(A, δ) -modules, Hochschild homology and higher derivations

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Received: 15 December 2018 / Accepted: 18 March 2019 / Published online: 28 March 2019
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Abstract

In this paper, we develop the theory of modules over (A, δ) , where A is an algebra and $\delta : A \rightarrow A$ is a derivation. Our approach is heavily influenced by Lie derivative operators in noncommutative geometry, which make the Hochschild homologies $HH_\bullet(A)$ of A into a module over (A, δ) . We also consider modules over (A, Δ) , where $\Delta = \{\Delta^n\}_{n \geq 0}$ is a higher derivation on A . Further, we obtain a Cartan homotopy formula for an arbitrary higher derivation on A .

Keywords A, δ -modules · Hochschild homology · Higher derivations

Mathematics Subject Classification 13N15 · 16W25

1 Introduction

Let k be a commutative ring having characteristic zero and let A be a commutative k -algebra equipped with a k -linear derivation $\delta : A \rightarrow A$. An (A, δ) -module is a pair (M, δ_M) which consists of an A -module M and a k -linear map $\delta_M : M \rightarrow M$ satisfying the condition

$$\delta_M(a \cdot m) = \delta(a) \cdot m + a \cdot \delta_M(m) \quad (1.1)$$

for any $a \in A$ and any $m \in M$. The basics of homological algebra over such objects were developed by Tanaka [17], who referred to them as “pre- (A, δ) -modules.” The purpose of this paper is to proceed as if the pair (A, δ) is a commutative ring and to build a theory of (A, δ) -modules similar to that of modules over commutative rings. Our approach is heavily influenced by noncommutative geometry, especially by the observation that the pair $(HH_\bullet(A), L_\delta)$ consisting of the Hochschild homology $HH_\bullet(A)$ equipped with the Lie

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derivative L_δ induced by the derivation δ on A becomes an (A, δ) -module in the sense of (1.1).

We now describe the paper in more detail. We begin in Sect. 2 by describing the category $(A, \delta)\text{-mod}$ of (A, δ) -modules. In particular, we show that $(A, \delta)\text{-mod}$ is a closed symmetric monoidal category as well as give several examples of (A, δ) -modules constructed from polynomial rings and their quotients. In Sect. 3, we explain how the Lie derivative L_δ on Hochschild homologies makes $(HH_\bullet(A), L_\delta)$ into an (A, δ) -module. Further, we show that Galois descent of Hochschild homology of commutative rings extends to the (A, δ) -case.

Thereafter, in Sect. 4, we present a descent result for (A, δ) -modules that extends Beauville–Laszlo descent (see [4, 5]) from commutative algebra. For a nonzero divisor $s \in A$, the construction recovers an (A, δ) -module given its localization over (A_s, δ_s) and its completion over $(\hat{A}, \hat{\delta})$ with respect to the (s) -adic topology. We should mention that the setup of Beauville and Laszlo is one of the handful of examples of descent that does not involve any noetherian assumptions.

It was shown by the first author in [1] that a higher derivation $\Delta = \{\Delta^n\}_{n \geq 0}$ on A (see Definition 5.1) induces Lie derivative operators $L_\Delta = \{L_\Delta^l\}_{l \geq 0}$ on the Hochschild homologies of A . In Sect. 5, this motivates us to define and study modules over (A, Δ) and develop them in a manner that extends our study of (A, δ) -modules in previous sections. Our first main result for (A, Δ) -modules is that every (A, Δ) -module corresponds to a unique sequence of modules over (A, δ^n) , $n \geq 1$, where $\{\delta^n\}_{n \geq 1}$ is the sequence of ordinary derivations on A determining the higher derivation Δ (see [13]). We continue in Sect. 6 by taking advantage of this fact to produce a closed symmetric monoidal structure on the category of (A, Δ) -modules.

In the last two sections of this paper, we work in the completely general setup where A is not assumed to be commutative. We recall that for a k -linear derivation δ on A , the Lie derivative L_δ on the normalized Hochschild complex $\hat{C}_\bullet^h(A)$ satisfies the Cartan homotopy formula (see for instance, [12, § 4.1.8])

$$L_\delta = [e_\delta, B] + [E_\delta, b] \quad (1.2)$$

Here b is the differential on the Hochschild complex, while B is the Connes differential used to define cyclic homology. In the context of noncommutative geometry, the operator L_δ is the analogue of the derivation induced by a vector field, while the operator e_δ of degree -1 plays a role similar to that of the interior product (see [12, § 4.1.7]). The formula in (1.2) is due to Goodwillie [9] (see also Rinehart [14]).

In this paper, the operators L_Δ^k , $k \geq 1$ from [1] corresponding to the higher derivation Δ are recast in a more convenient format by using homomorphisms of formal power series rings. As a consequence, it follows easily that if $\Delta = \{\Delta^n\}_{n \geq 0}$ and $\Delta' = \{\Delta'^n\}_{n \geq 0}$ are two higher derivations on A that are equivalent in the sense of Roy and Sridharan [15], they must induce identical operators $L_\Delta^k = L_{\Delta'}^k$ on Hochschild homology groups for each $k \geq 1$. In particular, we see that the Lie derivative induced by an inner higher derivation must be zero. This fully answers the question that the first author had in mind during the writing of [1], where the latter fact was proven only for one specific inner higher derivation on A .

In addition to constructing the Lie derivative L_Δ for a higher derivation Δ , it is natural to ask whether there also exist higher-order versions of the operators e_δ and E_δ appearing in (1.2), so that we can obtain a full Cartan homotopy formula for higher derivations. The recasting of the Lie derivative L_Δ using formal power series gives us exactly the intuition we need. In other words, for every $k \geq 1$, we define operators e_Δ^k and E_Δ^k satisfying the following higher version of Cartan homotopy formula (see Sect. 8)

$$[e_\Delta^k, B] + [E_\Delta^k, b] = L_\Delta^k \quad (1.3)$$

on the normalized Hochschild complex $\bar{C}_\bullet^h(A)$ of A .

We mention here that in future we also hope to explore possible relations between (A, δ) -modules and modules over the Lie algebra $Der(A)$ of derivations on A .

2 Category of (A, δ) -modules

Let k be a commutative ring having characteristic zero. Unless otherwise mentioned, we let A be a commutative unital k -algebra equipped with a k -derivation $\delta : A \rightarrow A$.

Definition 2.1 An (A, δ) -module is a pair (M, δ_M) consisting of the following data:

- (a) An A -module M .
- (b) A morphism $\delta_M : M \rightarrow M$ of k -modules satisfying the condition:

$$\delta_M(a \cdot m) = \delta(a) \cdot m + a \cdot \delta_M(m)$$

for any $a \in A$ and any $m \in M$.

A morphism $f : (M, \delta_M) \rightarrow (N, \delta_N)$ of (A, δ) -modules consists of a morphism $f : M \rightarrow N$ of A -modules satisfying the condition $\delta_N \circ f = f \circ \delta_M : M \rightarrow N$. We will refer to these as δ -morphisms, and when there is no danger of confusion, their collection will be denoted by $\text{Hom}_\delta(M, N)$. The category of (A, δ) -modules will be denoted by $(A, \delta)\text{-mod}$.

In other words, the objects that we call (A, δ) -modules in Definition 2.1 are those that Tanaka [17, Definition 3.1] called “pre- (A, δ) -modules” (see also Bland [6]). A similar notion, in the context of “rings with several objects” has been studied in [3].

We should mention here that the derivation δ on A may be used to define a skew polynomial ring $A[x; \delta]$ (see [8, p. 27]). As a k -module, $A[x; \delta]$ is identical to $A[x]$. The multiplication on $A[x; \delta]$ is determined by setting $xa = ax + \delta(a)$ for every $a \in A$. Then, if M is an object of the category $A[x; \delta]\text{-mod}$ (resp. $\text{mod} - A[x; \delta]$) of left (resp. right) modules over $A[x; \delta]$, it may be easily verified that setting $\delta_M(m) := xm$ (resp. $\delta_M(m) := -mx$) for each $m \in M$ makes (M, δ_M) into an (A, δ) -module. This identifies (A, δ) -modules with left (resp. right) modules over the ring $A[x; \delta]$.

However, in the context of this paper, we want to work as if the pair (A, δ) is a commutative ring and we are interested in properties that $(A, \delta)\text{-mod}$ shares with modules over commutative rings. In general, the skew polynomial ring $A[x; \delta]$ is not commutative. Further, even when $\delta = 0$ and hence $A[x; \delta] = A[x]$ as an algebra, the tensor structure on $A[x]\text{-mod}$ is different from the tensor structure $_ \otimes_A _$ on $(A, 0)\text{-mod}$ that we will describe below. In Sect. 3, we extend Galois descent for Hochschild homology (see [7]) of commutative algebras to the (A, δ) -case. In Sect. 4, we extend the descent result of Beauville and Laszlo (see [4, 5]) involving localizations and completions from commutative algebra to the (A, δ) -case. As such, in our study of (A, δ) -modules, we will typically not look at the module structure over the skew polynomial ring $A[x; \delta]$.

We begin by showing that $(A, \delta)\text{-mod}$ forms a symmetric monoidal category. Given (A, δ) -modules (M, δ_M) and (N, δ_N) , we set:

$$\delta_{(M \otimes N)}(m \otimes n) := \delta_M(m) \otimes n + m \otimes \delta_N(n) \quad (2.1)$$

for any $m \in M$ and $n \in N$. It may be easily verified that $(M, \delta_M) \otimes_{(A, \delta)} (N, \delta_N) := (M \otimes_A N, \delta_{M \otimes N})$ is an (A, δ) -module.

We also define an internal Hom object in (A, δ) -mod as follows: Given $(M, \delta_M), (N, \delta_N) \in (A, \delta)$ -mod, we set

$$\underline{\text{Hom}}_{(A, \delta)}((M, \delta_M), (N, \delta_N)) := (\text{Hom}_A(M, N), \underline{\delta}) \quad (2.2)$$

where $\text{Hom}_A(M, N)$ is the collection of A -module morphisms from M to N and

$$\underline{\delta}(f) := \delta_N \circ f - f \circ \delta_M \quad (2.3)$$

for any $f \in \text{Hom}_A(M, N)$. When there is no danger of confusion, this internal Hom object will often be denoted simply by $\underline{\text{Hom}}_\delta(M, N)$. Using the expression in (2.3), it is easily verified that $\underline{\delta}(f) \in \text{Hom}_A(M, N)$ for any $f \in \text{Hom}_A(M, N)$ and that $\underline{\delta}(a \cdot f) = \delta(a) \cdot f + a \cdot \underline{\delta}(f)$.

