of (5.33) to conclude. However, to make the proof self-contained, we provide a simple argument to justify (5.31).
Integrating the inequality (5.33) over a ball

$$\int_{B_{|x|}(0)} -\Delta_p v_\varepsilon(y) \mathrm{d}y \leq \int_{B_{|x|}(0)} V_*(y) v_\varepsilon^{p-1}(y) \mathrm{d}y,$$

and by the divergence theorem, taking into account the monotonicity of v_ε with respect to $|x|$ we have

$$\begin{aligned} \int_{B_{|x|}(0)} -\Delta_p v_\varepsilon(y) \mathrm{d}y &= \int_{\partial B_{|x|}(0)} -|\nabla v_\varepsilon(\sigma)|^{p-2} \nabla v_\varepsilon(\sigma) \cdot \nu \, \mathrm{d}\sigma \\ &= |\nabla v_\varepsilon(x)|^{p-1} \int_{\partial B_{|x|}(0)} \mathrm{d}\sigma = C_1 |\nabla v_\varepsilon(x)|^{p-1} |x|^{N-1}. \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{B_{|x|}(0)} V_*(y) v_\varepsilon^{p-1}(y) \mathrm{d}y &= C_2 \int_0^{|x|} r^{-\frac{pp^*}{l}+N-1} v_\varepsilon^{p-1}(r) \mathrm{d}r \\ &\leq C_2 |v_\varepsilon(0)|^{p-1} \int_0^{|x|} r^{-\frac{pp^*}{l}+N-1} \mathrm{d}r = C_3 |v_\varepsilon(0)|^{p-1} |x|^{-\frac{pp^*}{l}+N}, \end{aligned}$$

since $-\frac{pp^*}{l} + N > 0$. Hence

$$|\nabla v_\varepsilon(x)|^{p-1} \le \frac{C_4}{|x|^{N-1}} \int\limits_{B_{|x|}(0)} V_*(r) v_\varepsilon^{p-1}(r)\mathrm{d}r \le C_5 |v_\varepsilon(0)|^{p-1} |x|^{1-pp^*/l}, \tag{5.34}$$

for some $C_4, C_5 > 0$ independent of ε and x. Integrating again from 0 to x_0 after writing (5.34) in this form

$$-\frac{\mathrm{d}}{\mathrm{d}r} v_\varepsilon(|x|) \le C_6 |v_\varepsilon(0)| |x|^{(1-pp^*/l)/(p-1)},$$

we have

$$v_\varepsilon(0) \le v_\varepsilon(x_0) + C_7 v_\varepsilon(0) |x_0|^{\frac{p(l-p^*)}{l(p-1)}}, \tag{5.35}$$

for some C_7 independent of ε and x. We pick A small enough such that for all $|x_0| \le A$ we have

$$v_\varepsilon(0)\left(1 - C_7 A^{\frac{p(l-p^*)}{l(p-1)}}\right) \le v_\varepsilon(0)\left(1 - C_7 |x_0|^{\frac{p(l-p^*)}{l(p-1)}}\right) \le v_\varepsilon(x_0).$$

Then

$$C_8 v_\varepsilon(0) \le v_\varepsilon(x_0), \quad \text{for all } \ x_0, \ |x_0| < A,$$

where $C_8 = 1 - C_7 A^{\frac{p(l-p^*)}{l(p-1)}}$. Hence by taking the power l and integrating we obtain

$$\int\limits_{|x|<A} C_9 |v_\varepsilon(0)|^l \mathrm{d}x \le \int\limits_{|x|<A} |v_\varepsilon(x)|^l \mathrm{d}x.$$

which by Corollary 5.17 immediately concludes the proof. □

By elliptic estimates for the p-Laplacian, we have the following

Corollary 5.20 *Let either* $1 < p < \sqrt{N}$, *or* $N \ge 4$ *and* $\sqrt{N} \le p < \frac{N+1}{2}$. *It holds that* $v_\varepsilon \to W_1$ *in* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $L^s(\mathbb{R}^N)$ *for any* $s \ge p^*$. *In particular,*

$$v_\varepsilon(0) \simeq U_1(0).$$

Proof As a consequence of the L^∞ bound of Lemma 5.19 and the convergence of v_ε to the Sobolev minimiser W_1 in $D^{1,p}(\mathbb{R}^N)$ via the compactness result in Lemma A.5 we obtain the convergence in $L^s(\mathbb{R}^N)$ for any $s \ge p^*$. Since we can write (R^*_ε) in the form

$$-\Delta_p v_\varepsilon = g_\varepsilon(v_\varepsilon),$$

and by Lemma 5.19 we have

$$||g_\varepsilon(v_\varepsilon)||_{L^\infty_{\mathrm{loc}}(\mathbb{R}^N)} < C,$$

uniformly with respect to ε, then by [11, Theorem 2] we have

$$||v_\varepsilon||_{C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} < C,$$

uniformly with respect to ε. It follows that by the classical Arzelá–Ascoli theorem that for a suitable sequence $\varepsilon \to 0$ we have

$$v_\varepsilon \to W_1 \quad \text{in } C^{1,\alpha'}_{\mathrm{loc}}(\mathbb{R}^N),$$

where $\alpha < \alpha'$. □

Proof of Theorem 2.3 The proof follows immediately from Lemmas 5.5 and 5.6, which yield the upper and lower estimates on λ_ε. □

