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Asymptotic behavior of extremals for fractional Sobolev inequalities associated with singular problems

G. Ercole[1] · **G. A. Pereira**[1] · **R. Sanchis**[1]

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Abstract

Let Ω be a smooth, bounded domain of $\mathbb{R}^N$, ω be a positive, L^1-normalized function, and $0 < s < 1 < p$. We study the asymptotic behavior, as $p \to \infty$, of the pair $\left(\sqrt[p]{\Lambda_p}, u_p\right)$, where Λ_p is the best constant C in the Sobolev-type inequality

$$C \exp\left(\int_\Omega (\log |u|^p)\omega \mathrm{d}x\right) \le [u]_{s,p}^p \quad \forall\, u \in W_0^{s,p}(\Omega)$$

and u_p is the positive, suitably normalized extremal function corresponding to Λ_p. We show that the limit pairs are closely related to the problem of minimizing the quotient $|u|_s \,/\, \exp\left(\int_\Omega (\log |u|)\omega \mathrm{d}x\right)$, where $|u|_s$ denotes the s-Hölder seminorm of a function $u \in C_0^{0,s}(\overline{\Omega})$.

Keywords Asymptotic behavior · Fractional p-Laplacian · Singular problem · Viscosity solution

Mathematics Subject Classification 35D40 · 35R11 · 35J60

1 Introduction

Let Ω be a smooth (at least Lipschitz) domain of $\mathbb{R}^N$, and consider the fractional Sobolev space

$$W_0^{s,p}(\Omega) := \left\{u \in L^p(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \quad \text{and} \quad [u]_{s,p} < \infty\right\}, \quad 0 < s < 1 < p,$$

✉ G. Ercole
grey@mat.ufmg.br

G. A. Pereira
gilbertoapereira@yahoo.com.br

R. Sanchis
rsanchis@mat.ufmg.br

1 Universidade Federal de Minas Gerais, Belo Horizonte, MG 30.123-970, Brazil

where

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

It is well known that the Gagliardo seminorm $[\cdot]_{s,p}$ is a norm in $W_0^{s,p}(\Omega)$ and that this Banach space is uniformly convex. Actually,

$$W_0^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{[\cdot]_{s,p}}.$$

Let ω be a nonnegative function in $L^1(\Omega)$ satisfying $\|\omega\|_{L^1(\Omega)} = 1$, and define

$$\mathcal{M}_p := \left\{ u \in W_0^{s,p}(\Omega) : \int_\Omega (\log |u|) \omega \mathrm{d}x = 0 \right\}$$

and

$$\Lambda_p := \inf \left\{ [u]_{s,p}^p : u \in \mathcal{M}_p \right\}. \tag{1}$$

In the recent paper [9], it is proved that $\Lambda_p > 0$ and that

$$\Lambda_p \exp \left(\int_\Omega (\log |u|^p) \omega \mathrm{d}x \right) \leq [u]_{s,p}^p \quad \forall\, u \in W_0^{s,p}(\Omega), \tag{2}$$

provided that $\Lambda_p < \infty$. Moreover, the equality in this Sobolev-type inequality holds if, and only if, u is a scalar multiple of the function $u_p \in \mathcal{M}_p$ which is the only weak solution of the problem

$$\begin{cases} (-\Delta_p)^s u = \Lambda_p u^{-1} \omega & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3}$$

Here, $(-\Delta_p)^s$ is the s-fractional p-Laplacian, formally defined by

$$(-\Delta_p)^s u(x) = -2 \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2} (u(y) - u(x))}{|y - x|^{N+sp}} \mathrm{d}y.$$

We recall that a weak solution of the equation in (3) is a function $u \in W_0^{s,p}(\Omega)$ satisfying

$$\left\langle (-\Delta_p)^s u, \varphi \right\rangle = \Lambda_p \int_\Omega u^{-1} \varphi \omega \mathrm{d}x \quad \forall\, \varphi \in W_0^{s,p}(\Omega),$$

where

$$\left\langle (-\Delta_p)^s u, \varphi \right\rangle := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y$$

is the expression of $(-\Delta_p)^s$ as an operator from $W_0^{s,p}(\Omega)$ into its dual.

The purpose of this paper is to determine both the asymptotic behavior of the pair $\left(\sqrt[p]{\Lambda_p}, u_p\right)$, as $p \to \infty$, and the corresponding limit problem of (3). In our study $s \in (0, 1)$ is kept fixed.

After introducing, in Sect. 2, the notation used throughout the paper, we prove in Sect. 3 that $\Lambda_p < \infty$ by constructing a function $\xi \in C_0^{0,1}(\overline{\Omega}) \cap \mathcal{M}_p$. In the simplest case $\omega \equiv |\Omega|^{-1}$ this was made in [10] where the inequality (2) corresponding to the standard Sobolev Space $W_0^{1,p}(\Omega)$ has been derived.

In Sect. 4, we show that the limit problem is closely related to the problem of minimizing the quotient

$$Q_s(u) := \frac{|u|_s}{\exp\left(\int_\Omega (\log|u|)\omega \mathrm{d}x\right)}$$

on the Banach space $\left(C_0^{0,s}(\overline{\Omega}), |\cdot|_s\right)$ of the s-Hölder continuous functions in $\overline{\Omega}$ that are zero on the boundary $\partial\Omega$. Here, $|u|_s$ denotes the s-Hölder seminorm of u (see (6)).

We prove that if $p_n \to \infty$ then (up to a subsequence)

$$u_{p_n} \to u_\infty \in C_0^{0,s}(\overline{\Omega}) \text{ uniformly in } \overline{\Omega}, \quad \text{and} \quad \sqrt[p_n]{\Lambda_{p_n}} \to |u_\infty|_s\,.$$

Moreover, the limit function u_∞ satisfies

$$\int_\Omega (\log|u_\infty|)\omega \mathrm{d}x \geq 0 \quad \text{and} \quad Q_s(u_\infty) \leq Q_s(u) \quad \forall\, u \in C_0^{0,s}(\overline{\Omega})\setminus\{0\}$$

and the only minimizers of the quotient Q_s are the scalar multiples of u_∞.

One of the difficulties we face in Sect. 4 is that $C_c^\infty(\Omega)$ is not dense in $\left(C_0^{0,s}(\Omega), |\cdot|_s\right)$. This makes it impossible to directly exploit the fact that u_p is a weak solution of (3). We overcome this issue by using a convenient technical result proved in [18, Lemma 3.2] and employed in [2] to deal with a similar approximation matter.

In Sect. 5, motivated by [3,13,17], we derive the limit problem of (3). Assuming that ω is continuous and positive in Ω, we prove that u_∞ is a viscosity solution of

$$\begin{cases} \mathcal{L}_\infty^- u + |u|_s = 0 \text{ in } \Omega \\ u = 0 \qquad\qquad \text{in} \quad \mathbb{R}^N\setminus\Omega \end{cases}$$

where

$$\left(\mathcal{L}_\infty^- u\right)(x) := \inf_{y\in\mathbb{R}^N\setminus\{x\}} \frac{u(y)-u(x)}{|y-x|^s}.$$

We also show u_∞ is a viscosity supersolution of

$$\begin{cases} \mathcal{L}_\infty u = 0 \text{ in } \Omega \\ u = 0 \quad\ \text{in} \quad \mathbb{R}^N\setminus\Omega \end{cases}$$

where

$$\mathcal{L}_\infty := \mathcal{L}_\infty^+ + \mathcal{L}_\infty^-$$

and

$$\left(\mathcal{L}_\infty^+ u\right)(x) := \sup_{y\in\mathbb{R}^N\setminus\{x\}} \frac{u(y)-u(x)}{|y-x|^s}.$$

This fact guarantees that $u_\infty > 0$ in Ω.