Proposition 2.2 *Let $(M, \delta_M), (N, \delta_N)$ and (P, δ_P) be (A, δ) -modules. Then, we have natural isomorphisms:*

$$\text{Hom}_\delta(M \otimes_A N, P) \xrightarrow{\cong} \text{Hom}_\delta(M, \underline{\text{Hom}}_\delta(N, P)) \quad (2.4)$$

Proof We already have an isomorphism $\text{Hom}_A(M \otimes_A N, P) \cong \text{Hom}_A(M, \text{Hom}_A(N, P))$ of A -modules. Suppose that $f \in \text{Hom}_\delta(M \otimes_A N, P)$. Then, f corresponds to a morphism $g : M \rightarrow \text{Hom}_A(N, P)$ given by

$$g(m)(n) := f(m \otimes n) \quad (2.5)$$

for any $m \in M, n \in N$. Considering $\underline{\delta} : \text{Hom}_A(N, P) \rightarrow \text{Hom}_A(N, P)$, we have:

$$\begin{aligned} (\underline{\delta} \circ g)(m)(n) &= \underline{\delta}(g(m))(n) = (\delta_P \circ g(m) - g(m) \circ \delta_N)(n) \\ &= \delta_P(f(m \otimes n)) - f(m \otimes \delta_N(n)) \\ &= f(\delta_{M \otimes N}(m \otimes n)) - f(m \otimes \delta_N(n)) = f(\delta_M(m) \otimes n) \end{aligned} \quad (2.6)$$

On the other hand, we also have:

$$(g \circ \delta_M)(m)(n) = g(\delta_M(m))(n) = f(\delta_M(m) \otimes n) \quad (2.7)$$

This shows that $g \in \text{Hom}_\delta(M, \underline{\text{Hom}}_\delta(N, P))$. Conversely, we may also verify easily that if $g' : M \rightarrow \text{Hom}_A(N, P)$ is an A -module map such that $g' \in \text{Hom}_\delta(M, \underline{\text{Hom}}_\delta(N, P))$, the corresponding morphism $f' : M \otimes_A N \rightarrow P$ also lies in $\text{Hom}_\delta(M \otimes_A N, P)$. $\square$

Theorem 2.3 *The category of (A, δ) -modules is a closed symmetric monoidal category.*

Proof The symmetric monoidal structure is defined in (2.1), and the internal Hom object is defined in (2.2) and (2.3). The required adjointness property involving the internal Hom object is shown in Proposition 2.2. $\square$

Let $f : A \rightarrow A'$ be a morphism of commutative k -algebras. Suppose further that A' is equipped with a k -linear derivation $\delta' : A' \rightarrow A'$ that satisfies $\delta'(f(a)) = f(\delta(a))$ for any $a \in A$, in which case f extends to a morphism $A[x; \delta] \rightarrow A'[x; \delta']$ of skew polynomial rings. Then, it is clear that (A', δ') is an (A, δ) -module

$$\delta'(f(a)a') = \delta'(f(a))a' + f(a)\delta'(a) = f(\delta(a))a' + f(a)\delta'(a) \quad a \in A, a' \in A' \quad (2.8)$$

In this situation, we will say that (A', δ') is an (A, δ) -algebra. Accordingly, we have a pair of adjoint functors given by extension and restriction of scalars. For our purposes, we will describe these functors in the following result.

Proposition 2.4 *In the above setup, we have:*

- (a) *A functor $f^* : (A, \delta)\text{-mod} \rightarrow (A', \delta')\text{-mod}$ that is an ‘extension of scalars.’*
- (b) *A functor $f_* : (A', \delta')\text{-mod} \rightarrow (A, \delta)\text{-mod}$ that is a ‘restriction of scalars.’*
- (c) *The functor f^* is left adjoint to f_* .*

Proof Let $(M, \delta_M) \in (A, \delta)\text{-mod}$ and $(N, \delta_N) \in (A', \delta')\text{-mod}$. We have observed above that (A', δ') may be treated as an object of $(A, \delta)\text{-mod}$. As such, we set $f^*(M, \delta_M) := (M, \delta_M) \otimes_{(A, \delta)} (A', \delta')$. On the other hand, it is clear that $(N, \delta_N) \in (A', \delta')\text{-mod}$ restricts to an object of $(A, \delta)\text{-mod}$. We already know that

$$\text{Hom}_{A'\text{-mod}}(M \otimes_A A', N) = \text{Hom}_{A\text{-mod}}(M, N) \quad (2.9)$$

Suppose that $g : M \otimes_A A' \rightarrow N$ is an A' -module morphism. Then, g corresponds to an A -module morphism $h : M \rightarrow N$ given by $h(m) = g(m \otimes 1)$ for any $m \in M$. If $g : M \otimes_A A' \rightarrow N$ is additionally a δ' -morphism, we see that h must be a δ -morphism:

$$\delta_N(h(m)) = \delta_N(g(m \otimes 1)) = g(\delta_M(m) \otimes 1 + m \otimes \delta'(1)) = g(\delta_M(m) \otimes 1) = h(\delta_M(m))$$

Conversely if $h : M \rightarrow N$ is an A -module morphism, it corresponds to the A' -module morphism $g : M \otimes_A A' \rightarrow N$ given by $g(m \otimes a') = a'h(m)$ for any $m \in M, a' \in A'$. Additionally, if h is a δ -morphism, we see that g is also a δ' -morphism:

$$\begin{aligned} \delta_N(g(m \otimes a')) &= \delta_N(a'h(m)) = \delta'(a')h(m) + a'\delta_N(h(m)) \\ &= g(m \otimes \delta'(a')) + a'h(\delta_M(m)) \\ &= g(m \otimes \delta'(a')) + g(\delta_M(m) \otimes a') \end{aligned} \quad (2.11)$$

This proves the result. $\square$

We now present several examples of (A, δ) -modules. Another important example, involving Hochschild homology, will be described in detail in the next section.

Example 2.5 The category of triples (M, ϕ, δ) where:

- (1) M is an A -module,
- (2) $\phi : M \rightarrow M$ is a morphism of A -modules and
- (3) $\delta : M \rightarrow M$ is an A -linear map satisfying the condition that

$$\delta \circ \phi = 1_M + \phi \circ \delta : M \rightarrow M \quad (2.12)$$

A morphism from (M, ϕ, δ) to (M', ϕ', δ') is an A -module map $f : M \rightarrow M'$ such that $f \circ \phi = \phi' \circ f$ and $f \circ \delta = \delta' \circ f$.

In fact, this is the category of modules over $(A[x], \frac{d}{dx})$, where $\frac{d}{dx}$ denotes the formal derivative with respect to x . Then, each pair (M, ϕ) corresponds to an $A[x]$ -module while the A -linear map $\delta : M \rightarrow M$ along with the condition in (2.12) makes (M, ϕ, δ) into an $(A[x], \frac{d}{dx})$ -module. In terms of skew polynomial rings, the $(A[x], \frac{d}{dx})$ -modules correspond to modules over the ring $A[x, y; \frac{d}{dx}]$ of polynomials in two variables with multiplication determined by $yx = xy + 1$.

Example 2.6 Fix a positive integer n . Consider the category of triples (M, ϕ, δ) where:

- (1) M is an A -module,
- (2) $\phi : M \rightarrow M$ is a morphism of A -modules satisfying $\phi^n = 0$ and
- (3) $\delta : M \rightarrow M$ is an A -linear map such that $\delta \circ \phi = \phi \circ (1_M + \delta) : M \rightarrow M$.

By reasoning similar to Example 2.5, the above may be viewed as the category of modules over $(\frac{A[x]}{(x^n)}, x \frac{d}{dx})$. In terms of skew polynomial rings, the $(\frac{A[x]}{(x^n)}, x \frac{d}{dx})$ -modules correspond to modules over the ring $\frac{A[x]}{(x^n)}[y; x \frac{d}{dx}]$ of polynomials with multiplication determined by $yx = xy + x$.

Example 2.7 Fix a pair (A, δ) . Consider the category of triples (M, ϕ, δ_M) where:

- (1) (M, δ_M) is an (A, δ) -module,
- (2) $\phi : M \rightarrow M$ is a morphism of A -modules and
- (3) $\delta_M : M \rightarrow M$ satisfies $\phi \circ \delta_M = \delta_M \circ \phi$.

Indeed, this may be viewed as the category of modules over $(A[x], \delta[x])$, where $\delta[x]$ is the extended derivation on $A[x]$ defined by $(\delta[x])(\sum a_i x^i) = \sum \delta(a_i) x^i$ for any polynomial $\sum a_i x^i \in A[x]$. In terms of skew polynomial rings, the $(A[x], \delta[x])$ -modules correspond to modules over the ring $A[x, y; \delta[x]]$ of polynomials with multiplication determined by $yx = xy$ and $ya = ay + \delta(a)$ for all $a \in A$.

3 Hochschild homology, Lie derivative and descent

For us, a compelling example of an (A, δ) -module comes from noncommutative geometry and more precisely, from the Hochschild homology groups. We recall (see for instance, [12, § 1.1]) that if A is an algebra over a commutative ring k , its Hochschild homology groups $HH_\bullet(A)$ are defined to be the homology groups of the complex $(C_\bullet^h(A), b)$, where $C_n^h(A) := A \otimes_k A^{\otimes n}$ and the differentials are given by

$$\begin{aligned} b : C_n^h(A) &\rightarrow C_{n-1}^h(A) \\ b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &:= (a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \end{aligned} \quad (3.1)$$

If A is commutative, the differential b is always a map of A -modules and it follows that each Hochschild homology group $HH_n(A)$, $n \geq 0$, becomes an A -module. Further, a k -derivation $\delta : A \rightarrow A$ on A gives rise to a linear operator

$$L_\delta^n : HH_n(A) \rightarrow HH_n(A) \quad (3.2)$$

for each $n \geq 0$. The operators $\{L_\delta^n\}_{n \geq 0}$ play the role of the Lie derivative in noncommutative geometry (see [12, § 4]).

