High-order Bahri–Lions Liouville-type theorems

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Received: 13 December 2018 / Accepted: 20 February 2019 / Published online: 28 March 2019
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Abstract

Consider the following polyharmonic equations:

$$(-\Delta)^r u = f(u) \quad \text{in } \mathcal{O}, \quad (0.1)$$

where $\mathcal{O} = \mathbb{R}^n$ or $\mathcal{O} = \mathbb{R}_+^n$ with the Dirichlet boundary conditions, $r \in \mathbb{N}^*$ and $n \geq 2r + 1$. We prove some Liouville-type theorems classifying stable (or stable at infinity) solutions, possibly unbounded and sign-changing. Regarding the class of stable solutions, we focus on the case of superlinear nonlinearities f with subcritical or critical growth **near zero**, like

$$f(s) = |s|^{p-1}s(1 + c_0|s|^q) \text{ or } f(s) = |s|^{p-1}s \exp(s^2),$$

$$\text{where } 1 < p \leq \frac{n+2r}{n-2r}, \quad q > 0 \text{ and } c_0 \geq 0.$$

Our approach to get the main integral estimates makes use of delicate analysis with appropriate test functions and weighted seminorms. We also establish a variant of Pohozaev identity (Pohozaev in *Sov Math Dokl* 5:1408–1411, 1965). This permits us to get classification result for stable at infinity solutions under the global subcritical condition:

$$\frac{2n}{n-2r} F(s) - f(s)s > 0, \quad \forall s \neq 0, \quad \text{where } F(s) = \int_0^s f(t)dt.$$

Our assumptions can be verified by many nonlinearities very close to the critical growth, like

$$f(s) = c|s|^{p-1}s + \frac{|s|^{\frac{4r}{n-2r}}s}{\ln^q(s^2 + a)},$$

where $1 < p < \frac{n+2r}{n-2r}$, $q \geq 0$, $c \geq 0$ and $a > 1$, with $\frac{4r}{n-2r} - \frac{2q}{a \ln(a)} > 0$; or $c > 0$ if $q = 0$.

Keywords Polyharmonic equations · Liouville theorems · Morse index · Pohozaev identity · Blow-up argument · Universal estimate · L^∞ -estimates

Mathematics Subject Classification Primary 35G20 · 35G30; Secondary 35B05 · 35B09 · 35B53

In Memory of my Great Professor Abbas Bahri, with Gratitude

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1 Introduction

Consider the following polyharmonic semilinear problems:

$$(-\Delta)^r u = f(u) \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

or the Dirichlet boundary problems on the half-space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n, x_n > 0\}$:

$$\begin{cases} (-\Delta)^r u = f(u) & \text{in } \mathbb{R}_+^n, \\ u = \frac{\partial}{\partial x_n} u = \dots = \frac{\partial^{r-1}}{\partial x_n^{r-1}} u = 0 & \text{on } \partial \mathbb{R}_+^n, \end{cases} \quad (1.2)$$

where $r \in \mathbb{N}^*$, $n \geq 2r + 1$, $f \in C^1(\mathbb{R})$ and $u \in C^{2r}(\mathbb{R}^n)$ (respectively $u \in C^{2r}(\overline{\mathbb{R}_+^n})$). Let us first comment on some previous Liouville theorems dealing with the homogeneous case $f(s) =: |s|^{p-1}s$. The nonexistence of positive solutions of (1.1) has been established by Wei-Xu for $1 < p < \frac{n+2r}{n-2r}$ [22], which extends the celebrated theorem of Gidas and Spruck [9]. The corresponding Liouville theorem on the half-space is more harder to achieve; it has been proved by Sirakov [20] and Reichel and Weth [19] under, respectively, the Navier and Dirichlet boundary conditions for $1 < p \leq \frac{n+2r}{n-2r}$. In a very interesting paper [2], Bahri–Lions explored the case of finite Morse index sign-changing solutions for $r = 1$ and $1 < p < \frac{n+2}{n-2}$. In [8], Farina made a significant progress and obtained a complete classification, which states that when $r = 1$, nontrivial finite Morse index solutions to (1.1) exist if and only if $p \geq p_{JL}$ and $n \geq 11$; or $p = \frac{n+2}{n-2}$ and $n \geq 3$. Here p_{JL} is the Joseph–Lundgren exponent. For the biharmonic equations, i.e., $r = 2$, the generalization of Moser’s iteration technique combined with Souplet’s inequality [21] allow only to classify positive stable solutions [4, 11, 23]. Recently, [6], Dávila–Dupaigne–Wang–Wei obtained sharp classifications for sign-changing solutions which are stable at infinity, by using a monotonicity formula and blow-down argument. Similar Liouville theorems have been obtained for the triharmonic equation in [14], where the so-called sixth-order Joseph–Lundgren exponent was computed.

There are few papers considering the nonhomogeneous nonlinearities (see [12] and the references therein). Particularly, in an elegant paper, Dupaigne–Farina investigated the non-homogeneous convex nonlinearities for $r = 1$ [7]. They provided several Liouville-type theorems in the subcritical and **supercritical** cases.

We will establish various nonexistence results for stable solutions of (1.1) (respectively (1.2)) or solutions which are stable outside a compact set, for a large class of global superlinear nonlinearities. The proof of the integral estimate for $r \leq 3$ is closely related to an iterative use of Young and Hölder’s inequalities (see [6, 8, 11, 12, 14, 23]), but these kinds of arguments seem to be difficultly extendable to the general higher-order operator. Our approach relies on a crucial idea, borrowed from [10, 19], where an appropriate family of test functions and weighted seminorms are used to obtain some local L^p - $W^{2r,p}$ estimates for solutions of the linear polyharmonic problems. However, our situation is very different from [19] since we have to combine the nonlinear equations (1.1) or (1.2) with the associated quadratic form which need more involved analysis. To describe our results more accurately, we first introduce some notations and definitions. We denote confusedly $\mathbb{R}^n$ and $\mathbb{R}_+^n$ by $\mathcal{O}$. B_λ stands for the ball of radius $\lambda > 0$ centered at the origin. Let $j = (j_1, j_2, \dots, j_n)$ be a multi-index of order $|j| \leq r$. The j th partial derivative of $v \in C^r(\mathcal{O})$ is $D^j v = \frac{\partial^{|j|} v}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}$, and the q th gradient

of v is denoted by $\nabla^q v$ and $|\nabla^q v|^2 = \sum_{|j|=q} (\mathcal{D}^j v)^2$. Let $v, w \in C^r(\mathcal{O})$, we denote

$$\begin{cases} D^r v = \Delta^k v, & D^r v D^r w = \Delta^k v \Delta^k w, & \text{if } r = 2k, \\ D^r v = \nabla \Delta^k v, & D^r v D^r w = \nabla \Delta^k v \cdot \nabla \Delta^k w, & \text{if } r = 2k + 1, \\ |D^r v|^2 = D^r v D^r v. \end{cases}$$

Note that $D^r v$ is an N -vectorial operator when r is odd and $N > 1$. Let Ω be a domain of $\mathbb{R}^n$, $f \in C^1(\mathbb{R})$ and $u \in C^{2r}(\Omega)$ a solution of the following polyharmonic problem

$$(-\Delta)^r u = f(u) \text{ in } \Omega.$$

The associated quadratic form is defined by

$$Q_u(h) = \int_{\Omega} |D^r h|^2 - \int_{\Omega} f'(u) h^2, \quad \forall h \in C_c^r(\Omega). \quad (1.3)$$

u is said to be stable if $Q_u(h) \geq 0$, $\forall h \in C_c^r(\Omega)$. The Morse index of u is defined as the maximal dimension of all subspaces V of $C_c^r(\Omega)$ such that $Q_u(h) < 0$, $\forall h \in V \setminus \{0\}$. More generally, u is said stable outside a compact set $\mathcal{K} \subset \Omega$, if $Q_u(h) \geq 0$ for any $h \in C_c^r(\Omega \setminus \mathcal{K})$. It is well know that any finite Morse index solution u is stable outside a compact set.

