Pluripotential theory on the support of closed positive currents and applications to dynamics in $\mathbb{C}^n$

Frédéric Protin¹

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Abstract

We extend certain classical theorems in pluripotential theory to a class of functions defined on the support of a $(1, 1)$ -closed positive current T , analogous to plurisubharmonic functions, called T -plurisubharmonic functions. These functions are defined as limits, on the support of T , of sequences of plurisubharmonic functions decreasing on this support. We study these functions by means of a class of measures, so-called pluri-Jensen measures, which prove to be numerous on the support of $(1, 1)$ -closed positive currents. For any fat compact set, we obtain an expression of its relative Green's function in terms of pluri-Jensen measures and deduce a characterization of the polynomially convex fat compact sets and of pluripolar sets. These tools are then used to study dynamics of a class of automorphisms of $\mathbb{C}^n$ which naturally generalize Hénon's automorphisms. We obtain an equidistribution result for the convergence of pull-back of certain measures toward an ergodic invariant measure with compact support.

Keywords Pluricomplex Green's function · Closed positive current · Pluripolar set · Polynomial convexity · Complex dynamics · Jensen measure

Mathematics Subject Classification 32U15 · 32E20 · 32U35 · 32U40 · 32U05 · 32H50

1 Introduction

The main goal of this article is to develop a pluripotential theory for T -plurisubharmonic functions (abbreviated T -psh), i.e., functions that are limits, on the support of a $(1, 1)$ -closed positive current T , of sequences of plurisubharmonic functions decreasing on this support. The T -psh functions were introduced in [3] (see also [11]), and a more general class of functions, called plurisubharmonic in the direction of the current T , is considered in [3] and [27] from a different point of view. T -psh functions naturally arise in complex dynamics, as in [22], or in [7] and [5] where they appear with a different name. The second goal of this article is to apply these results to the study of the dynamics of a certain class of automorphisms of $\mathbb{C}^n$

✉ Frédéric Protin
fredprotin@yahoo.fr

¹ INSA de Toulouse, Institut de Mathématiques de Toulouse, 135 avenue de Rangueil, 31400 Toulouse, France

which generalizes Hénon's automorphisms of $\mathbb{C}^2$, that we call Hénon-like automorphisms. The article is structured as follows.

In Sect. 2, we develop a pluripotential theory for continuous T -psh functions. We proceed as follows. In Sect. 2.1, we give a more intrinsic characterization of such functions. More precisely, we prove that a continuous function $u : \text{supp}(T) \rightarrow \mathbb{R}$ is T -psh if and only if it satisfies $u(x) \leq \int_{\text{supp}(T)} u \, d\mu_x$ for every x in the support $\text{supp}(T)$ of T and every pluri-Jensen measure μ_x relative to x and with support in $\text{supp}(T)$. We have equality in the previous relation when we take the infimum over the set of all such pluri-Jensen measures. We also prove that there are sufficiently many such measures in order to construct a potential theory, analogous to the one for plurisubharmonic functions. We deduce some results, as for example a maximum principle for T -psh functions, and the fact that a support of a closed positive $(1, 1)$ -current is nowhere pluri-thin.

In Sect. 2.2, for an open connected bounded hyperconvex set $U \subset \mathbb{C}^n$, we express the value at any $x \in U$ of the Green's function relative to U of a fat compact set $A \subset U$ as the infimum of $-\mu_x(\hat{A})$ over all pluri-Jensen measures μ_x relative to x with support in U . We deduce a characterization of polynomially convex fat compact sets and of pluripolar sets in terms of pluri-Jensen measures. We further prove that the poles of T -psh functions form a pluripolar set.

In Sect. 2.3, we pursue the analogy between T -psh functions and plurisubharmonic functions. In particular, we obtain a Hartogs-like theorem, an extension property across certain small sets, and a Chern–Levine–Nirenberg inequality. We also prove that, under minor conditions, the limit superior of a locally uniformly bounded from above sequence of T -psh functions is a T -psh function, except on a pluripolar set.

In Sect. 3, we apply the previous results to the study of the dynamics of Hénon-like automorphisms. In Sect. 3.2, we give an equidistribution lemma for the pull-backs of certain closed currents by a Hénon-like automorphism, that we further exploit to obtain information about its Julia set.

In Sect. 3.3, we give an equidistribution result for the pull-backs of certain measures by a Hénon-like automorphism. More precisely, we prove that the pull-backs of a probability measure defined on the support of certain currents and which does not charge pluripolar sets, converge in a Cesàro sense toward an unique canonical invariant measure. This measure is ergodic, with compact support.

Notations. We will denote $d := \partial + \bar{\partial}$ the usual exterior derivative and $d^c := \frac{1}{2i\pi}(\partial - \bar{\partial})$. Thus, $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. These operators will be understood in the sense of currents.

A continuous function u from an open set of $\mathbb{C}^n$ into $\mathbb{R}$ is pluriharmonic (harmonic if $n = 1$) if and only if $dd^c u = 0$. We denote by $\|S\| := S \wedge \omega^{n-1}$ the trace measure of a closed positive $(1, 1)$ -current S on $\mathbb{C}^n$, where $\omega := \frac{1}{2}dd^c \log(1 + \|z\|^2)$ is the Fubini–Study form. Then, the mass of the current S is $\|S\|(\mathbb{C}^n)$.

2 Study of T -plurisubharmonic functions

2.1 Fundamental results on T -psh functions

In the sequel, we denote by $\hat{A}$ the polynomially convex envelope of a compact A . Following [11], we define the T -psh functions as follows:

Definition 1 Given a closed positive $(1, 1)$ -current T defined in an open set $V \subset \mathbb{C}^n$ with support $\mathcal{S} \subset V$, a function $u : \mathcal{S} \rightarrow \mathbb{R} \cup \{-\infty\}$ is called T -plurisubharmonic (abbreviated

T -psh) if it is locally $\|T\|$ -integrable and if there exists an open set $U \subset V$ containing S , and a sequence of plurisubharmonic functions $u_m : U \rightarrow \mathbb{R}$ decreasing on S , such that $(u_m|_S)_m$ converges simply toward u on $S \cap U$. Up to shrinking U , we may suppose that the functions u_m are continuous.

More generally, if we replace S by a compact $K \subset \mathbb{C}^n$ in the previous definition, the function $u : K \rightarrow \mathbb{R} \cup \{-\infty\}$ will be said *K-plurisubharmonic*.

The elements of a family $(v_m)_{m \in \mathbb{N}}$ of T -psh functions will be said *co-T-psh* if for every $m \in \mathbb{N}$, there exists an open set $U \subset V$ containing S , and a sequence of plurisubharmonic functions $(v_{m,i})_i : U \rightarrow \mathbb{R}$ decreasing on $S \cap U$ toward v_m .

The current $T \wedge dd^c u := dd^c(uT)$ is then well defined, positive and closed (see Lemma 4–13 in [11]).

The main purpose of this section is to present a more intrinsic characterization of continuous T -psh functions. First let us recall some definitions and notations.

A *pluri-Jensen measure* μ_x relative to $x \in \mathbb{C}^n$ is a regular probability measure with compact support such that, for any plurisubharmonic function v defined on an open set U containing x , we have

$$v(x) \leq \int_U v d\mu_x. \quad (1)$$

If v is only assumed subharmonic, μ_x is a *Jensen measure relative to x*. (Therefore, it is also a pluri-Jensen measure.) The reader should pay attention to the fact that a pluri-Jensen measure in our terminology is called Jensen measure in [28,29], and [17].

Remark 1 It is sufficient to suppose that (1) holds when v is a continuous plurisubharmonic function in order to ensure that μ_x is a pluri-Jensen measure, thanks to the monotone convergence theorem, since a plurisubharmonic function is a limit of a decreasing sequence of continuous plurisubharmonic functions in a neighborhood of the support of μ_x . It is even sufficient to suppose that v is of the form $\log |f|$ for some holomorphic function f , since a classical theorem of Bremermann ensures that a continuous plurisubharmonic function can be approximated locally uniformly by functions of the form $\max_{i=1,\dots,m} c_i \log |f_i|$, $c_i > 0$, where f_i are holomorphic functions.

Let $B \subset \mathbb{C}^n$. We denote by $PJ_B(x)$ the set of pluri-Jensen measures relative to x with support in B . Note that $\delta_x \in PJ_B(x)$ if $x \in B$. Moreover, for a given compact set $A \subset \mathbb{C}^n$, for any $x \in \hat{A}$, there exists a measure $\mu_x \in PJ_A(x)$. (This follows from Theorem 1.2.9, Theorem 1.2.14 and Remark 1.2.18 in [28], or even directly from Theorem 2.2.5 in [28], see also [8].) Note also that an automorphism $Q : \mathbb{C}^n \rightarrow \mathbb{C}^n$ induces a bijection $Q_* : PJ_{\mathbb{C}^n}(x) \rightarrow PJ_{\mathbb{C}^n}(Q(x))$.

The following lemma ensures the existence of sufficiently many pluri-Jensen measures on the support of a closed positive current, allowing us to establish a potential theory on it.

Lemma 1 *Let T be a closed positive $(1, 1)$ -current in $\mathbb{C}^n$, of support S . Then, $\forall r > 0$, $\forall x \in S$, there exists a pluri-Jensen measure relative to x with support in $\partial B(x, r) \cap S$, where $B(x, r)$ denotes the open ball of center x and radius r .*

Proof Let $y \in \partial B(x, r) \cap S$. Then, y is a *peak point* for the uniform algebra $\mathcal{A}$ of the functions on $\overline{B}(x, r) \cap S$ with complex values that are restrictions of holomorphic functions defined in a neighborhood of $\overline{B}(x, r)$: in other words, there exists $f \in \mathcal{A}$ such that $f(y) = 1$ and $|f(z)| < 1$ for $z \in (B(x, r) \cap S) \setminus \{y\}$.

Indeed, assuming that x is the origin, the restriction of the function $f(z) = \frac{1}{2}(1 + \langle z, \frac{y}{\|y\|} \rangle)$ to $\overline{B}(x, r) \cap \mathcal{S}$ fulfills these conditions. ($\langle \cdot, \cdot \rangle$ denotes here the canonical hermitian scalar product on $\mathbb{C}^n$.)

In particular, y is a *strong boundary point* for $\mathcal{A}$, i.e., for any neighborhood V of y for the topology induced on $\partial B(x, r) \cap \mathcal{S}$, there exists $f \in \mathcal{A}$ such that $\sup_V |f| = f(y) = 1$ and $|f(z)| < 1$ when $z \in (B(x, r) \cap \mathcal{S}) \setminus V$.