Proposition 3.1 *Let A be a commutative k -algebra equipped with a k -linear derivation $\delta : A \rightarrow A$. Then, for any $n \geq 0$, the pair $(HH_n(A), L_\delta^n)$ is an (A, δ) -module.*

Proof We know that the Lie derivative on the Hochschild homology groups is induced by the following endomorphism of the Hochschild complex (see [12, § 4.1.4]):

$$\begin{aligned} L_\delta : C_\bullet^h(A) &\rightarrow C_\bullet^h(A) \\ L_\delta(a_0 \otimes \cdots \otimes a_n) &:= \sum_{i \geq 0} (a_0 \otimes \cdots \otimes \delta(a_i) \otimes \cdots \otimes a_n) \end{aligned} \quad (3.3)$$

From (3.3), it is clear that for any $a \in A$, we have:

$$\begin{aligned} L_\delta(aa_0 \otimes \cdots \otimes a_n) &= (\delta(aa_0) \otimes a_1 \otimes \cdots \otimes a_n) + \sum_{i \geq 1} (aa_0 \otimes \cdots \otimes \delta(a_i) \otimes \cdots \otimes a_n) \\ &= (\delta(a)a_0 \otimes a_1 \otimes \cdots \otimes a_n) + \sum_{i \geq 0} a \cdot (a_0 \otimes \cdots \otimes \delta(a_i) \otimes \cdots \otimes a_n) \\ &= \delta(a) \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n) + a \cdot L_\delta(a_0 \otimes \cdots \otimes a_n) \end{aligned} \quad (3.4)$$

The result now follows from (3.4). $\square$

We now consider a multiplicatively closed set $S \subseteq A$ and the corresponding localization A_S . The derivation δ on A may be extended to A_S by setting:

$$\delta_S\left(\frac{a}{s}\right) := \frac{\delta(a)s - a\delta(s)}{s^2} \quad (3.5)$$

for any $a \in A$ and $s \in S$. The derivation δ_S gives rise to Lie derivatives on the Hochschild homology groups of the algebra A_S and we may accordingly consider the (A_S, δ_S) -modules $(HH_n(A_S), L_{\delta_S}^n)$, $n \geq 0$. The following result shows, in particular, that the (A, δ) -modules $(HH_n(A), L_\delta^n)$ are well behaved with respect to localization.

Proposition 3.2 *Let $S \subseteq A$ be a multiplicatively closed set and let δ_S be the extension of the derivation $\delta : A \rightarrow A$ to the localization A_S . Then, for each $n \geq 0$, we have an isomorphism:*

$$(A_S, \delta_S) \otimes_{(A, \delta)} (HH_n(A), L_\delta^n) \cong (HH_n(A_S), L_{\delta_S}^n) \quad (3.6)$$

Proof The canonical morphism $A \rightarrow A_S$ induces a map on the Hochschild complexes $C_\bullet^h(A) \rightarrow C_\bullet^h(A_S)$. Since $\delta_S : A_S \rightarrow A_S$ extends the derivation $\delta : A \rightarrow A$, it is clear that the following diagram is commutative for each $n \geq 0$:

$$\begin{array}{ccc} C_n^h(A) & \longrightarrow & C_n^h(A_S) \\ L_\delta \downarrow & & L_{\delta_S} \downarrow \\ C_n^h(A) & \longrightarrow & C_n^h(A_S) \end{array} \quad (3.7)$$

It follows that each of the canonical maps $C_n^h(A) \rightarrow C_n^h(A_S)$ is a δ -morphism. Using Proposition 2.4, we obtain a δ_S -morphism $(A_S, \delta_S) \otimes_{(A, \delta)} (C_n^h(A), L_\delta^n) \rightarrow (C_n^h(A_S), L_{\delta_S}^n)$. Since A_S is a flat A -module, this induces a δ_S -morphism $(A_S, \delta_S) \otimes_{(A, \delta)} (HH_n(A), L_\delta^n) \rightarrow (HH_n(A_S), L_{\delta_S}^n)$ on the homology groups. From [12, § 1.1.17], we know that the underlying A_S -module map $A_S \otimes_A HH_n(A) \rightarrow HH_n(A_S)$ is an isomorphism. This proves the result. $\square$

The argument in Proposition 3.2 can be pushed further. We recall the classical result (see [10, Corollaire 20.5.8]) that if $A \rightarrow B$ is an étale extension of k -algebras, then the modules of relative differentials satisfy

$$\Omega_{A|k} \otimes_A B \cong \Omega_{B|k} \quad (3.8)$$

The derivation $\delta : A \rightarrow A$, when composed with the given étale map $A \rightarrow B$, corresponds to an element of $\text{Hom}_A(\Omega_{A|k}, B) \cong \text{Hom}_B(\Omega_{A|k} \otimes_A B, B) \cong \text{Hom}_B(\Omega_{B|k}, B)$. Hence, the derivation δ on A may be extended to a derivation δ_B on B .

Theorem 3.3 *Let A be a k -algebra equipped with a k -linear derivation $\delta : A \rightarrow A$. Let $f : A \rightarrow B$ be an étale extension of algebras and let $\delta_B : B \rightarrow B$ be the extension of the derivation δ to the k -algebra B . Then, for each $n \geq 0$, we have an isomorphism of (B, δ_B) -modules:*

$$(B, \delta_B) \otimes_{(A, \delta)} (HH_n(A), L_\delta^n) \cong (HH_n(B), L_{\delta_B}^n) \quad (3.9)$$

Proof We consider the induced morphism on the Hochschild complexes $C_\bullet^h(A) \rightarrow C_\bullet^h(B)$. Since δ_B extends the derivation δ on A , as in the proof of Proposition 3.2, we know that the following diagram is commutative

$$\begin{array}{ccc} C_n^h(A) & \longrightarrow & C_n^h(B) \\ L_\delta \downarrow & & L_{\delta_B} \downarrow \\ C_n^h(A) & \longrightarrow & C_n^h(B) \end{array} \quad (3.10)$$

and hence each of the maps $C_n^h(A) \rightarrow C_n^h(B)$ is a δ -morphism. The result of Proposition 2.4 now gives a δ_B -morphism $(B, \delta_B) \otimes_{(A, \delta)} (C_n^h(A), L_\delta^n) \rightarrow (C_n^h(B), L_{\delta_B}^n)$. Since $f : A \rightarrow B$ is an étale extension, we get a δ_B -morphism on the homologies $(B, \delta_B) \otimes_{(A, \delta)} (HH_n(A), L_\delta^n) \rightarrow (HH_n(B), L_{\delta_B}^n)$. From [7, § 2], we know that the underlying morphism on homologies $B \otimes_A HH_n(A) \rightarrow HH_n(B)$ induced by the map of complexes $B \otimes_A C_\bullet^h(A) \rightarrow C_\bullet^h(B)$ is an isomorphism. This proves the result. $\square$

4 A descent result for (A, δ) -modules

The purpose of this section is to prove a descent result for (A, δ) -modules, involving the localization and the completion of the ring A . Our result is an extension of a result of Beauville and Laszlo for ordinary rings (see [4, 5]). The fact that makes the Beauville–Laszlo descent theorem so exciting is that it is an example of nonnoetherian descent. A version of the result of Beauville and Laszlo for commutative monoidal objects in symmetric monoidal categories was proved by the first author in [2].

We begin with an element $s \in A$ that is not a zero divisor and suppose that $\delta(s) = 0$. Let A_s be the localization of A with respect to s . Throughout this section, we let $\hat{A}$ be the completion of A in the (s) -adic topology. Since $\delta(s) = 0$, the derivation δ extends to a derivation δ_s on A_s given by

$$\delta_s \left(\frac{a}{s^k} \right) := \frac{\delta(a)}{s^k} \quad \forall a \in A, k \geq 0 \quad (4.1)$$

On the other hand, any element of the completion $\hat{A}$ may be represented by a sequence $(a_1, a_2, \dots)$ of elements from A such that $(a_i - a_j)$ lies in the principal ideal (s^i) for any $j \geq i$. As such, the derivation δ extends to a derivation $\hat{\delta}$ on $\hat{A}$ determined by:

$$\hat{\delta}(a_1, a_2, \dots) := (\delta(a_1), \delta(a_2), \dots) \quad (4.2)$$

We also have a derivation $\hat{\delta}_s$ on $\hat{A}_s := \hat{A} \otimes_A A_s$ defined by setting

$$\hat{\delta}_s \left(\frac{\hat{a}}{s^k} \right) := \frac{\delta(\hat{a})}{s^k} \quad \forall \hat{a} \in \hat{A}, k \geq 0 \quad (4.3)$$

It is clear that (A_s, δ_s) , $(\hat{A}, \hat{\delta})$ and $(\hat{A}_s, \hat{\delta}_s)$ are all (A, δ) -modules. Then, we have a commutative diagram:

$$\begin{array}{ccc} (A, \delta) - \text{mod} & \xrightarrow{-\otimes_{(A, \delta)} (A_s, \delta_s)} & (A_s, \delta_s) - \text{mod} \\ \downarrow -\otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) & & \downarrow -\otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \\ (\hat{A}, \hat{\delta}) - \text{mod} & \xrightarrow{-\otimes_{(A, \delta)} (A_s, \delta_s)} & (\hat{A}_s, \hat{\delta}_s) - \text{mod} \end{array} \quad (4.4)$$

We now make the following convention: If B is an A -algebra and M is a B -module, we will say that M is s -regular if the morphism $s_M : M \rightarrow M$ induced by the action of s is injective.

We also recall from Tanaka [17, Lemma 5.2] that if $f : (M, \delta_M) \rightarrow (N, \delta_N)$ is a morphism of (A, δ) -modules, then the objects $\text{Ker}(f)$, $\text{Im}(f)$ and $\text{Coker}(f)$ automatically carry the structure of (A, δ) -modules. As such, a short exact sequence

$$0 \rightarrow (M', \delta_{M'}) \rightarrow (M, \delta_M) \rightarrow (M'', \delta_{M''}) \rightarrow 0 \quad (4.5)$$

in (A, δ) -mod is simply a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in A -mod such that all the maps are δ -morphisms.

Proposition 4.1 *Let $s \in A$ be an element that is not a zero divisor. Suppose that we have an (A_s, δ_s) -module (P, δ_P) and an $(\hat{A}, \hat{\delta})$ -module (Q, δ_Q) along with an isomorphism*

$$\varphi : (P, \delta_P) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \xrightarrow{\cong} (Q, \delta_Q) \otimes_{(A, \delta)} (A_s, \delta_s) \quad (4.6)$$

in the category of $(\hat{A}_s, \hat{\delta}_s)$ -modules.

Suppose that Q is s -regular. Let $((M, \delta_M), \alpha, \beta)$ be such that $(M, \delta_M) \in (A, \delta)$ -mod, M is s -regular and α, β are isomorphisms:

$$\alpha : (M, \delta_M) \otimes_{(A, \delta)} (A_s, \delta_s) \xrightarrow{\cong} (P, \delta_P) \quad \beta : (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \xrightarrow{\cong} (Q, \delta_Q) \quad (4.7)$$

in the categories (A_s, δ_s) -mod and $(\hat{A}, \hat{\delta})$ -mod, respectively. Then, the tuple $((M, \delta_M), \alpha, \beta)$ is unique up to isomorphism.