The existing literature on the asymptotic behavior (as $p \to \infty$) of solutions of problems involving the p-Laplacian is most focused on the local version of the operator, that is, on the problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the standard p-Laplacian. This kind of asymptotic behavior has been studied for at least three decades (see [1,14,16]) and many new results, adding the dependence of p in the term $f(x,u)$, are still being produced (see [4–6,8]). The solutions of (4) are obtained in the natural Sobolev space $W_0^{1,p}(\Omega)$, and an important property related to this space, crucial in the study of the asymptotic behavior of the corresponding family of solutions $\{u_p\}$, is the inclusion

$$W_0^{1,p_2}(\Omega) \subset W_0^{1,p_1}(\Omega) \quad \text{whenever} \quad 1 < p_1 < p_2.$$

It allows us to show that any uniform limit function u_∞ of the sequence $\{u_{p_n}\}$ (with $p_n \to \infty$) is admissible as a test function in the weak formulation of (4), so that u_∞ inherits certain properties of the functions of $\{u_{p_n}\}$.

Since the inclusion $W_0^{s,p_2}(\Omega) \subset W_0^{s,p_1}(\Omega)$ does not hold when $0 < s < 1 < p_1 < p_2$ (see [19]), the asymptotic behavior, as $p \to \infty$, of the solutions of the problem

$$\begin{cases} (-\Delta_p)^s u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega \end{cases} \tag{5}$$

is more difficult to be determined. For example, in the case considered in the present paper ($f(x,u) = \omega(x)/u$) we cannot ensure that the property

$$\int_\Omega (\log |u_{p_n}|)\omega \mathrm{d}x = 0$$

is inherited by the limit function u_∞ (see Remark 12). Actually, we are able to prove only that

$$\int_\Omega (\log u_\infty)\omega \mathrm{d}x \geq 0.$$

As a consequence, the limit functions of the family $\{u_p\}_{p>1}$ might not be unique.

The study of the asymptotic behavior, as $p \to \infty$, of the solutions of (5) is quite recent and restricted to few works. In [17] the authors considered $f(x,u) = \lambda_p |u|^{p-2} u$ where λ_p is the first eigenvalue of the s-fractional p-Laplacian. Among other results, they proved that

$$\lim_{p\to\infty} \sqrt[p]{\lambda_p} = R^{-s},$$

where R is the radius of the largest ball inscribed in Ω, and that limit function u_∞ of the family $\{u_p\}$ is a positive viscosity solution of

$$\max\left\{\mathcal{L}_\infty u\ ,\ \mathcal{L}_\infty^- u + R^{-s} u\right\} = 0.$$

The equation in (5) with $f = 0$ and under the nonhomogeneous boundary condition $u = g$ in $\mathbb{R}^N \backslash \Omega$ was first studied in [3]. It is shown that the limit function is an optimal s-Hölder extension of $g \in C^{0,s}(\partial\Omega)$ and also a viscosity solution of the equation

$$\mathcal{L}_\infty u = 0 \quad \text{in } \partial\Omega.$$

Moreover, some tools for studying the behavior as $p \to \infty$ of the solutions of (5) are developed there.

In [13], also under the boundary condition $u = g$ in $\mathbb{R}^N \backslash \Omega$, the cases $f = f(x)$ and $f = f(u) = |u|^{\theta(p)-2} u$ with $\Theta := \lim_{p\to\infty} \theta(p)/p < 1$ are studied. In the first case, different limit equations involving the operators $\mathcal{L}_\infty$, $\mathcal{L}_\infty^+$ and $\mathcal{L}_\infty^-$ are derived according to the sign of the function $f(x)$, what resembles the known results obtained in [1], where the standard p-Laplacian is considered. For example, the limit function u_∞ is a viscosity solution of

$$-\mathcal{L}_\infty^- u = 1 \quad \text{in } \{f > 0\}.$$

As for the second case, the limit equation is

$$\min\left\{-\mathcal{L}_\infty^- u - u^\Theta, -\mathcal{L}_\infty u\right\} = 0$$

which is consistent with the limit equation obtained in [4] for the standard p-Laplacian and $f(u) = |u|^{\theta(p)-2} u$ satisfying $\Theta := \lim_{p\to\infty} \theta(p)/p < 1$.

2 Notation

The ball centered at $x \in \mathbb{R}^N$ with radius ρ is denoted by $B(x, \rho)$, and δ stands for the distance function to the boundary $\partial\Omega$, defined by

$$\delta(x) := \min_{y\in\partial\Omega} |x - y|, \quad x \in \overline{\Omega}.$$

We recall that $\delta \in C_0^{0,1}(\overline{\Omega})$ and satisfies $|\nabla\delta| = 1$ a.e. in Ω. Here,

$$C_0^{0,\beta}(\overline{\Omega}) := \left\{u \in C^{0,\beta}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\}, \quad 0 < \beta \le 1,$$

where $C^{0,\beta}(\overline{\Omega})$ is the well-known β-Hölder space endowed with the norm

$$\|u\|_{0,\beta} = \|u\|_\infty + |u|_\beta$$

with $\|u\|_\infty$ denoting the sup norm of u and $|u|_\beta$ denoting the β-Hölder seminorm, that is,

$$|u|_\beta := \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta}. \tag{6}$$

We recall that $\left(C_0^{0,\beta}(\overline{\Omega}), |\cdot|_\beta\right)$ is a Banach space. The fact that the β-Hölder seminorm $|\cdot|_\beta$ is a norm in $C_0^{0,\beta}(\overline{\Omega})$ equivalent to $\|u\|_{0,\beta}$ is a consequence of the estimate

$$\|u\|_\infty \le |u|_\beta \|\delta\|_\infty^\beta \quad \forall u \in C_0^{0,\beta}(\overline{\Omega}),$$

which in turn follows from the following

$$|u(x)| = |u(x) - u(y_x)| \le |u|_\beta |x - y_x|^\beta = |u|_\beta \delta(x)^\beta \quad \forall x \in \Omega, \tag{7}$$

where $y_x \in \partial\Omega$ is such that $\delta(x) = |x - y_x|$.

We also define

$$C_c^\infty(\Omega) := \left\{u \in C^\infty(\Omega) : \operatorname{supp}(f) \subset\subset \Omega\right\}$$

where

$$\operatorname{supp}(u) := \{x \in \Omega : u(x) \neq 0\}$$

is the support of u and $X \subset\subset Y$ means that $\overline{X}$ is a compact subset of Y. Analogously, we define E_c if E is a space of functions (e.g., $C_c(\mathbb{R}^N)$, $C_c(\mathbb{R}^N; \mathbb{R}^N)$, $C_c^{0,\beta}(\overline{\Omega})$).

3 Finiteness of Λ_p

Let us recall the Federer's co-area formula (see [12])

$$\int_\Omega g(x)\,|\nabla f(x)|\,\mathrm{d}x = \int_{-\infty}^{\infty}\left(\int_{f^{-1}\{t\}} g(x)\mathrm{d}\mathcal{H}_{N-1}\right)\mathrm{d}t,$$

which holds whenever $g \in L^1(\Omega)$ and $f \in C^{0,1}(\overline{\Omega})$. (In this formula $\mathcal{H}_{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure).

In the particular case $f = \delta$, the above formula becomes

$$\int_\Omega g(x)\mathrm{d}x = \int_0^{\|\delta\|_\infty}\left(\int_{\delta^{-1}\{t\}} g(x)\mathrm{d}\mathcal{H}_{N-1}\right)\mathrm{d}t. \tag{8}$$

Proposition 1 *Let $\omega \in L^1(\Omega)$ such that*

$$\int_\Omega \omega\mathrm{d}x = 1 \quad \text{and} \quad \omega \geq 0 \quad \text{a.e. in } \Omega. \tag{9}$$

There exists a nonnegative function $\xi \in C(\overline{\Omega})$ that vanishes on the boundary $\partial\Omega$ and satisfies

$$\int_\Omega (\log|\xi|)\omega\mathrm{d}x = 0.$$

If, in addition,

$$K_\epsilon := \underset{0\leq t\leq\epsilon}{\mathrm{ess}} \int_{\delta^{-1}\{t\}} \omega\mathrm{d}\mathcal{H}_{N-1} < \infty \tag{10}$$

for some $\epsilon > 0$, then $\xi \in C_0^{0,1}(\overline{\Omega})$.

Proof Let $\sigma : [0, \|\delta\|_\infty] \to [0, 1]$ be the ω-distribution associated with δ, that is,

$$\sigma(t) := \int_{\Omega_t} \omega\mathrm{d}x, \quad t \in [0, \|\delta\|_\infty]$$

where

$$\Omega_t := \{x \in \Omega : \delta(x) > t\}$$

is the t-superlevel set of δ.