Proof of Theorem 2.5 The proof follows from the sharp upper bound on λ_ε in Corollaries 5.14–5.16, and from Corollary 5.20. In particular, since from Corollary 5.20 and in view of (3.8), we have

$$u_\varepsilon(0) \sim \lambda_\varepsilon^{-\frac{N-p}{p}} v_\varepsilon(0),$$

then by the sharp estimate of λ_ε we have the exact rate of the groundstate $u_\varepsilon(0)$ in the present critical case

$$u_\varepsilon(0) \sim \begin{cases} \varepsilon^{\frac{l}{(l-p)}} & 1 < p < \sqrt{N},\ N \geq 2, \\ \varepsilon^{\frac{N-p}{p[(l-p^*)(p-1)+p]}} & \sqrt{N} < p < \frac{N+1}{2},\ N \geq 4, \\ \left(\varepsilon(\log\frac{1}{\varepsilon})\right)^{\frac{l}{(l-p)}} & p = \sqrt{N},\ N \geq 4. \end{cases} \tag{5.36}$$

□

6 Proof of Theorem 2.8: supercritical case $q > p^*$

In this section, we consider the supercritical case $q > p^*$ and prove Theorem 2.8 stated in Introduction, which essentially says that for $q > p^*$ groundstate solutions u_ε converge as $\varepsilon \to 0$ to a non-trivial radial groundstate solution of the formal limit equation (P_0). This result extends [24, Theorem 2.3] to $p \neq 2$.

6.1 The limiting PDE

From the results of Sect. 4 we know that for $q > p^*$ the limit equation

$$-\Delta_p u - u^{q-1} + u^{l-1} = 0 \quad \text{in } \mathbb{R}^N, \tag{P_0}$$

admits positive radial groundstates solutions $u_0 \in D^{1,p}(\mathbb{R}^N) \cap L^l(\mathbb{R}^N)$, which are, since they are radial, fast decaying, namely such that

$$u_0(x) \sim |x|^{-\frac{N-p}{p-1}} \quad \text{as} \quad |x| \to \infty. \tag{6.1}$$

Note that by construction $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover u_0 admits a variational charachterization in the Sobolev space $D^{1,p}(\mathbb{R}^N)$ via the rescaling

$$u_0(x) := w_0\Big(\frac{x}{\sqrt[p]{S_0}}\Big),$$

where w_0 is a positive radial minimizer of the constrained minimization problem

$$S_0 := \inf\left\{\int\limits_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x \,\middle|\, w \in D^{1,p}(\mathbb{R}^N),\quad p^* \int\limits_{\mathbb{R}^N} \tilde{F}_0(w)\mathrm{d}x = 1\right\}, \tag{S_0}$$

where

$$\tilde{F}_0(w) = \int_0^w \tilde{f}_0(s)\mathrm{d}s,$$

and $\tilde{f}_0(s)$ is a truncation of the nonlinearity

$$f_0(s) = |s|^{q-2}s - |s|^{l-2}s,$$

as described in Sect. 4. Then the minimization problem (S_0) is well defined on $D^{1,p}(\mathbb{R}^N)$. The minimizer w_0 satisfies the Euler–Lagrange equation

$$-\Delta_p w_0 = S_0(w_0^{q-1} - w_0^{l-1}).$$

Moreover, w_0 satisfies Nehari's identity

$$\int\limits_{\mathbb{R}^N} |\nabla w_0|^p \mathrm{d}x = S_0 \left(\int\limits_{\mathbb{R}^N} |w_0|^q \mathrm{d}x - \int\limits_{\mathbb{R}^N} |w_0|^l \mathrm{d}x \right),$$

which yields

$$1 = ||w_0||_q^q - ||w_0||_l^l. \tag{6.2}$$

From the Pohožaev identity

$$\int\limits_{\mathbb{R}^N} |\nabla w_0|^p \mathrm{d}x = S_0 p^* \left(\frac{1}{q} \int\limits_{\mathbb{R}^N} |w_0|^q \mathrm{d}x - \frac{1}{l} \int\limits_{\mathbb{R}^N} |w_0|^l \mathrm{d}x \right),$$

we have

$$1 = \frac{p^*}{q} ||w_0||_q^q - \frac{p^*}{l} ||w_0||_l^l. \tag{6.3}$$

Hence from (6.2) and (6.3) we obtain the relation

$$||w_0||_q^q - ||w_0||_l^l = \frac{p^*}{q} ||w_0||_q^q - \frac{p^*}{l} ||w_0||_l^l = 1,$$

from which we obtain the expressions

$$||w_0||_q^q = \frac{q(l-p^*)}{p^*(l-q)}, \quad ||w_0||_l^l = \frac{l(q-p^*)}{p^*(l-q)}.$$

6.2 Energy estimates and groundstate asymptotics

The relations between $\mathcal{S}_\varepsilon$ and $\mathcal{S}_0$ is provided by introducing the convenient scaling-invariant quotient

$$\mathcal{S}_0(w) := \frac{\int\limits_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x}{\left(p^* \int\limits_{\mathbb{R}^N} \tilde{F}_0(w) \mathrm{d}x \right)^{(N-p)/N}}, \quad w \in \mathcal{M}_0, \tag{6.4}$$

where

$$\mathcal{M}_0 := \left\{ w \in D^{1,p}(\mathbb{R}^N), \int\limits_{\mathbb{R}^N} \tilde{F}_0(w) \mathrm{d}x > 0 \right\}.$$