On the other hand, there is no strong boundary point for $\mathcal{A}$ in $B(x, r) \cap \mathcal{S}$. Indeed, suppose that there exists such a point $z \in B(x, r) \cap \mathcal{S}$. The point z cannot be a peak point, since it follows from Proposition 2.5 in [6] that the algebra $\mathcal{A}$ cannot have local peak points in $B(x, r) \cap \mathcal{S}$. Thus, for any neighborhood V of y , a function $f \in \mathcal{A}$ such that $\sup_V |f| = f(y) = 1$ and $|f(z)| < 1$ for $z \in (B(x, r) \cap \mathcal{S}) \setminus \{y\}$ takes the value 1 on $V \setminus \{z\}$. By the Theorem of isolated zeros applied to the holomorphic function whose restriction is f , it follows that f is constant. Consequently, z cannot be a strong boundary point.

Therefore, the strong boundary points of $\mathcal{A}$ belong to $\partial B(x, r) \cap \mathcal{S}$. But by Corollary 7.24 in [29], the Shilov boundary Γ for $\mathcal{A}$ is the closure of the set of strong boundary points of $\mathcal{A}$, and thus $\Gamma \subset \partial B(x, r) \cap \mathcal{S}$. Then, by Lemma 6.2 in [1] and Remark 1, there exists a pluri-Jensen measure relative to x with support in Γ (see also Remark 1.2.18 in [28]). $\square$

Remark 2 Let T be a closed positive $(1, 1)$ -current in an open subset of $\mathbb{C}^n$, with support $\mathcal{S}$, and $u : \mathcal{S} \rightarrow \mathbb{R}$ a T -psh function. The monotone convergence theorem and Lemma 1 imply that $\forall x \in \mathcal{S}, \forall v_x \in PJ_{\mathcal{S}}(x)$, we have $u(x) \leq \int_{\mathcal{S}} u dv_x$. Since $\delta_x \in PJ_{\mathcal{S}}(x)$, we obtain

$$u(x) = \inf_{v_x \in PJ_{\mathcal{S}}(x)} \int_{\mathcal{S}} u dv_x. \quad (2)$$

We deduce a maximum principle for T -psh functions which improves Lemma 4.3 in [22] (see also Theorem 3.2 in [27]):

Corollary 1 Let T be a closed positive $(1, 1)$ -current in $\mathbb{C}^n$, of support $\mathcal{S}$, and $u : \mathcal{S} \rightarrow \mathbb{R}$ a T -psh function. Let $E \subset \mathcal{S}$ a bounded set, $x \in E$, and $r \in]0, +\infty[$ such that $E \subset \subset B(x, r)$. Then,

$$\sup_E u \leq \sup_{\partial B(x, r) \cap \mathcal{S}} u.$$

In particular, u does not have a strict local maximum.

Proof Let $x \in E$. By Lemma 1, we can find a pluri-Jensen measure μ_x with support in $\partial B(x, r) \cap \mathcal{S}$. Then, Remark 2 gives $u(x) \leq \int_{\partial B(x, r) \cap \mathcal{S}} u d\mu_x \leq \sup_{\partial B(x, r) \cap \mathcal{S}} u$. $\square$

Note that, by virtue of Proposition 4.8.2 in [18], the previous lemma implies that the support of a closed positive $(1, 1)$ -current is nowhere *pluri-thin*, in the terminology of [18].

For a subset $A \in \mathbb{C}^n$ and a function $w : A \rightarrow \mathbb{R}$, we will denote w^* its upper semi-continuous regularization, defined in A by $w^*(x) := \limsup_{y \rightarrow x, y \in A} w(y)$.

We deduce from Lemma 1 the following result:

Lemma 2 Let T be a closed positive current defined in an open set $U \subset \mathbb{C}^n$. Let $(u_n)_n$ a locally uniformly bounded from above sequence of continuous co- T -psh functions. Then, the function $\sup_n^* u_n$ is T -psh.

Moreover, given an open set $W \subset \mathbb{C}^n$ and a holomorphic map $f : W \rightarrow U$, we have

$$\left(\sup_n^* u_n \right) \circ f = \sup_n^* (u_n \circ f).$$

Proof Fix an open set $V \subset\subset U$. For each $n \in \mathbb{N}$, there exists a sequence $(v_{n,i})_i$ of continuous plurisubharmonic functions defined in V , decreasing toward u_n on $\mathcal{S} \cap V$. Thanks to Dini's Lemma, we may suppose $|v_{n,i} - u_n| < \frac{1}{n}$ on $\mathcal{S} \cap \bar{V}$ for all $n, i \in \mathbb{N}$. Then, the sequence of plurisubharmonic functions $(\sup_n^* v_{n,i})_i$ decreases on $\mathcal{S} \cap V$. (The upper regularization is done here in V .) We will prove that its limit on $\mathcal{S} \cap V$ is precisely $\sup_n^* (u_n|_{\mathcal{S} \cap V})$, which suffices to conclude.

For this purpose, it suffices to prove that $\forall i \in \mathbb{N}$, $(\sup_n^* v_{n,i})|_{\mathcal{S} \cap V} = \sup_n^* (v_{n,i}|_{\mathcal{S} \cap V})$. Suppose, by contradiction, that there exists $x_0 \in \mathcal{S} \cap V$ and $i \in \mathbb{N}$ such that $(\sup_n^* v_{n,i})|_{\mathcal{S} \cap V}(x_0) > \sup_n^* (v_{n,i}|_{\mathcal{S} \cap V})(x_0)$. Then, by Proposition 4.8.2 in [18], there exists a plurisubharmonic function $v : V \rightarrow \mathbb{R}$ such that $\limsup_{x \rightarrow x_0, x \in \mathcal{S} \cap V \setminus \{x_0\}} v(x) < v(x_0)$. But this contradicts Corollary 1 applied to the T -psh function $v|_{\mathcal{S} \cap V} : \mathcal{S} \cap V \rightarrow \mathbb{R}$.

An adaptation of the idea of the proof of Proposition 2.9.16 in [18] establishes the second statement, as follows. Denote by J_f the Jacobian of f . For all $i \in \mathbb{N}$, the functions $(\sup_n^* u_{n,i}) \circ f$ and $\sup_n^* (u_{n,i} \circ f)$ (the upper regularization is done here in W) are two plurisubharmonic functions defined in W which are equal in $W \setminus \{J_f = 0\}$, hence everywhere. The previous argument shows that they decrease on $\mathcal{S} \cap W$ when $i \rightarrow +\infty$, respectively, toward the functions $(\sup_n^* u_n) \circ f$ and $\sup_n^* (u_n \circ f)$, which are therefore equal. $\square$

The following lemma gives an alternative definition for continuous T -psh functions.

Lemma 3 Let T be a closed positive $(1, 1)$ -current with support $\mathcal{S}$ in an open set $U \subset \mathbb{C}^n$, and let $u : \mathcal{S} \rightarrow \mathbb{R}$ be a continuous function. Then, u is T -psh if and only if $u(x) \leq \int_U u d\Lambda_x$, $\forall x \in \mathcal{S}$ and $\forall \Lambda_x \in P J_{\mathcal{S}}(x)$.

Proof Necessary condition. Consider a sequence $(v_n)_n$ of plurisubharmonic functions defined in an open set containing $\mathcal{S}$, such that $(v_n)_n$ decreases toward u on $\mathcal{S}$. Let $x \in \mathcal{S}$ and $\Lambda_x \in P J_{\mathcal{S}}(x)$. The monotone convergence theorem gives

$$v(x) = \lim_{n \rightarrow +\infty} v_n(x) \leq \lim_{n \rightarrow +\infty} \int_U v_n d\Lambda_x = \int_U v d\Lambda_x. \quad (3)$$

Sufficient condition. Let $V \subset\subset U$ be an open set intersecting $\mathcal{S}$, and let $K := \mathcal{S} \cap \bar{V}$. First let us recall an abstract duality theorem due to Edwards [9]. Let Z be a compact metric space, and $\mathcal{R}$ a cone of continuous functions from Z to $\mathbb{R}$, such that $\mathcal{R}$ separates points and contains the constants. Given $x \in Z$, we denote by I_x the set of probability measures Λ on Z which satisfy $v(x) \leq \int_K v d\Lambda$ for all $v \in \mathcal{R}$. Let $\phi : Z \rightarrow \mathbb{R}$ be a continuous function. Then, Edwards' Theorem ensures that for all $x \in Z$,

$$\sup \{v(x) : v \in \mathcal{R}, v \leq \phi\} = \inf \left\{ \int_Z \phi d\Lambda : \Lambda \in I_x \right\}.$$

Now let $u : \mathcal{S} \rightarrow \mathbb{R}$ be a continuous function such that $\forall x \in \mathcal{S}, \forall \Lambda_x \in PJ_{\mathcal{S}}(x), u(x) \leq \int_U u d\Lambda_x$. Let also $(A_n)_n$ be a compact exhaustion of U , and fix $m \in \mathbb{N}$. Denote $K_m := A_m \cap \mathcal{S}$. We apply Edwards' theorem with $\mathcal{R}$ being the set of continuous K_m -psh functions, $Z = K_m$ and $\phi = u$. Note that $I_x = PJ_{K_m}(x)$. Indeed, (3) implies that $PJ_{K_m}(x) \subset I_x$, while the inverse inclusion is trivial. We have $\forall x \in K_m$,

$$\sup \{v(x) | v \in \mathcal{R}, v \leq u\} \leq u(x) \leq \inf \left\{ \int_{K_m} u d\Lambda | \Lambda \in I_x \right\}.$$

Then, Edwards' theorem gives $\forall x \in K_m$,

$$u(x) = \sup \{v(x) | v \in \mathcal{R}, v \leq u\} = \inf_{\Lambda_x \in PJ_{K_m}(x)} \int_{K_m} u d\Lambda_x.$$

We deduce that $u = u^* = \sup^* \{v | v \in \mathcal{R}, v \leq u\}$ on K_m . But thanks to Lemma 2.3.4 in [18] and our Lemma 2, $\sup^* \{v | v \in \mathcal{R}, v \leq u\}$ is T -psh in K_m . The result follows by taking the limit $m \rightarrow +\infty$. $\square$

2.2 Expression for relative Green's functions and applications

Denote by $\Delta \subset \mathbb{C}$ the unit disk of $\mathbb{C}$. Let $U \subset \mathbb{C}^n$ be an open set. Recall that an *analytic disk in U with center $x \in U$* is a continuous function $h : \overline{\Delta} \rightarrow \mathbb{C}^n$, analytic in Δ , such that $h(0) = x$. If λ is the normalized Lebesgue measure on $\partial\Delta$, an *analytic-disk measure on U for x* is a measure of the form $g_*\lambda$, where $g : \overline{\Delta} \rightarrow U$ is an analytic disk in U with center x .