We remark that σ is continuous at each point $t \in [0, \|\delta\|_\infty]$ since the t-level set $\delta^{-1}\{t\}$ has Lebesgue measure zero. This follows, for example, from the Lebesgue density theorem (see [11], where the distance function to a general closed set in $\mathbb{R}^N$ is considered).

Thus, there exists a nonincreasing sequence $\{t_n\} \subset [0, \|\delta\|_\infty]$ such that

$$\sigma(t_n) = 1 - \frac{1}{2^n}.$$

Now, choose a nondecreasing, piecewise linear function $\varphi \in C([0, \|\delta\|_\infty])$ satisfying

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(t_n) = \frac{1}{2^n},$$

and take the function

$$\xi_1 := \varphi \circ \delta \in C_0(\overline{\Omega}).$$

Taking into account that

$$t_{n+1} \le \delta(x) \le t_n \quad \text{a.e. } x \in \Omega_{t_{n+1}} \backslash \Omega_{t_n},$$

one has

$$\frac{1}{2^{n+1}} = \varphi(t_{n+1}) \le \xi_1(x) \le \varphi(t_n) = \frac{1}{2^n} \quad \text{a.e. } x \in \Omega_{t_{n+1}} \backslash \Omega_{t_n}.$$

Consequently,

$$\begin{aligned}
\int_\Omega |\xi_1|^\epsilon \, \omega \mathrm{d}x &\ge \int_{\Omega_{t_1}} |\xi_1|^\epsilon \, \omega \mathrm{d}x + \sum_{k=1}^{n} \int_{\Omega_{t_{k+1}} \backslash \Omega_{t_k}} |\xi_1|^\epsilon \, \omega \mathrm{d}x \\
&\ge \frac{1}{2^\epsilon} \int_{\Omega_{t_1}} \omega \mathrm{d}x + \sum_{k=1}^{n} \frac{1}{2^{\epsilon(k+1)}} \int_{\Omega_{t_{k+1}} \backslash \Omega_{t_k}} \omega \mathrm{d}x \\
&= \frac{1}{2^\epsilon} \sigma(t_1) + \sum_{k=1}^{n} \frac{1}{2^{\epsilon(k+1)}} \left(\sigma(t_{k+1}) - \sigma(t_k)\right) \\
&= \frac{1}{2^\epsilon} \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2^{\epsilon(k+1)}} \frac{1}{2^{k+1}} = \sum_{k=1}^{n+1} \left((1/2)^{\epsilon+1}\right)^k .
\end{aligned}$$

It follows that

$$\lim_{\epsilon \to 0} \left(\int_\Omega |\xi_1|^\epsilon \, \omega \mathrm{d}x \right)^{\frac{1}{\epsilon}} \ge \lim_{\epsilon \to 0} \left(\sum_{k=1}^{\infty} \left((1/2)^{\epsilon+1}\right)^k \right)^{\frac{1}{\epsilon}} = \lim_{\epsilon \to 0} \left(\frac{(1/2)^{\epsilon+1}}{1 - (1/2)^{\epsilon+1}} \right)^{\frac{1}{\epsilon}} = \frac{1}{4}.$$

Taking $\xi := k\xi_1$ with

$$k = \lim_{\epsilon \to 0} \left(\int_\Omega |\xi_1|^\epsilon \, \omega \mathrm{d}x \right)^{-\frac{1}{\epsilon}}$$

we obtain, by L'Hôpital's rule,

$$1 = \lim_{\epsilon \to 0^+} \left(\int_\Omega |\xi|^\epsilon \, \omega \mathrm{d}x \right)^{\frac{1}{\epsilon}} = \exp\left(\int_\Omega (\log |\xi|) \omega \mathrm{d}x \right).$$

Hence,

$$\int_\Omega (\log |\xi|) \omega \mathrm{d}x = 0.$$

We now prove that $\xi_1 \in C^{0,1}(\overline{\Omega})$ under the additional hypothesis (10). Since the nondecreasing function φ can be chosen such that φ' is bounded in any closed interval contained in $(0, \|\delta\|_\infty]$, we can assume that $\nabla \xi_1 \in L^\infty_{\mathrm{loc}}(\Omega)$ (note that $|\nabla \xi_1| = \left|\varphi'(\delta) \nabla \delta\right| = \left|\varphi'(\delta)\right|$ a.e. in Ω).

Thus, it suffices to show that the quotient

$$Q(x, y) := \frac{|\xi_1(x) - \xi_1(y)|}{|x - y|}$$

is bounded uniformly with respect to $y \in \partial\Omega$ and $x \in \Omega^c_\epsilon := \left\{x \in \overline{\Omega} : \delta(x) \le \epsilon\right\}$, where ϵ is given by (10).

Let $x \in \Omega_\epsilon^c$ and $y \in \partial\Omega$ be fixed and chose $n \in \mathbb{N}$ sufficiently large such that

$$t_{n+1} < \delta(x) \leq t_n \leq \epsilon.$$

Since $\xi_1(y) = 0$ and φ is nondecreasing, one has

$$|\xi_1(x) - \xi_1(y)| = \xi_1(x) \leq \varphi(t_n) = \frac{1}{2^n}.$$

Moreover,

$$t_{n+1} < \delta(x) \leq |x - y|.$$

Hence,

$$Q(x, y) \leq \frac{1}{2^n t_{n+1}} \quad \text{whenever } y \in \partial\Omega \text{ and } x \in \Omega_\epsilon^c.$$

Applying the co-area formula (8) with $g = \omega$ and $\Omega = \Omega_{t_n+1}^c$, we find

$$\frac{1}{2^{n+1}} = \int_{\Omega_{t_n+1}^c} \omega \mathrm{d}x = \int_0^{t_{n+1}} \left(\int_{\delta^{-1}\{t\}} \omega \mathrm{d}\mathcal{H}_{N-1} \right) \mathrm{d}t \leq K_\epsilon t_{n+1}.$$

It follows that

$$Q(x, y) \leq \frac{1}{2^n t_{n+1}} \leq \frac{K_\epsilon 2^{n+1}}{2^n} = 2K_\epsilon \quad \text{whenever } y \in \partial\Omega \text{ and } x \in \Omega_\epsilon^c, \tag{11}$$

concluding thus the proof that $\xi_1 \in C^{0,1}(\overline{\Omega})$. □

Remark 2 The estimate (11) can also be obtained from the Weyl's Formula (see [15]) provided that ω is bounded on an ϵ-tubular neighborhood of $\partial\Omega$.

In the remaining of this section, ξ denotes the function obtained in Proposition 1 extended as zero outside Ω. So,

$$\xi \in C_0^{0,1}(\overline{\Omega}) \quad \text{and} \quad \int_\Omega (\log|\xi|)\omega \mathrm{d}x = 0.$$

Since $C_0^{0,1}(\overline{\Omega}) \subseteq W_0^{1,p}(\Omega) \subseteq W_0^{s,p}(\Omega)$, we have $\xi \in \mathcal{M}_p$ (for a proof of the second inclusion see [7]). Therefore,

$$\Lambda_p \leq [\xi]_{s,p}^p \quad \forall\, p > 1. \tag{12}$$

Combining (12) with the results proved in [9, Section 4] (which requires $\omega \in L^r(\Omega)$, for some $r > 1$), we have the following theorem.

Theorem 3 *Let ω be a function in $L^r(\Omega)$, for some $r > 1$, satisfying (9)–(10). For each $p > 1$, the infimum Λ_p in (1) is attained by a function $u_p \in \mathcal{M}_p$ which is the only positive weak solution of*

$$\left(-\Delta_p\right)^s u = \Lambda_p u^{-1}\omega, \quad u \in W_0^{s,p}(\Omega).$$

Summarizing,

$$\left[u_p\right]_{s,p}^p = \Lambda_p := \min\left\{[u]_{s,p}^p : u \in \mathcal{M}_p\right\} \leq [\xi]_{s,p}^p \quad \forall\, p > 1, \tag{13}$$

and u_p is the unique function in $W_0^{1,p}(\Omega)$ satisfying

$$u_p > 0 \quad \text{in}\,\Omega \quad \text{and} \quad \left\langle \left(-\Delta_p\right)^s u_p, \phi \right\rangle = \Lambda_p \int_\Omega \omega (u_p)^{-1} \phi \mathrm{d}x \quad \forall \phi \in W_0^{s,p}(\Omega).$$

We also have

$$0 < \sqrt[p]{\Lambda_p} \le \frac{[u]_{s,p}}{\exp\left(\int_\Omega (\log|u|)\omega \mathrm{d}x\right)} \quad \forall u \in W_0^{s,p}(\Omega),$$

since the quotient is homogeneous.