Note that, by a rescaling argument, this is equivalent to (S_0) :

$$S_0 = \inf_{w \in \mathcal{M}_0} \mathcal{S}_0(w).$$

Lemma 6.1 *For all* $1 < p < N$*, it holds that*

$$0 < S_\varepsilon - S_0 \to 0, \quad as \quad \varepsilon \to 0.$$

Proof To show that $\mathcal{S}_0 < \mathcal{S}_\varepsilon$, simply note that

$$S_0 \le \mathcal{S}_0(w_\varepsilon) < \mathcal{S}_\varepsilon(w_\varepsilon) = S_\varepsilon. \tag{6.5}$$

To estimate S_ε from above we test (S_ε) with the minimizer w_0. By (6.1), we have $w_0 \in L^p(\mathbb{R}^N)$ if and only if $1 < p < \sqrt{N}$. We break the proof by analysing the higher and lower dimensions separately.
Case $1 < p < \sqrt{N}$. Using w_0 as a test function for (S_ε), we obtain

$$S_\varepsilon \le \mathcal{S}_\varepsilon(w_0) \le \frac{S_0}{(1 - \frac{\varepsilon p^*}{p}||w_0||^p_{L^p(\mathbb{R}^N)})^{\frac{N-p}{N}}} \le S_0 + O(\varepsilon), \tag{6.6}$$

which proves the statement for $1 < p < \sqrt{N}$.

In the cases $p = \sqrt{N}$ and $\sqrt{N} < p < N$, given $R > 1$ we pick a cutoff function $\eta_R \in C_0^\infty(\mathbb{R})$ such that $\eta_R(r) = 1$ for $|r| < R$, $0 < \eta_R < 1$ for $R < |r| < 2R$, $\eta_R = 0$ for $|r| > 2R$ and $|\eta_R'| \le 2/R$. By (6.1), for $s > \frac{N}{N-p}$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla(\eta_R w_0(x)|^p \mathrm{d}x &= S_0 + O\left(R^{-\frac{N-p}{p-1}}\right), \\
\int_{\mathbb{R}^N} |\eta_R w_0(x)|^s \mathrm{d}x &= ||w_0||^s_{L^s(\mathbb{R}^N)}\left(1 - O\left(R^{-\frac{(N-p)s}{p-1}+N}\right)\right), \\
\int_{\mathbb{R}^N} |\eta_R w_0(x)|^p \mathrm{d}x &= \begin{cases} O(\log(R)), & p = \sqrt{N}, \\ O\left(R^{\frac{p^2-N}{p-1}}\right), & \sqrt{N} < p < N. \end{cases}
\end{aligned}$$

Case $p = \sqrt{N}$. Let $R = \varepsilon^{-1}$. Testing (S_ε) with $\eta_R w_0$ and since $q > p^*$, we get

$$\begin{aligned}
S_\varepsilon \le \mathcal{S}_\varepsilon(\eta_R w_0) \le &\left(S_0 + O\left(R^{-\frac{N-p}{p-1}}\right)\right) \\
&\div \left(\frac{p^*}{q}||w_0||^q_q\left(1 - O\left(R^{-\frac{(N-p)q}{p-1}+N}\right)\right) - \frac{\varepsilon p^*}{p} O(\log R)\right. \\
&\left. - \frac{p^*}{l}||w_0||^l_l\left(1 - O\left(R^{-\frac{(N-p)l}{p-1}+N}\right)\right)\right)^{\frac{N-p}{N}} \\
= &\frac{S_0 + O(\varepsilon^{\frac{N-p}{p-1}})}{\left(1 - o(\varepsilon^{\frac{N}{p-1}}) - O(\varepsilon \log \frac{1}{\varepsilon})\right)^{\frac{N-p}{N}}} \le S_0 + O\left(\varepsilon \log \frac{1}{\varepsilon}\right),
\end{aligned}$$

from which the claim follows.

Case $\sqrt{N} < p < N$. Let $R = \varepsilon^{-\frac{1}{p}}$. We test ($S_\varepsilon$) with $\eta_R w_0$ and as $q > p^*$, we obtain

$$\begin{aligned} S_\varepsilon \le \mathcal{S}_\varepsilon(\eta_R w_0) \le & \left(S_0 + O\left((R)^{-\frac{N-p}{p-1}} \right) \right) \div \\ & \left(\frac{p^*}{q} ||w_0||_q^q \left(1 - O\left(R^{-\frac{(N-p)q}{p-1} + N} \right) \right) - \frac{\varepsilon p^*}{p} O\left(R^{\frac{p^2-N}{p-1}} \right) \right. \\ & \left. - \frac{p^*}{l} ||w_0||_l^l \left(1 - O\left((R)^{-\frac{(N-p)l}{p-1} + N} \right) \right) \right)^{\frac{N-p}{N}} \\ \le & \frac{S_0 + O(\varepsilon^{\frac{N-p}{p(p-1)}})}{\left(1 - o(\varepsilon^{\frac{N}{p(p-1)}}) - O(\varepsilon^{-\frac{p^2-N}{p(p-1)} + 1}) \right)^{\frac{N-p}{N}}} \le S_0 + O\left(\varepsilon^{\frac{N-p}{p(p-1)}} \right), \end{aligned}$$

which completes the proof. □

Lemma 6.2 *It holds that* $||w_\varepsilon||_\infty \le 1$ *and* $||w_\varepsilon||_s \lesssim 1$ *for all* $s > p^*$.