Proof Since $P \in A_s$ -mod and $Q \in \hat{A}$ -mod and $M \in A$ -mod is an s -regular module such that $M \otimes_A A_s \cong P$, $M \otimes_A \hat{A} \cong Q$, it follows directly from the result of Beauville and Laszlo [5] that the triple (M, α, β) must be unique up to isomorphism. Additionally, we know from [5] that the module M fits into a certain short exact sequence in A -mod:

$$0 \rightarrow M \xrightarrow{\psi} P \xrightarrow{\bar{\varphi}} Q_s/Q \rightarrow 0 \quad (4.8)$$

We will show that the morphisms in (4.8) are both δ -morphisms by describing ψ and $\bar{\varphi}$. First of all, it is clear that the canonical morphism

$$(M, \delta_M) = (M, \delta_M) \otimes_{(A, \delta)} (A, \delta) \rightarrow (M, \delta_M) \otimes_{(A, \delta)} (A_s, \delta_s) = (M_s, \delta_{M_s}) \quad (4.9)$$

is a δ -morphism. Since α is given to be a δ -morphism, we can compose it with the morphism in (4.9) to give a δ -morphism:

$$\begin{aligned} \psi : (M, \delta_M) &= (M, \delta_M) \otimes_{(A, \delta)} (A, \delta) \rightarrow (M, \delta_M) \otimes_{(A, \delta)} (A_s, \delta_s) \\ &= (M_s, \delta_{M_s}) \xrightarrow{\alpha} (P, \delta_P) \end{aligned} \quad (4.10)$$

On the other hand, we have a δ -morphism

$$(Q, \delta_Q) \xrightarrow{\beta^{-1}} (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \longrightarrow (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}_s, \hat{\delta}_s) \xrightarrow{\beta_s} (Q_s, \delta_{Q_s}) \quad (4.11)$$

It follows that the cokernel of the morphism in (4.11) is also an (A, δ) -module, giving us a δ -morphism:

$$(Q_s, \delta_{Q_s}) \longrightarrow (Q_s, \delta_{Q_s}) / (Q, \delta_Q) \quad (4.12)$$

We also have the δ -morphism:

$$(P, \delta_P) \xrightarrow{\alpha^{-1}} (M, \delta_M) \otimes_{(A, \delta)} (A_s, \delta_s) \longrightarrow (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \otimes_{(A, \delta)} (A_s, \delta_s) \xrightarrow{\beta_s} (Q_s, \delta_{Q_s}) \quad (4.13)$$

Composing the morphisms in (4.12) and (4.13), we obtain a δ -morphism $\bar{\varphi} : (P, \delta_P) \longrightarrow (Q_s, \delta_{Q_s}) / (Q, \delta_Q)$. Hence, (4.8) is a short exact sequence of (A, δ) -modules. It follows that δ_M must coincide with the restriction of δ_P to the submodule $M = \text{Ker}(\bar{\varphi}) \subseteq P$. This proves the result. $\square$

Theorem 4.2 *Let $s \in A$ be an element that is not a zero divisor. Suppose that we have an (A_s, δ_s) -module (P, δ_P) and an $(\hat{A}, \hat{\delta})$ -module (Q, δ_Q) along with an isomorphism*

$$\varphi : (P, \delta_P) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \xrightarrow{\cong} (Q, \delta_Q) \otimes_{(A, \delta)} (A_s, \delta_s) \quad (4.14)$$

in the category of $(\hat{A}_s, \hat{\delta}_s)$ -modules. Suppose that Q is s -regular. Then, there exists an (A, δ) -module $(M, \delta_M) \in (A, \delta)\text{-mod}$, with M an s -regular A -module and isomorphisms α, β :

$$\alpha : (M, \delta_M) \otimes_{(A, \delta)} (A_s, \delta_s) \xrightarrow{\cong} (P, \delta_P) \quad \beta : (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \xrightarrow{\cong} (Q, \delta_Q) \quad (4.15)$$

Further, the tuple $((M, \delta_M), \alpha, \beta)$ is unique up to isomorphism.

Proof We consider the δ -morphism

$$\bar{\varphi} : (P, \delta_P) \longrightarrow (P, \delta_P) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \xrightarrow{\varphi} (Q_s, \delta_{Q_s}) \longrightarrow (Q_s, \delta_{Q_s}) / (Q, \delta_Q) \quad (4.16)$$

Then, the kernel of $\bar{\varphi}$ is a (A, δ) -module and we set $(M, \delta_M) := \text{Ker}(\bar{\varphi})$. Then, the canonical map $(M, \delta_M) \hookrightarrow (P, \delta_P)$ is a δ -morphism and hence so is the induced morphism $\alpha : (M, \delta_M) \otimes_{(A, \delta)} (A_s, \delta_s) \longrightarrow (P, \delta_P) \otimes_{(A, \delta)} (A_s, \delta_s) = (P, \delta_P)$ (notice that $P \otimes_A A_s = P$). From [5], we know that α is an isomorphism and hence α becomes an isomorphism in $(A_s, \delta_s)\text{-mod}$.

Since $(M, \delta_M) = \text{Ker}(\bar{\varphi})$, we have an induced δ -morphism:

$$\begin{aligned} (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) &\xrightarrow{\psi_1} \text{Ker}((P, \delta_P) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta})) \\ &= (Q_s, \delta_{Q_s}) \longrightarrow ((Q_s, \delta_{Q_s}) / (Q, \delta_Q)) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \end{aligned} \quad (4.17)$$

There are also canonical δ -morphisms:

$$\begin{aligned} (Q, \delta_Q) &\xrightarrow{\psi_2} \text{Ker}((Q_s, \delta_{Q_s}) \longrightarrow (Q_s, \delta_{Q_s}) / (Q, \delta_Q)) \\ (Q_s, \delta_{Q_s}) / (Q, \delta_Q) &\xrightarrow{\psi_3} ((Q_s, \delta_{Q_s}) / (Q, \delta_Q)) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \end{aligned} \quad (4.18)$$

From [5, § 3], we know that the δ -morphisms ψ_1, ψ_2, ψ_3 are all isomorphisms of ordinary modules. From (4.17) and (4.18), we conclude that there is an isomorphism $\beta : (M, \delta_M) \otimes_{(A, \delta)} (\hat{A}, \hat{\delta}) \rightarrow (Q, \delta_Q)$ in $(\hat{A}, \hat{\delta})$ -mod. $\square$

5 Higher derivations and (A, Δ) -modules

We continue with A being a commutative k -algebra. We begin this section by recalling the notion of a higher derivation (or Hasse–Schmidt derivation) on A (see [11]).

Definition 5.1 A sequence $\Delta = \{\Delta^n\}_{n \geq 0}$ of k -linear maps $\Delta^n : A \rightarrow A$ with $\Delta^0 = \text{id}$ is said to be a higher (or Hasse–Schmidt) derivation on A , if for each $n \geq 0$, we have

$$\Delta^n(ab) = \sum_{i=0}^n \Delta^i(a) \Delta^{n-i}(b) \quad \forall a, b \in A.$$

We have assumed in this paper that k has characteristic zero and hence, for any ordinary derivation $\Delta : A \rightarrow A$, the sequence $\{\frac{1}{n!} \Delta^n\}_{n \geq 0}$ forms a higher derivation on A . However, there are several other interesting examples (including in the case where the ground ring k does not have characteristic zero) and the topic of higher derivations is well studied in the literature (see for example, [16, 18]). A complete characterization of higher derivations in the characteristic zero case may be found in [13].

We now fix a higher derivation Δ on A and consider the pair (A, Δ) . Our purpose is to work with the category of (A, Δ) -modules in a manner similar to what we have done with (A, δ) -modules in previous sections.

Definition 5.2 An (A, Δ) -module is a pair (M, Δ_M) consisting of the following data:

- (a) An A -module M .
- (b) A sequence $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ of morphisms $\Delta_M^n : M \rightarrow M$ of k -modules satisfying, for each $n \geq 0$, the condition:

$$\Delta_M^n(a \cdot m) = \sum_{i=0}^n \Delta^i(a) \cdot \Delta_M^{n-i}(m)$$

for any $a \in A$ and any $m \in M$.

A morphism $f : (M, \Delta_M) \rightarrow (N, \Delta_N)$ of (A, Δ) -modules consists of a morphism $f : M \rightarrow N$ of A -modules satisfying the condition $\Delta_N^n \circ f = f \circ \Delta_M^n : M \rightarrow N$ for each $n \geq 0$. We will refer to these as Δ -morphisms and when there is no danger of confusion, their collection will be denoted by $\text{Hom}_\Delta(M, N)$. The category of (A, Δ) -modules will be denoted by (A, Δ) -mod.

As in Sect. 3, the basic example of an (A, Δ) -module will come from the action of “Lie derivative operators” on Hochschild homology. For a higher derivation $\Delta = \{\Delta^n\}_{n \geq 0}$ on A , the Lie derivative operators on Hochschild homology were defined by the first author in [1]. For this, we fix an integer $l \geq 1$ and consider the collection of operators $\{L_{\Delta, n}^l : C_n^h(A) \rightarrow C_n^h(A)\}_{n \geq 0}$ given by

$$L_{\Delta, n}^l(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{j_0 + j_1 + \cdots + j_n = l} (\Delta^{j_0}(a_0) \otimes \Delta^{j_1}(a_1) \otimes \cdots \otimes \Delta^{j_n}(a_n)). \quad (5.1)$$

Here, the sum in (5.1) is taken over all ordered tuples of nonnegative integers $(j_0, \dots, j_n)$ that satisfy $j_0 + \dots + j_n = l$. We now recall the following result.

Proposition 5.3 (See [1, Proposition 2.2]) *Let $\Delta = \{\Delta^n\}_{n \geq 0}$ be a higher derivation on A . For any fixed integer $l \geq 1$, the collection $\{L_{\Delta, n}^l : C_n^h(A) \rightarrow C_n^h(A)\}_{n \geq 0}$ defines a morphism $L_{\Delta}^l : C_{\bullet}^h(A) \rightarrow C_{\bullet}^h(A)$ of degree zero on the Hochschild complex of A . In particular, for each $l \geq 0$, we have a morphism $L_{\Delta}^l : HH_{\bullet}(A) \rightarrow HH_{\bullet}(A)$ on the Hochschild homology groups of A .*

The following result now gives us a natural example of an (A, Δ) -module coming from Hochschild homology.