Remark 4 It is worth pointing out that

$$\int_\Omega (\log|u|)\omega \mathrm{d}x = -\infty \tag{14}$$

for any function $u \in L^\infty(\Omega)$ whose supp u is a proper subset of supp ω. Indeed, in this case we have

$$0 \le \exp\left(\int_\Omega (\log|u|)\omega \mathrm{d}x\right) = \lim_{t \to 0^+} \left(\int_\Omega |u|^t \omega \mathrm{d}x\right)^{\frac{1}{t}} \le \|u\|_\infty \lim_{t \to 0^+} \left(\int_{\mathrm{supp}|u|} \omega \mathrm{d}x\right)^{\frac{1}{t}} = 0.$$

Thus, if $\omega > 0$ almost everywhere in Ω then (14) holds for every $u \in C_c^\infty(\Omega) \setminus \{0\}$.

4 The asymptotic behavior as $p \to \infty$

In this section, we assume that the weight ω satisfies the hypothesis of Theorem 3. Our goal is to relate the asymptotic behavior (as $p \to \infty$) of the pair $\left(\sqrt[p]{\Lambda_p}, u_p\right)$ with the problem of minimizing the homogeneous quotient $Q_s : C_0^{0,s}(\overline{\Omega}) \setminus \{0\} \to (0, \infty)$ defined by

$$Q_s(u) := \frac{|u|_s}{k(u)} \quad \text{where} \quad k(u) := \exp\left(\int_\Omega (\log|u|)\omega \mathrm{d}x\right).$$

Note that $k(u) = 0$ if, and only if, u satisfies (14). In particular, according to Remark 4,

$$\omega > 0 \quad \text{a.e. in}\,\Omega \Longrightarrow Q_s(u) = \infty \quad \forall u \in C_c^\infty(\Omega) \setminus \{0\}.$$

We also observe that

$$0 \le k(u) \le \int_\Omega |u|\, \omega \mathrm{d}x < \infty \quad \forall u \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}, \tag{15}$$

where the second inequality is consequence of the Jensen's inequality (since the logarithm is concave):

$$\int_\Omega (\log|u|)\omega \mathrm{d}x \le \log\left(\int_\Omega |u|\, \omega \mathrm{d}x\right). \tag{16}$$

Now, let us define

$$\mu_s := \inf_{u \in C_0^{0,s}(\overline{\Omega}) \setminus \{0\}} Q_s(u).$$

Thanks to the homogeneity of Q_s, we have

$$\mu_s = \inf_{u \in \mathcal{M}_s} |u|_s$$

where

$$\mathcal{M}_s := \left\{ u \in C_0^{0,s}(\overline{\Omega}) : k(u) = 1 \right\}.$$

Combining (15) and (7), we obtain

$$1 \leq \int_\Omega |u|\, \omega \mathrm{d}x \leq |u|_s \int_\Omega \delta^s \omega \mathrm{d}x \quad \forall\, u \in \mathcal{M}_s,$$

what yields the following positive lower bound to μ_s

$$\left(\int_\Omega \delta^s \omega \mathrm{d}x \right)^{-1} \leq \mu_s.$$

In the sequel we show that μ_s is in fact a minimum, attained at a unique nonnegative function. Before this, let us make an important remark.

Remark 5 If v minimizes $|\cdot|_s$ in $\mathcal{M}_s$ the same holds for $|v|$, since the function $w = |v|$ belongs to $\mathcal{M}_s$ and satisfies $|w|_s \leq |v|_s$.

Proposition 6 *There exists a unique nonnegative function $v \in \mathcal{M}_s$ such that*

$$\mu_s = |v|_s\,.$$

Proof Let $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_s$ be such that

$$\lim_{n\to\infty} |v_n|_s = \mu_s. \tag{17}$$

Since the function $w_n = |v_n|$ belongs to $\mathcal{M}_s$ and satisfies $|w_n|_s \leq |v_n|_s$, we can assume that $v_n \geq 0$ in Ω.

It follows from (17) that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $C_0^{0,s}(\overline{\Omega})$. Hence, the compactness of the embedding $C_0^{0,s}(\overline{\Omega}) \hookrightarrow C_0(\overline{\Omega})$ allows us to assume (by renaming a subsequence) that $\{v_n\}_{n\in\mathbb{N}}$ converges uniformly to a function $v \in C_0(\overline{\Omega})$. Of course, $v \geq 0$ in Ω.

Letting $n \to \infty$ in the inequality

$$|v_n(x) - v_n(y)| \leq |v_n|_s\, |x - y|^s \quad \forall\, x, y \in \overline{\Omega}$$

and taking (17) into account, we obtain

$$|v(x) - v(y)| \leq \mu_s\, |x - y|^s \quad \forall\, x, y \in \overline{\Omega}.$$

This implies that $v \in C_0^{0,s}(\overline{\Omega})$ and

$$|v|_s \leq \mu_s. \tag{18}$$

Thus, to prove that $\mu_s = |v|_s$ it suffices to verify that $v \in \mathcal{M}_s$. Since

$$1 = k(v_n) = \lim_{\epsilon\to 0^+} \left(\int_\Omega |v_n|^\epsilon\, \omega \mathrm{d}x \right)^{\frac{1}{\epsilon}} \leq \left(\int_\Omega |v_n|^t\, \omega \mathrm{d}x \right)^{\frac{1}{t}} \quad \forall\, t > 0$$

the uniform convergence $v_n \to v$ yields

$$1 \leq \left(\int_\Omega |v|^t\, \omega \mathrm{d}x \right)^{\frac{1}{t}} \quad \forall\, t > 0.$$

Hence,

$$1 \leq \lim_{t\to 0^+} \left(\int_\Omega |v|^t\, \mathrm{d}x \right)^{\frac{1}{t}} = k(v).$$

Thus, noticing that $(k(v))^{-1}v \in \mathcal{M}_s$ and taking (18) into account, we obtain

$$\mu_s \leq \left|(k(v))^{-1}v\right|_s = (k(v))^{-1} |v|_s \leq |v|_s \leq \mu_s.$$

Therefore, $k(v) = 1$, $v \in \mathcal{M}_s$ and $|v|_s = \mu_s$.

Now, let $u \in \mathcal{M}_s$ be a nonnegative minimizer of $|\cdot|_s$ and consider the convex combination

$$w := \theta u + (1-\theta)v \quad \text{with} \quad 0 < \theta < 1.$$

Since the logarithm is a concave function, we have

$$\begin{aligned}\int_\Omega (\log w)\omega \mathrm{d}x &\geq \int_\Omega (\theta \log(u) + (1-\theta)\log(v))\omega \mathrm{d}x \\ &= \theta \int_\Omega (\log u)\omega \mathrm{d}x + (1-\theta)\int_\Omega (\log v)\omega \mathrm{d}x = 0.\end{aligned}$$

This implies that $c^{-1}w \in \mathcal{M}_s$ where $c := k(w) \geq 1$.

Hence,

$$\mu_s \leq c^{-1}|w|_s \leq |w|_s \leq \theta |u|_s + (1-\theta)|v|_s = \theta\mu_s + (1-\theta)\mu_s = \mu_s.$$

It follows that $c = 1$ and the convex combination w minimizes $|\cdot|_s$ in $\mathcal{M}_s$. Consequently,

$$0 = \int_\Omega \left[\log(\theta u + (1-\theta)v)\right]\omega \mathrm{d}x \geq \int_\Omega \left[\theta\log(u) + (1-\theta)\log(v)\right]\omega \mathrm{d}x = 0.$$

Since the concavity of the logarithm is strict, one must have $u = Cv$ for some positive constant C. Taking account that $1 = k(u) = Ck(v) = C$, we have $u = v$. □

From now on, $v_s \in \mathcal{M}_s$ denotes the only nonnegative minimizer of $|\cdot|_s$ on $\mathcal{M}_s$, given by Proposition 6. The main result of this section, proved in the sequence, shows that if $p_n \to \infty$ then a subsequence of $\{u_{p_n}\}_{n\in\mathbb{N}}$ converges uniformly to a scalar multiple of v_s, say $u_\infty = k_\infty v_s$ where $k_\infty \geq 1$.