Proof Note that by (3.8) we have

$$||w_\varepsilon||_\infty = ||u_\varepsilon||_\infty \le 1.$$

By Sobolev's inequality and Lemma 6.1, we have

$$||w_\varepsilon||_{p^*}^p \le S_*^{-1} ||\nabla w_\varepsilon||_p^p = S_*^{-1} S_\varepsilon = S_*^{-1} S_0 \big(1 + o(1) \big).$$

Hence for every $s > p^*$,

$$||w_\varepsilon||_s^s \le ||w_\varepsilon||_{p^*}^{p^*},$$

which concludes the proof. □

Lemma 6.3 *For all* $1 < p < N$, *we have*

$$\varepsilon ||w_\varepsilon||_p^p \to 0 \quad as \quad \varepsilon \to 0.$$

Proof Observing that w_ε is an optimizer to (S_ε), it follows that

$$1 = p^* \int\limits_{\mathbb{R}^N} F_\varepsilon(w_\varepsilon) \mathrm{d}x = p^* \int\limits_{\mathbb{R}^N} F_0(w_\varepsilon) - p^* \frac{\varepsilon}{p} ||w_\varepsilon||_p^p. \tag{6.7}$$

Hence

$$\mathcal{S}_0(w_\varepsilon) = \frac{\int\limits_{\mathbb{R}^\mathbb{N}} |\nabla w_\varepsilon|^p \mathrm{d}x}{\left(p^* \int\limits_{\mathbb{R}^\mathbb{N}} F_0(w_\varepsilon) \mathrm{d}x \right)^{(N-p)/N}} = \frac{S_\varepsilon}{\left(1 + \frac{p^*}{p} \varepsilon ||w_\varepsilon||_p^p \right)^{(N-p)/N}}.$$

If by contradiction we had $\limsup_{\varepsilon \to 0} \varepsilon ||w_\varepsilon||_p^p = m > 0$, then by Lemma 6.1 for any sequence $\varepsilon_n \to 0$, we would obtain

$$S_0 \le \mathcal{S}_0(w_{\varepsilon_n}) = \frac{S_{\varepsilon_n}}{\left(1 + \frac{p^*}{p} \varepsilon_n ||w_{\varepsilon_n}||_p^p \right)^{(N-p)/N}} \le \frac{S_0 \big(1 + o(1) \big)}{1 + \frac{p^*}{p} m} < S_0,$$

and this, as it is clearly a contradiction, concludes the proof. □

Theorem 6.4 *Let* $1 < p < N$ *and* $q > p^*$. *As* $\varepsilon \to 0$, *the family of groundstates* u_ε *converges to a groundstate* u_0 *in* $D^{1,p}(\mathbb{R}^N)$, $L^l(\mathbb{R}^N)$ *and* $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ *to* (P_0). *In particular*

$$u_\varepsilon(0) \simeq u_0(0).$$

Furthermore u_0 *is fast decaying, namely*

$$u_0(x) \sim |x|^{-\frac{N-p}{p-1}} \quad as \quad |x| \to \infty.$$

Proof Since the family w_ε is bounded in $D^{1,p}(\mathbb{R}^N)$, then there exists a subsequence w_{ε_n} such that

$$w_{\varepsilon_n} \rightharpoonup \tilde{w} \quad \text{in} \quad D^{1,p}(\mathbb{R}^N) \quad \text{and} \quad w_{\varepsilon_n} \to \tilde{w} \quad \text{a.e} \quad \text{in} \ \mathbb{R}^N, \quad \text{as } n \to \infty$$

where $\tilde{w} \in D^{1,p}(\mathbb{R}^N)$ is a radial function. By Sobolev's inequality, the sequence (w_{ε_n}) is bounded in $L^{p^*}(\mathbb{R}^N)$. Using Lemma A.5 we conclude that

$$w_{\varepsilon_n} \to \tilde{w} \quad \text{in} \quad L^s(\mathbb{R}^N \setminus B_r(0)) \quad \text{for } r > 0 \text{ and } s \in (p^*, \infty).$$

Taking into account Lemma 6.3 and (6.7) we also obtain

$$\int_{\mathbb{R}^N} F_0(\tilde{w})\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} F_0(w_{\varepsilon_n})\mathrm{d}x = \lim_{n\to\infty} \left(1 + p^* \frac{\varepsilon_n}{p} ||w_{\varepsilon_n}||_p^p\right) = 1.$$

By the weak lower semicontinuity property of the norm we also have that

$$||\nabla\tilde{w}||_p^p \le \liminf_{n\longrightarrow\infty} ||\nabla w_{\varepsilon_n}||_p^p = S_0,$$

i.e. $\tilde{w}$ is a minimizer for (S_0). We now claim that

$$\nabla w_{\varepsilon_n} \to \nabla\tilde{w} \quad a.e. \quad \text{on } \mathbb{R}^N, \tag{6.8}$$

and then by Brezis–Lieb Lemma [4], (w_{ε_n}) converges strongly to $\tilde{w}$ in $D^{1,p}(\mathbb{R}^N)$. In fact, arguing as in [22, Theorem 3.3] (see also [21, Proposition 2.3], define a bounded function

$$T := \begin{cases} s & \text{if} \quad |s| \le 1, \\ \frac{s}{|s|} & \text{if} \quad |s| > 1, \end{cases}$$