1.1 Main results

To provide the main integral estimate, we assume :

(H_1): $f(s)s > 0$, $\forall s \neq 0$ and there exists $\mu > 0$ such that $f'(s)s^2 - (1 + \mu)f(s)s \geq 0$, $\forall s \in \mathbb{R}$.

Assumption (H_1) signifies that f has a global superlinear growth (see Remark 1.1 for further comments). The following Proposition is the key step to derive nonexistence results.

Proposition 1.1 *Let $u \in C^{2r}(\mathbb{R}^n)$ be a solution of (1.1) (respectively, $u \in C^{2r}(\overline{\mathbb{R}_+^n})$ solution of (1.2)) which is stable outside a compact set $\mathcal{K} \subset \mathcal{O} \cap B_{R_0}$ for some $R_0 > 0$. Assume that f satisfies (H_1), then for every $\alpha \in [\frac{1}{2}, 1)$ and any $R > 4R_0$, we have*

$$\int_{\mathcal{O} \cap B_{\alpha R}} |\nabla^r u|^2 + \int_{\mathcal{O} \cap B_{\alpha R}} f(u)u \leq (1 - \alpha)^{-2r} \left(C_0 + C R^{-2r} \int_{\mathcal{O} \cap B_R} u^2 \right). \quad (1.4)$$

Here $C = C(n, r, \mu)$ and $C_0 = C_0(u, R_0, n, r, \mu)$ are positive constants independent of R and α .

If u is a stable solution, then $C_0 = 0$. Moreover, we have

$$\int_{\mathcal{O} \cap B_{\frac{1}{2}R}} f(u)u \leq C \left(\int_{\mathcal{O} \cap A_R} |\nabla^r u|^2 + \left(\int_{\mathcal{O} \cap A_R} |u|^{\frac{2n}{n-2r}} \right)^{\frac{n-2r}{n}} \right), \quad (1.5)$$

where $A_R = \{\frac{1}{2}R < |x| < \frac{3}{4}R\}$.

Remark that for general nonlinearities, we cannot always use the doubling lemma technique as in the homogeneous case (see [17,21,23]) to reduce the classification problem only for bounded solutions. Therefore, we shall pay special attention to the delicate case of unbounded solutions. In fact, since (1.4) is weaker than the similar integral estimates obtained in [6,8,12,14,23] for $r \leq 4$, then to provide nonexistence result we have to adapt the Moser iteration technique which requires that u evolves less rapidly than an exponential growth:

(U_1): There exist $b_0, b_1 > 0$ and $0 < b < \frac{\ln(p+1)}{\ln(2)} - 1$ such that $|u(x)| \leq b_0 \exp(b_1|x|^b)$, $\forall x \in \mathcal{O}$.

To investigate the nonexistence of stable solutions, we assume:

(H_2): There exist $1 < p \leq \frac{n+2r}{n-2r}$ and a positive constant C such that $C|s|^{p+1} \leq f(s)s$, $\forall s \in \mathbb{R}$.

Of course, the additional assumption (U_1) is removed if we discuss the case of **bounded** solutions. Also, we could weaken (H_2) by the following condition only near 0:

(H'_2): There exist $s_0 > 0$ and a positive constant C such that $C|s|^{\frac{2n}{n-2r}} \leq f(s)s$, $\forall |s| \leq s_0$.

However, for the sake of simplicity, we assume often the condition (H_2) (see Remark 1.1 for more comments).

Theorem 1.1 Let $u \in C^{2r}(\mathbb{R}^n)$ (respectively $u \in C^{2r}(\overline{\mathbb{R}^n_+})$) be a stable solution of (1.1) (respectively of (1.2)) satisfying (U_1). We have

1. If f satisfies (H_1)–(H_2), then $u \equiv 0$.
2. If $f(s) = |s|^{p-1}s$, $p > 1$, then there exists a positive constant $C = C(n, p, r)$ such that

$$\int_{\mathcal{O} \cap B_R} |\nabla^r u|^2 + \int_{\mathcal{O} \cap B_R} |u|^{p+1} \leq C R^{n-2r} \frac{p+1}{p-1}, \quad \forall R > 1.$$

Next, set $F(s) = \int_0^s f(t)dt$ and $G(s) = \frac{2n}{n-2r}F(s) - sf(s)$. In addition to (H_1)–(H_2), we need the following assumption to classify solutions which are stable outside a compact set.

(H_3): $G(s) > 0$, $\forall s \neq 0$.

Theorem 1.2 Let $u \in C^{2r}(\mathbb{R}^n)$ (respectively, $u \in C^{2r}(\overline{\mathbb{R}^n_+})$) be a solution of (1.1) (respectively, of (1.2)) which is stable outside a compact of $\mathcal{O}$ and satisfying (U_1). We have

1. If f satisfies (H_1)–(H_2), then

$$\int_{\mathcal{O}} |D^r u|^2 = \int_{\mathcal{O}} f(u)u = \frac{2n}{n-2r} \int_{\mathcal{O}} F(u) < \infty.$$

2. If in addition f satisfies (H_3), then $u \equiv 0$.

In the following, we collect some constructive examples of nonlinearities satisfying our assumptions.

Remark 1.1 1. In view of (H_1), we have $\left(\frac{f(s)s}{|s|^{\mu+2}}\right)' > 0$, so according to (H_2) we derive

$$\begin{cases} C|s|^{\frac{2n}{n-2r}} \leq f(s)s \leq \max(f(1), -f(-1))|s|^{\mu+2}, & \forall |s| \leq 1, \\ f(s)s \geq \min(f(1), -f(-1))|s|^{\mu+2}, & \forall |s| \geq 1, \end{cases}$$

which yields that $\mu + 1 \leq \frac{n+2r}{n-2r}$. Therefore, we require that f has a subcritical or critical growth near zero and only a superlinear growth at infinity. As for example the following nonlinearities satisfy assumption (H_1)–(H_2) of Theorem 1.2:

$$f(s) = |s|^{p-1}s(1 + c_0|s|^q) \text{ or } f(s) = |s|^{p-1}s \exp(s^2),$$

$$\text{where } 1 < p \leq \frac{n+2r}{n-2r}, q > 0 \text{ and } c_0 \geq 0.$$

2. From (H_1) (respectively, (H_3)), we have

$$\text{and } (\mu+2)F(s) < sf(s), \quad \forall s \neq 0 \text{ and } \min(F(1), F(-1))|s|^{\frac{2n}{n-2r}} \leq F(s), \quad \forall |s| \leq 1, \quad (1.6)$$

which means that (H_1) and (H_3) imply (H'_2) . Thus for **bounded** solutions, Theorem 1.2 holds under (H_1) and (H_3) only.

3. Consider the following nonlinearities (which are very close to the critical growth):

$$f(s) = c|s|^{p-1}s + \frac{|s|^{\frac{4r}{n-2r}}s}{\ln^q(s^2+a)}, \quad c \geq 0, \quad 1 < p < \frac{n+2r}{n-2r}, \quad q > 0 \text{ and } a > 1.$$

We have f satisfies $f(s)s > 0$ and (H_3) as

$$\frac{2n}{n-2r}F(s) - sf(s) > 2q \int_0^s \frac{|t|^{\frac{2n}{n-2r}}t}{(t^2+a)(\ln(t^2+a))^{q+1}} dt > 0, \quad \forall s \neq 0.$$

Also, f satisfies, respectively, (H_2) if $c > 0$ and only (H'_2) if $c = 0$. Assume now that $\frac{4r}{n-2r} - \frac{2q}{a \ln(a)} > 0$, then f satisfies (H_1) for all $0 < \mu < \min(\frac{4r}{n-2r} - \frac{2q}{a \ln(a)}, p-1)$. In conclusion, Theorems 1.1 and 1.2 hold for bounded or unbounded solutions if $c > 0$ but only for bounded solutions if $c = 0$.

4. Set $f(s) = |s|^{\frac{4r}{n-2r}}s + c|s|^{p-1}s$. Clearly, f satisfies assumptions (H_1) – (H_3) for all $1 < p < \frac{n+2r}{n-2r}$ and $c > 0$.