In the next section (see (37)), we show that u_∞ is strictly positive in Ω, implying thus that $-v_s$ and v_s are the only minimizers of $|\cdot|_s$ on $\mathcal{M}_s$. As consequence, the minimizers of Q_s on $C_0^{0,s}(\overline{\Omega})\setminus\{0\}$ are precisely the scalar multiples of v_s (or, equivalently, the scalar multiples of u_∞). Further, we derive an equation satisfied by v_s and μ_s in the viscosity sense (see Corollary 16).

Lemma 7 *Let $u \in C_0^{0,s}(\overline{\Omega})$ be extended as zero outside Ω. If $u \in W^{s,q}(\Omega)$ for some $q > 1$, then $u \in W_0^{s,p}(\Omega)$ for all $p \geq q$ and*

$$\lim_{p\to\infty} [u]_{s,p} = |u|_s\,. \tag{19}$$

Proof First, note that the inequality

$$|u(x) - u(y)| \leq |u|_s\,|x-y|^s$$

is valid for all $x, y \in \mathbb{R}^N$, not only for those $x, y \in \overline{\Omega}$. In fact, this is obvious when $x, y \in \mathbb{R}^N\setminus\overline{\Omega}$. Now, if $x \in \Omega$ and $y \in \mathbb{R}^N\setminus\overline{\Omega}$ then take $y_1 \in \partial\Omega$ such that $|x - y_1| \leq |x - y|$ (such y_1 can be taken on the straight line connecting x to y). Since $u(y) = u(y_1) = 0$, we have

$$|u(x) - u(y)| = |u(x)| = |u(x) - u(y_1)| \leq |u|_s\,|x-y_1|^s \leq |u|_s\,|x-y|^s\,.$$

For each $p > q$, we have

$$[u]_{s,p}^{p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-q}}{|x-y|^{s(p-q)}} \frac{|u(x)-u(y)|^{q}}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \le (|u|_s)^{(p-q)}\,[u]_{s,q}^{q} .$$

Thus, $u \in W_0^{s,p}(\Omega)$ and

$$\limsup_{p\to\infty}\,[u]_{s,p} \le \lim_{p\to\infty} |u|_s^{(p-q)/p}\,[u]_{s,q}^{q/p} = |u|_s . \tag{20}$$

Now, noticing that (by Fatou's lemma)

$$\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{s}}\right)^{q} \mathrm{d}x\mathrm{d}y \le \liminf_{p\to\infty} \int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\right)^{q} \mathrm{d}x\mathrm{d}y$$

and (by Hölder's inequality)

$$\begin{aligned}\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\right)^{q} \mathrm{d}x\mathrm{d}y &\le |\Omega|^{2(1-\frac{q}{p})} \left(\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\right)^{p} \mathrm{d}x\mathrm{d}y\right)^{\frac{q}{p}}\\ &\le |\Omega|^{2(1-\frac{q}{p})}\,[u]_{s,p}^{q} ,\end{aligned}$$

we obtain

$$\left(\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{s}}\right)^{q} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}} \le |\Omega|^{2/q} \liminf_{p\to\infty}\,[u]_{s,p} .$$

Hence, taking into account that

$$|u|_s = \lim_{q\to\infty} \left(\int_\Omega\int_\Omega \left(\frac{|u(x)-u(y)|}{|x-y|^{s}}\right)^{q} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}}$$

we arrive at

$$|u|_s \le \lim_{q\to\infty} |\Omega|^{2/q} \left(\liminf_{p\to\infty}\,[u]_{s,p}\right) = \liminf_{p\to\infty}\,[u]_{s,p} .$$

This estimate combined with (20) leads us to (19). □

It is known (see [7, Theorem 8.2]) that if $p > \dfrac{N}{s}$ then there exists of a positive constant C such that

$$\|u\|_{C^{0,\beta}(\overline{\Omega})} \le C\,[u]_{s,p} \quad \forall\, u \in W_0^{s,p}(\Omega), \tag{21}$$

where $\beta := s - \dfrac{N}{p} \in (0, 1)$. As pointed out in [13, Remark 2.2] the constant C in (21) can be chosen uniform with respect to p.

We remark that the family of positive numbers $\left\{\sqrt[p]{\Lambda_p}\right\}_{p>1}$ is bounded. Indeed, combining (12) with the previous lemma we obtain

$$\limsup_{p\to\infty} \sqrt[p]{\Lambda_p} \le |\xi|_s .$$

The next lemma, where Id stands for the identity function, is extracted of the proof of [18, Lemma 3.2]. It helps us to overcome the fact that $C_c^\infty(\Omega)$ is not dense in $C_0^{0,s}(\overline{\Omega})$.

Lemma 8 *[see [18, Lemma 3.2]]Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain. There exist $\phi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $0 < \tau_0 < (|\phi|_1)^{-1}$ such that, for each $0 \le \tau \le \tau_0$, the map*

$$\Phi_\tau := \operatorname{Id} + \tau\phi : \mathbb{R}^N \to \mathbb{R}^N$$

is a diffeomorphism satisfying

1. $\Phi_\tau(\overline{\Omega}) \subset\subset \Omega$,
2. $\Phi_\tau \to \operatorname{Id}$ *and* $(\Phi_\tau)^{-1} \to \operatorname{Id}$ *as* $\tau \to 0^+$ *uniformly on* $\mathbb{R}^N$,
3. $\left|(\Phi_\tau)^{-1}(x) - (\Phi_\tau)^{-1}(y)\right| \le \dfrac{|x-y|}{1-\tau\,|\phi|_1}$.

Lemma 9 *Let $u \in C_0^{0,s}(\overline{\Omega})$ be a nonnegative function extended as zero outside Ω. There exists a sequence of nonnegative functions $\{u_k\}_{k\in\mathbf{N}} \subset C_0^{0,s}(\overline{\Omega}) \cap W_0^{s,p}(\Omega)$, for all $p > 1$, converging uniformly to u in $\overline{\Omega}$ and such that*

$$\limsup_{k\to\infty} |u_k|_s \le |u|_s\,.$$

Proof For each $k \in \mathbb{N}$ let Ψ_k denote the inverse of $\Phi_{1/k}$, given by Lemma 8, and set

$$\Omega_k := \Phi_{1/k}(\overline{\Omega}).$$

Since $\Omega_k \subset\subset \Omega$ there exists U_k, a subdomain of Ω, such that

$$\overline{\Omega_k} \subset U_k \subset \overline{U_k} \subset \Omega.$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be a standard convolution kernel: $\eta(z) > 0$ if $|z| < 1$, $\eta(z) = 0$ if $|z| \ge 1$ and $\int_{|z|\le 1} \phi(z)\mathrm{d}z = 1$.

Define the function

$$u_k = (u \circ \Psi_k) * \eta_k \in C^\infty(\mathbb{R}^N),$$

where

$$\eta_k(x) := (\epsilon_k)^{-N}\eta\Big(\frac{x}{\epsilon_k}\Big), \quad x \in \mathbb{R}^N$$

and $\epsilon_k < \operatorname{dist}(\Omega_k, \partial U_k)$. Note that $\epsilon_k \to 0$.

Since

$$B(x,\epsilon_k) \subset \mathbb{R}^N\backslash\Omega_k \quad \forall x \in \mathbb{R}^N\backslash U_k,$$

we have

$$\Psi_k(B(x,\epsilon_k)) \subset \mathbb{R}^N\backslash\Omega \quad \forall x \in \mathbb{R}^N\backslash U_k.$$

Hence, observing that

$$u_k(x) = \int_{\mathbb{R}^N} \eta_k(x-z)u(\Psi_k(z))\mathrm{d}z = \int_{B(0,1)} \eta(z)u(\psi_k(x-\epsilon_k z))\mathrm{d}z \quad \forall x \in \mathbb{R}^N$$

and that

$$|x - \epsilon_k z - x| \le \epsilon_k \quad \forall z \in B(0,1)$$

we conclude that

$$u_k(x) = 0 \quad \forall x \in \mathbb{R}^N\backslash U_k.$$

Therefore, $u_k \in C_c^\infty(\Omega) \subset W_0^{1,p}(\Omega)$ for all $p > 1$.