We will denote by $\Lambda_U(x)$ the set of analytic disks which are pluri-Jensen measures relative to x with support in an open set U . By Theorem 6.3 in [25], $PJ_U(x) = \overline{\Lambda_U(x)}$.

We will need the following theorem of Poletsky (Corollary p. 170 in [20], see also [21]), generalized by Rosay [26]. Let U be a connected complex manifold of dimension ≥ 1 . We denote by $\mathcal{H}_{z,U}$ the set of holomorphic functions $h : V_h \rightarrow U$ from some neighborhood V_h of $\overline{\Delta} = \{z \in \mathbb{C}, |z| \leq 1\}$ (possibly depending on h) into U such that $h(0) = z$. We also denote by $PSH(U)$ the set of plurisubharmonic functions defined on U .

Theorem 1 *Let $u : U \rightarrow \mathbb{R}$ be an upper semi-continuous function. With the previous notations, the function defined in U by*

$$\tilde{u}(z) := \frac{1}{2\pi} \inf_{f \in \mathcal{H}_{z,U}} \int_0^{2\pi} u(f(e^{i\theta})) d\theta,$$

if it is not everywhere equal to $-\infty$, belongs to $PSH(U)$ and verifies $\tilde{u} \leq u$. Moreover, this function $\tilde{u}$ is maximal among all the functions in $PSH(U)$ verifying this inequality.

If u is only assumed borelian, we still have $\tilde{u} \leq u$, and $\tilde{u}^$ is plurisubharmonic.*

The last statement in the above theorem is Lemma 2.2.12 in [28]. We prove below a version of this theorem, useful in our context (see also Corollary 1 in [24]). Let us recall some useful definitions.

An open bounded set $U \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a continuous plurisubharmonic function $\rho : U \rightarrow]-\infty, 0[$ such that $\forall c \in]-\infty, 0[, \{z \in \mathbb{C}^n : \rho(z) < c\} \subset\subset U$. In order to simplify the presentation, we will also suppose here that a hyperconvex set is connected.

Let $U \subset \mathbb{C}^n$, $n \geq 1$, be a bounded open set, and let $A \subset \mathbb{C}^n$ be a compact set. The *Green's function of $A \subset U$ relative to U* is defined by

$$G_{A,U} := \sup^* \{v \in PSH(U) : v \leq 0, v|_A \leq -1\},$$

where $\sup^*$ denotes the upper semi-continuous regularization of the upper envelope. A compact set $K \subset \mathbb{C}^n$ is called *fat* if $\hat{K} = K$. In this case, if K is included into a bounded hyperconvex open set $U \subset \mathbb{C}^n$, the function $G_{K,U}$ is continuous by Proposition 4.5.3 of [18].

Theorem 2 *Let $w : U \rightarrow \mathbb{R}$ be an upper semi-continuous function on a connected open set $U \subset \mathbb{C}^n$. Then, the function defined in U by*

$$\hat{w}(x) := \inf_{\mu_x \in PJ_U(x)} \int_U w d\mu_x \quad (4)$$

is maximal among all the plurisubharmonic functions defined in U and bounded from above by w (if there are any).

In particular, if the open set U is bounded and hyperconvex, and if $A \subset U$ is a fat compact set, the Green's function A relative to U equals

$$G_{A,U}(x) = \inf_{\mu_x \in PJ_U(x)} \int_U (-1_A)^* d\mu_x = - \sup_{\mu_x \in PJ_U(x)} \mu_x(\hat{A}). \quad (5)$$

In the case where the set $A \subset U$ is only assumed relatively compact, if the open set U is bounded and hyperconvex, we have

$$G_{A,U}(x) \geq \inf_{\mu_x \in PJ_U(x)} \int_U (-1_{\bar{A}}) d\mu_x. \quad (6)$$

Proof We denote again $\mathcal{H}_{z,U}$ the set of holomorphic functions $h : V_h \rightarrow U$ from a neighborhood V_h of $\bar{\Delta}$ (possibly depending on h) into a neighborhood U of $z \in \mathbb{C}^n$, such that $h(0) = z$.

Let $\mu_x = g_* \lambda$ with $g : \bar{\Delta} \rightarrow \mathbb{C}^n$ an analytic disk. By Mergelyan's Theorem, we can find a sequence $(f_n)_n$ of $\mathcal{H}_{z,U}$ which converges to g uniformly on $\bar{\Delta}$. The upper semi-continuity of w implies that $\forall \theta \in [0, 2\pi]$,

$$\limsup_{n \rightarrow +\infty} w \circ f_n(\theta) \leq w \circ g(\theta).$$

It follows, by Fatou's Lemma, that:

$$\limsup_{n \rightarrow +\infty} \frac{1}{2\pi} \int_0^{2\pi} w \circ f_n(e^{i\theta}) d\theta \leq \int_U w \mu_x.$$

(Note that this inequality also follows from Lemma 3.4.4 in [18].) We deduce that

$$\frac{1}{2\pi} \inf_{f \in \mathcal{H}_{z,U}} \int_0^{2\pi} w \circ f(e^{i\theta}) d\theta \leq \inf_{\nu_x \in \Lambda_U(x)} \int_U w d\nu_x.$$

The inequality in the other direction being trivial, we finally have

$$\inf_{\nu_x \in \Lambda_U(x)} \int_U w d\nu_x = \frac{1}{2\pi} \inf_{f \in \mathcal{H}_{z,U}} \int_0^{2\pi} w \circ f(e^{i\theta}) d\theta. \quad (7)$$

On the other hand, recall that every pluri-Jensen measure is a limit of analytic disks. For $\nu_x \in PJ_U(x)$, denote $(\nu_{x,n})_n$ a sequence of analytic disks, weakly convergent to ν_x . Thanks

to the upper semi-continuity of w (see, e.g., Lemma 2.8 in [4], or Lemma 3.4.4 in [18]), we have $\forall v_x \in PJ_U(x)$,

$$\inf_{\mu_x \in \Lambda_U(x)} \int_U w d\mu_x \leq \lim_{n \rightarrow +\infty} \int_U w dv_{x,n} \leq \int_U w dv_x. \quad (8)$$

Bringing together equations (7) and (8), we obtain

$$\begin{aligned} \inf_{v_x \in PJ_U(x)} \int_U w dv_x &\geq \inf_{v_x \in \Lambda_U(x)} \int_U w dv_x = \frac{1}{2\pi} \inf_{f \in \mathcal{H}_{z,U}} \int_0^{2\pi} w \circ f(e^{i\theta}) d\theta \\ &\geq \inf_{v_x \in PJ_U(x)} \int_U w dv_x, \end{aligned}$$

and hence the above inequalities are all equalities. Theorem 1 concludes the proof of (4).

Recall that $G_{A,U}$ is continuous by Proposition 4.5.3 of [18]. Then, (5) comes from (4), since

$$G_{A,U}(x) = \inf_{\mu_x \in PJ_U(x)} \int_U (-1_{\hat{A}}) d\mu_x \geq \inf_{\mu_x \in PJ_U(x)} \int_U (G_{A,U}) d\mu_x = G_{A,U}(x).$$

Let us now show (6). Denote by $\text{dist}(\cdot, \cdot)$ the euclidian distance. Take $\epsilon > 0$ such that $U_\epsilon := \{z \in U, \text{dist}(z, \partial U) < \epsilon\}$ is non-empty and contains A . Take $x \in U_\epsilon$. Let $A_m := \{z \in \mathbb{C}^n, \text{dist}(z, A) \leq \frac{1}{m}\}$ for every $m \in \mathbb{N}^*$ sufficiently large such that $A_m \subset \subset U_\epsilon$. Let then $(v_{x,n,k})_k$ be a weakly convergent sequence of measures in $PJ_{U_\epsilon}(x)$ such that

$$\inf_{v_x \in PJ_{U_\epsilon}(x)} \int_{U_\epsilon} (-1_{A_n}) dv_x = \lim_{k \rightarrow +\infty} \int_{U_\epsilon} (-1_{A_n}) dv_{x,k,n}.$$

Denote $v'_{x,n}$ its limit, and note that $v'_{x,n} \in PJ_{U_\epsilon}(x)$ (see Theorem 6.3 in [25]).

Let also $v'_x \in PJ_{U_\epsilon}(x)$ be an accumulation point of the sequence $(v'_{x,n})_n$, toward which we may suppose that it converges. Note that the sequence $(-1_{A_n})_n$ is a non-decreasing sequence of lower semi-continuous functions. Lemma 2.8 in [4] used twice implies that

$$\begin{aligned} G_{A,U_\epsilon}(x) &\geq \lim_{n \rightarrow +\infty} G_{A_n,U_\epsilon}(x) = \lim_{n \rightarrow +\infty} \lim_{k \rightarrow +\infty} \int_{U_\epsilon} (-1_{A_n})^* dv_{x,k,n} \\ &\geq \limsup_{n \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \int_{U_\epsilon} (-1_{A_n}) dv_{x,k,n} \geq \limsup_{n \rightarrow +\infty} \int_{U_\epsilon} (-1_{A_n}) dv'_{x,n} \\ &\geq \int_{U_\epsilon} (-1_{\hat{A}}) dv'_x \geq \inf_{\mu_x \in PJ_{U_\epsilon}(x)} \int_{U_\epsilon} (-1_{\hat{A}}) d\mu_x \geq \inf_{\mu_x \in PJ_U(x)} \int_U (-1_{\hat{A}}) d\mu_x. \end{aligned}$$

Now, since U is assumed hyperconvex, Proposition 4.5.7 in [18] allows us to pass to the limit

$$G_{A,U}(x) = \lim_{\epsilon \rightarrow 0} G_{A,U_\epsilon}(x) \geq \inf_{\mu_x \in PJ_U(x)} \int_U (-1_{\hat{A}}) d\mu_x.$$