At last, consider the following polyharmonic problems:

$$(-\Delta)^r u = g(u), \quad \text{in } \Omega, \quad (1.7)$$

where Ω is a domain of $\mathbb{R}^n$, $u \in C^{2r}(\Omega)$ and $g \in C^1(\mathbb{R})$. If Ω is a smooth bounded domain, we assume in addition that u satisfies the Dirichlet boundary conditions:

$$u = \frac{\partial}{\partial \nu} u = \dots = \left(\frac{\partial}{\partial \nu} \right)^{r-1} u = 0 \quad \text{on } \partial\Omega, \quad (1.8)$$

where ν is the unit exterior normal on $\partial\Omega$.

Inspired by [2], Yang improved Bahri–Lions's *a priori* bounds and obtained a first explicit L^∞ -estimate via the Morse index [24] in a bounded smooth domain. The general higher-order case is harder to achieve since we need to carefully handle some local interior estimate, especially near the boundary (see [13] for the biharmonic and triharmonic cases). However, following [2,5,8,9,15,19,23], by the virtue of Theorem 1.2 we can derive L^∞ -bounds and universal estimate. As usual, we require that g behaves like $|s|^{p-1}s$ as $s \rightarrow \infty$ for some subcritical power $1 < p < \frac{n+2r}{n-2r}$. More precisely we assume: (H') : $g \in C^1(\mathbb{R})$ and $\lim_{|s| \rightarrow \infty} g'(s)|s|^{1-p} = c_0$, where $c_0 > 0$.

Denote by $i(u)$ the Morse index of u , $d_x = \text{dist}(x, \partial\Omega)$ and fix $m_0 \in \mathbb{N}^*$. We have,

Theorem 1.3 1. Assume that Ω is a proper domain. Then there exists a positive constant $C = C(n, r, p, m_0)$ independent of Ω such that for every finite Morse index solution u of (1.7) with $i(u) \leq m_0$, we have

$$\sum_{q=0}^{2r-1} |\nabla^q u(x)|^{\frac{2r}{2r+q(p-1)}} \leq C \left(1 + d_x^{\frac{-2r}{p-1}} \right), \quad \forall x \in \Omega. \quad (1.9)$$

2. Assume that Ω is a smooth bounded domain. Let (u_j) be a sequence of solutions of (1.7) and (1.8). Then (u_j) is bounded in $L^\infty(\Omega)$ if and only if the sequence of their Morse indices is bounded.

This paper is organized as follows. In Sect. 2, we give the proof of Proposition 1.1. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we establish a variant of Pohozaev identity and we develop the proof of Theorem 1.2. Lastly, in Sect. 5 we prove Theorem 1.3. The proofs of Proposition 1.1 and Theorems 1.1, 1.2 will be only done in the delicate case of the Dirichlet boundary problems (1.2), which needs more careful analysis.

2 Proof of Proposition 1.1

2.1 Preliminary results

For $\lambda > 0$, set $\Omega_\lambda = B_\lambda^+ = B_\lambda \cap \mathbb{R}_+^n$ or $\Omega_\lambda = A_\lambda^+ = \{x \in \mathbb{R}_+^n, a_1\lambda < |x| < a_2\lambda\}$, for some $0 < a_1 < a_2$. The following two inequalities follow from standard estimates for $\lambda = 1$ and an obvious dilation argument.

Interpolation inequality [1] There exists a positive constant C depending on (n, r) only, such that for every $\lambda > 0$, $\varepsilon > 0$ and $1 \leq q \leq r - 1$ we have

$$\lambda^{2(q-r)} \int_{\Omega_\lambda} |\nabla^q v|^2 \leq \varepsilon \int_{\Omega_\lambda} |\nabla^r v|^2 + C \varepsilon^{\frac{-q}{r-q}} \lambda^{-2r} \int_{\Omega_\lambda} v^2, \quad \forall v \in H^r(\Omega_\lambda). \quad (2.1)$$

Higher-order Calderon–Zygmund’s inequality (see appendix of [3]) There exists a positive constant $C = C(n, r)$ such that

$$\int_{B_\lambda^+} |\nabla^r v|^2 \leq C \int_{B_\lambda^+} |D^r v|^2, \quad \forall v \in H_0^r(B_\lambda^+). \quad (2.2)$$

Recall also the following elementary inequality which will be used a number of times later. Let $\beta_q \in \mathbb{R}$, $0 \leq q \leq r - 1$, then for any $\varepsilon > 0$, we have

$$\left| \beta_0 \sum_{q=1}^{r-1} \beta_q \right| \leq \varepsilon \beta_0^2 + \frac{r}{\varepsilon} \sum_{q=1}^{r-1} \beta^2. \quad (2.3)$$

Choice of appropriate test functions Let u be a solution of (1.2) which is stable outside a compact set $\mathcal{K} \subset B_{R_0}^+$. For all $R > 4R_0$ and $\alpha \in [\frac{1}{2}, 1)$, we have $2R_0 < \alpha R$. Set $\alpha' = \frac{1+\alpha}{2} \in [\frac{3}{4}, 1)$, then we can define a family of test functions $\psi = \psi_{(R, R_0, \alpha)} \in C_c^r(\mathbb{R}^n)$ satisfying

$$\begin{cases} 0 \leq \psi \leq 1, \\ \psi \equiv 1 \text{ if } 2R_0 < |x| < \alpha R, \quad \psi \equiv 0 \text{ if } |x| < R_0 \text{ or } |x| > \alpha' R, \\ |\nabla^q \psi| \leq C R_0^{-q}, \text{ if } R_0 < |x| < 2R_0, \quad q \leq r, \\ |\nabla^q \psi| \leq C((1-\alpha)R)^{-q}, \text{ if } \alpha R < |x| < \alpha' R, \text{ and } q \leq r. \end{cases} \quad (2.4)$$

If u is a stable solution, we use $\psi = \psi_{(R, \alpha)}$ verifying (2.4) with $R_0 = 0$ (that is $\psi = 1$, $|x| < \alpha R$).

Introduce the following family of weighted seminorms (see [10, 19] for similar definitions):

$$\Phi_q^2(u) = \Phi_{q,R}^2(u) = \sup \left\{ (1-\alpha)^{2q} \int_{B_{\alpha R}^+} |\nabla^q u|^2, \frac{1}{2} \leq \alpha < 1 \right\}, \quad 0 \leq q \leq r. \quad (2.5)$$

Therefore, Φ_q satisfies the following interpolation inequality which is essential to prove Proposition 1.1: There exists a positive constant $C = C(n, r) > 0$ such that for any $\varepsilon > 0$, we have

$$R^{2(q-r)}\Phi_q^2(u) \leq \varepsilon\Phi_r^2(u) + C\varepsilon^{\frac{-q}{r-q}}R^{-2r} \int_{B_R} u^2, \quad \forall R > 4R_0. \quad (2.6)$$

Indeed, set $h(\alpha) = (1 - \alpha)^{2q} \int_{B_{\alpha R}^+} |\nabla^q u|^2$, $\alpha \in [\frac{1}{2}, 1]$. Recall that $u \in C^{2r}(\overline{\mathbb{R}_+^n})$, then we may see easily that h is a continuous function with $h(1) = 0$. Consequently, there exists $\alpha_0 = \alpha(u, R, q) \in [\frac{1}{2}, 1)$ such that

$$\Phi_q^2(u) = (1 - \alpha_0)^{2q} \int_{B_{\alpha_0 R}^+} |\nabla^q u|^2.$$

Thus, using (2.1) where we substitute ε by $\varepsilon((1 - \alpha_0))^{2(r-q)}$ and λ by $\alpha_0 R$, we obtain

$$\begin{aligned} R^{2(q-r)}(1 - \alpha_0)^{2q} \int_{B_{\alpha_0 R}^+} |\nabla^q u|^2 &\leq \varepsilon(1 - \alpha_0)^{2r} \int_{B_{\alpha_0 R}^+} |\nabla^r u|^2 + C\varepsilon^{\frac{-q}{r-q}}R^{-2r} \int_{B_{\alpha_0 R}^+} u^2 \\ &\leq \varepsilon\Phi_r^2(u) + C\varepsilon^{\frac{-q}{r-q}}R^{-2r} \int_{B_R^+} u^2. \end{aligned}$$

So, inequality (2.6) follows.