Now, let $x, y \in \overline{\Omega}$ be fixed. According to item 3 of Lemma 8,

$$
\begin{aligned}
|u_k(x) - u_k(y)| &\le \int_{B(0,1)} \eta(z)\, |u(\Psi_k(x - \epsilon_k z)) - u(\Psi_k(y - \epsilon_k z))|\, dz \\
&\le |u|_s \int_{B(0,1)} \eta(z)\, |\Psi_k(x - \epsilon_k z) - \Psi_k(y - \epsilon_k z))|^s\, dz \\
&\le \frac{|u|_s}{(1 - (1/k)\, |\phi|_1)^s} \int_{B(0,1)} \eta(z)\, |x - y|^s\, dz \\
&= \frac{|u|_s}{(1 - (1/k)\, |\phi|_1)^s}\, |x - y|^s .
\end{aligned}
$$

It follows that $u_k \in C_0^{0,s}(\overline{\Omega})$ and

$$
\limsup_{k\to\infty} |u_k|_s \le \lim_{k\to\infty} \frac{|u|_s}{(1 - (1/k)\, |\phi|_1)^s} = |u|_s .
$$

Consequently, up to a subsequence, $u_k \to \widetilde{u} \in C(\overline{\Omega})$ uniformly in $\overline{\Omega}$. Hence, $\widetilde{u} = u$ since item 2 of Lemma 8 implies that

$$
\lim_{k\to\infty} u_k(x) = \int_{B(0,1)} \eta(z) u(\lim_{k\to\infty} \Psi_k(x - \epsilon_k z)) dz = u(x) \int_{B(0,1)} \eta(z) dz = u(x).
$$

□

Theorem 10 *Let $p_n \to \infty$. Up to a subsequence, $\{u_{p_n}\}_{n\in\mathbb{N}}$ converges uniformly to a nonnegative function $u_\infty \in C_0^{0,s}(\overline{\Omega})$ such that*

$$
|u_\infty|_s = \lim_{n\to\infty} \sqrt[p_n]{\Lambda_{p_n}}.
$$

Furthermore,

$$
v_s = (k_\infty)^{-1} u_\infty \tag{22}
$$

where

$$
k_\infty := k(u_\infty) = \exp\left(\int_\Omega (\log |u_\infty|) \omega dx \right) \ge 1. \tag{23}
$$

Proof Let $p_0 > \dfrac{N}{s}$ be fixed and take $\beta_0 = s - \frac{N}{p_0}$. For each $(x, y) \in \Omega \times \Omega$, with $x \neq y$, we obtain from (21)

$$
\begin{aligned}
\frac{|u_p(x) - u_p(y)|}{|x - y|^{s - \frac{N}{p_0}}} &= \frac{|u_p(x) - u_p(y)|}{|x - y|^{s - \frac{N}{p}}}\, |x - y|^{N(\frac{1}{p_0} - \frac{1}{p})} \\
&\le C\, [u_p]_{s,p}\, \text{diam}(\Omega)^{N(\frac{1}{p_0} - \frac{1}{p})}, \quad \forall\, p \ge p_0,
\end{aligned}
$$

where C is uniform with respect to p and $\text{diam}(\Omega)$ is the diameter of Ω. Hence, in view of (13) and (12) the family $\{u_p\}_{p\ge p_0}$ is bounded in $C_0^{0,\beta_0}(\overline{\Omega})$, implying that, up to a subsequence, $u_{p_n} \to u_\infty \in C(\overline{\Omega})$ uniformly in $\overline{\Omega}$. Of course, the limit function u_∞ is nonnegative in Ω and vanishes on $\partial\Omega$.

Letting $n \to \infty$ in the inequality (which follows from (21))

$$
\frac{|u_{p_n}(x) - u_{p_n}(y)|}{|x - y|^{s - \frac{N}{p_n}}} \le C\, [u_{p_n}]_{s,p_n} = C\, \sqrt[p_n]{\Lambda_{p_n}}
$$

and taking (12) into account, we conclude that $u_\infty \in C_0^{0,s}(\overline{\Omega})$.

Up to another subsequence, we can assume that

$$\sqrt[p_n]{\Lambda_{p_n}} \to L.$$

Let $q > \dfrac{N}{s}$ be fixed. By Fatou's Lemma and Hölder's inequality,

$$\begin{aligned}
&\int_\Omega \int_\Omega \left(\frac{|u_\infty(x) - u_\infty(y)|}{|x-y|^s} \right)^q \mathrm{d}x\mathrm{d}y \\
&\leq \liminf_{n\to\infty} \int_\Omega \int_\Omega \left(\frac{\left|u_{p_n}(x) - u_{p_n}(y)\right|}{|x-y|^{\frac{N}{p_n}+s}} \right)^q \mathrm{d}x\mathrm{d}y \\
&\leq \liminf_{n\to\infty} |\Omega|^{2(1-\frac{q}{p_n})} \left(\int_\Omega \int_\Omega \left(\frac{\left|u_{p_n}(x) - u_{p_n}(y)\right|}{|x-y|^{\frac{N}{p_n}+s}} \right)^{p_n} \mathrm{d}x\mathrm{d}y \right)^{\frac{q}{p_n}} \\
&\leq |\Omega|^2 \liminf_{n\to\infty} \left[u_{p_n}\right]_{s,p_n}^q = |\Omega|^2 \lim_{n\to\infty} (\sqrt[p_n]{\Lambda_{p_n}})^q = |\Omega|^2 L^q.
\end{aligned}$$

Therefore,

$$|u_\infty|_s = \lim_{q\to\infty} \left(\int_\Omega \int_\Omega \left(\frac{|u_\infty(x) - u_\infty(y)|}{|x-y|^s} \right)^q \mathrm{d}x\mathrm{d}y \right)^{1/q} \leq \lim_{q\to\infty} |\Omega|^{\frac{2}{q}} L = L. \tag{24}$$

To prove that $k_\infty \geq 1$, we first note that

$$\lim_{t\to 0^+} \left(\int_\Omega \left|u_{p_n}\right|^t \omega\mathrm{d}x \right)^{\frac{1}{t}} = \inf_{0<t<1} \left(\int_\Omega \left|u_{p_n}\right|^t \omega\mathrm{d}x \right)^{\frac{1}{t}} \leq \left(\int_\Omega \left|u_{p_n}\right|^\epsilon \omega\mathrm{d}x \right)^{\frac{1}{\epsilon}} \quad \forall \epsilon \in (0,1).$$

Consequently,

$$1 = k(u_{p_n}) = \lim_{t\to 0^+} \left(\int_\Omega \left|u_{p_n}\right|^t \omega\mathrm{d}x \right)^{\frac{1}{t}} \leq \left(\int_\Omega \left|u_{p_n}\right|^\epsilon \omega\mathrm{d}x \right)^{\frac{1}{\epsilon}}.$$

The uniform convergence $u_{p_n} \to u_\infty$ then yields

$$1 \leq \lim_{n\to\infty} \left(\int_\Omega \left|u_{p_n}\right|^\epsilon \omega\mathrm{d}x \right)^{\frac{1}{\epsilon}} = \left(\int_\Omega |u_\infty|^\epsilon \omega\mathrm{d}x \right)^{\frac{1}{\epsilon}}.$$

Therefore,

$$k_\infty = k(u_\infty) = \lim_{\epsilon\to 0^+} \left(\int_\Omega |u_\infty|^\epsilon \omega\mathrm{d}x \right)^{\frac{1}{\epsilon}} \geq 1.$$

It follows that $(k_\infty)^{-1}u_\infty \in \mathcal{M}_s$, so that

$$\mu_s \leq \left|(k_\infty)^{-1}u_\infty\right|_s = (k_\infty)^{-1} |u_\infty|_s. \tag{25}$$