□

Remark 3 It follows from Theorem 2 that for a bounded hyperconvex open set $U \subset \mathbb{C}^n$, if $A \subset U$ is a fat compact set such that $\hat{A} \subset U$, then $\forall x \in U$, $\sup_{v_x \in PJ_U(x)} v_x(\hat{A}) = \sup_{v_x \in PJ_U(x)} v_x(\overset{\circ}{A})$. Indeed, $\forall x \in U$, we have

$$\inf_{v_x \in PJ_U(x)} \int_U (-1_{\hat{A}})^* dv_x = G_{A,U}(x) = G_{\hat{A},U}(x) = \inf_{v_x \in PJ_U(x)} \int_U (-1_{\hat{A}})^* dv_x.$$

(The equality $G_{A,U} = G_{\hat{A},U}$ comes from Theorem 5.1.7 in [18] and Corollary 5.3.4 in [18].) Thus, A is polynomially convex if and only if $\forall x \notin A$,

$$\sup_{\nu_x \in PJ_U(x)} \nu_x(\hat{A}) < 1.$$

Finally, since $PJ_{\overline{U}}(x)$ is closed (Lemma 7.2 in [17], see also Theorem 6.3 in [25]), Lemma 3.4.4 in [18] implies

$$A = \hat{A} \iff \forall x \notin A, \forall \nu_x \in PJ_U(x), \nu_x(\hat{A}) < 1.$$

Application 1 In particular, Remark 3 allows to deduce that, if $u : \mathbb{C}^n \rightarrow \mathbb{R}^+$ is a continuous plurisubharmonic function, then the set $\{u = 0\}$ is polynomially convex, provided that it is bounded and fat.

Indeed, suppose that $\{u = 0\}$ is bounded and fat but not polynomially convex. By Remark 3, one can find $y \in \mathbb{C}^n \setminus \{u = 0\}$ and a measure $\mu_y \in PJ_y(\mathbb{C}^n)$ whose support is included in $\{u = 0\}$. Then,

$$0 < u(y) \leq \int_{\mathbb{C}^n} u d\mu_y \leq \int_{\mathbb{C}^n} 0 d\mu_y = 0.$$

We have thus proved by contradiction that $\{u = 0\}$ is polynomially convex, if it is bounded and fat.

Corollary 2 Let $U \subset \mathbb{C}^n$ be an open set, and let $A \subset U$ be a complete pluripolar set, i.e., $A = \{u = -\infty\}$ for a plurisubharmonic function $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$. Then, $\forall x \in U \setminus A$, the set A is of null measure for every pluri-Jensen measure in $PJ_U(x)$.

Conversely, suppose U bounded and hyperconvex. If there exists $x \in U \setminus \overline{A}$ such that the set A is of null measure for every pluri-Jensen measure in $PJ_U(x)$, then A is pluripolar.

Proof Let $x \in \mathbb{C}^n \setminus A$, and $\mu_x \in PJ_U(x)$. We have $\mu_x(A) = 0$, since

$$-\infty < u(x) \leq \int_U u d\mu_x.$$

Conversely, we may suppose that A is compact. Suppose $\exists x \in U \setminus \overline{A}$, $\forall \nu_x \in PJ_U(x)$, we have $\nu_x(A) = 0$. Then, Theorem 1 implies that

$$0 = \inf_{\nu_x \in PJ_U(x)} \int_U (-1_{\overline{A}}) d\nu_x \leq G_{A,U}(x).$$

The function $G_{A,U}$ is null in a point outside $\overline{A}$, hence everywhere on U , and thus A is pluripolar. $\square$

Remark 4 Note that, given a borelian function $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$ such that the set $\{u = -\infty\}$ is complete pluripolar, we deduce, by applying Theorem 1 to $U \setminus \{u = -\infty\}$, that the statement of Theorem 1 remains valid when replacing $u : U \rightarrow \mathbb{R}$ by $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$. Similarly, the statement of Theorem 2 remains valid when replacing $w : U \rightarrow \mathbb{R}$ by $w : U \rightarrow \mathbb{R} \cup \{-\infty\}$.

We obtain the following lemma, which we will often use in the sequel:

Lemma 4 Let T be a closed positive $(1, 1)$ -current defined in a connected open set $U \subset \mathbb{C}^n$. Let $v : S \rightarrow \mathbb{R}^-$ be a T -psh function. If $u : U \rightarrow \mathbb{R}^-$ is the extension of v to U with the value 0, we have $\forall x \in S$,

$$\inf_{\nu_x \in PJ_U(x)} \int_U u d\mu_x = \inf_{\nu_x \in PJ_S(x)} \int_S v d\mu_x = v(x).$$

Proof First suppose that $v = w|_{\mathcal{S}}$ for some plurisubharmonic function $w : U \rightarrow \mathbb{R}$ such that $w|_{\mathcal{S}} < 0$. Denote by W the open set $\{w < 0\} \subset U$, and note that $\mathcal{S} \subset W$. Then, Theorem 2 and (2) imply that $\forall x \in \mathcal{S}$ we have

$$\begin{aligned} v(x) &= \inf_{v_x \in PJ_U(x)} \int_U w dv_x \leq \inf_{v_x \in PJ_W(x)} \int_U w dv_x \\ &\leq \inf_{v_x \in PJ_W(x)} \int_U u dv_x \leq \inf_{v_x \in PJ_{\mathcal{S}}(x)} \int_U u dv_x = v(x). \end{aligned}$$

Now suppose only that $v : \mathcal{S} \rightarrow \mathbb{R}^-$ is a T -psh function as in the statement. Let $(v_n)_n$ be a sequence of plurisubharmonic functions defined in U , decreasing on $\mathcal{S}$ toward v . Take $x \in \mathcal{S}$ and $\mu_x \in PJ_U(x)$. Take $\epsilon > 0$. Then, there exists n_0 such that $v_n(x) - \epsilon < 0$ for $n \geq n_0$. There also exists an open neighborhood $O \subset U$ of x such that $O \cap \mathcal{S} \subset \bigcap_{n \geq n_0} \{v_n - \epsilon < 0\}$. Up to increasing n_0 , we may suppose that the compact support of μ_x is included in O . Denote by $u_{n,\epsilon}$ the extension to U with the value 0 of $(v_n - \epsilon)|_{\mathcal{S}}$, and denote by u_n the extension to U with the value 0 of $v_n|_{\mathcal{S}}$. By the previous step,

$$v_n(x) - \epsilon = \inf_{v_x \in PJ_O(x)} \int_U u_{n,\epsilon} dv_x \leq \int_U u_n d\mu_x.$$

The monotone convergence theorem and the arbitrary choice of $\epsilon > 0$ give $v(x) \leq \int_U u d\mu_x$.

Then, $v(x) = \inf_{\mu_x \in PJ_U(x)} \int_U u d\mu_x$ since $\delta_x \in PJ_U(x)$. $\square$

Let us now introduce a new notion, which we will use in the next result.

Definition 2 Let $\mathcal{S}$ be the support of a closed positive $(1, 1)$ -current T . A set $E \subset \mathcal{S}$ is called T -pluripolar if there exists a T -psh function $v : \mathcal{S} \rightarrow \mathbb{R} \cup \{-\infty\}$ such that e^u is continuous and $E \subset \{v = -\infty\}$.

We prove the following result.

Proposition 1 A set $F \subset \mathcal{S}$ which is T -pluripolar is pluripolar.

Proof Without loss of generality, we may suppose that F is bounded, since a locally pluripolar set is pluripolar. Let $u : \mathcal{S} \rightarrow \mathbb{R} \cup \{-\infty\}$ be a T -psh function with e^u continuous, and $E = \{u = -\infty\} \subset \mathcal{S}$ a T -pluripolar set containing F . Let $U \subset \mathbb{C}^n$ be an open bounded hyperconvex set containing F , and let $(u_n)_n$ be a sequence of continuous plurisubharmonic functions defined in U , decreasing toward u on $\mathcal{S}$. Let $v_n := u_n|_{\mathcal{S}}$ extended to U with the value 0. Let $K \subset U \cap \mathcal{S}$ be a non-pluripolar compact set containing also F . Up to shrinking U , we may suppose that $u_n \leq 0$ on $U \cap \mathcal{S}$ for $n \geq n_0$. Theorem 2 and Lemma 4 imply that $\forall n \geq n_0, \forall c \in \mathbb{R}^- \setminus \{0\}$, we have $\forall x \in U \cap \mathcal{S}$,

$$\begin{aligned} G_{\{u_n \leq c\} \cap K, U}(x) &\geq \inf_{\mu_x \in PJ_U(x)} \int_U (-\mathbf{1}_{\{u_n \leq c\} \cap K}) d\mu_x \\ &\geq \inf_{\mu_x \in PJ_U(x)} \int_{\{u_n \leq c\} \cap \mathcal{S}} (-\mathbf{1}_{\{u_n \leq c\}}) d\mu_x \\ &\geq \frac{1}{|c|} \inf_{\mu_x \in PJ_U(x)} \int_{\{u_n \leq c\} \cap \mathcal{S}} v_n d\mu_x \geq \frac{1}{|c|} \inf_{\mu_x \in PJ_U(x)} \int_{U \cap \mathcal{S}} v_n d\mu_x = \frac{u_n(x)}{|c|}. \end{aligned}$$

Corollary 4.7.8 in [18] allows us to pass to the limit : $\forall c \in \mathbb{R}^- \setminus \{0\}, \forall x \in U \cap \mathcal{S}$,

$$G_{\{u \leq c\} \cap K, U}(x) = \lim_{n \rightarrow +\infty} G_{\{u_n \leq c\} \cap K, U}(x) \geq \frac{u(x)}{|c|}.$$

Now let $y \in (U \cap S) \setminus E$. We have

$$G_{E \cap K, U}(y) \geq \lim_{c \rightarrow -\infty} G_{\{u \leq c\} \cap K, U}(y) \geq 0.$$

Since we obtain $G_{E \cap K, U} = 0$ on $(U \cap S) \setminus E$, hence on U , it follows that the set $E \cap K$ is pluripolar. Thus, $F \subset E \cap K$ is pluripolar. $\square$

Remark 5 It is not always true that a pluripolar set is T -pluripolar: for instance, $B(0, 1) \cap \{z_1 = 0\}$ is pluripolar in $\mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$. This set is not T -pluripolar for $T = dd^c \log |z_1|$, because $(z_1, z_2) \mapsto \log |z_1| \notin L^1_{\text{loc}}(\|T\|)$.