Set $m = r + 1$, we need also the following inequality (see the proof in Appendix):

$$|\nabla^l \psi^{2m}| \leq C\psi^m((1 - \alpha)R)^{-l}, \quad \forall x \in A_R^+ \text{ and } l \leq r. \quad (2.7)$$

In the sequel, $C = C(n, r, \mu, \varepsilon)$ (respectively, $C_0 = C_0(u, R_0, r, n, \mu, \varepsilon)$) denotes generic positive (respectively, nonnegative) constants which are **independent of R and α** . In the following technical Lemma, we do not assume any stability condition on u .¹

Lemma 2.1 *Let $u \in C^{2r}(\overline{\mathbb{R}_+^n})$ be a solution of (1.2) and $h \in C_c^r(\mathbb{R}^n)$. Consider $v \in C^r(\overline{\mathbb{R}_+^n})$ and satisfying the Dirichlet boundary conditions*

$$v = \frac{\partial}{\partial x_n} v = \dots = \frac{\partial^{r-1}}{\partial x_n^{r-1}} v = 0 \text{ on } \partial\mathbb{R}_+^n. \quad (2.8)$$

We have,

$$\left| \int_{\mathbb{R}_+^n} h D^r u D^r v - f(u) v h \right| \leq \int_{\mathbb{R}_+^n} |D^r u| \sum_{q=0}^{r-1} |\nabla^{r-q} h| |\nabla^q v|. \quad (2.9)$$

For every $0 < \varepsilon < \frac{1}{3}$, there exist two constants $C > 0$ and $C_0 \geq 0$ such that

$$(1 - 3\varepsilon) \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 - \int_{B_{\alpha'R}^+} f(u) u \psi^{2m} \leq C_0 + CE(u, R, \alpha), \quad (2.10)$$

where

$$\psi = \psi_{(R, R_0, \alpha)} \text{ and } E(u, R, \alpha) = \sum_{q=0}^{r-1} ((1 - \alpha)R)^{2(q-r)} \int_{A_R^+} |\nabla^q u|^2.$$

In particular, $C_0 = 0$ if $\psi = \psi_{(R, \alpha)}$, i.e., $R_0 = 0$.

¹ Inequality (2.9) is also useful to prove Lemma 4.1 in Section 4.

Proof of Lemma 2.1. First, from Leibnitz's formula [1], we have²

$$D^r(vh) = hD^r v + A(h, v), \text{ where } |A(h, v)| \leq C_{r,n} \sum_{q=0}^{r-1} |\nabla^{r-q} h| |\nabla^q v|. \quad (2.11)$$

Observe that vh satisfies also (2.8), then multiplying (1.2) by vh and integrating by parts we obtain

$$\int_{\mathbb{R}_+^n} D^r u D^r(vh) = \int_{\mathbb{R}_+^n} f(u)vh. \quad (2.12)$$

Multiply now (2.11) by $D^r u$ and integrate over $\mathbb{R}_+^n$, we have

$$\left| \int_{\mathbb{R}_+^n} h D^r u D^r v - D^r u D^r(vh) \right| \leq C_{n,r} \int_{\mathbb{R}_+^n} |D^r u| \sum_{q=0}^{r-1} |\nabla^{r-q} h| |\nabla^q v|.$$

Hence, (2.9) follows from by combining inequality with (2.12). $\square$

Proof of (2.10) First, for $v = u$ rewrite (2.11) as $(D^r u)h = D^r(uh) - A(h, u)$, we derive

$$|D^r u|^2 h^2 - |D^r(uh)|^2 \leq 2|D^r(uh)||A(h, u)| + |A(h, u)|^2.$$

Using (2.3) and integrating over $\mathbb{R}_+^n$, we obtain

$$\left| \int_{\mathbb{R}_+^n} |D^r u|^2 h^2 - |D^r(uh)|^2 \right| \leq \varepsilon \int_{\mathbb{R}_+^n} |D^r(uh)|^2 + C_{n,r,\varepsilon} \int_{\mathbb{R}_+^n} \sum_{q=0}^{r-1} |\nabla^{r-q} h|^2 |\nabla^q u|^2. \quad (2.13)$$

Substitute h by $h = \psi^m$ in (2.13), then from (2.4) we get³

$$\left| \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 - \int_{B_{\alpha'R}^+} |D^r u|^2 \psi^{2m} \right| \leq C_0 + \varepsilon \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 + CE(u, R, \alpha). \quad (2.14)$$

Using now (2.9) with $v = u$ and $h = \psi^{2m}$, so in view of (2.7) we derive⁴

$$\int_{B_{\alpha'R}^+} |D^r u|^2 \psi^{2m} - \int_{B_{\alpha'R}^+} f(u)u\psi^{2m} \leq C_0 + C \sum_{q=0}^{r-1} ((1-\alpha)R)^{q-r} \int_{A_R^+} (|D^r u| \psi^m) |\nabla^q u|.$$

Apply again (2.3), yields

$$\int_{B_{\alpha'R}^+} |D^r u|^2 \psi^{2m} - \int_{B_{\alpha'R}^+} f(u)u\psi^{2m} \leq C_0 + \varepsilon \int_{B_{\alpha'R}^+} |D^r u|^2 \psi^{2m} + CE(u, R, \alpha). \quad (2.15)$$

² Note that $A(h, v)$ is an N -vectorial operator when r is odd and $N > 1$.

³ $C_0 = C \sum_{q=0}^{r-1} R_0^{2(q-r)} \int_{A_0^+} |\nabla^q u|^2$, where $A_0^+ = \{x \in \mathbb{R}_+^n, R_0 \leq |x| \leq 2R_0\}$ and $C_0 = 0$ if $R_0 = 0$.

⁴ $C_0 = C \sum_{q=0}^{r-1} R_0^{q-r} \int_{A_0^+} |D^r u| |\nabla^q u|$ and $C_0 = 0$ if $R_0 = 0$.

Since (2.14) (with $\varepsilon = 1$) implies

$$\int_{B_{\alpha'R}^+} |D^r u|^2 \psi^{2m} \leq C_0 + 2 \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 + CE(u, R, \alpha),$$

then inequality (2.15) gives

$$\int_{B_{\alpha'R}^+} |D^r u|^2 \psi^{2m} - \int_{B_{\alpha'R}^+} f(u) u \psi^{2m} \leq C_0 + 2\varepsilon \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 + CE(u, R, \alpha).$$

Combine the above inequality with (2.14), we obtain the desired inequality (2.10). $\square$

2.2 End of the proof of Proposition 1.1.

Let $u \in C^{2r}(\overline{\mathbb{R}_+^n})$ be a solution of (1.2) which is stable outside $\mathcal{K} \subset B_{R_0}^+$. Clearly $u\psi^m \in H_0^r(B_{\alpha'R}^+ \setminus B_{R_0}^+)$, so after a standard approximation argument, the main inequality of stability (1.3) implies

$$\int_{B_{\alpha'R}^+} f'(u) u^2 \psi^{2m} dx - \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 \leq 0, \quad \text{for all } \alpha \in [\frac{1}{2}, 1) \text{ and } R > 4R_0. \quad (2.16)$$

Also, we have $u\psi^m \in H_0^r(B_{\alpha'R}^+)$ with $\psi = 1$ for $2R_0 < |x| < \alpha R$, then according to (2.2) we derive

$$\int_{B_{\alpha R}^+} |\nabla^r u|^2 - C_0 = \int_{2R_0 < |x| < \alpha R} |\nabla^r u|^2 \leq \int_{B_{\alpha'R}^+} |\nabla^r(u\psi^m)|^2 \leq C \int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2, \quad (2.17)$$

where $C_0 = \int_{B_{2R_0}} |\nabla^r u|^2$.