and consider a sequence (B_k) of open subsets of $\mathbb{R}^N$ such that $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^N$. Then if

$$\lim_{n\to\infty} \int_{B_k} (|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla\tilde{w}|^{p-2}\nabla\tilde{w}) \cdot \nabla T(w_{\varepsilon_n} - \tilde{w})\mathrm{d}x \to 0, \tag{6.9}$$

for every k, then

$$\nabla w_{\varepsilon_n} \to \nabla\tilde{w} \quad a.e. \quad \text{on } B_k,$$

and hence by a Cantor diagonal argument, (6.8) is satisfied.
To show (6.9), we introduce a cutoff function

$$\rho(x) := \begin{cases} 1 & \text{if} \quad |x| \le k, \\ 0 & \text{if} \quad |x| \ge k+1, \end{cases}$$

and since

$$\big(|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla\tilde{w}|^{p-2}\nabla\tilde{w}\big)\nabla T(w_{\varepsilon_n} - \tilde{w}) \ge 0,$$

then

$$\begin{aligned}
0 &\le \int\limits_{B_k} \big(|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla \tilde{w}|^{p-2}\nabla \tilde{w}\big)\nabla T(w_{\varepsilon_n} - \tilde{w})\mathrm{d}x \\
&\le \int\limits_{B_{k+1}} \Big[\big(|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla \tilde{w}|^{p-2}\nabla \tilde{w}\big)\nabla T(w_{\varepsilon_n} - \tilde{w})\Big]\rho(x)\mathrm{d}x \\
&\le \Big|\int\limits_{B_{k+1}} \big(|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla \tilde{w}|^{p-2}\nabla \tilde{w}\big)\nabla\big(\rho T(w_{\varepsilon_n} - \tilde{w})\big)\mathrm{d}x\Big| \\
&\quad + \Big|\int\limits_{B_{k+1}} \big(|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla \tilde{w}|^{p-2}\nabla \tilde{w}\big)T(w_{\varepsilon_n} - \tilde{w})\nabla\rho\mathrm{d}x\Big| \to 0,
\end{aligned}$$

as $n \to \infty$. In fact

$$\begin{aligned}
&\Big|\int\limits_{B_{k+1}} \big(|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla w_0|^{p-2}\nabla \tilde{w}\big)\nabla\big(\rho T(w_{\varepsilon_n} - \tilde{w})\big)\mathrm{d}x\Big| \\
&\le \Big|\int\limits_{\mathbb{R}^N} |\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n}\nabla\big(\rho T(w_{\varepsilon_n} - \tilde{w})\big)\mathrm{d}x\Big| + \Big|\int\limits_{\mathbb{R}^N} |\nabla \tilde{w}|^{p-2}\nabla \tilde{w}\nabla\big(\rho T(w_{\varepsilon_n} - \tilde{w})\big)\mathrm{d}x\Big| \\
&= \Big|\int\limits_{\mathbb{R}^N} f_\varepsilon(w_{\varepsilon_n})\rho T(w_{\varepsilon_n} - \tilde{w})\mathrm{d}x\Big| + \Big|\int\limits_{\mathbb{R}^N} f(\tilde{w})\rho T(w_{\varepsilon_n} - \tilde{w})\mathrm{d}x\Big| \to 0,
\end{aligned}$$

by local compactness. Moreover, by Hölder's inequality and since T is bounded and $w_{\varepsilon_n} - \tilde{w} \to 0$ a.e. on $\mathbb{R}^N$, then by dominated convergence theorem, we have

$$\begin{aligned}
&\Big|\int\limits_{B_{k+1}} (|\nabla w_{\varepsilon_n}|^{p-2}\nabla w_{\varepsilon_n} - |\nabla \tilde{w}|^{p-2}\nabla \tilde{w})T(w_{\varepsilon_n} - \tilde{w})\nabla\rho\mathrm{d}x\Big| \\
&\quad \le C\Big(\int\limits_{B_{k+1}} |T(w_{\varepsilon_n} - \tilde{w})|^p|\nabla\rho|^p\mathrm{d}x\Big)^{\frac{1}{p}} \to 0,
\end{aligned}$$

and hence (6.9) follows. As a consequence (w_{ε_n}) converges to $\tilde{w}$ in $D^{1,p}(\mathbb{R}^N)$ and in $L^s(\mathbb{R}^N)$ for any $s \ge p^*$, where $\tilde{w}$ is a minimizer of (S_0) satisfying the constraint. Similarly to the proof of Corollary 5.20, using Lemma 6.2, by uniform elliptic estimates we conclude that (w_{ε_n}) converges to $\tilde{w}_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. The decay follows from Lemma 4.7. This concludes the proof. □

Proof of Theorem 2.8 The statement follows directly from Theorem 6.4 and Lemma 6.3. □

7 Proof of Theorem 2.2: subcritical case $p < q < p^*$

In this section, we consider the subcritical case $p < q < p^*$ and prove Theorem 2.2 showing that, after the canonical rescaling (1.2), the groundstate solutions u_ε converge as $\varepsilon \to 0$ to the unique non-trivial radial groundstate solution to the limit equation (R_0). This result extends [24, Theorem 2.1] to $p \neq 2$.