Using Proposition 1, the T -pluripolar sets are the pluripolar subsets of $\text{supp}(T)$ which are included in a set of the form $\{u = -\infty\}$, for a continuous plurisubharmonic function $u \in L^1_{\text{loc}}(\|T\|)$. This is the case, for example, if the set $\{u = -\infty\} \cap \text{supp}(T)$ is relatively compact in a ball contained in the domain of definition of u , by Proposition 3.1 in [4], or if it is of Hausdorff measure $\mathcal{H}_1(E) = 0$, by Theorem 3.5 in [4].

We will further need the following lemma:

Lemma 5 *Let $U \subset \mathbb{C}^n$ be an open bounded hyperconvex set, and $A \subset\subset U$ a non-pluripolar fat compact set. Then, for every $\epsilon > 0$, there exists a neighborhood $V \subset\subset U$ of A such that for every $x \in V$, there exists a pluri-Jensen measure $\nu_x \in P J_U(x)$ with $\nu_x(A) \geq 1 - \epsilon$.*

Proof By Theorem 2,

$$\inf_{\nu_x \in P J_U(x)} \int_U (-1_A) d\nu_x \leq G_{A, U}(x).$$

It suffices thus to take $V := \{G_{A, U} < -1 + \frac{\epsilon}{2}\}$. $\square$

2.3 Properties analogous to the ones for plurisubharmonic functions

We prove here several results which transpose to T -psh functions certain properties of plurisubharmonic functions.

Lemma 6 *Let T be a closed positive current in an open subset of $\mathbb{C}^n$. Let $(u_m)_m$ be a locally uniformly bounded from above sequence of continuous co- T -psh functions. Suppose that the upper semi-continuous regularization $u_\infty^* : S \rightarrow \mathbb{R}$ of the function $u_\infty := \limsup_{m \rightarrow +\infty} u_m$ is continuous. Then, the set $\{u_\infty < u_\infty^*\}$ is pluripolar.*

Proof First suppose that u_∞^* is constant. Up to replacing u_m by $u_m - u_\infty^*$, we may suppose that $u_\infty^* = 0$. Let V be an open hyperconvex subset of $\mathbb{C}^n$ intersecting S , and $K \subset V \cap S$ a non-pluripolar compact set. Let a constant $\epsilon > 0$. For every $m \in \mathbb{N}$, by Dini's Lemma, up to shrinking V , there exists a continuous plurisubharmonic function $v_m : V \rightarrow \mathbb{R}$, with $v_m \geq u_m$ on S and $|v_m - u_m| \leq \frac{1}{m}$ on S . We can also suppose that for every $m \in \mathbb{N}$ we have $v_m < \epsilon$ on $S \cap V$. Then, $u_\infty = \limsup_{m \rightarrow +\infty} v_m|_S$ on $V \cap S$.

Let a constant $c < -\epsilon$. Let $x \in V \cap S \setminus \{u_\infty < 0\}$. There exists an extracted subsequence $(v_{m_j})_j$ of $(v_m)_m$ such that $(v_{m_j}(x))_j$ converges to 0. Then, $\forall j \in \mathbb{N}$, a computation similar to the one in the proof of Proposition 1 gives $\forall j \in \mathbb{N}$,

$$G_{\{\sup_{m \geq m_j} v_m - \epsilon \leq c\} \cap K, V}(x) \geq G_{\{v_{m_j} - \epsilon \leq c\} \cap K, V}(x) \geq \frac{v_{m_j}(x) - \epsilon}{|c|}.$$

Corollary 4.7.8 in [18] allows us to pass to the limit

$$G_{\{u_\infty - \epsilon \leq c\} \cap K, V}(x) = \lim_{j \rightarrow +\infty} G_{\{\sup_{m \geq j} v_m - \epsilon \leq c\} \cap K, V}(x) \geq \lim_{j \rightarrow +\infty} \frac{v_{m_j}(x) - \epsilon}{|c|} = \frac{-\epsilon}{|c|}.$$

Corollary 4.7.8 in [18] allows us to take the limit $\epsilon \rightarrow 0$, and to obtain

$$G_{\{u_\infty \leq c\} \cap K, V}(x) = 0. \quad (9)$$

The function $G_{\{u_\infty \leq c\} \cap K, V}$ is null at x , hence on V . Finally, Corollary 4.7.8 in [18] allows again to pass to the limit

$$G_{\{u_\infty < 0\} \cap K, V} = \lim_{c \rightarrow 0^-} G_{\{u_\infty \leq c\} \cap K, V} = 0.$$

Thus, by the arbitrary choice of K , the set $\{u_\infty < 0\}$ is pluripolar.

Now we suppose only that u_∞^* is continuous. Remark that in this case the previous reasoning shows that for every compact set $K \subset S$ and every real number $c < u_\infty^*$ on K , the set $\{u_\infty < c\} \cap K$ is pluripolar. Indeed, under these conditions, (9) is still true.

Let $(s_m)_m$ be an increasing sequence of step functions, locally uniformly convergent toward u_∞^* . By the previous remark, for all $m \in \mathbb{N}$, the set $\{\phi_\infty < s_m\}$ is pluripolar. Thus, the set $\{\phi_\infty < \phi_\infty^*\} = \bigcup_{m \in \mathbb{N}} \{\phi_\infty < s_m\}$ is pluripolar. $\square$

We further present a version of Hartogs's Theorem for T -psh functions.

Theorem 3 *Let S be the support of a closed positive $(1, 1)$ -current T in an open bounded set $U \subset \mathbb{C}^n$. Let also $Q \subset \mathbb{C}^n$ be a fat compact set such that $K := Q \cap S$ is non-pluripolar, and $E \subset K$ a set whose closure $\bar{E}$ in K is of empty interior for the induced topology.*

Let $(u_n)_n : U \cap S \rightarrow \mathbb{R}$ be a sequence of continuous co- T -psh functions uniformly bounded

from above on U , such that $\lim_{n \rightarrow +\infty} \left(\sup_{i \geq n} u_i \right)^ \leq f$ on $K \setminus E$ for some continuous function $f : S \cap U \rightarrow \mathbb{R}$. Then, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, and a neighborhood $V \subset U$ of K such that $u_n \leq f + \epsilon$ on $V \cap S$ for all $n \geq n_0$. In particular, $\limsup_{n \rightarrow +\infty} u_n \leq C$ on K .*

Proof First suppose that f is equal to a constant C . We may suppose, up to subtracting a constant, that $u_n < 0$ in $U \cap S$ and that $C < 0$. Let us denote $u'_n := (\sup_{i \geq n} u_i)^*$. Then, the sequence

of upper semi-continuous functions $(u'_n)_n$ decreases on S toward $u := \lim_{n \rightarrow +\infty} \left(\sup_{i \geq n} u_i \right)^*$.

By Lemma 5, there exists a neighborhood V of K such that $\forall x \in V \cap S$, we can find $v_x \in PJ_U(x)$ such that $v_x(K) > 1 - \frac{\epsilon}{2|C+\epsilon|}$.

By Egoroff's Theorem, there exists a set $A \subset K$, with $v_x(A) < \frac{\epsilon}{2|C+\epsilon|}$, such that $(u'_n)_n$ converges uniformly toward u on $K \setminus A$. Fix $\epsilon > 0$ such that $C + 2\epsilon < 0$. Then, $\exists n_0 \in \mathbb{N}$ such that $\forall z \in K \setminus (A \cup E)$, $\forall n \geq n_0$: $u_n(z) \leq u'_n(z) \leq C + \epsilon$. The continuity of the functions u_n then implies that $\forall z \in K \setminus A$, $\forall n \geq n_0$ we have

$$u_n(z) \leq C + \epsilon.$$

Extend each u_n to U with the value 0. By the negativity of the functions u_n on $S \cap U$, we have for $n \geq n_0$ and $x \in V \cap S$, thanks to (2) and Lemma 4,

$$\begin{aligned} u_n(x) &= \inf_{\mu_x \in PJ_{U \cap S}(x)} \int_U u_n d\mu_x = \inf_{\mu_x \in PJ_U(x)} \int_U u_n d\mu_x \leq \int_U u_n d\nu_x \\ &\leq \int_{K \setminus A} u_n d\nu_x \leq \nu_x(K \setminus A) \sup_{K \setminus A} u_n \leq \left(1 - \frac{\epsilon}{|C + \epsilon|}\right) (C + \epsilon) = C + 2\epsilon. \end{aligned}$$

Now suppose that f is a continuous function. Let $(s_m)_m$ be a sequence of step functions decreasing toward f , uniformly on K . There exists $m_0 \in \mathbb{N}$ such that $s_{m_0} - f \leq \epsilon$. The previous step implies that, for m sufficiently large we have on K :

$$u_m \leq s_{m_0} \leq f + \epsilon,$$

which proves the stated result. $\square$

We deduce the following result:

Proposition 2 *Let S be the support, supposed non-pluripolar, of a closed positive $(1, 1)$ -current T in an open set $U \subset \mathbb{C}^n$. A T -psh function $u : S \rightarrow \mathbb{R}$, constant outside a set E whose closure in S is of empty interior, is constant everywhere.*

Proof Let us suppose that $u = C$ outside E . The upper semi-continuity of u implies that $E = \{u \neq C\} = \{u > C\}$. Let $(u_n)_n$ be a sequence of continuous plurisubharmonic functions decreasing on S toward u . Theorem 3 applied to $(u_n)_n$ on compact subsets of S concludes. $\square$

This result allows us to weaken the conditions required in the statement of Theorem 3:

Corollary 3 *In Theorem 3, if the current T does not charge pluripolar sets, we can replace the condition $\lim_{n \rightarrow +\infty} \left(\sup_{i \geq n} u_i \right)^* \leq f$ by the less restrictive condition $\lim_{n \rightarrow +\infty} \left(\sup_{i \geq n} u_i \right) \leq f$.*

Proof We use the same notations as those in the statement of Theorem 3. As in the proof of this theorem, we can suppose that the continuous function f is equal to a constant $C \in \mathbb{R}$. It suffices to show that

$$\lim_{n \rightarrow +\infty} \sup_{i \geq n} u_i \leq C \Rightarrow \lim_{n \rightarrow +\infty} \left(\sup_{i \geq n} u_i \right)^* \leq C.$$

It is sufficient again, by Lemma 2 and Proposition 2, to show that $\lim_{n \rightarrow +\infty} \sup_{i \geq n} u_i \leq C$

implies that $\lim_{n \rightarrow +\infty} \max \left(\left(\sup_{i \geq n} u_i \right)^*, C \right) = C$ on $S \setminus E$, where $E \subset S$ is a closed set of empty interior for the induced topology. Suppose that the last condition is false. By Baire's theorem, there exists $k \in \mathbb{N} \setminus \{0\}$ such that

$$\lim_{n \rightarrow +\infty} \left(\sup_{i \geq n} u_i \right)^* \geq C + \frac{1}{k}$$

in an open set (for the induced topology) $W \subset S$. Then, $\left(\sup_{i \geq m} u_i \right)^* \geq C + \frac{1}{k}$ in W , $\forall m \in \mathbb{N}$.