We shall combine inequality (2.10) of Lemma 2.1 with (2.16) by using assumption (H_1) as follows. Set $\mu_0 = \frac{1+\varepsilon_0}{1-3\varepsilon_0} - 1$ and choose ε_0 small enough such that $\mu_0 < \mu$, then in view of (H_1) we have

$$0 \leq (\mu - \mu_0) f(s) s \leq f'(s) s^2 - (1 + \mu_0) f(s) s, \quad \forall s \in \mathbb{R}.$$

Therefore, multiplying (2.10) by $1 + \mu_0$ and summing with (2.16), we derive

$$\int_{B_{\alpha'R}^+} |D^r(u\psi^m)|^2 + \int_{B_{\alpha'R}^+} f(u) u \psi^{2m} \leq C_0 + CE(u, R, \alpha). \quad (2.18)$$

Insert now (2.17) into the left-hand side of (2.18), we get

$$\int_{B_{\alpha R}^+} |\nabla^r u|^2 + \int_{B_{\alpha'R}^+} f(u) u \psi^{2m} \leq C_0 + CE(u, R, \alpha). \quad (2.19)$$

Taking into account that $A_R^+ \subset B_{\alpha'R}^+$ and $(1 - \alpha)^{2r} = 2^{4r}(1 - \alpha')^{2r}$, and multiply (2.19) by $(1 - \alpha)^{2r}$, we obtain

$$\begin{aligned} (1 - \alpha)^{2r} \left(\int_{B_{\alpha'R}^+} |\nabla^r u|^2 + \int_{B_{\alpha'R}^+} f(u)u \right) &\leq C_0 + C \sum_{q=0}^{r-1} R^{2(q-r)} (1 - \alpha')^{2q} \int_{B_{\alpha'R}^+} |\nabla^q u|^2 \\ &\leq C_0 + C \sum_{q=0}^{r-1} R^{2(q-r)} \Phi_q^2(u), \end{aligned} \quad (2.20)$$

$\forall \alpha \in [\frac{1}{2}, 1)$, $R > 4R_0$ and where $\Phi_q^2(u)$ are the weighted seminorms defined in (2.5). Invoke now the crucial interpolation inequality (2.6) and recall that $f(s)s \geq 0$, $\forall s \in \mathbb{R}$ (see (H_1)), we derive

$$\Phi_r^2(u) \leq C_0 + C \sum_{q=0}^{r-1} R^{2(q-r)} \Phi_q^2(u) \leq \varepsilon \Phi_r^2(u) + C_\varepsilon \int_{B_R^+} u^2, \quad \forall R > 4R_0, \quad (2.21)$$

which for $\varepsilon = \frac{1}{2}$ implies

$$\frac{1}{2} \Phi_r^2(u) \leq C_0 + CR^{-2r} \int_{B_R^+} u^2, \quad \forall R > 4R_0. \quad (2.22)$$

Plugging now (2.6) in the right-hand side of (2.20) (with $\varepsilon = \frac{1}{2}$) and using (2.22), we obtain

$$\begin{aligned} (1 - \alpha)^{2r} \left(\int_{B_{\alpha'R}^+} |\nabla^r u|^2 + \int_{B_{\alpha'R}^+} f(u)u \right) &\leq C_0 + \frac{1}{2} \Phi_r^2(u) + CR^{-2r} \int_{B_R^+} u^2 \\ &\leq C_0 + CR^{-2r} \int_{B_R^+} u^2, \end{aligned} \quad (2.23)$$

for all $\alpha \in [\frac{1}{2}, 1)$ and $R > 4R_0$. Thus, the main integral estimate (1.4) is well proved.

Proof of (1.5). We comeback to (2.19) with $C_0 = 0$ as u is a stable solution, then for $\alpha = \frac{1}{2}$ and $A_R^+ = \{\frac{1}{2}R < |x| < \frac{3}{4}R\}$ we derive

$$\int_{B_{\frac{1}{2}R}^+} f(u)u \leq C \sum_{q=0}^{r-1} (R)^{2(q-r)} \int_{A_R^+} |\nabla^q u|^2.$$

Using (2.1) (with $\varepsilon = 1$) and Hölder's inequality, we obtain

$$\begin{aligned} \int_{B_{\frac{1}{2}R}^+} f(u)u &\leq C \left(\int_{A_R^+} |\nabla^r u|^2 + R^{-2r} \int_{A_R^+} u^2 \right) \\ &\leq C \left[\int_{A_R^+} |\nabla^r u|^2 + \left(\int_{A_R^+} |u|^{\frac{2n}{n-2r}} \right)^{\frac{n-2r}{n}} \right]. \end{aligned}$$

The desired inequality (1.5) follows and the proof of Proposition 1.1 is well completed. $\square$

3 Proof of Theorem 1.1

3.1 Preliminary Lemma

We adapt the Moser iteration method to establish the following technical Lemma.

Lemma 3.1 *Let $p > 1$ and $v \in L_{loc}^{p+1}(\mathbb{R}_+^n)$ satisfying (U_1) . Assume that there exist $C > 0$ and $R'_0 > 0$ such that*

$$\int_{B_R^+} |v|^{p+1} \leq CR^{-2r} \int_{B_{2R}^+} v^2, \quad \forall R > R'_0. \quad (3.1)$$

Then, there exists a positive constant C depending on (n, p) only such that

$$\int_{B_R^+} |v|^{p+1} \leq CR^{\delta_p \frac{p+1}{p-1}}, \quad \forall R > R'_0, \quad (3.2)$$

where $\delta_p = n \frac{p-1}{p+1} - 2r$. Moreover, we have

$$R^{-2r} \int_{B_R^+} v^2 \leq CR^{\delta_p \frac{p+1}{p-1}}, \quad \forall R > R'_0. \quad (3.3)$$

Proof of (3.2). Substituting R by $2^j R$ in (3.1), and using Hölder's inequality, we obtain

$$\int_{B_{2^j R}^+} |v|^{p+1} \leq CR^{-2r} \int_{B_{2^{j+1} R}^+} v^2 \leq C_n 2^{(j+1)\delta_p} R^{\delta_p} \left(\int_{B_{2^{j+1} R}^+} |v|^{p+1} \right)^{\beta}, \quad \forall R > R'_0 \text{ and } j \in \mathbb{N},$$

where $\beta = \frac{2}{p+1}$. Thus, a simple iteration of the above inequality gives

$$\int_{B_R^+} |v|^{p+1} \leq C_n^{s_j} 2^{(\delta_p s'_j)} R^{\delta_p s_j} \left(\int_{B_{2^{j+1} R}^+} |v|^{p+1} \right)^{\beta^{j+1}}, \quad \forall R > R'_0, \quad (3.4)$$

where $s_j = \sum_{q=0}^j \beta^q$ and $s'_j = \sum_{q=1}^j q\beta^q$. According to assumption (U_1) , we can find a positive constant $C' = C'(n, p, b_0)$ such that

$$\left(\int_{B_{2^{j+1} R}^+} |v|^{p+1} \right)^{\beta^{j+1}} \leq (C' R^n)^{\beta^{j+1}} \exp \left(b_1(p+1)(\beta 2^b)^{j+1} R^b \right),$$

which combined with (3.4) imply

$$\int_{B_R^+} |v|^{p+1} \leq C_n^{s_j} 2^{(\delta_p s'_j)} R^{\delta_p s_j} (C' R^n)^{\beta^{j+1}} \exp \left(b_1(p+1)(\beta 2^b)^{j+1} R^b \right), \quad \forall R > R'_0 \text{ and } j \in \mathbb{N}^*. \quad (3.5)$$

Since $s_j \rightarrow \frac{p+1}{p-1}$, $s'_j \rightarrow \frac{2(p+1)}{(p-1)^2}$, $(C' R^n)^{\beta^{j+1}} \rightarrow 1$, also we have $\exp(b_1(p+1)(\beta 2^b)^{j+1} R^b) \rightarrow 1$ as $\beta 2^b < 1$ (see (U_1)). Passing to the limit in (3.5), we derive $\int_{B_R^+} |v|^{p+1} \leq CR^{\delta_p \frac{p+1}{p-1}}$, $\forall R > 0$, where $C > 0$ is independent of b_0, b_1, b and depending on (n, p) only. So (3.2) is well proved. $\square$