Proposition 5.4 *Let A be a commutative k -algebra equipped with a k -linear higher derivation Δ . Then, the collection $L_{\Delta} = \{L_{\Delta}^l\}_{l \geq 0}$ of Lie derivative operators makes $HH_{\bullet}(A)$ into an (A, Δ) -module.*

Proof For a commutative k -algebra A , we know that there is a shuffle product structure

$$sh_{p+q}^{\mu} : HH_p(A) \otimes HH_q(A) \rightarrow HH_{p+q}(A) \quad \forall p, q \geq 0 \quad (5.2)$$

on the Hochschild homology groups of A . Since $HH_0(A) = A$, the A -module structure on $HH_{\bullet}(A)$ is obtained by setting $p = 0$ in (5.2). It now follows from [1, Proposition 2.4] that the collection $L_{\Delta} = \{L_{\Delta}^l\}_{l \geq 0}$ makes $HH_{\bullet}(A)$ into an (A, Δ) -module. $\square$

Given a higher derivation $\{\Delta^n\}_{n \geq 0}$ on A , we know from the work of Mirzavaziri [13] that there exists a sequence $\{\delta^n\}_{n \geq 1}$ of ordinary derivations on A such that $(n+1)\Delta^{n+1} = \sum_{k=0}^n \delta^{k+1} \Delta^{n-k}$ for each $n \geq 1$. In other words, every higher derivation (over a ring of characteristic zero) is determined by a sequence of ordinary derivations. In the rest of this section, we will show in a manner similar to [13] that every (A, Δ) -module corresponds to a sequence of (A, δ^n) -modules, where $\{\delta^n\}_{n \geq 1}$ is the sequence of derivations on A corresponding to the higher derivation Δ .

Theorem 5.5 *Let (M, Δ_M) be an (A, Δ) -module and let $\{\delta^n\}_{n \geq 1}$ be the sequence of ordinary derivations on A determining the higher derivation Δ . Then, there exists a sequence $\{\delta_M^n\}_{n \geq 1}$ of k -linear morphisms on M such that:*

- (a) *For each $n \geq 1$, (M, δ_M^n) is an (A, δ^n) -module.*
- (b) *The morphisms in the sequence $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ may be expressed as follows:*

$$(n+1)\Delta_M^{n+1} = \sum_{k=0}^n \delta_M^{k+1} \Delta_M^{n-k}$$

Proof We will define each $\delta_M^n : M \rightarrow M$ inductively. For $n = 0$, we have

$$\Delta_M^1(am) = \Delta^1(a)\Delta_M^0(m) + a\Delta_M^1(m) = \delta^1(a)m + a\Delta_M^1(m). \quad (5.3)$$

for any $a \in A$ and $m \in M$. As such, we set $\delta_M^1 := \Delta_M^1$. Then, (M, δ_M^1) is an (A, δ^1) module.

Now suppose that δ_M^k is defined for $1 \leq k \leq n$ and set $\delta_M^{n+1} := (n+1)\Delta_M^{n+1} - \sum_{k=0}^{n-1} \delta_M^{k+1} \Delta_M^{n-k}$. In order to prove the result, it suffices to check that (M, δ_M^{n+1}) is an (A, δ^{n+1}) module. For any $a \in A$ and $m \in M$, we now have

$$\begin{aligned}
\delta_M^{n+1}(am) &= (n+1)\Delta_M^{n+1}(am) - \sum_{k=0}^{n-1} \delta_M^{k+1} \Delta_M^{n-k}(am) \\
&= \sum_{k=0}^{n+1} (n+1)\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \delta_M^{k+1} \left(\sum_{t=0}^{n-k} \Delta^t(a)\Delta_M^{n-k-t}(m) \right) \\
&= \sum_{k=0}^{n+1} k\Delta^k(a)\Delta_M^{n+1-k}(m) + \sum_{k=0}^{n+1} \Delta^k(a)(n+1-k)\Delta_M^{n+1-k}(m) \\
&\quad - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \delta^{k+1}(\Delta^t(a))\Delta_M^{n-k-t}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \Delta^t(a)\delta_M^{k+1}(\Delta_M^{n-k-t}(m)) \\
&= \left(\sum_{k=0}^{n+1} k\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \delta^{k+1}(\Delta^t(a))\Delta_M^{n-k-t}(m) \right) \\
&\quad + \left(\sum_{k=0}^{n+1} \Delta^k(a)(n+1-k)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \Delta^t(a)\delta_M^{k+1}(\Delta_M^{n-k-t}(m)) \right)
\end{aligned} \tag{5.4}$$

Let us consider

$$L = \sum_{k=0}^{n+1} k\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \delta^{k+1}(\Delta^t(a))\Delta_M^{n-k-t}(m)$$

and

$$M = \sum_{k=0}^{n+1} \Delta^k(a)(n+1-k)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \Delta^t(a)\delta_M^{k+1}(\Delta_M^{n-k-t}(m))$$

which gives us $\delta_M^{n+1}(am) = L + M$. We now write

$$\begin{aligned}
L &= \sum_{k=0}^{n+1} k\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \delta^{k+1}(\Delta^t(a))\Delta_M^{n-k-t}(m) \\
&= (n+1)\Delta^{n+1}(a)m + \sum_{k=0}^n k\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \delta^{k+1}(\Delta^t(a))\Delta_M^{n-k-t}(m)
\end{aligned} \tag{5.5}$$

If we put $r = k + t$ and rewrite the second sum in (5.5), we get

$$\begin{aligned}
L &= (n+1)\Delta^{n+1}(a)m + \sum_{k=0}^n k\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{r=0}^n \sum_{k=0, k \neq n}^r \delta^{k+1}(\Delta^{r-k}(a))\Delta_M^{n-r}(m) \\
&= (n+1)\Delta^{n+1}(a)m + \sum_{k=1}^n k\Delta^k(a)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \delta^{k+1}(\Delta^{n-k}(a))m \\
&\quad - \sum_{r=0}^{n-1} \sum_{k=0}^r \delta^{k+1}(\Delta^{r-k}(a))\Delta_M^{n-r}(m)
\end{aligned}$$

$$\begin{aligned}
&= \left((n+1)\Delta^{n+1}(a) - \sum_{k=0}^{n-1} \delta^{k+1}(\Delta^{n-k}(a)) \right) m + \sum_{r=0}^{n-1} (r+1)\Delta^{r+1}(a)\Delta_M^{n-r}(m) \\
&\quad - \sum_{r=0}^{n-1} \sum_{k=0}^r \delta^{k+1}(\Delta^{r-k}(a))\Delta_M^{n-r}(m) \\
&= \delta^{n+1}(a)m + \sum_{r=0}^{n-1} \left((r+1)\Delta^{r+1}(a) - \sum_{k=0}^r \delta^{k+1}(\Delta^{r-k}(a)) \right) \Delta_M^{n-r}(m) \\
&= \delta^{n+1}(a)m.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
M &= \sum_{k=0}^{n+1} \Delta^k(a)(n+1-k)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} \sum_{t=0}^{n-k} \Delta^t(a)\delta_M^{k+1}(\Delta_M^{n-k-t}(m)) \\
&= a(n+1)\Delta_M^{n+1}(m) + \sum_{k=1}^{n+1} \Delta^k(a)(n+1-k)\Delta_M^{n+1-k}(m) \\
&\quad - \sum_{t=0}^n \sum_{k=0, k \neq n}^{n-t} \Delta^t(a)\delta_M^{k+1}(\Delta_M^{n-k-t}(m)) \\
&= a(n+1)\Delta_M^{n+1}(m) + \sum_{k=1}^n \Delta^k(a)(n+1-k)\Delta_M^{n+1-k}(m) - \sum_{k=0}^{n-1} a\delta_M^{k+1}(\Delta_M^{n-k}(m)) \\
&\quad - \sum_{t=1}^n \sum_{k=0}^{n-t} \Delta^t(a)\delta_M^{k+1}(\Delta_M^{n-k-t}(m)) \\
&= a \left((n+1)\Delta_M^{n+1}(m) - \sum_{k=0}^{n-1} \delta_M^{k+1}(\Delta_M^{n-k}(m)) \right) + \sum_{r=0}^{n-1} \Delta^{r+1}(a)(n-r)\Delta_M^{n-r}(m) \\
&\quad - \sum_{r=0}^{n-1} \sum_{k=0}^{n-r-1} \Delta^{r+1}(a)\delta_M^{k+1}(\Delta_M^{n-k-r-1}(m)) \\
&= a\delta_M^{n+1}(m) + \sum_{r=0}^{n-1} \Delta^{r+1}(a) \left((n-r)\Delta_M^{n-r}(m) - \sum_{k=0}^{n-r-1} \delta_M^{k+1}(\Delta_M^{n-k-r-1}(m)) \right)
\end{aligned}$$

By induction, for $1 \leq i \leq n-1$, $(i+1)\Delta_M^{i+1} = \sum_{k=0}^i \delta_M^{k+1}\Delta_M^{i-k}$. Hence, we have

$$M = a\delta_M^{n+1}(m) + \sum_{r=0}^{n-1} \Delta^{r+1}(a).0 = a\delta_M^{n+1}(m)$$

We now see that $\delta_M^{n+1}(am) = \delta^{n+1}(a)m + a\delta_M^{n+1}(m)$. This proves the result. $\square$

Corollary 5.6 Let $\{\delta^n\}_{n \geq 1}$ be the sequence of ordinary derivations on A determining the higher derivation Δ on A . Let M be an A -module. Then, there is a one-one correspondence between the following:

- (a) Sequences $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ of k -module morphisms $\Delta_M^n : M \longrightarrow M$ making (M, Δ_M) into an (A, Δ) -module.

- (b) Sequences $\{\delta_M^n\}_{n \geq 1}$ of k -module morphisms $\delta_M^n : M \longrightarrow M$ such that each (M, δ_M^n) is an (A, δ^n) -module.

Proof From Theorem 5.5, we know that the sequences $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ and $\{\delta_M^n\}_{n \geq 1}$ are related as follows

$$(n+1)\Delta_M^{n+1} = \sum_{k=0}^n \delta_M^{k+1} \Delta_M^{n-k} \quad (5.6)$$

It is clear from (5.6) that the sequences $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ and $\{\delta_M^n\}_{n \geq 0}$ determine each other uniquely. It may also be verified easily that given a sequence $\{\delta_M^n\}_{n \geq 0}$ making each (M, δ_M^n) an (A, δ^n) -module, the sequence $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ as defined by (5.6) makes (M, Δ_M) into an (A, Δ) -module. $\square$

6 Symmetric monoidal category of (A, Δ) -modules

We continue with $\Delta = \{\Delta^n\}_{n \geq 0}$ being a higher derivation on A which corresponds to a sequence $\{\delta^n\}_{n \geq 1}$ of ordinary k -linear derivations on A . We now make the convention that if (M, Δ_M) is an (A, Δ) -module, then $\{\delta_M^n\}_{n \geq 1}$ is the corresponding sequence of k -module morphisms $\delta_M^n : M \longrightarrow M$ such that each (M, δ_M^n) is an (A, δ^n) -module.