Hence there exists a non-empty set $O \subset W$ dense in W (for the induced topology) such that $\forall m \in \mathbb{N}$, $\sup_{i \geq m} u_i \geq C + \frac{1}{2k}$ on O . This implies that $\lim_{n \rightarrow +\infty} \sup_{i \geq n} u_n > C$ on O , which finishes the proof. $\square$

We can now state a result which improves Corollary 1 for certain currents:

Proposition 3 *Let T be a closed positive $(1, 1)$ -current in an open set $U \subset \mathbb{C}^n$, with support $S \subset U$. Let $(v_m)_m$ be a sequence of T -psh functions, such that $\limsup_{m \rightarrow +\infty} v_m \leq C$ for some $C \in \mathbb{R}$. If $\lim_{m \rightarrow +\infty} v_m(x) = C$ for some $x \in S$, then $(v_m)_m$ converges toward C outside a pluripolar set in S .*

In particular, if $\|T\|$ does not charge pluripolar sets, then any T -psh function that reaches its maximum is constant.

Proof Let us suppose $C = 0$. Note that the function $\limsup_{m \rightarrow +\infty} v_m : S \rightarrow \mathbb{R}$ is T -psh by Lemma 2. Extend each function v_m to U with the value 0. Fatou's lemma and Lemma 4 imply that $\forall v_x \in PJ_U(x)$,

$$0 = \lim_{m \rightarrow +\infty} v_m(x) \leq \liminf_{m \rightarrow +\infty} \int_U v_m dv_x \leq \limsup_{m \rightarrow +\infty} \int_U v_m dv_x \leq \int_U \limsup_{m \rightarrow +\infty} v_m dv_x \leq 0.$$

Finally $\forall v_x \in PJ_U(x)$, $\lim_{m \rightarrow +\infty} \int_U v_m dv_x = 0$. Thus, $(v_m)_m$ extended to U with the value 0 converges v_x -a.e. toward 0 on the support of every measure $v_x \in PJ_U(x)$. Since $PJ_U(x)$ is invariant by rotations and homotheties with center x , Lemma 1 and Corollary 2 imply that $(v_n)_n$ converges toward 0 on $S \setminus E$, where $E \subset S$ is a pluripolar set. In particular, if $\|T\|$ does not charge pluripolar sets, then any T -psh function that reaches its maximum is constant outside such a set E , hence everywhere by Proposition 2. $\square$

Finally, let us prove an inequality of Chern–Levine–Nirenberg type for T -psh functions:

Proposition 4 *Let $r_0 > 0$ be a real number. If u is a continuous T -psh function, then there exists a constant $c \geq 0$ such that $\forall r < \frac{r_0}{2}$,*

$$T \wedge dd^c u(B(z, r)) \leq \frac{c}{r^{2n}} \|T\|(B(z, 2r)) \sup_{\text{supp}(T) \cap B(z, 2r)} u.$$

Proof It suffices to prove the above formula when u is the restriction of a plurisubharmonic function to the support S of T . The monotone convergence theorem then allows us to extend the formula to the general case.

Let ξ be a positive step function equal to 1 in $B(z, 1)$ and to 0 in $B(z, 2)$. Let us denote $\xi_r(z) := \xi(\frac{z}{r})$. Then, $\forall r \in]0, 1[$,

$$\begin{aligned} T \wedge dd^c u(B(z, r)) &\leq \int_{\mathbb{C}^n} \xi_r T \wedge dd^c u = \int_{\mathbb{C}^n} u T \wedge dd^c \xi_r \\ &\leq \frac{c}{r^{2n}} \|T\|(B(z, 2r)) \sup_{S \cap B(z, 2r)} u. \end{aligned}$$

$\square$

3 Applications to dynamics in $\mathbb{C}^n$

3.1 Background

Given a polynomial application $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, we extend it to a meromorphic endomorphism $[P_1, \dots, P_n] : \mathbb{P}^n \rightarrow \mathbb{P}^n$, where $P_1, \dots, P_n$ are homogeneous polynomials. The choice of

$\mathbb{P}^n$ as compactification of $\mathbb{C}^n$ is practical in our case because the group $H^{1,1}(\mathbb{P}^n, \mathbb{R})$ is of dimension 1, and the classes which compose it are Kählerian, in the sense that they contain a strictly positive smooth $(1, 1)$ -form. Denote by $H \subset \mathbb{P}^n$ the hyperplane at infinity. Denote by $I_f \subset H$ the *indetermination set* of f , i.e., the set of points in the neighborhood of which f is not holomorphic. We denote $X_f := \overline{f(H \setminus I_f)} \subset H$. This is an analytic set (see for example [14] for details).

Let ω be a Kählerian form of mass 1 on $\mathbb{P}^n$. We define the *dynamical degrees* $\lambda_1(f), \dots, \lambda_n(f) \geq 1$ of f by

$$\lambda_k(f) := \liminf_{m \rightarrow +\infty} \left(\int_{\mathbb{C}^n} (f^{m*} \omega^k) \wedge \omega^{n-k} \right)^{\frac{1}{m}}.$$

Note that they are independent of the choice of the Kähler form ω . When there is no risk of ambiguity, we will simply denote them by λ_k , and put $\lambda := \lambda_1$.

The pull-back operation of a current induces a linear action f^* on the cohomology spaces, defined by $f^*\{S\} := \{f^*S\}$. A polynomial application $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ extended to $\mathbb{P}^n$ is called *algebraically stable* if the linear action induced by f^* on the cohomology spaces $H^{k,k}(\mathbb{P}^n, \mathbb{R})$ (of dimension 1) commutes with the composition operation, i.e., when $(f^n)^* = (f^*)^n$ for all $n \in \mathbb{N}$. When f is algebraically stable, the spectral radius of the linear action induced in cohomology $f^* : H^{k,k}(\mathbb{P}^n, \mathbb{R}) \rightarrow H^{k,k}(\mathbb{P}^n, \mathbb{R})$ equals the k -th dynamical degree λ_k . By duality, it also equals the spectral radius λ'_{n-k} of the action induced by f_* on $H^{n-k,n-k}(\mathbb{P}^n, \mathbb{R})$. In order for f to be algebraically stable, it suffices to have $I_f \cap X_f = \emptyset$.

We will be interested in polynomial applications verifying $\lambda_1 > \lambda_2$, whose properties will be seen to generalize those of the Hénon's applications of $\mathbb{C}^2$. The concavity inequalities fulfilled by the λ_i (Theorem 2.4.a in [14]) imply that λ_1 is a dynamical degree strictly dominant ($\lambda_1 > \max_{j \neq 1} \lambda_j$).

Given an algebraically stable polynomial endomorphism f of $\mathbb{P}^n$ of dynamical degree λ_1 strictly dominant, and ω a Kählerian form of mass 1, the sequence $\frac{1}{\lambda_1^m} f^{m*} \omega$ converges, in the weak sense of currents, toward a closed positive $(1, 1)$ -current T^+ (see, e.g., Theorem 1.1 in [2] for a general statement). The current T^+ is invariant by $\frac{1}{\lambda_1} f^*$ and does not depend on the choice of ω . We denote by J its support.

If moreover f is an automorphism, we can define a canonical positive $(n-1, n-1)$ -current $\frac{1}{\lambda_1} f_*$ -invariant $T^- := \lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} f^m_* \omega^{n-1}$. The measures $\|T^+\|$ and $\|T^-\|$ do not charge pluripolar sets.

More generally, even if f is not an automorphism, the current T^- is still well defined if f satisfies $\lim_{z \in \mathcal{C}_f, \|z\| \rightarrow +\infty} f(z) \in X_f$, where $\mathcal{C}_f$ is the critical set of f (see section 2.2 in [16], see also [15]).

The canonical invariant measure $T^+ \wedge T^-$ is then well defined. It does not charge pluripolar sets. We will denote by J^- the support of such current T^- .

If f is a polynomial endomorphism of $\mathbb{P}^n$ which is algebraically stable and verifies $\lambda_1 > \lambda_2$, then there exists a plurisubharmonic function G defined on $\mathbb{C}^n \subset \mathbb{P}^n$, called *dynamical Green's function of f* , such that $T^+ = dd^c G$. It can be written as the limit in L^1_{loc} of the sequence of term $\frac{1}{\lambda_1^m} \log(1 + \|f^m\|)$. The function G is continuous and pluriharmonic outside the support J of T^+ .

The current T^+ intersects the divisor at infinity exactly on the indetermination set of f , i.e., on the points where f cannot be holomorphically extended at infinity. This indetermination

set is an analytical subset of $\mathbb{P}^n$ of (complex) dimension at most $n - 2$. (See Theorem 4.17 in [14] for more details.)

The invariance property $G \circ f = \lambda_1 G$ implies that f sends the sublevel set $\{G < \epsilon\}$ onto the sublevel set $\{G < \lambda_1 \epsilon\}$. In other words, the set $K^+ := \{G = 0\}$, which we call *filled Julia set*, is included in an open set dilated by f . Therefore, every $y \notin K^+$ verifies $\lim_{m \rightarrow +\infty} \|f^m(y)\| = +\infty$. We thus define:

Definition 3 A polynomial automorphism $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $n \geq 2$, algebraically stable and such that $\lambda_1 > \lambda_2$, will be called *quasi-Hénon* if $X_f \cap I_f = \emptyset$ and the set I_f is f^{-1} -attracting.

Example 1 The Hénon's applications $f(x, y) = (x^2 + ay, x)$, $a \in \mathbb{C}^*$ are quasi-Hénon automorphisms of $\mathbb{C}^2$ if $|a| < 1$.

In the following subsections, $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ will be supposed to be a quasi-Hénon map. We then have $\lambda_n = 1$ and $f_* = (f^{-1})^*$ on $\mathbb{C}^n$.