Proof of (3.3) Observe that $1 + \frac{2}{p-1} = \frac{p+1}{p-1}$ and using again Hölder's inequality in (3.2), we deduce $R^{-2r} \int_{B_R^+} v^2 \leq C_n R^{\delta_p} \left(\int_{B_R^+} |v|^{p+1} \right)^{\beta} \leq C'_n R^{\delta_p(1+\frac{2}{p-1})} \leq C R^{\delta_p \frac{p+1}{p-1}}$. Thus, Lemma 3.1 is well proved. $\square$

3.2 End of the proof of Theorem 1.1

Proof of point 1 Since we assume that u is a stable solution, then the integral estimate (1.4) holds with $C_0 = 0$. So for $\alpha = \frac{1}{2}$ and R substituted by $2R$, assumption (H_2) implies

$$C \int_{B_R^+} |u|^{p+1} \leq \int_{B_R^+} |\nabla^r u|^2 + \int_{B_R^+} f(u)u \leq C R^{-2r} \int_{B_{2R}^+} u^2, \quad \forall R > 0, \quad (3.6)$$

where $1 < p \leq \frac{n+2r}{n-2r}$ (see (H_2)), which implies that u satisfies assumption (3.1) of Lemma 3.1. So, by the virtue of (3.2), we deduce that $u \equiv 0$ if $1 < p < \frac{n+2r}{n-2r}$. If $p = \frac{n+2r}{n-2r}$, according to (3.3) we have $R^{-2r} \int_{B_R^+} u^2 \leq C, \quad \forall R > 0$, which combined with (4.15) imply $\int_{\mathbb{R}_+^n} |\nabla^r u|^2 + \int_{\mathbb{R}_+^n} |u|^{\frac{2n}{n-2r}} < \infty$. Consequently, one has $\int_{A_R^+} |\nabla^r u|^2 + \int_{A_R^+} |u|^{\frac{2n}{n-2r}} = o(1)$, as $R \rightarrow \infty$, where $A_R^+ = \{x \in \mathbb{R}_+^n, \frac{1}{2}R \leq |x| \leq \frac{3}{4}R\}$, and by invoking the integral estimate (1.5) of Proposition 1.1 we obtain $\int_{B_{\frac{1}{2}R}^+} f(u)u = o(1)$. As $f(s)s > 0, \quad \forall s \in \mathbb{R}^*$ (see (H_1)), letting $R \rightarrow \infty$, we conclude that $u \equiv 0$. $\square$

Proof of point 2 Observe that $f(s) = |s|^{p-1}s, \quad p > 1$ satisfies (H_1) , then (1.4) implies

$$\int_{B_R^+} |\nabla^r u|^2 + \int_{B_R^+} |u|^{p+1} \leq C R^{-2r} \int_{B_{2R}^+} u^2, \quad \forall R > 0.$$

Hence, u satisfies assumption (3.1) of Lemma 3.1, and point 2 follows from (3.3). The proof of Theorem 1.1 is well completed. $\square$

4 Proof of Theorem 1.2

4.1 A variant of Pohozaev identity

Consider $u \in C^{2r}(\overline{\mathbb{R}_+^n})$ a solution of (1.2) satisfying the following evanescence condition: $(\overline{U})$: $\int_{A_R^+} |\nabla^r u|^2 + R^{-2r} \int_{A_R^+} u^2 = o(1)$, as $R \rightarrow \infty$ where $A_R^+ = \{x \in \mathbb{R}_+^n, \quad R < |x| < 2R\}$. Our approach consists in testing the equation (1.2) against $\phi \nabla u \cdot x$ where $\phi = \phi_R \in C_c^r(\mathbb{R}^n)$ is a cutoff function satisfying

$$\begin{cases} \phi \equiv 1 & \text{if } |x| < R, \quad \phi \equiv 0 & \text{if } |x| > 2R, \\ |\nabla^q \phi| \leq C R^{-q}, & \text{if } A_R = \{R < |x| < 2R\}, \text{ and } q \leq r. \end{cases} \quad (4.1)$$

We employ here cutoff function to avoid the spherical integrals raised in the Pohozaev identity (see [18,22]), which are very difficult to control.

Lemma 4.1 Assume that $f \in C(\mathbb{R})$. Consider $u \in C^{2r}(\overline{\mathbb{R}_+^n})$ a solution of (1.2) satisfying $(\overline{U})$, then

$$\int_{B_{2R}} (\phi |D^r u|^2 - \phi f(u)u) = o(1), \text{ as } R \rightarrow \infty, \quad (4.2)$$

and

$$\int_{B_{2R}^+} \phi G(u) = -\frac{2}{n-2r} \int_{A_R^+} F(u) \nabla \phi \cdot x + o(1). \quad (4.3)$$

Remark 4.1 Note that we can choose ϕ such that⁵

$$\nabla \phi \cdot x \leq 0, \quad \forall x \in \mathbb{R}^n. \quad (4.4)$$

Consequently, if f has a supercritical growth, i.e., $G(s) < 0$, for all $s \neq 0$ and u satisfies $(\overline{U})$, then from (4.4) and (4.3), we have $0 \leq \int_{B_R^+} -G(u) \leq \int_{B_{2R}^+} -G(u)\phi \leq o(1)$, and therefore $\int_{B_R^+} -G(u) = o(1)$. Consequently, $\int_{\mathbb{R}_+^n} -G(u) = 0$, that is $u \equiv 0$. $\square$

However, condition $(\overline{U})$ is so difficult to obtain in the supercritical growth case even if u is a stable solution and $f(s) = |s|^{p-1}s$.

Proof of Lemma 4.1. We divide the proof in Three steps.

Step 1. We establish (4.2) and the following estimate:

$$\int_{B_{2R}^+} \phi D^r u D^r (\nabla u \cdot x) - \int_{B_{\beta_{2R}^+}^+} \phi f(u) \nabla u \cdot x = o(1). \quad (4.5)$$

Invoke inequality (2.9) with $h = \phi$ and $v \in C^r(\overline{\mathbb{R}_+^n})$ satisfying the Dirichlet boundary conditions (2.8) (see the statement of Lemma 2.1). So apply (2.3) (with $\varepsilon = 1$) in the right-hand side of (2.9) and using (4.1), we obtain

$$\left| \int_{B_{\beta_{2R}^+}} (\phi D^r u D^r v - \phi f(u)v) \right| \leq C \left(\int_{A_R^+} |\nabla^r u|^2 + \int_{\mathbb{R}_+^n} \sum_{q=0}^{r-1} R^{2(q-r)} |\nabla^q v|^2 \right). \quad (4.6)$$

Choose $v = u$ (which satisfies (2.8) as a solution of (1.2)) and apply (2.1) (with $\varepsilon = 1$) in the right-hand side of (4.6), we derive

$$\left| \int_{B_{2R}^+} \phi |D^r u|^2 - \int_{B_{2R}^+} \phi f(u)u \right| \leq C \left(\int_{A_R^+} |\nabla^r u|^2 + R^{-2r} \int_{A_R^+} u^2 \right). \quad (4.7)$$

Thanks to assumption $(\overline{U})$, we deduce (4.2). Choose now $v = \nabla u \cdot x$, we have $v \in C^r(\overline{\mathbb{R}_+^n})$ as $u \in C^{2r}(\overline{\mathbb{R}_+^n})$ and a simple computation gives

$$|\nabla^q (\nabla u \cdot x)|^2 \leq C (R^2 |\nabla^{q+1} u|^2 + |\nabla^q u|^2). \quad (4.8)$$

⁵ In fact, set $\eta(t) = 1$ if $0 \leq t \leq 1$, $\eta(t) = \exp\left(\left(\frac{t-1}{t-2}\right)^{2r+1}\right)$ if $1 \leq t < 2$ and $\eta(t) = 0$ if $t \geq 2$. Clearly $\phi(x) = \eta\left(\frac{|x|}{R}\right) \in C_c^r(\mathbb{R}^n)$ and satisfies (4.1). As $\left(\left(\frac{t-1}{t-2}\right)^{2r+1}\right)' = \frac{-1}{(t-2)^2} \left(\frac{t-1}{t-2}\right)^{2r} < 0$, then (4.4) follows from a simple computation.