Lemma 6.1 Let (M, Δ_M) and (N, Δ_N) be (A, Δ) -modules. Let $\Delta_{(M \otimes N)} = \{\Delta_{(M \otimes N)}^n\}_{n \geq 0}$, where

$$\Delta_{(M \otimes N)}^n(x \otimes y) := \sum_{i+j=n} \Delta_M^i(x) \otimes \Delta_N^j(y) \quad x \in M, y \in N \quad (6.1)$$

Then, the pair $(M \otimes_A N, \Delta_{(M \otimes N)})$ is an (A, Δ) -module.

Proof For any $a \in A, x \in M$ and $y \in N$, we see that:

$$\begin{aligned} \Delta_{(M \otimes N)}^n(a \cdot (x \otimes y)) &= \Delta_{(M \otimes N)}^n((a \cdot x) \otimes y) = \Delta_{(M \otimes N)}^n(x \otimes (a \cdot y)) \\ &= \sum_{i+j=n} \Delta_M^i(a \cdot x) \otimes \Delta_N^j(y) \\ &= \sum_{i+j+k=n} (\Delta^i(a) \cdot \Delta_M^j(x)) \otimes \Delta_N^k(y) \\ &= \sum_{i+j=n} \Delta^i(a) \cdot \Delta_{(M \otimes N)}^j(x \otimes y) \end{aligned} \quad (6.2)$$

$\square$

Following Lemma 6.1, we put a monoidal structure on the category (A, Δ) -mod by setting

$$(M, \Delta_M) \otimes_{(A, \Delta)} (N, \Delta_N) := (M \otimes_A N, \Delta_{(M \otimes N)}) \quad (6.3)$$

Lemma 6.2 Let (M, Δ_M) and (N, Δ_N) be (A, Δ) -modules. Then, $\Delta_{(M \otimes N)} = \{\Delta_{(M \otimes N)}^n\}_{n \geq 0}$ corresponds to the sequence $\{\delta_{(M \otimes N)}^n\}_{n \geq 1}$ of k -module morphisms $\delta_{(M \otimes N)}^n : M \otimes_A N \longrightarrow M \otimes_A N$.

Proof For any $n \geq 0$, we see that

$$\begin{aligned}
 \sum_{k=0}^n \delta_{(M \otimes N)}^{k+1} \Delta_{(M \otimes N)}^{n-k} &= \sum_{k=0}^n \sum_{i=0}^{n-k} (\delta_M^{k+1} \Delta_M^i) \otimes (\Delta_N^{n-k-i}) \\
 &\quad + \sum_{k=0}^n \sum_{i=0}^{n-k} (\Delta_M^i) \otimes (\delta_N^{k+1} \Delta_N^{n-k-i}) \\
 &= \sum_{j=0}^n \left(\sum_{k=0}^j \delta_M^{k+1} \Delta_M^{j-k} \right) \otimes \Delta_N^{n-j} \\
 &\quad + \sum_{i=0}^n \Delta_M^i \otimes \left(\sum_{k=0}^{n-i} \delta_N^{k+1} \Delta_N^{n-k-i} \right) \\
 &= \sum_{j=0}^n (j+1) \Delta_M^{j+1} \otimes \Delta_N^{n-j} + \sum_{i=0}^n (n-i+1) \Delta_M^i \otimes \Delta_N^{n-i+1} \\
 &= (n+1) \sum_{j=0}^{n+1} \Delta_M^j \otimes \Delta_N^{n+1-j} = (n+1) \Delta_{(M \otimes N)}^{n+1}
 \end{aligned}$$

This proves the result. $\square$

The next step is to define an internal Hom in the category of (A, Δ) -modules. Let (M, Δ_M) , (N, Δ_N) be (A, Δ) -modules corresponding to sequences $\{\delta_M^n\}_{n \geq 1}$ and $\{\delta_N^n\}_{n \geq 1}$. For each $n \geq 1$, we know that (M, δ_M^n) and (N, δ_N^n) are (A, δ^n) -modules. From the construction in Sect. 2, this gives us for each $n \geq 1$ an (A, δ^n) -module $\underline{\text{Hom}}_{(A, \delta^n)}((M, \delta_M^n), (N, \delta_N^n)) = (\text{Hom}_A(M, N), \underline{\delta}^n)$ given by

$$\underline{\delta}^n(f) := \delta_N^n \circ f - f \circ \delta_M^n \quad \forall f \in \text{Hom}_A(M, N) \quad (6.4)$$

Using Corollary 5.6, we define $\underline{\text{Hom}}_{(A, \Delta)}((M, \Delta_M), (N, \Delta_N))$ to be the (A, Δ) -module corresponding to the sequence $\{(\text{Hom}_A(M, N), \underline{\delta}^n)\}_{n \geq 1}$. For the sake of convenience, we will often denote this (A, Δ) -module simply by $\underline{\text{Hom}}_\Delta(M, N)$.

We also make the following simple observation: From the relation between $\Delta_M = \{\Delta_M^n\}_{n \geq 0}$ and $\{\delta_M^n\}_{n \geq 1}$ in (5.6), it is clear that any linear map $f : M \rightarrow N$ satisfies $\Delta_N^n \circ f = f \circ \Delta_M^n$ for each $n \geq 0$ if and only if it satisfies $\delta_N^n \circ f = f \circ \delta_M^n$ for each $n \geq 1$. In particular, a morphism $f : (M, \Delta_M) \rightarrow (N, \Delta_N)$ of (A, Δ) -modules is an A -module morphism $f : M \rightarrow N$ that satisfies $\delta_N^n \circ f = f \circ \delta_M^n$ for each $n \geq 1$.

Theorem 6.3 *The category of (A, Δ) -modules is a closed symmetric monoidal category.*

Proof Let (M, Δ_M) , (N, Δ_N) , (P, Δ_P) be (A, Δ) -modules. In order to prove the result, it suffices to show that we have natural isomorphisms

$$\text{Hom}_\Delta(M \otimes_A N, P) \cong \text{Hom}_\Delta(M, \underline{\text{Hom}}_\Delta(N, P)) \quad (6.5)$$

From the discussion above, we know that an A -module morphism $f : M \otimes_A N \rightarrow P$ lies in $\text{Hom}_\Delta(M \otimes_A N, P)$ if and only if $f \in \text{Hom}_{\delta^n}((M, \delta_M^n) \otimes_{(A, \delta^n)} (N, \delta_N^n), (P, \delta_P^n))$ for each $n \geq 1$. Applying Proposition 2.2, we know that

$$\mathrm{Hom}_{\delta^n}((M, \delta_M^n) \otimes_{(A, \delta^n)} (N, \delta_N^n), (P, \delta_P^n)) \cong \mathrm{Hom}_{\delta^n}((M, \delta_M^n), \underline{\mathrm{Hom}}_{\delta^n}((N, \delta_N^n), (P, \delta_P^n))) \quad (6.6)$$

for each $n \geq 1$. As such, $f \in \mathrm{Hom}_\Delta(M \otimes_A N, P)$ if and only if the corresponding A -module map $M \rightarrow \mathrm{Hom}_A(N, P)$ lies in $\mathrm{Hom}_{\delta^n}((M, \delta_M^n), \underline{\mathrm{Hom}}_{\delta^n}((N, \delta_N^n), (P, \delta_P^n)))$ for each $n \geq 1$. By definition, we know that $\underline{\mathrm{Hom}}_\Delta(N, P)$ is the (A, Δ) -module corresponding to the sequence $\{\underline{\mathrm{Hom}}_{\delta^n}((N, \delta_N^n), (P, \delta_P^n))\}_{n \geq 1}$. This proves the result. $\square$

We suppose now that $f : A \rightarrow A'$ is a morphism of commutative k -algebras. Suppose that there is a higher derivation Δ' on A' such that $\Delta' \circ f = f \circ \Delta$. It is clear that (A', Δ') is an (A, Δ) -module. In this case, we will say that (A', Δ') is an (A, Δ) -algebra. This gives rise to the functor:

$$f^* : (A, \Delta)\text{-mod} \rightarrow (A', \Delta')\text{-mod} \quad (M, \Delta_M) \mapsto (M, \Delta_M) \otimes_{(A, \Delta)} (A', \Delta') \quad (6.7)$$

as well as a forgetful functor:

$$f_* : (A', \Delta')\text{-mod} \rightarrow (A, \Delta)\text{-mod} \quad (6.8)$$

Proposition 6.4 *For an (A, Δ) -algebra (A', Δ') , the functor f^* is left adjoint to f_* .*

Proof Let $(M, \Delta_M) \in (A, \Delta)\text{-mod}$ and $(N, \Delta_N) \in (A', \Delta')\text{-mod}$. We already know that

$$\mathrm{Hom}_{A'\text{-mod}}(M \otimes_A A', N) = \mathrm{Hom}_{A\text{-mod}}(M, N) \quad (6.9)$$

Suppose that $g : M \otimes_A A' \rightarrow N$ is an A' -module morphism. Then, g corresponds to an A -module morphism $h : M \rightarrow N$ given by $h(m) = g(m \otimes 1)$ for any $m \in M$. If $g : M \otimes_A A' \rightarrow N$ is additionally a Δ' -morphism, we see that h must be a Δ -morphism, i.e., for each $n \geq 1$:

$$\Delta_N^n(h(m)) = \Delta_N^n(g(m \otimes 1)) = g(\Delta_{(M \otimes_A A')}^n(m \otimes 1)) = g(\Delta_M^n(m) \otimes 1) = h(\Delta_M^n(m)) \quad (6.10)$$

Conversely if $h : M \rightarrow N$ is an A -module morphism, it corresponds to the A' -module morphism $g : M \otimes_A A' \rightarrow N$ given by $g(m \otimes a') = a'h(m)$ for any $m \in M, a' \in A'$. Additionally, if h is a Δ -morphism, we see that g is also a Δ' -morphism, i.e., for each $n \geq 1$:

$$\begin{aligned} \Delta_N^n(g(m \otimes a')) &= \Delta_N^n(a'h(m)) = \sum_{i=0}^n \Delta_N^i(a') \cdot \Delta_N^{n-i}(h(m)) \\ &= \sum_{i=0}^n \Delta_N^i(a') \cdot h(\Delta_M^{n-i}(m)) \\ &= \sum_{i=0}^n g(\Delta_M^{n-i}(m) \otimes \Delta_N^i(a')) \\ &= g(\Delta_{(M \otimes_A A')}^n(m \otimes a')) \end{aligned} \quad (6.11)$$

This proves the result. $\square$

7 Equivalence relations on higher derivations and Lie derivatives

In the last two sections of this paper, we will use power series rings to give a new description of the Lie derivative on Hochschild homology of an algebra A induced by a higher derivation

Δ . This new description has two clear advantages compared to the description given by the first author in [1]. First, it enables us to show that the Lie derivatives corresponding to two higher derivations that are equivalent (in the sense of Roy and Sridharan [15]) are equal. Second, it enables us to extend the Cartan homotopy formula in noncommutative geometry from ordinary derivations to higher derivations.