3.2 Equidistribution result for currents

The following result, which generalizes a theorem of [3] (see also Theorem 7.17 in [12]), will allow us to study locally the current $dd^c G$ in dimension $n > 1$.

Lemma 7 Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a quasi-Hénon map. Let S be a closed positive $(1, 1)$ -current on $\mathbb{P}^n$, with mass 1, without Lelong numbers, and let $\xi \geq 0$ be a C^∞ function of compact support in $\mathbb{C}^n \subset \mathbb{P}^n$. Then, denoting $c := \int_{\mathbb{P}^n} \xi S \wedge T^- \geq 0$, we have

$$\lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} f^{m*}(\xi S) = c \cdot T^+.$$

On the other hand, denoting $c' := \int_{\mathbb{P}^n} \xi S \wedge T^+ \geq 0$, we have

$$\lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} f_*^m(\xi T^-) = \lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} f_*^m(\xi \omega^{n-1}) = c' \cdot T^-.$$

Proof We will give the proof in the case $n > 2$, the case $n = 2$ being only a simplified version. We may suppose, without loss of generality, that $0 < \xi \leq 1$. Let ω be the Fubini-Study $(1, 1)$ -form. Let ϕ be a closed $(0, 1)$ -form.

We put $S_m := \frac{1}{\lambda_1^m} f^{m*}(\xi S)$. Let us first show that the sequence of currents $(S_m)_m$ is bounded. It follows, using the fact that the currents $\frac{1}{\lambda_1^m} f_*^m \omega^{n-1}$ are cohomologous, that $\forall m \in \mathbb{N}$,

$$\int_{\mathbb{P}^n} S_m \wedge \omega = \int_{\mathbb{P}^n} \xi S \wedge \frac{1}{\lambda_1^m} f_*^m \omega^{n-1} \leq \int_{\mathbb{P}^n} S \wedge \frac{1}{\lambda_1^m} f_*^m \omega^{n-1} = \int_{\mathbb{P}^n} S \wedge \omega^{n-1} < +\infty.$$

The mass of the currents S_m is thus uniformly bounded.

Let us now show that an accumulation point S_∞ of the sequence $(S_m)_m$ is closed. The Cauchy-Schwarz inequality gives:

$$\begin{aligned}
 \lim_{m \rightarrow +\infty} \left| \int_{\mathbb{P}^n} \partial S_m \wedge \phi \wedge \omega^{n-2} \right| &= \lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} \left| \int_{\mathbb{C}^n} \partial \xi \wedge S \wedge f_*^m(\omega^{n-2}) \wedge f_*^m \phi \right| \\
 &\leq \lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} \left| \int_{\mathbb{C}^n} f_*^m(\omega^{n-2}) \wedge S \wedge \partial \xi \wedge i \bar{\partial} \xi \right|^{1/2} \cdot \left| \int_{\mathbb{C}^n} f_*^m(\omega^{n-2} \wedge \phi \wedge i \bar{\phi}) \wedge S \right|^{1/2} \\
 &= \lim_{m \rightarrow +\infty} \left(\frac{\lambda'_{n-2}}{\lambda_1} \right)^{m/2} \left| \int_{\mathbb{C}^n} \left(\frac{1}{\lambda'_{n-2}} \right)^m f_*^m(\omega^{n-2}) \wedge S \wedge \partial \xi \wedge i \bar{\partial} \xi \right|^{1/2} \\
 &\quad \cdot \left| \int_{\mathbb{C}^n} \left(\frac{1}{\lambda_1} \right)^m f_*^m(\omega^{n-2} \wedge \phi \wedge i \bar{\phi}) \wedge S \right|^{1/2} \\
 &= \lim_{m \rightarrow +\infty} \left(\frac{\lambda_2}{\lambda_1} \right)^{m/2} \left| \int_{\mathbb{C}^n} \left(\frac{1}{\lambda_2} \right)^m f_*^m(\omega^{n-2}) \wedge S \wedge \partial \xi \wedge i \bar{\partial} \xi \right|^{1/2} \\
 &\quad \cdot \left| \int_{\mathbb{C}^n} \omega^{n-2} \wedge \phi \wedge i \bar{\phi} \wedge \left(\frac{1}{\lambda_1} \right)^m f^{m*} S \right|^{1/2} \\
 &= \lim_{m \rightarrow +\infty} O \left(\left(\frac{\lambda_2}{\lambda_1} \right)^{m/2} \right) = 0.
 \end{aligned}$$

We will further show that $S_\infty = c \cdot T^+$. By virtue of Theorem 1.1 in [2], we have $\frac{1}{\lambda_1^m} f^{m*} S \rightarrow T^+$ in the sense of currents. Since $S_m \leq \frac{1}{\lambda_1^m} f^{m*} S$ for all m , due to the positivity of the operator f^* , we have $S_\infty \leq T^+$. The extremality of T^+ (Theorem 2.1 in [16]) leads to $S_\infty = c \cdot T^+$, with $c := \int_{\mathbb{P}^n} \xi S \wedge T^- \geq 0$. (Note that for $n = 2$, the assumption that S does not have Lelong numbers can be removed thanks to Theorem A in [23].)

In order to show that $\lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} f_*^m(\xi S) = c \cdot T^-$ when $S = \omega$ or T^- , the proof proceeds similarly, with the following precision. If S_∞ is an accumulation point of the sequence of term $\frac{1}{\lambda_1^m} f_*^m(\xi S)$, then $S_\infty = c' \cdot T^-$ follows from Theorem 2.6 in [16]. $\square$

Let us give some consequences. Recall that we denote $K^+ := \{G = 0\}$.

Proposition 5 *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $n \geq 1$ be a quasi-Hénon map. Then, ∂K^+ is the support of T^+ .*

Proof Recall that we denote $J := \text{supp}(T^+) = \text{supp}(dd^c G)$. Note that the sets K^+ , ∂K^+ and J are invariant by f and by f^{-1} .

There exists a C^∞ function $\xi \geq 0$ with compact support in $\mathring{K}^+$ whose interior intersects J^- . Using Lemma 7, if ω is the Fubini-Study $(1, 1)$ -form, there exists $c > 0$ such that:

$$\lim_{m \rightarrow +\infty} \frac{1}{\lambda_1^m} f^{m*}(\xi \omega) = c \cdot T^+.$$

It follows that $J \subset K^+$, and more precisely that $J \subset \partial K^+$.

Conversely, suppose by contradiction that there exists $x \in \partial K^+ \setminus J$. The function G is harmonic in a small neighborhood W of x , where it has a minimum $G(x) = 0$, hence it vanishes in W , which is impossible. Thus, $J = \partial K^+$. $\square$

Lemma 8 *The filled Julia set $\overline{K^+}$ of a quasi-Hénon map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $n \geq 2$, if it is of non-empty interior, satisfies $\mathring{K}^+ = K^+$.*

Proof Let $\xi\omega$ be a $(1, 1)$ -form of support in $\mathring{K}^+$, where ω is the Fubini-Study form and $\xi \geq 0$ is a C^∞ function of compact support in $\mathring{K}^+$ intersecting J^- . Note that K^+ is invariant by f and by f^{-1} , thus $\mathring{K}^+$ is also. Note that, by Proposition 5, $J = \partial K^+$. The support of $f^{m*}(\xi\omega)$ being in $\mathring{K}^+$ for all m , by Lemma 7, every open subset of $\mathbb{C}^n$ intersecting $J = \partial K^+$ also intersects $\mathring{K}^+$, thus $\mathring{K}^+ = K^+$. $\square$

3.3 Equidistribution result for measures

We give conditions under which the means of pull-backs $\mu_m := \frac{1}{m} \sum_{i=0}^m f^{i*}\mu$ converge to the invariant measure $T^+ \wedge T^-$. Recall that we denote by J and, respectively, J^- the supports of the currents T^+ and T^- .

Lemma 9 *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a quasi-Hénon map. Let U be an open set intersecting $J \cap J^-$. Then, $\bigcup_{m \geq 0} f^m(U|_{J^-})$, and even $\bigcap_{i \geq 0} \bigcup_{m \geq i} f^m(U|_{J^-})$, are dense in J^- for the induced topology.*

More precisely, for every open set V intersecting J^- , there exists $\epsilon > 0$ such that

$$\|T^-\| (f^m(U|_{J^-}) \cap V) > \epsilon$$

for m sufficiently large.

Proof Let ξ be a positive C^∞ function with compact support Z , such that $U := \mathring{Z}$ intersects $\text{supp}(T^+ \wedge T^-) \subset J \cap J^-$. By Lemma 7, the sequence of currents $\left(\frac{1}{\lambda_1^m} f^m_*(\xi T^-)\right)_m$ converges to $c \cdot T^-$, with $c = \int \xi T^- \wedge T^+ > 0$, when $m \rightarrow +\infty$.

But the support of $\frac{1}{\lambda_1^m} f^m_*(\xi T^-)$ is $f^m(Z) \cap J^-$, hence for every open set V intersecting J^- , $\exists m_0 \in \mathbb{N}$ such that $f^m(U|_{J^-}) \cap V \neq \emptyset$ when $m \geq m_0$. In other words, $f^m(U|_{J^-}) \cap V|_{J^-} \neq \emptyset$ when $m \geq m_0$. Therefore, $\bigcup_{m \geq 0} f^m(U|_{J^-})$ is dense in J^- for the induced topology, and even $\bigcap_{i \geq 0} \bigcup_{m \geq i} f^m(U|_{J^-})$.

More precisely, it follows from the convergence of $\left(\frac{1}{\lambda_1^m} f^m_*(\xi T^-)\right)_m$ to $c \cdot T^-$ when $m \rightarrow +\infty$ that there exists $\epsilon > 0$ such that $\|T^-\| (f^m(U|_{J^-}) \cap V) > \epsilon$ when $m \geq m_0$. Indeed, let us suppose that $\max_{x \in \mathbb{C}^n} \xi(x) = 1$, and consider a test function ρ positive on V , such that $\max_{x \in \mathbb{C}^n} \rho(x) = 1$. Then,

$$\begin{aligned} \|T^-\| (f^m(U|_{J^-}) \cap V) &= \frac{1}{\lambda_1^m} \int_{f^m(U) \cap V} (f^m_* T^-) \wedge \omega \geq \frac{1}{\lambda_1^m} \int_{\mathbb{C}^n} (f^m_* \xi T^-) \wedge \rho \omega \\ &\xrightarrow{m \rightarrow +\infty} c \int_{\mathbb{C}^n} T^- \wedge \rho \omega > 0. \end{aligned}$$

$\square$

We can now give the following convergence result (see for example [30] or [19] for the definition of an ergodic measure). The fact that the support of $T^+ \wedge T^-$ is $J \cap J^-$ is proved in dimension 2 in [10] in a little different context.