Denote a generic point of $\mathbb{R}_+^n$ by $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$, $x_n \in (0, \infty)$. Since u satisfies (2.8), then $\nabla_{x'}(\frac{\partial^i}{\partial x_n^i} u)(x', 0) = 0$, $\forall 0 \leq i \leq r-1$. Using Leibnitz's formula and interchanging the order of taking partial derivatives, we derive

$$\begin{aligned} \left(\frac{\partial^i}{\partial x_n^i} (\nabla u \cdot x) \right) (x', 0) &= \nabla_{x'} \left(\frac{\partial^i}{\partial x_n^i} u \right) (x', 0) \cdot x' + \left(\frac{\partial^{i+1}}{\partial x_n^{i+1}} u \right) (x', 0) \cdot 0 \\ &\quad + i \left(\frac{\partial^i}{\partial x_n^i} u \right) (x', 0) = 0, \end{aligned} \quad (4.9)$$

for all $0 \leq i \leq r-1$. As a consequence, $v = \nabla u \cdot x$ satisfies (2.8) and thus (4.6) holds. Precisely, we have

$$\left| \int_{B_{\beta_2 R}} (\phi D^r u D^r (\nabla u \cdot x) - \phi f(u) \nabla u \cdot x) \right| \leq C \left(\int_{A_R^+} |\nabla^r u|^2 + \int_{\mathbb{R}_+^n} \sum_{q=0}^{r-1} R^{2(q-r)} |\nabla^q (\nabla u \cdot x)|^2 \right).$$

Using now (4.8) and (2.1), we obtain

$$\left| \int_{B_{2R}^+} \phi D^r u D^r (\nabla u \cdot x) - \int_{B_{2R}^+} \phi f(u) \nabla u \cdot x \right| \leq C \left(\int_{A_R^+} |\nabla^r u|^2 + R^{-2r} \int_{A_R^+} u^2 \right).$$

According to assumption $(\overline{U})$, we deduce (4.5).

Step 2 we establish the following equality (which is trivial for $r = 1$):

$$D^r u D^r (\nabla u \cdot x) = \frac{1}{2} \nabla (|D^r u|^2) \cdot x + r |D^r u|^2. \quad (4.10)$$

First, for $r = 2k$, i.e., $D^r = \Delta^k$, apply Leibnitz's formula, we have

$$\Delta^k (\nabla u \cdot x) = \nabla (\Delta^k u) \cdot x + 2k \Delta^k u. \quad (4.11)$$

Multiplying (4.11) by $\Delta^k u$ and taking into account that $\Delta^k u \nabla (\Delta^k u) \cdot x = \frac{1}{2} \nabla ((\Delta^k u)^2) \cdot x$, then (4.10) follows.

If $r = 2k + 1$, that is $D^r = \nabla \Delta^k$, using again (4.11) we derive

$$D^r u D^r (\nabla u \cdot x) = \nabla \Delta^k \cdot \nabla (\nabla (\Delta^k u) \cdot x) + (r-1) |D^r u|^2.$$

Notice that $\nabla w \cdot \nabla (\nabla w \cdot x) = \frac{1}{2} \nabla (|\nabla w|^2) \cdot x + |\nabla w|^2$, $\forall w \in C^2(\overline{\mathbb{R}_+^n})$, which for $w = \Delta^k u$ implies (4.10).

Step 3 We end the proof of Lemma 4.1. Since $f(u) \nabla u = \nabla (F(u))$ and $F(u(x', 0)) = F(0) = 0$, then integrating by parts we get

$$\int_{B_{2R}^+} f(u) (\nabla u \cdot x) \phi = - \int_{B_{2R}^+} (n F(u) \phi + F(u) \nabla \phi \cdot x). \quad (4.12)$$

Using (4.10) and integrate by parts, yields

$$\int_{B_{2R}^+} \phi D^r u D^r (\nabla u \cdot x) = - \frac{n-2r}{2} \int_{B_{2R}^+} \phi |D^r u|^2 - \frac{1}{2} \int_{B_{2R}^+} |D^r u|^2 \nabla \phi \cdot x. \quad (4.13)$$

From $(\overline{U})$ and (4.1), we have $\int_{B_{2R}^+} |D^r u|^2 \nabla \phi \cdot x = o(1)$. Hence, plugging (4.13) and (4.12) in (4.5), we obtain

$$\frac{2n}{n-2r} \int_{B_{2R}^+} F(u) \phi - \int_{B_{2R}^+} \phi |D^r u|^2 = -\frac{2}{n-2r} \int_{B_{2R}^+} F(u) \nabla \phi \cdot x + o(1). \quad (4.14)$$

Therefore, the desired estimate (4.3) follows by combining (4.2) and (4.14). $\square$

End of the proof of Theorem 1.2. As in the proof of Theorem 1.1, Proposition 1.1 and assumption (H_2) imply

$$C \int_{B_R^+} |u|^{p+1} \leq \int_{B_R^+} |\nabla^r u|^2 + \int_{B_R^+} f(u)u \leq C_0 + CR^{-2r} \int_{B_{2R}^+} u^2, \quad \forall R > 4R_0. \quad (4.15)$$

First, we claim that there exist a sequence $R_j \rightarrow \infty$ and a positive constant C' independent of j such that

$$R_j^{-2r} \int_{B_{2R_j}^+} u^2 \leq C', \quad \forall j \in \mathbb{N}^*. \quad (4.16)$$

Assume by contradiction that $R^{-2r} \int_{B_{2R}^+} u^2 \rightarrow \infty$ as $R \rightarrow \infty$. Then, we can find R'_0 large enough such that $\int_{B_R^+} |u|^{p+1} \leq (1+C)R^{-2r} \int_{B_{2R}^+} u^2$, $\forall R > R'_0$. It turns out that u satisfies (3.1) of Lemma 3.1 and thus (3.3) holds. As $1 < p \leq \frac{n+2r}{n-2r}$ then $\delta_p = n \frac{p-1}{p+1} - 2r \leq 0$. Therefore, $R^{-2r} \int_{B_{2R}^+} u^2$ is uniformly bounded, so the claim is well proved.

Substitute R by R_j in (4.15) and letting $j \rightarrow \infty$, we derive

$$\int_{\mathbb{R}_+^n} |\nabla^r u|^2, \int_{\mathbb{R}_+^n} f(u)u \text{ and } \int_{\mathbb{R}_+^n} |u|^{p+1} < \infty. \quad (4.17)$$

As $\delta_p \leq 0$, then Hölder's inequality implies $R^{-2r} \int_{A_R^+} u^2 = o(1)$, which with (4.17) imply that u satisfies assumption $(\overline{U})$. Therefore inequality (4.2) of Lemma 4.1 holds and the dominated convergence theorem gives

$$\int_{\mathbb{R}_+^n} |D^r u|^2 = \int_{\mathbb{R}_+^n} f(u)u.$$

Recall that $0 \leq (\mu + 2)F(s) \leq sf(s)$, $\forall s \in \mathbb{R}$ (see (1.6)), which with (4.17) imply $\int_{\mathbb{R}_+^n} F(u) < \infty$. Consequently $\int_{A_R^+} F(u) \nabla \phi \cdot x = o(1)$, and as above (4.3) implies

$$\frac{2n}{n-2r} \int_{\mathbb{R}_+^n} F(u) = \int_{\mathbb{R}_+^n} f(u)u, \quad (4.18)$$

which ends the proof of point 1. $\square$

Proof of point 2 According to (4.18), we have $\int_{\mathbb{R}_+^n} G(u) = 0$, then from assumption (H_3) we derive that $u \equiv 0$. Theorem 1.2 is well proved. $\square$

5 Proof of Theorem 1.3

Proof of Point 1 Set $M(x) = \left(\sum_{q=0}^{2r-1} |\nabla^q u(x)|^{\frac{2r}{2r+q(p-1)}} \right)^{\frac{p-1}{2r}}$, $x \in \Omega$. Working by contradiction and applying the doubling lemma [17], we can find sequences u_j , Ω_j and $x_j \in \Omega_j$ such that u_j is a solution of (1.7) in Ω_j with $i(u_j) \leq m_0$ and satisfying

$$M(x_j) \geq j, M(x_j)d(x_j) \geq j \text{ and } M(z) \leq 2M(x_j), \forall z \in \left\{ \Omega_j, |z - x_j| \leq \frac{d_{x_j}}{2} \right\},$$

where $d_{x_j} = \text{dist}(x_j, \partial\Omega_j)$. (5.1)