Since by Pohožaev's identity the equation (P_0) has no positive finite energy solutions, to understand the asymptotic behaviour of the groundstates u_ε we consider the rescaling in (1.2), which transforms (P_ε) into (R_ε), whose limit problem as $\varepsilon \to 0$ is (R_0).
Pick $G_\varepsilon : \mathbb{R} \to \mathbb{R}$, a bounded truncated function such that

$$G_\varepsilon(w) = \frac{1}{q}|w|^q - \frac{1}{p}|w|^p - \frac{\varepsilon^{\frac{l-q}{q-p}}}{l}|w|^l,$$

for $0 < w \leq \varepsilon^{-\frac{1}{q-p}}$, $G_\varepsilon(w) \leq 0$ for $w > \varepsilon^{-\frac{1}{q-p}}$ and $G_\varepsilon(w) = 0$ for $w \leq 0$. For $\varepsilon \in [0, \varepsilon^*)$, we set

$$S'_\varepsilon := \inf\left\{ \int\limits_{\mathbb{R}^N} |\nabla w|^p \mathrm{d}x \;\middle|\; w \in W^{1,p}(\mathbb{R}^N), \quad p^* \int\limits_{\mathbb{R}^N} G_\varepsilon(w)\mathrm{d}x = 1 \right\}, \qquad (S'_\varepsilon)$$

a well-defined family of constrained minimization problems, which share, together with the limit problem (S'_0), the same functional setting $W^{1,p}(\mathbb{R}^N)$. By Theorem 3.2, (S'_ε) possesses a radial positive minimizer w_ε for every $\varepsilon \in [0, \varepsilon^*)$. The rescaled function

$$v_\varepsilon(x) := w_\varepsilon\Big(\frac{x}{\sqrt[p]{S'_\varepsilon}}\Big),$$

is a radial groundstate of (R_ε).
We estimate (S'_ε) by means of the dilation invariant representation

$$\mathcal{S}'_\varepsilon(w) := \frac{\int\limits_{\mathbb{R}^{\mathbb{N}}} |\nabla w|^p \mathrm{d}x}{\Big(p^* \int\limits_{\mathbb{R}^{\mathbb{N}}} G_\varepsilon(w)\mathrm{d}x\Big)^{(N-p)/N}}, \quad w \in \mathcal{M}'_\varepsilon,$$

where $\mathcal{M}'_\varepsilon := \{0 \leq w \in W^{1,p}(\mathbb{R}^N), \int\limits_{\mathbb{R}^{\mathbb{N}}} G_\varepsilon(w)\mathrm{d}x > 0\}$. We have

$$S'_\varepsilon = \inf_{w \in \mathcal{M}'_\varepsilon} \mathcal{S}'_\varepsilon(w),$$

and for ε small enough we have

$$S'_0 \leq \mathcal{S}'_0(w_\varepsilon) < \mathcal{S}'_\varepsilon(w_\varepsilon) = S'_\varepsilon. \tag{7.1}$$

This follows by observing that as $p^* \int\limits_{\mathbb{R}^N} G_\varepsilon(w_\varepsilon)\mathrm{d}x = 1$ and $G_\varepsilon(s)$ is a decreasing function of ε for each $s > 0$, we have $w_\varepsilon \in \mathcal{M}'_0$, and the second inequality follows again by monotonicity. Observe that by continuity $w_0 \in \mathcal{M}'_\varepsilon$ for sufficiently small ε. As a consequence, by testing(S'_ε) with w_0, that for ε small enough, we have that

$$S'_\varepsilon \leq \mathcal{S}'_\varepsilon(w_0) = \frac{S'_0}{\Big(1 - \frac{p^*}{l}\varepsilon^{\frac{l-q}{q-p}}||w_0||_l^l\Big)^{(N-p)/N}} \leq S'_0 + O(\varepsilon^{\frac{l-q}{q-p}}).$$

Hence $S'_\varepsilon \to S'_0$. Reasoning as in Lemma 5.2 , we obtain that

$$||w_\varepsilon||_q^q = \frac{(l - p^*)q}{(l - q)p^*} + \frac{q(l - p)}{p(l - q)}||w_\varepsilon||_p^p.$$

Inserting this identity into the definition of $S'_0(w_\varepsilon)$ and using the convergence of S'_ε to S'_0, one can easily check that

$$\lim_{\varepsilon\to 0}||w_\varepsilon||_p^p = \frac{p(p^*-q)}{p^*(q-p)}, \quad \lim_{\varepsilon\to 0}||w_\varepsilon||_q^q = \frac{q(p^*-p)}{p^*(q-p)}. \tag{7.2}$$

Therefore $p^* \int\limits_{\mathbb{R}^N} G_0(w_\varepsilon)\mathrm{d}x \to 1 \quad \text{as} \quad \varepsilon \to 0$. We have then achieved that a rescaling $\lambda_\varepsilon \to 1$ exists such that $p^* \int\limits_{\mathbb{R}^N} G_0(\tilde{w}_\varepsilon)\mathrm{d}x = 1$ and $\mathcal{S}'_\varepsilon(\tilde{w}_\varepsilon) \to S'_0$ with $\tilde{w}_\varepsilon(x) := w_\varepsilon(\lambda_\varepsilon x)$. It follows that $(\tilde{w}_\varepsilon)$ is a minimizing one parameter family for (S'_0) that satisfies the constraint used in the method which yields Theorem 3.2. By Theorem 3.2 we conclude that for a suitable sequence $\varepsilon_n \to 0$, it holds $\tilde{w}_{\varepsilon_n} \to \tilde{w}$ strongly in $W^{1,p}(\mathbb{R}^N)$, and since $\lambda_\varepsilon \to 1$, it holds that $w_{\varepsilon_n} \to \tilde{w}$, where $\tilde{w}$ is the minimizer of (S'_0) satisfying the constraint. By the uniqueness of minimizer of (R_0), we have $\tilde{w} = w_0$. An obvious modification of the proof of Lemma 5.19, using $||w_\varepsilon||_{p^*}$, yields that $||w_\varepsilon||_\infty \lesssim 1$ as $\varepsilon \to 0$. By uniform elliptic estimates we conclude that w_ε converges to w_0 in $L^s(\mathbb{R}^N)$ for any $s \geq p$ and in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, and therefore the proof of Theorem 2.2 is complete.