Proposition 6 *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a quasi-Hénon map. If μ is a probability measure with compact support on J^- that does not charge pluripolar sets, then the sequence of measures $(\mu_m := \frac{1}{m} \sum_{i=0}^m f^{i*}\mu)_m$ converges weakly to the invariant probability measure $T^+ \wedge T^-$. Moreover, the measure $T^+ \wedge T^-$ is ergodic, and its support is the compact set $J \cap J^-$.*

Proof Let us denote by μ_f an accumulation point of the sequence of probability measures $(\frac{1}{m} \sum_{i=0}^m f^{i*} \mu)_m$. By construction, we have $f^* \mu_f = \mu_f$, with $\text{supp}(\mu_f) \subset J \cap J^-$. It remains to show the inclusion in the opposite direction.

(1) Let us show that the support of $T^+ \wedge T^-$ is exactly $J \cap J^-$. Let $U \subset \mathbb{C}^n$ be an open set intersecting $J \cap J^-$. Denote by ξ a positive test function such that $\xi \equiv 1$ on U . By Lemma 7, we have

$$\frac{1}{\lambda_1^m} f^m_*(\xi T^-) \rightarrow c \cdot T^-$$

for a constant $c \geq 0$. We will show that $c > 0$. Since $c = \int_{\mathbb{C}^n} \xi d(T^+ \wedge T^-)$, this will imply that the support of $T^+ \wedge T^-$ is included in $J \cap J^-$.

Note that $\forall m \in \mathbb{N}$, we have

$$\frac{1}{\lambda_1^m} f^m_*(\xi T^-) \geq \frac{1}{\lambda_1^m} f^m_* T^-|_U = T^-|_{f^m(U)}.$$

But by Lemma 9, for every open set V intersecting J^- , $\exists \epsilon > 0$, such that for m sufficiently large, we have

$$\|T^-\| (f^m(U|_{J^-}) \cap V) > \epsilon. \quad (10)$$

Thus, $c > 0$. Finally $\text{supp}(T^+ \wedge T^-) \subset J \cap J^-$. The inverse inclusion is trivial.

(2) Let ρ be a test function whose support U intersects $J^+ \cap J^-$. We can decompose ρ into the difference of two positive continuous plurisubharmonic functions $\rho = u - v$. (For example, we can choose $u = \rho + A \log(1 + \|z\|^2)$ and $v = A \log(1 + \|z\|^2)$ for A sufficiently large.) Consider the sequence of functions $(\phi_m)_m := (\frac{1}{m} \sum_{i=1}^m f^{i*} u|_{J^-})_m$ defined on $\mathcal{S}$.

Note that this sequence is bounded on every compact of J^- : indeed, let U be an open set containing $J \cap J^-$, and denote H the compact set $J^- \cap \{G = 0\}$. Then, $U|_{J^-} \subset \subset f(U)|_{J^-} \cup H$. Consequently, we have $\forall m \in \mathbb{N}$, $\sup_{U|_{J^-}} \phi_m \leq \sup_{U|_{J^-} \cup H} u$.

Denote $\phi_\infty := \limsup_{m \rightarrow +\infty} \phi_m$. Then, by Lemma 2, ϕ_∞^* is a T -psh function. Since ϕ_∞ satisfies by construction $f_* \phi_\infty = \phi_\infty$, we have by Lemma 2:

$$f_* \phi_\infty^* = \phi_\infty^*. \quad (11)$$

(3) Let us show that ϕ_∞^* is constant on J^- . Let $c := \max_{J \cap J^-} \phi_\infty^*$ and $\epsilon > 0$. Then, $J \cap J^-$ is included in the open set $U := \{\phi_\infty^* < c + \epsilon\}$. Then, (11) implies that $\forall m \in \mathbb{N}$,

$$f^m(U) = \{f^m_* \phi_\infty^* < c + \epsilon\} = U.$$

We have $\phi_\infty^* \leq c + \epsilon$ on $U = \bigcup_{m \geq 0} f^m(U)$, and this open set being dense in J^- for the induced topology by Lemma 9, Theorem 3 and Corollary 3 imply $\phi_\infty^* \leq c + \epsilon$ on J^- . The constant $\epsilon > 0$ being arbitrary, it follows that $\phi_\infty^* \leq c$ on J^- . Instead of invoking Proposition 3, let us pursue with some dynamical arguments.

Let $d := \inf_{J \cap J^-} \phi_\infty^*$ and $\epsilon > 0$. Let $V := \{\phi_\infty^* < d - \epsilon\}$. Then, $J \cap J^- \cap \bar{V} = \emptyset$. As previously we have $V = f(V)$; since for any bounded set W containing $J \cap J^-$ there exists $m_0 \in \mathbb{N}$ such that $f^m(V) \cap W \neq \emptyset$ for $m \geq m_0$, we obtain $V = \emptyset$. Therefore, $\phi_\infty^* \geq d$ on J^- .

If $\phi_\infty^*|_{J \cap J^-}$ is not constant on $J \cap J^-$, there exists $\epsilon > 0$ such that the open set $W := \{\phi_\infty^* < c - \epsilon\}$ is non-empty and intersects $J \cap J^-$. Moreover, we have again $f(W) = W$. By Lemma 9, $W = \bigcup_{m \geq 0} f^m(W)$ is an open set dense in J^- . By Theorem 3 and Corollary 3,

$\phi_\infty^* < c - \epsilon$ in $J \cap J^-$, which contradicts the definition of $c := \max_{J \cap J^-} \phi_\infty^*$. Therefore, $\phi_\infty^* = c$ on $J \cap J^-$. Thus, $c = d$, hence $\phi_\infty^* = c$ on J^- .

(4) *Conclusion.* The sequence $(\phi_m)_m$ converges in $L^1(T^+ \wedge T^-)$ by Birkhoff's ergodic theorem. The set $\{\phi_\infty < \phi_\infty^*\}$ is pluripolar by Lemma 6. Since the measure $T^+ \wedge T^-$ does not charge pluripolar sets, we deduce that the sequence $(\phi_m)_m$ converges $T^+ \wedge T^-$ -a.e. toward $c = \limsup_{m \rightarrow +\infty} \phi_m$. Proposition 3 implies that this sequence converges $\|T^-\|$ -a.e. toward c on J^- . Let us denote by $(\mu_{m_i})_i$ a sequence of measures, extracted from $(\mu_m)_m$, which converges to μ_f . Since the measure μ does not charge, by hypothesis, the pluripolar sets, thanks to the dominated convergence theorem, there exists a constant $C \in \mathbb{R}$ such that

$$\langle \mu_f, \rho \rangle = \lim_{i \rightarrow +\infty} \left\langle \mu, \frac{1}{m_i} \sum_{j=1}^{m_i} f^j *_i u \right\rangle - \lim_{i \rightarrow +\infty} \left\langle \mu, \frac{1}{m_i} \sum_{j=1}^{m_i} f^j *_i v \right\rangle = C.$$

On the other hand, since the measure $T^+ \wedge T^-$ does not charge pluripolar sets (see, e.g., [14]), we have

$$\begin{aligned} \langle T^+ \wedge T^-, \rho \rangle &= \lim_{i \rightarrow +\infty} \left\langle T^+ \wedge T^-, \frac{1}{m_i} \sum_{j=1}^{m_i} f^j *_i u \right\rangle \\ &\quad - \lim_{i \rightarrow +\infty} \left\langle T^+ \wedge T^-, \frac{1}{m_i} \sum_{j=1}^{m_i} f^j *_i v \right\rangle = C. \end{aligned}$$

Thus, $T^+ \wedge T^- = \mu_f$. The ergodicity of this invariant measure follows directly from Theorem 4.4.1 in [19]. $\square$

Remark 6 Let us emphasize that, in the proof of Proposition 6, we have obtained that if ρ is a test function, the sequence of functions $(\frac{1}{m} \sum_{i=1}^m f^i *_i \rho)_m$ converges toward a constant C outside a pluripolar set, and therefore converges in $L^1_{\text{loc}}(T^+ \wedge T^-)$ toward C thanks to the dominated convergence theorem. Moreover, $C > 0$ if and only if the support of $\rho > 0$ intersects $J \cap J^-$.

We thus deduce an alternative proof of the point 1) of the proof of Proposition 6 above. Let $\xi > 0$ be a test function with support intersecting $J \cap J^-$. Then, $\forall m \in \mathbb{N}$,

$$\int_{\mathbb{C}^n} \xi d(T^+ \wedge T^-) = \int_{\mathbb{C}^n} \frac{1}{m} \sum_{i=1}^m f^i *_i \xi d(T^+ \wedge T^-).$$

By taking $m \rightarrow +\infty$, since $T^+ \wedge T^-$ does not charge pluripolar sets, the dominated convergence theorem implies that $\int_{\mathbb{C}^n} \xi d(T^+ \wedge T^-) > 0$, which concludes.

The following remark generalizes Theorem 6.6 in [13].

Remark 7 We deduce from Remark 6 that for every open set U intersecting $J \cap J^-$, the set $J^- \setminus \bigcup_{m \geq 0} f^m(U|_{J^-})$ is pluripolar.

Proof Indeed, let ρ be a strictly positive test function of support included in U , and whose interior intersects $J \cap J^-$. Then,

$$\phi_\infty := \limsup_{m \rightarrow +\infty} \frac{1}{m} \sum_{i=1}^m f^i *_i \rho \leq \limsup_{m \rightarrow +\infty} \frac{1}{m} \sum_{i=1}^m f^i *_i \mathbf{1}_U.$$

On the other hand, Remark 6 shows that ϕ_∞ is equal, outside a pluripolar set, to a constant $C > 0$. Thus,

$$J^- \setminus \bigcup_{m \geq 0} f^m(U|_{J^-}) \subset \left\{ \limsup_{m \rightarrow +\infty} \frac{1}{m} \sum_{i=1}^m f^i_* \mathbf{1}_U = 0 \right\} \subset \{\phi_\infty = 0\} \subset \{\phi_\infty < C\}.$$

Thus, all these sets are pluripolar, which proves the statement. $\square$

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