In fact, in these two last sections, we work in a more general setup where the k -algebra A is not assumed to be commutative.

We now let $A[[t]]$ be the power series ring in a single indeterminate t over the ring A . Then, a higher derivation $\Delta = \{\Delta^n\}_{n \geq 0}$ on A corresponds to a homomorphism of rings (see, for instance, [15]):

$$\Phi_\Delta : A \longrightarrow A[[t]] \quad a \mapsto \sum_{i=0}^{\infty} \Delta^i(a)t^i$$

Let $e_0 : A[[t]] \longrightarrow A$ be the “evaluation at 0,” i.e., the homomorphism that picks out the constant term of any power series with coefficients in A . It is clear that any homomorphism $\Phi : A \longrightarrow A[[t]]$ that satisfies $e_0 \circ \Phi = id : A \longrightarrow A$ corresponds to a higher derivation on A .

From [1, Proposition 2.2], we know that the higher derivation Δ induces Lie derivative operators $\{L_\Delta^k\}_{k \geq 0}$ on the Hochschild homology $HH_\bullet(A)$. Our first main aim in this section is to reinterpret these operators in a more convenient fashion using the power series ring $A[[t]]$. It is clear that the formal derivative

$$\frac{\partial}{\partial t} : A[[t]] \longrightarrow A[[t]] \quad (7.1)$$

is a derivation on $A[[t]]$. Since the ground ring is assumed to be of characteristic zero, the sequence $\{\frac{1}{n!} \frac{\partial^n}{\partial t^n}\}_{n \geq 0}$ gives a higher derivation on $A[[t]]$ which we denote by ∂ . For the sake of convenience, we put $\partial^n := \frac{1}{n!} \frac{\partial^n}{\partial t^n}$.

It now follows that for each $k \geq 1$, the higher derivation $\partial = \{\partial^n\}_{n \geq 0}$ induces operators $L_\partial^k : HH_\bullet(A[[t]]) \longrightarrow HH_\bullet(A[[t]])$ on the Hochschild homologies of $A[[t]]$.

Proposition 7.1 *Let $\Delta = \{\Delta^n\}_{n \geq 0}$ be a higher derivation on A and let $\Phi_\Delta : A \longrightarrow A[[t]]$ be the ring homomorphism corresponding to Δ . Let $e_0 : A[[t]] \longrightarrow A$ be the homomorphism given by evaluation at 0. Then, for any fixed $k \geq 1$, we have*

$$L_\Delta^k = HH(e_0) \circ L_\partial^k \circ HH(\Phi_\Delta) : HH_\bullet(A) \longrightarrow HH_\bullet(A) \quad (7.2)$$

where $HH(\Phi_\Delta)$ and $HH(e_0)$ are the morphisms on Hochschild homology induced by the ring homomorphisms $\Phi_\Delta : A \longrightarrow A[[t]]$ and $e_0 : A[[t]] \longrightarrow A$, respectively.

Proof We pick $n \geq 0$ and consider $C_n^h(A) = A \otimes A^{\otimes n}$. The ring homomorphism Φ_Δ induces a map

$$\begin{aligned} C_n^h(\Phi_\Delta) : C_n^h(A) &\longrightarrow C_n^h(A[[t]]) \\ (a_0 \otimes a_1 \otimes \cdots \otimes a_n) &\mapsto (\Phi_\Delta(a_0) \otimes \Phi_\Delta(a_1) \otimes \cdots \otimes \Phi_\Delta(a_n)) \\ &= \left(\left(\sum_{i=0}^{\infty} \Delta^i(a_0)t^i \right) \otimes \left(\sum_{i=0}^{\infty} \Delta^i(a_1)t^i \right) \otimes \cdots \otimes \left(\sum_{i=0}^{\infty} \Delta^i(a_n)t^i \right) \right). \end{aligned} \quad (7.3)$$

Applying the operator L_{∂}^k now gives:

$$\begin{aligned} & L_{\partial,n}^k(\Phi_{\Delta}(a_0) \otimes \Phi_{\Delta}(a_1) \otimes \cdots \otimes \Phi_{\Delta}(a_n)) \\ &= \sum_{j_0+j_1+\cdots+j_n=k} \left(\partial_{j_0} \left(\sum_{i=0}^{\infty} \Delta^i(a_0)t^i \right) \otimes \partial_{j_1} \left(\sum_{i=0}^{\infty} \Delta^i(a_1)t^i \right) \otimes \cdots \otimes \partial_{j_n} \left(\sum_{i=0}^{\infty} \Delta^i(a_n)t^i \right) \right) \\ &= \sum_{j_0+j_1+\cdots+j_n=k} \left(\left(\sum_{i=0}^{\infty} \binom{i+j_0}{j_0} \Delta^{i+j_0}(a_0)t^i \right) \otimes \cdots \otimes \left(\sum_{i=0}^{\infty} \binom{i+j_n}{j_n} \Delta^{i+j_n}(a_n)t^i \right) \right). \end{aligned}$$

Finally, evaluating the above expression at 0, we obtain

$$\begin{aligned} & C_n^h(e_0) \circ L_{\partial,n}^k \circ C_n^h(\Phi_{\Delta})(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &= \sum_{j_0+j_1+\cdots+j_n=k} (\Delta^{j_0}(a_0) \otimes \Delta^{j_1}(a_1) \otimes \cdots \otimes \Delta^{j_n}(a_n)). \end{aligned} \quad (7.4)$$

This proves the result. $\square$

Following Roy and Sridharan [15, § 1], we say that two higher derivations $\Delta = \{\Delta^n\}_{n \geq 0}$ and $\Delta' = \{\Delta'^n\}_{n \geq 0}$ are equivalent, written $\Delta \sim \Delta'$, if there exists a unit $u \in A[[t]]$ such that

$$\Phi_{\Delta}(a) = u^{-1} \cdot \Phi_{\Delta'}(a) \cdot u = (T_u \circ \Phi_{\Delta'})(a) \quad \forall a \in A, \quad (7.5)$$

where T_u denotes the ring automorphism of $A[[t]]$ given by $T_u(x) := u^{-1}xu$. The new description of the Lie derivatives in Proposition 7.1 will enable us to show that $L_{\Delta}^k = L_{\Delta'}^k$ whenever $\Delta \sim \Delta'$.

Proposition 7.2 *Let $\Delta = \{\Delta^n\}_{n \geq 0}$ and $\Delta' = \{\Delta'^n\}_{n \geq 0}$ be two higher derivations on A such that $\Delta \sim \Delta'$. Then, for each fixed k , we have:*

$$L_{\Delta}^k = L_{\Delta'}^k : HH_{\bullet}(A) \longrightarrow HH_{\bullet}(A).$$

Proof From Proposition 7.1, we know that:

$$L_{\Delta}^k = HH(e_0) \circ L_{\partial}^k \circ HH(\Phi_{\Delta}) \quad L_{\Delta'}^k = HH(e_0) \circ L_{\partial}^k \circ HH(\Phi_{\Delta'}). \quad (7.6)$$

Since $\Delta \sim \Delta'$, it follows by definition that we can find a unit $u \in A[[t]]$ such that $\Phi_{\Delta} = T_u \circ \Phi_{\Delta'}$. However, we know from [12, § 4.1.2] that the conjugation T_u induces an identity map on Hochschild homology groups and the result now follows immediately from the expressions in (7.6). $\square$

In particular, consider the higher derivation $\{\Delta^0 = id, \Delta^n = 0 \text{ for } n \geq 1\}$ on A . Following [15, § 1], we will refer to any higher derivation equivalent to it as an inner higher derivation.

Corollary 7.3 *Let $\Delta' = \{\Delta'^n\}_{n \geq 0}$ be an inner higher derivation on A . Then, for each $k \geq 1$, the Lie derivative operator $L_{\Delta'}^k = 0$.*

Proof We know that Δ' is equivalent to the higher derivation $\{\Delta^0 = id, \Delta^n = 0 \text{ for } n \geq 1\}$. The maps induced by the latter on Hochschild homology groups are clearly zero. The result now follows from Proposition 7.2. $\square$

Remark 7.4 Suppose that δ is an ordinary inner derivation on A , i.e., there is an element $a \in A$ such that $\delta(x) = [a, x] = ax - xa$ for all $x \in A$. Then, it is known that the corresponding Lie derivative L_δ is the zero operator on Hochschild homology (see [12, § 4.1.5]). During the writing of [1], the first author had wondered if there was a similar result for Lie derivatives coming from inner higher derivations. However, in [1, Proposition 2.7], this was successfully proved only for inner higher derivations of the form $\{\Delta^0 = id, \Delta^n = (-1)^n a^{n-1}[a, \cdot], n \geq 1\}$ for some given $a \in A$. In this paper, we realized that the new description of the Lie derivative in Proposition 7.1 enables us to prove the general result for all inner higher derivations as a simple consequence.

8 Cartan homotopy formulas for higher derivations

We continue with A being an associative algebra over a commutative ground ring k of characteristic zero. We set $\bar{A} := A/k$ and $\bar{C}_n^h(A) := A \otimes \bar{A}^{\otimes n}$ for $n \geq 0$. Then, the terms $\bar{C}_n^h(A)$ form a normalized Hochschild complex $\bar{C}_\bullet^h(A)$ with differentials induced by the Hochschild differentials $b : C_n^h(A) \rightarrow C_{n-1}^h(A)$. It is well known that the canonical map $C^h(A) \rightarrow \bar{C}^h(A)$ is a quasi-isomorphism of complexes (see [12, § 1.1.14]). In particular, the Hochschild homology groups of A may also be computed by taking the homology groups of the complex $\bar{C}^h(A)$.