A Radial functions

We recall that for $u \in L^1(\mathbb{R}^N)$, the radially decreasing rearrangement of a function u is denoted by u^* and it is such that for any $\alpha > 0$ it holds that

$$\Big|x \in \mathbb{R}^N : u(x)^* \geq \alpha\Big| = \Big|x \in \mathbb{R}^N : |u(x)| \geq \alpha\Big|,$$

where $\Big|\cdot\Big|$ denotes the Lebesgue measure in $\mathbb{R}^N$. We recall that

$$\int_{\mathbb{R}^N} F(u)\mathrm{d}x = \int_{\mathbb{R}^N} F(u^*)\mathrm{d}x,$$

for every continuous F such that $F(u)$ is summable.
The following fundamental properties of rearrangements can be found, e.g. in [43]:

Lemma A.1 *Let $1 \leq p < \infty$ and $u, v \in L^p(\mathbb{R}^N)$. Then $u^*, v^* \in L^p(\mathbb{R}^N)$ and*

$$||u^*||_p = ||u||_p, \qquad ||u^* - v^*||_p \leq ||u - v||_p.$$

Lemma A.2 *Let $1 < p < N$ and $u \in D^{1,p}(\mathbb{R}^N)$ (respectively, in $W^{1,p}(\mathbb{R}^N)$). Then u^* belongs to $D^{1,p}(\mathbb{R}^N)$ (respectively, in $W^{1,p}(\mathbb{R}^N)$), and we have*

$$\int_{\mathbb{R}^N} |\nabla u^*(x)|^p \mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla u(x)|^p \mathrm{d}x.$$

We will be frequently using the following well-known decay and compactness properties of radial functions on $\mathbb{R}^N$.

Lemma A.3 [36] *Assume that $1 < p < N$. Then there exists $C = C(N, p) > 0$ such that for all $u \in D^{1,p}_r(\mathbb{R}^N)$,*

$$|u(x)| \leq C|x|^{-\frac{N-p}{p}} ||\nabla u||_{L^p(\mathbb{R}^N)}. \tag{A.1}$$

Lemma A.4 (Compactness of the radial embedding [36]) *Let $1 < p < N$. Then we have the following continuous embedding*

$$W^{1,p}_r(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \tag{A.2}$$

for $p \leq q \leq p^* := \frac{pN}{N-p}$ *when* $p^* < \infty$ *and for* $p \leq q < \infty$ *when* $p^* = \infty$. *Furthermore, the embedding is compact for* $p < q < p^*$.

Lemma A.5 (1) *Let* $s \geq 1$ *and let* $u \in L^s(\mathbb{R}^N)$ *be a radial nonincreasing function. Then for every* $x \neq 0$,

$$|u(x)| \leq C|x|^{\frac{-N}{s}} ||u||_s, \tag{A.3}$$

where $C = C(s, N)$*, see, e.g.* [3].

(2) *Let* $u_n \in D^{1,p}(\mathbb{R}^N)$ *be a sequence of radial functions such that* $u_n \rightharpoonup u$ *in* $D^{1,p}(\mathbb{R}^N)$. *Then, passing if necessary to a subsequence, it holds that*

$$u_n \to u \quad in\ L^\infty(\mathbb{R}^N \backslash B_r(0)) \quad and \quad L^s(\mathbb{R}^N \backslash B_r(0)) \quad \forall\, r > 0,\ s > p^*.$$

Proof Since $(u_n)_{n\in\mathbb{N}} \in D^{1,p}(\mathbb{R}^N)$ is a radial sequence, setting $f_n(|x|) = u_n(x)$ from the fundamental theorem of calculus and Hölder's inequality for all $|x| > |y| > r > 0$ it holds that

$$|u_n(x) - u_n(y)| \leq \int_{|y|}^{|x|} |f_n'(t)| \mathrm{d}t \leq |x - y|^{1/p'} |y|^{(1-N)/p} |\mathbb{S}^{N-1}|^{-1/p} \|\nabla u_n\|_{L^p(\mathbb{R}^N)}$$

and as a consequence, since $u_n \rightharpoonup u$ is bounded, for all $x, y \in \mathbb{R}^N \setminus B_r(0)$ and a uniform constant $C > 0$ we have that

$$|u_n(x) - u_n(y)| \leq C r^{(1-N)/p} |x - y|^{1/p'}. \tag{A.4}$$

Namely, $(u_n)_{n\in\mathbb{N}}$ is bounded in $C^{0,1/p'}(\mathbb{R}^N \setminus B_r(0))$ and by the locally compact embedding it is strongly convergent to u in $L^\infty_{\text{loc}}(\mathbb{R}^N \setminus B_r(0))$. This and Lemma A.3 yield the convergence in $L^s(\mathbb{R}^N \setminus B_r(0))$. □