In the next step, we prove that

$$\int_\Omega \frac{u}{u_\infty} \omega\mathrm{d}x \leq \frac{|u|_s}{L} \quad \forall u \in C_0^{0,s}(\overline{\Omega}). \tag{26}$$

According to Lemma 9, there exists a sequence of nonnegative functions $\{u_k\}_{k\in\mathbf{N}} \subset C_0^{0,s}(\overline{\Omega}) \cap W_0^{s,p}(\Omega)$, for all $p > 1$, converging uniformly to u in $C(\overline{\Omega})$ and such that

$$\limsup_{k\to\infty} |u_k|_s \leq |u|_s.$$

Since u_p is the weak solution of (3) and $\Lambda_p = [u_p]_{s,p}^p$, we use Hölder's inequality to get

$$\Lambda_p \int_\Omega \frac{u_k}{u_p}\omega \mathrm{d}x = \langle (-\Delta_p)^s u_p, u_k \rangle \leq [u_p]_{s,p}^{p-1} [u_k]_{s,p} = (\Lambda_p)^{\frac{p-1}{p}} [u_k]_{s,p} .$$

It follows that

$$\sqrt[p_n]{\Lambda_{p_n}} \int_\Omega \frac{u_k}{u_{p_n}}\omega \mathrm{d}x \leq [u_k]_{s,p_n} .$$

Combining Fatou's lemma with the uniform convergence $u_{p_n} \to u_\infty$ and Lemma 7, we obtain

$$L \int_\Omega \frac{u_k}{u_\infty}\omega \mathrm{d}x \leq L \liminf_{n\to\infty} \int_\Omega \frac{u_k}{u_{p_n}}\omega \mathrm{d}x \leq \liminf_{n\to\infty} [u_k]_{s,p_n} = |u_k|_s ,$$

that is,

$$L \int_\Omega \frac{u_k}{u_\infty}\omega \mathrm{d}x \leq |u_k|_s .$$

Letting $k \to \infty$ and applying Fatou's lemma again, we arrive at (26):

$$L \int_\Omega \frac{u}{u_\infty}\omega \mathrm{d}x \leq L \liminf_{k\to\infty} \int_\Omega \frac{u_k}{u_\infty}\omega \mathrm{d}x \leq \liminf_{k\to\infty} |u_k|_s \leq |u|_s .$$

Taking $u = u_\infty$ in (26), we obtain

$$L \leq |u_\infty|_s$$

and combining this with (24) we conclude that

$$L = |u_\infty|_s . \tag{27}$$

Now, let $0 \leq u \in \mathcal{M}_s$ be fixed. Then (16) yields

$$\begin{aligned} -\int_\Omega (\log u_\infty)\omega \mathrm{d}x &= \int_\Omega (\log u)\omega \mathrm{d}x - \int_\Omega (\log u_\infty)\omega \mathrm{d}x \\ &= \int_\Omega (\log(\frac{u}{u_\infty}))\omega \mathrm{d}x \leq \log\left(\int_\Omega \frac{u}{u_\infty}\omega \mathrm{d}x\right). \end{aligned}$$

Hence, (26) and (27) imply that

$$(k_\infty)^{-1} \leq \int_\Omega \frac{u}{u_\infty}\omega \mathrm{d}x \leq \frac{|u|_s}{|u_\infty|_s} \quad \text{whenever} \quad 0 \leq u \in \mathcal{M}_s. \tag{28}$$

Combining these estimates at $u = v_s$ with (25), we obtain

$$(k_\infty)^{-1} \leq \int_\Omega \frac{v_s}{u_\infty}\omega \mathrm{d}x \leq \frac{|v_s|_s}{|u_\infty|_s} = \frac{\mu_s}{|u_\infty|_s} \leq (k_\infty)^{-1},$$

which leads us to conclude that

$$\mu_s = \left|(k_\infty)^{-1} u_\infty\right|_s \quad \text{and} \quad (k_\infty)^{-1} = \int_\Omega \frac{v_s}{u_\infty}\omega \mathrm{d}x.$$

Since v_s is the only nonnegative minimizer of $|\cdot|_s$ on $\mathcal{M}_s$, we get (22). □

Corollary 11 *The following inequalities hold*

$$k(u) \leq \int_\Omega \frac{|u|}{v_s}\omega \mathrm{d}x \leq \frac{|u|_s}{\mu_s} \quad \forall\, u \in C_0^{0,s}(\overline{\Omega}). \tag{29}$$

Proof Since we already know that $L = |u_\infty|_s$ and $u_\infty = k_\infty v_s$, the second inequality in (29) follows from (26), with u replaced with $w = |u|$ (note that $|w|_s \le |u|_s$). The first inequality in (29) is obvious when $k(u) = 0$ and, when $k(u) > 0$, it follows from the first inequality in (28), with $w = (k(u))^{-1}|u| \in \mathcal{M}_s$. □

Remark 12 In contrast with what happens in similar problems driven by the standard p-Laplacian, we are not able to prove that $u_\infty \in W_0^{s,q}(\Omega)$ for some $q > 1$. Such a property would guarantee that $u_\infty = v_s$ and, consequently,

$$\lim_{p\to\infty} u_p = v_s$$

(that is, v_s would be the only limit point of the family $\{u_p\}_{p>1}$, as $p \to \infty$). Indeed, if $u_\infty \in W_0^{s,q}(\Omega)$ for some $q > 1$ then, according to Lemma 7, $u_\infty \in W_0^{s,p_n}(\Omega)$ for all n sufficiently large (such that $p_n \ge q$) and

$$\lim_{n\to\infty} [u_\infty]_{s,p_n} = |u_\infty|_s\,.$$

Hence, proceeding as in the proof of Theorem 10, we would arrive at

$$1 \le k_\infty \le \int_\Omega \frac{u_\infty}{u_{p_n}} \omega \mathrm{d}x \le \frac{[u_\infty]_{s,p_n}}{\sqrt[p_n]{\Lambda_{p_n}}}.$$

Since $\lim_{n\to\infty}[u_\infty]_{s,p_n} = \lim_{n\to\infty}\sqrt[p_n]{\Lambda_{p_n}} = |u_\infty|_s$ we would conclude that $k_\infty = 1$ and $u_\infty = v_s$.

5 The limit problem

For a matter of compatibility with the viscosity approach, we add the hypotheses of continuity and strict positiveness to the weight ω. So, we assume in this section that

$$\omega \in C(\Omega) \cap L^r(\Omega),\ r > 1, \quad \omega > 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \int_\Omega \omega \mathrm{d}x = 1.$$

Note that such ω satisfies the hypotheses of Theorem 3.

For $1 < p < \infty$ we write the s-fractional p-Laplacian, in its integral version, as $(-\Delta_p)^s = -\mathcal{L}_p$ where

$$(\mathcal{L}_p u)(x) := 2\int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|y - x|^{N+sp}} \mathrm{d}y. \tag{30}$$

Corresponding to the case $p = \infty$, we define operator $\mathcal{L}_\infty$ by

$$\mathcal{L}_\infty := \mathcal{L}_\infty^+ + \mathcal{L}_\infty^-, \tag{31}$$

where

$$\left(\mathcal{L}_\infty^+ u\right)(x) := \sup_{y\in\mathbb{R}^N\setminus\{x\}} \frac{u(y) - u(x)}{|y - x|^s} \quad \text{and} \quad \left(\mathcal{L}_\infty^- u\right)(x) := \inf_{y\in\mathbb{R}^N\setminus\{x\}} \frac{u(y) - u(x)}{|y - x|^s}. \tag{32}$$

In the sequel, we consider, in the viscosity sense, the problem

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{33}$$

where either $\mathcal{L}u = \mathcal{L}_p u + \Lambda_p u^{-1}\omega$, with $1 < p < \infty$, or

$$\mathcal{L}u = \mathcal{L}_\infty u \quad \text{or} \quad \mathcal{L}u = \mathcal{L}_\infty^- u + |u_\infty|_s \,.$$

We recall some definitions related to the viscosity approach for the problem (33).