Denote B_j the ball of radius $\frac{j}{2}$ centered at the origin, set $\beta = \frac{2r}{p-1}$, $\lambda_j = M^{-1}(x_j)$ and $g_j(s) = \lambda_j^{\beta} g(\lambda_j^{-\beta} s)$. Define $\tilde{u}_j(y) = \lambda_j^{\beta} u(x_j + \lambda_j y)$, according to (5.1) $\tilde{u}_j$ is well defined in B_j with $\|\tilde{u}_j\|_{L^\infty(B_j)} \leq 2$, and satisfies

$$(-\Delta)^r \tilde{u}_j = g_j(\tilde{u}_j), \text{ in } B_j. \quad (5.2)$$

From assumption (H') , one has $\|g_j(\tilde{u}_j)\|_{L^\infty(B_j)} \leq C_0$, $C_0 > 0$, and as $\lambda_j^{-1} \rightarrow \infty$, then for all $R > 0$ we have $\tilde{u}_j$ is well defined in B_{3R} for j large enough. Invoking Corollary 6 of [19], we can see that $\tilde{u}_j$ is bounded in $C^{2r-1, \alpha}(B_{2R})$, and using the mean value theorem we derive that g_j is bounded in $C^{0, \alpha}(B_{2R})$ for some $\alpha \in (0, 1)$. Therefore, apply the local Schauder estimates and Ascoli–Arzela’s theorem, then modulo a subsequence $\tilde{u}_j \rightarrow u$ in the $C^{2r}(B_R)$ topology and according to (H') , we have $g_j \rightarrow \frac{c_o}{p} |u|^{p-1} u$ in the $C^{2r}(B_R)$ topology. Consequently, from a standard diagonal subsequence argument u satisfies $(-\Delta)^r u = \frac{c_o}{p} |u|^{p-1} u$, in $\mathbb{R}^n$. Also, (H') implies that $g'_j(\tilde{u}_j) \rightarrow c_o |u|^{p-1}$ pointwise and as $i(\tilde{u}_j) \leq m_0$ we derive that u has finite Morse index (see [5] for more detail). Since $1 < p < \frac{n+2r}{n-2r}$ then $f(s) = \frac{c_o}{p} |u|^{p-1} u$ satisfies assumptions (H_1) – (H_3) of Theorem 1.2 and

therefore $u \equiv 0$. To reach a contradiction observe that $\sum_{q=0}^{2r-1} |\nabla^q \tilde{u}_j(x_j)|^{\frac{2r}{2r+q(p-1)}} = 1$

which implies that $\sum_{q=0}^{2r-1} |\nabla^q u(0)|^{\frac{2r}{2r+q(p-1)}} = 1$. □

Proof of Point 2. We shall give an outline of the proof. Assume by contradiction that there exists $m_0 \in \mathbb{N}^*$ such that $i(u_j) \leq m_0$, $\forall j \in \mathbb{N}$ but $\|u_j\|_{L^\infty(\Omega)} \rightarrow \infty$. Let $x_j \in \Omega$ such that $\|u_j\|_{L^\infty(\Omega)} = |u_j(x_j)|$ and set $\lambda_j = |u_j(x_j)|^{-\beta}$, and as above, define $\tilde{u}_j$ satisfying (5.2) in $\tilde{\Omega}_j = \lambda_j^{-1}(\Omega - x_j)$ with $\|\tilde{u}_j\|_{L^\infty(\tilde{\Omega}_j)} = |\tilde{u}_j(0)| = 1$ and $\|g_j\|_{L^\infty(\tilde{\Omega}_j)}$ is uniformly bounded. Since Ω is a bounded domain, then from point 1 of Theorem 1.3, we have $\lambda_j^{-1} \text{dist}(x_j, \partial\Omega) \leq C_{n,p,m_0}$ and up to selecting a subsequence we have $\lim x_j = x_0 \in \partial\Omega$. Consequently, we can exhibit a finite Morse index regular solution of

$$(-\Delta)^r u = c_o |u|^{p-1} u \text{ in } x_n > -d, \text{ and } u = \frac{\partial}{\partial x_n} u = \dots = \frac{\partial^{r-1}}{\partial x_n^{r-1}} u = 0, \text{ on } x_n = -d,$$

with $d > 0$ (see case 2.2 of the proof of theorem 3 in [19] for more details). As above, we deduce that $u \equiv 0$, and a contradiction follows since $d > 0$ and $|u(0)| = 1$.

Conversely, if $\|u_j\|_{L^\infty(\Omega)} \leq M_0$, since $g \in C^1(\mathbb{R})$, then there exists $M' > 0$ such that $\|g'(u_j)\|_{L^\infty(\Omega)} \leq M'$. Consider $0 < \lambda_1 < \lambda_2 \leq \lambda_j$ the eigenvalues of the operator $(-\Delta)^r$ under the Dirichlet boundary conditions, and φ_k the corresponding eigenfunction of λ_k satisfying

$$\int_{\Omega} \varphi_k^2 = 1, \quad \int_{\Omega} D^r \varphi_k D^r \varphi_j = 0, \quad j \neq k \text{ and } \int_{\Omega} |D^r \varphi_k|^2 = \lambda_k \int_{\Omega} \varphi_k^2 = \lambda_k.$$

Since $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$, we can choose $j_0 \in \mathbb{N}^*$ large enough such that $\lambda_{j_0} > M'$. Set $E_0 = \langle \varphi_1, \dots, \varphi_{j_0} \rangle$, we have $\dim E_0 = m_0$ and for every $h \in E_0^\perp$, we have $Q_u(h) = \int_{\Omega} |D^r h|^2 - \int_{\Omega} g'(u)h^2 \geq (\lambda_{j_0} - M') \int_{\Omega} h^2 > 0$, which implies that $i(u) \leq j_0$. The proof of Theorem 1.3 is completed. $\square$

Acknowledgements This work was partially done in the I.C.T.P, in July 2017. The author wishes to acknowledge the reviewer for his/her thorough review, which significantly contributed to improving the quality of the publication.

Appendix

Proof of (2.7) Let l, s be two integers such that $s \geq 2$ and $1 \leq l \leq \min(r, s-1)$, denote $A_R^+ = \{x \in \mathbb{R}_+^n, \alpha R < |x| < \alpha' R\}$. We shall first prove the following inequality (which is trivial for $l = 1$):

$$|\nabla^l \psi^s| \leq C \psi^{s-l} ((1-\alpha)R)^{-l}, \quad \forall x \in A_R^+ \text{ and } l \leq r. \quad (5.3)$$

Fix $j = (j_1, j_2, \dots, j_n)$ such that $|j| = l$, let $i_0 \in \{1, 2, \dots, n\}$ such that $j_{i_0} \neq 0$ and denote $j_- = (j_1, \dots, j_{i_0} - 1, \dots, j_n)$. Apply Leibnitz's formula, we obtain

$$\mathcal{D}^j \psi^s = s \mathcal{D}^{j_-} (\psi^{s-1} \frac{\partial \psi}{\partial x_{i_0}}) = s \psi^{s-1} \mathcal{D}^j \psi + s \sum_{\substack{d+k=j_0 \\ d \neq j_0}} a_{d,k} \mathcal{D}^k \psi^{s-1} \mathcal{D}^d \frac{\partial \psi}{\partial x_{i_0}}, \quad \text{where } a_{d,k} \in \mathbb{R}.$$

From (2.4), we derive

$$|\nabla^l \psi^s| \leq C \psi^{s-l} ((1-\alpha)R)^{-l} + \sum_{1 \leq i \leq l-1} ((1-\alpha)R)^{l-i} |\nabla^i \psi^{s-1}|, \quad \forall x \in A_R^+ \text{ and } l \leq r.$$

Thus, (5.3) follows by working inductively with respect l , so inequality (2.7) is a direct Consequence of (5.3). $\square$

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