B Comparison principle for the *p*-Laplacian

Let $G \subseteq \mathbb{R}^N$ be a domain. We say that $0 \leq v \in W^{1,p}_{\text{loc}}(G)$ satisfies condition (S) if:

(S) there exists $(\theta_n)_{n\in\mathbb{N}} \subset W^{1,\infty}_c(\mathbb{R}^N)$ such that $0 \leq \theta_n \to 1$ a.e. in $\mathbb{R}^N$ and

$$\int_G \mathcal{R}(\theta_n v, v) \to 0, \quad \text{as} \quad n \to +\infty.$$

where $\mathcal{R}$ is defined by

$$\mathcal{R}(w, v) := \left|\nabla w\right|^p - \nabla\Big(\frac{w^p}{v^{p-1}}\Big)\left|\nabla v\right|^{p-2}\nabla v. \tag{B.1}$$

Notice that if G is bounded and $v \in W^{1,p}(G)$ then condition (S) is trivially satisfied with $\theta = 1$ in G. In case of an unbounded domain G, condition (S) ensures that the subsolution v is sufficiently *small* at infinity, in order to respect the comparison principle (see [19]). Using condition (S), we formulate a version of comparison principle for a p-Laplacian with a general negative potential (see, e.g. [19,27,34]).

Theorem B.1 (Comparison principle for p-Laplacian) *Let* $0 < u \in W^{1,p}_{\text{loc}}(G) \cap C(\bar{G})$ *be a supersolution and* $v \in W^{1,p}_{\text{loc}}(G) \cap C(\bar{G})$ *a subsolution to the equation*

$$-\Delta_p u + V|u|^{p-2}u = 0 \quad in\ \ G, \tag{B.2}$$

where $V \in L^\infty_{\mathrm{loc}}(G)$. If G is an unbounded domain, assume in addition that $\partial G \neq \emptyset$ and v^+ satisfies condition (S). Then $u \geq v$ on ∂G implies $u \geq v$ in G.

Below we prove a simple sufficient condition for assumption (S) to hold.

Lemma B.2 *If $0 \leq v \in D^{1,p}_{rad}(\mathbb{R}^N)$ then v satisfies (S).*

Proof Following [19,34], define

$$\eta_R(r) = \begin{cases} 1, & 0 \leq r \leq R \\ \frac{\log \frac{R^2}{r}}{\log R}, & R \leq r \leq R^2, \\ 0, & r \geq R^2, \end{cases}$$

and note that $|\eta_R| \leq 1$ a.e. in $\mathbb{R}^N$ and $|\eta_R'| \leq \frac{c}{\log R} r^{-1}$. We are going to show that

$$\int\limits_{\mathbb{R}^N} \mathcal{R}(\eta_R v, v) \to 0 \quad \text{as} \quad R \to \infty.$$

Using the Picone's identity [1,10] and inequalities [34, Lemma 7.4], it is straightforward to deduce the inequalities

$$\mathcal{R}(\eta_R v, v) \leq c_1 |v\,(\eta_R)'_r|^p, \qquad (1 < p \leq 2), \tag{B.3}$$

$$\mathcal{R}(\eta_R v, v) \leq c_2 |\eta_R v'_r|^{p-2} |v(\eta_R)'_r|^2 + c_3 |v\,(\eta_R)'_r|^p, \qquad (p > 2). \tag{B.4}$$

Case $1 < p \leq 2$. Using (B.3) and Ni's decay estimate Lemma A.3 on $v \in D^{1,p}_{rad}(\mathbb{R}^N)$,

$$v \leq c|x|^{-\frac{N-p}{p}},$$

by a direct calculation we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}^N} \mathcal{R}(\eta_R v, v)\mathrm{d}x &\leq c_1 \int\limits_R^{R^2} |v(\eta_R)'_r|^p r^{N-1} \mathrm{d}r \leq c \int\limits_R^{R^2} \left| r^{-\frac{N-p}{p}} \frac{1}{\log R} r^{-1} \right|^p r^{N-1} \mathrm{d}r \\ &\leq \frac{C}{(\log R)^{p-1}} \to 0 \quad \text{as } R \to \infty. \end{aligned} \tag{B.5}$$

Case $p > 2$. By Hölder and (B.5) we conclude

$$\begin{aligned} &\int_0^{+\infty} |\eta_R v'|^{p-2} |v(\eta_R)'_r)|^2 r^{N-1} \mathrm{d}r \\ &\leq \left(\int_0^{+\infty} |\eta_R v'|^p r^{N-1} \mathrm{d}r \right)^{\frac{p-2}{p}} \left(\int_0^{+\infty} |v(\eta_R)'_r)|^p r^{N-1} \mathrm{d}r \right)^{\frac{2}{p}} \\ &\leq c \|v\|^{p-2}_{D^{1,p}(\mathbb{R}^N)} \left(\int_R^{R^2} |v(\eta_R)'_r)|^p r^{N-1} \mathrm{d}r \right)^{\frac{2}{p}} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

Taking into account (B.4) and once again (B.5), the conclusion follows. □

Remark B.3 While the statement of Lemma B.2 is sufficient for our purposes, it is far from optimal. See [19, Appendix B] for constructions of radial functions $v \notin D^{1,p}(\mathbb{R}^N)$ which satisfy assumption (S).

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