Definition 13 Let $u \in C(\mathbb{R}^N)$ such that $u > 0$ in Ω and $u = 0$ in $\mathbb{R}^N \backslash \Omega$. We say that u is a viscosity supersolution of Eq. (33) if

$$(\mathcal{L}\varphi)(x_0) \le 0$$

for all pair $(x_0, \varphi) \in \Omega \times C_0^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \le u(x) \quad \forall\, x \in \mathbb{R}^N.$$

Analogously, we say that u is a viscosity subsolution of (33) if

$$(\mathcal{L}\varphi)(x_0) \ge 0$$

for all pair $(x_0, \varphi) \in \Omega \times C_0^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \ge u(x) \quad \forall\, x \in \mathbb{R}^N.$$

We say that u is a viscosity solution of (33) if it is simultaneously a subsolution and a supersolution of (33).

The next lemma can be proved by following, step by step, the proof of Proposition 11 of [17].

Lemma 14 *Let $u \in W_0^{s,p}(\Omega) \cap C(\overline{\Omega})$ be a positive weak solution of (3). Then u is a viscosity solution of*

$$\begin{cases} \mathcal{L}_p u + \Lambda_p u^{-1}\omega = 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{34}$$

Our main result in this section is the following, where $u_\infty \in C_0^{0,s}(\overline{\Omega})$ is the function given by Theorem 10.

Theorem 15 *The function $u_\infty \in C_0^{0,s}(\overline{\Omega})$, extended as zero outside Ω, is both a viscosity supersolution of the problem*

$$\begin{cases} \mathcal{L}_\infty u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega \end{cases} \tag{35}$$

and a viscosity solution of the problem

$$\begin{cases} \mathcal{L}_\infty^- u + |u_\infty|_s = 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{36}$$

Moreover, u_∞ is strictly positive in Ω and the only minimizers of $|\cdot|_s$ on $\mathcal{M}_s$ are

$$-v_s \quad \text{and} \quad v_s. \tag{37}$$

Proof We begin by proving that u_∞ is a viscosity supersolution of (36). For this, let us fix $(x_0, \varphi) \in \Omega \times C_0^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = u_\infty(x_0) \quad \text{and} \quad \varphi(x) \le u_\infty(x) \quad \forall\, x \in \mathbb{R}^N. \tag{38}$$

Without loss of generality, we can assume that

$$\varphi(x) < u_\infty(x) \quad \forall\, x \in \mathbb{R}^N,$$

what allows us to assure that $u_{p_n} - \varphi$ assumes its minimum value at a point x_n, with $x_n \to x_0$.

Let $c_n := u_{p_n}(x_n) - \varphi(x_n)$. Of course, $c_n \to 0$ (due to the uniform convergence $u_{p_n} \to u_\infty$). By construction,

$$\varphi(x_n) + c_n = u_{p_n}(x_n) \quad \text{and} \quad \varphi(x) + c_n \le u_{p_n}(x) \quad \forall\, x \in \mathbb{R}^N.$$

According to the previous lemma, u_p is a viscosity supersolution of (34) since it is a viscosity solution of the same problem. Therefore,

$$(\mathcal{L}_{p_n}\varphi)(x_n) + \Lambda_{p_n}\frac{\omega(x_n)}{u_{p_n}(x_n)} = (\mathcal{L}_{p_n}(\varphi + c_n))(x_n) + \Lambda_{p_n}\frac{\omega(x_n)}{\varphi(x_n) + c_n} \le 0,$$

an inequality that can be rewritten as

$$A_n^{p_n-1} + C_n^{p_n-1} \le B_n^{p_n-1}$$

where

$$A_n^{p_n-1} = 2\int_{\mathbb{R}^N} \frac{|\varphi(y) - \varphi(x_n)|^{p_n-2}\,(\varphi(y) - \varphi(x_n))^+}{|y - x|^{N+sp_n}}\mathrm{d}y \ge 0,$$

$$B_n^{p_n-1} = 2\int_{\mathbb{R}^N} \frac{|\varphi(y) - \varphi(x_n)|^{p_n-2}\,(\varphi(y) - \varphi(x_n))^-}{|y - x|^{N+sp_n}}\mathrm{d}y \ge 0,$$

and

$$C_n^{p_n-1} = \Lambda_{p_n}\frac{\omega(x_n)}{u_{p_n}(x_n)} > 0.$$

(Here, $a^+ := \max\{a, 0\}$ and $a^- := \max\{-a, 0\}$, so that $a = a^+ - a^-$.)

According to Lemma 6.1 of [13], which was adapted from Lemma 6.5 of [3], we have

$$\lim_{n\to\infty} A_n = \left(\mathcal{L}_\infty^+\varphi\right)(x_0) \quad \text{and} \quad \lim_{n\to\infty} B_n = -\left(\mathcal{L}_\infty^-\varphi\right)(x_0).$$

Hence, noticing that

$$A_n^{p_n-1} \le A_n^{p_n-1} + C_n^{p_n-1} \le B_n^{p_n-1}$$

we conclude that

$$(\mathcal{L}_\infty\varphi)(x_0) = \left(\mathcal{L}_\infty^+\varphi\right)(x_0) + \left(\mathcal{L}_\infty^-\varphi\right)(x_0) \le 0$$

since

$$\left(\mathcal{L}_\infty^+\varphi\right)(x_0) = \lim_{n\to\infty} A_n \le \lim_{n\to\infty} B_n = -\left(\mathcal{L}_\infty^-\varphi\right)(x_0).$$

We have proved that u_∞ is a supersolution of (35). Therefore, by directly applying Lemma 22 of [17] we conclude $u_\infty > 0$ in Ω.

The strict positiveness of u_∞ in Ω and the uniqueness of the nonnegative minimizers of $|\cdot|_s$ on $\mathcal{M}_s$ imply that if $w \in \mathcal{M}_s$ is such that

$$|w|_s = \min_{u\in\mathcal{M}_s} |u|_s$$

then $|w| = v_s = (k_\infty)^{-1} u_\infty > 0$ in Ω (recall that $|w|$ is also a minimizer). The continuity of w then implies that either $w > 0$ in Ω or $w < 0$ in Ω. Consequently, $w = v_s$ or $w = -v_s$.

Now, recalling that

$$\lim_{n\to\infty} (\Lambda_{p_n})^{\frac{1}{p_n-1}} = |u_\infty|_s$$

and using that $\omega(x_0) > 0$ and $u_\infty(x_0) > 0$ we have

$$\lim_{n\to\infty} C_n = |u_\infty|_s$$

Hence, since

$$C_n^{p_n-1} \leq A_n^{p_n-1} + C_n^{p_n-1} \leq B_n^{p_n-1},$$

we obtain

$$|u_\infty|_s = \lim_{n\to\infty} C_n \leq \lim_{n\to\infty} B_n = -\left(\mathcal{L}_\infty^- \varphi\right)(x_0).$$

It follows that u_∞ is a viscosity supersolution of (36).

Now, let us take a pair $(x_0, \varphi) \in \Omega \times C_0^1(\mathbb{R}^N)$ satisfying

$$\varphi(x_0) = u_\infty(x_0) \quad \text{and} \quad \varphi(x) \geq u_\infty(x) \quad \forall\, x \in \mathbb{R}^N. \tag{39}$$

Since

$$-|u_\infty|_s \leq \frac{u_\infty(x) - u_\infty(x_0)}{|x - x_0|^s} \leq \frac{\varphi(x) - \varphi(x_0)}{|x - x_0|^s} \quad \forall\, x \in \mathbb{R}^N \setminus \{x_0\},$$

we have

$$-|u_\infty|_s \leq \inf_{x \in \mathbb{R}^N \setminus \{x_0\}} \frac{\varphi(x) - \varphi(x_0)}{|x - x_0|^s} = \left(\mathcal{L}_\infty^- \varphi\right)(x_0).$$

Therefore, u_∞ is a viscosity subsolution of (36). □

Since $v_s = (k_\infty)^{-1} u_\infty$ is the only positive minimizer of $|\cdot|_s$ on $C_0^{0,s}(\overline{\Omega}) \setminus \{0\}$ and $\mathcal{L}_\infty^-(ku) = k\mathcal{L}_\infty^- u$ for any positive constant k, the following corollary is immediate.

Corollary 16 *The minimizer v_s is a viscosity solution of the problem*

$$\begin{cases} \mathcal{L}_\infty^- u + \mu_s = 0 \text{ in } \Omega \\ u = 0 \qquad\qquad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

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