We continue to denote by b the differential on the normalized Hochschild complex $\bar{C}^h(A)$. Similarly, we denote by B the differential on the normalized Hochschild complex induced by the Connes differential $B : C_n^h(A) \rightarrow C_{n+1}^h(A)$ (see [12, § 2.1.9]). When it is necessary to specify the algebra A , we will refer to the differentials b and B as b_A and B_A , respectively.

We now consider an ordinary k -linear derivation δ on A . The Lie derivative operator on $C^h(A)$ induces an operator on the normalized Hochschild complex $\bar{C}^h(A)$ that we continue to denote by L_δ . Besides L_δ , the derivation δ induces two further operators on the normalized Hochschild complex $\bar{C}^h(A)$ (see [12, § 4.1.7])

$$e_\delta : A \otimes \bar{A}^{\otimes n} \rightarrow A \otimes \bar{A}^{\otimes n-1} \quad E_\delta : A \otimes \bar{A}^{\otimes n} \rightarrow A \otimes \bar{A}^{\otimes n+1} \quad (8.1)$$

given by the formulas

$$\begin{aligned} e_\delta(a_0 \otimes \cdots \otimes a_n) &= (-1)^{n+1} (\delta(a_n) a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) E_\delta(a_0 \otimes \cdots \otimes a_n) \\ &= \sum_{1 \leq i \leq j \leq n} (-1)^{in+1} (1 \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_{j-1} \otimes \delta(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}). \end{aligned} \quad (8.2)$$

The operators L_δ , e_δ and E_δ are related by the following formula, which appears as the ‘Cartan homotopy formula’ in noncommutative geometry (see [12, § 4.1.8])

$$[e_\delta, B] + [E_\delta, b] = L_\delta \quad (8.3)$$

It should be noted that the commutators appearing in formula (8.3) are graded commutators, i.e., $[e_\delta, B] = e_\delta \circ B + B \circ e_\delta$ and $[E_\delta, b] = E_\delta \circ b + b \circ E_\delta$ since b, B, e_δ and E_δ are operators of degrees $-1, +1, -1, +1$, respectively, on the normalized Hochschild complex.

Our aim in this section is to obtain a similar Cartan homotopy formula for a higher derivation $\Delta = \{\Delta^n\}_{n \geq 0}$ on A . We observe that the formal derivative $\partial = \frac{\partial}{\partial t} : A[[t]] \rightarrow A[[t]]$ with respect to t induces operators L_∂, e_∂ and E_∂ on the normalized Hochschild complex $\bar{C}^h(A[[t]])$.

Proposition 8.1 *Let $\Delta = \{\Delta^n\}_{n \geq 0}$ be a higher derivation on A . For each fixed integer $k \geq 1$, we set:*

$$\begin{aligned} e_{\Delta}^k &:= \frac{1}{2} \cdot \frac{1}{k!} \left(\bar{C}^h(e_0) \circ L_{\partial}^{k-1} \circ e_{\partial} \circ \bar{C}^h(\Phi_{\Delta}) + \bar{C}^h(e_0) \circ e_{\partial} \circ L_{\partial}^{k-1} \circ \bar{C}^h(\Phi_{\Delta}) \right) \\ E_{\Delta}^k &:= \frac{1}{2} \cdot \frac{1}{k!} \left(\bar{C}^h(e_0) \circ L_{\partial}^{k-1} \circ E_{\partial} \circ \bar{C}^h(\Phi_{\Delta}) + \bar{C}^h(e_0) \circ E_{\partial} \circ L_{\partial}^{k-1} \circ \bar{C}^h(\Phi_{\Delta}) \right), \end{aligned} \quad (8.4)$$

where $\bar{C}^h(e_0)$ and $\bar{C}^h(\Phi_{\Delta})$ are the morphisms on the normalized Hochschild complex induced by the ring homomorphisms $e_0 : A[[t]] \rightarrow A$ and $\Phi_{\Delta} : A \rightarrow A[[t]]$, respectively. Then, for each fixed integer $k \geq 1$, we have the following formula:

$$[e_{\Delta}^k, B_A] + [E_{\Delta}^k, b_A] = L_{\Delta}^k : \bar{C}^h(A) \rightarrow \bar{C}^h(A). \quad (8.5)$$

Proof The Cartan homotopy formula for the derivation ∂ on $A[[t]]$ gives:

$$[e_{\partial}, B_{A[[t]]}] + [E_{\partial}, b_{A[[t]]}] = L_{\partial}. \quad (8.6)$$

Since the Lie derivative L_{∂} commutes with the Hochschild differential $b_{A[[t]]}$ and the Connes differential $B_{A[[t]]}$, we see that for any $k \geq 1$:

$$[L_{\partial}^{k-1} \circ e_{\partial}, B_{A[[t]]}] + [L_{\partial}^{k-1} \circ E_{\partial}, b_{A[[t]]}] = L_{\partial}^k. \quad (8.7)$$

We consider the higher derivation $\partial = \{\frac{1}{n!} \frac{\partial^n}{\partial t^n}\}_{n \geq 0}$ on $A[[t]]$. It is easy to see that $L_{\partial}^k = \frac{1}{k!} L_{\partial}^k$. From the proof of Proposition 7.1, it is also clear that $L_{\Delta}^k = \bar{C}^h(e_0) \circ L_{\partial}^k \circ \bar{C}^h(\Phi_{\Delta})$. We obtain:

$$\frac{1}{k!} \cdot \left(\bar{C}^h(e_0) \circ (L_{\partial}^{k-1} \circ e_{\partial}, B_{A[[t]]}] + [L_{\partial}^{k-1} \circ E_{\partial}, b_{A[[t]]}]) \circ \bar{C}^h(\Phi_{\Delta}) \right) = L_{\Delta}^k. \quad (8.8)$$

Since the Hochschild differential b and the Connes differential B commute with ring homomorphisms, we get:

$$\begin{aligned} B_{A[[t]]} \circ \bar{C}^h(\Phi_{\Delta}) &= \bar{C}^h(\Phi_{\Delta}) \circ B_A & \bar{C}^h(e_0) \circ B_{A[[t]]} &= B_A \circ \bar{C}^h(e_0) \\ b_{A[[t]]} \circ \bar{C}^h(\Phi_{\Delta}) &= \bar{C}^h(\Phi_{\Delta}) \circ b_A & \bar{C}^h(e_0) \circ b_{A[[t]]} &= b_A \circ \bar{C}^h(e_0). \end{aligned} \quad (8.9)$$

Combining (8.8) and (8.9), we get:

$$\frac{1}{k!} \cdot \left([\bar{C}^h(e_0) \circ L_{\partial}^{k-1} \circ e_{\partial} \circ \bar{C}^h(\Phi_{\Delta}), B_A] + [\bar{C}^h(e_0) \circ L_{\partial}^{k-1} \circ E_{\partial} \circ \bar{C}^h(\Phi_{\Delta}), b_A] \right) = L_{\Delta}^k. \quad (8.10)$$

In a manner similar to (8.7), we can also obtain

$$[e_{\partial} \circ L_{\partial}^{k-1}, B_{A[[t]]}] + [E_{\partial} \circ L_{\partial}^{k-1}, b_{A[[t]]}] = L_{\partial}^k \quad (8.11)$$

and proceed similarly to obtain the expression

$$\frac{1}{k!} \cdot \left([\bar{C}^h(e_0) \circ e_{\partial} \circ L_{\partial}^{k-1} \circ \bar{C}^h(\Phi_{\Delta}), B_A] + [\bar{C}^h(e_0) \circ E_{\partial} \circ L_{\partial}^{k-1} \circ \bar{C}^h(\Phi_{\Delta}), b_A] \right) = L_{\Delta}^k. \quad (8.12)$$

The result is now clear from combining (8.10) and (8.12). $\square$

Acknowledgements We are grateful to the referee for important comments and suggestions and especially for pointing out the fact that (A, δ) -modules may be seen as modules over the skew polynomial ring $A[x; \delta]$. In this respect, the reference to [8] was given to us by the referee.

References

1. Banerjee, A.: Higher derivations and Hochschild homology. *C. R. Math. Acad. Sci. Soc. R. Can.* **33**(2), 33–43 (2011)
2. Banerjee, A.: Completions of monoid objects and descent results. *J. Algebra* **507**, 362–399 (2018)
3. Banerjee, A.: On differential torsion theories and rings with several objects. *Can. Math. Bull.* (to appear)
4. Beauville, A., Laszlo, Y.: Conformal blocks and generalized theta functions. *Commun. Math. Phys.* **164**, 385–419 (1994)
5. Beauville, A., Laszlo, Y.: Un lemme de descente. *C. R. Acad. Sci. Paris Sér. I Math.* **320**(3), 335–340 (1995)
6. Bland, P.E.: Differential torsion theory. *J. Pure Appl. Algebra* **204**, 1–8 (2006)
7. Geller, S.C., Weibel, C.A.: Étale descent for Hochschild and cyclic homology. *Comment. Math. Helv.* **66**(3), 368–388 (1991)
8. Goodearl, K.R., Warfield Jr., R.B.: An Introduction to Noncommutative Noetherian Rings. London Mathematical Society Student Texts, vol. 61, 2nd edn. Cambridge University Press, Cambridge (2004)
9. Goodwillie, T.G.: Cyclic homology, derivations, and the free loop space. *Topology* **24**(2), 187–215 (1985)
10. Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.* **20** (1964)
11. Hasse, H., Schmidt, F.K.: Noch eine Begründung der Theorie der höheren Differential-quotienten in einem algebraischen Funktionenkörper einer Unbestimmten. *J. Reine Angew. Math.* **177**, 215–237 (1937)
12. Loday, J.-L.: Cyclic Homology, Appendix E by María O. Ronco. *Grundlehren der Mathematischen Wissenschaften*, vol. 301. Springer, Berlin (1992)
13. Mirzavaziri, M.: Characterization of higher derivations on algebras. *Commun. Algebra* **38**, 981–987 (2010)
14. Rinehart, G.S.: Differential forms on general commutative algebras. *Trans. Am. Math. Soc.* **108**, 195–222 (1963)
15. Roy, A., Sridharan, R.: Higher derivations and central simple algebras. *Nagoya Math. J.* **32**, 21–30 (1968)
16. Sweedler, M.E.: Structure of inseparable extensions. *Ann. Math. (2)* **87**, 401–410 (1968)
17. Tanaka, M.: Locally nilpotent derivations on modules. *J. Math. Kyoto Univ.* **49**(1), 131–159 (2009)
18. Weisfeld, M.: Purely inseparable extensions and higher derivations. *Trans. Am. Math. Soc.* **116**, 435–449 (1965)

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