Group invariance of integrable Pfaffian systems

A. Kumpera¹

Received: 10 November 2018 / Accepted: 16 February 2019 / Published online: 28 March 2019
© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag GmbH Germany, part of Springer Nature 2019, corrected publication 2019

Abstract

Let $\mathcal{S}$ be an integrable Pfaffian system. When it is invariant under a transversally free infinitesimal action of a finite-dimensional real Lie algebra $\mathfrak{g}$ and consequently invariant under the local action of a Lie group G , we show that the *vertical* variational cohomology of $\mathcal{S}$ is equal to the Lie algebra cohomology of $\mathfrak{g}$ with values in the space of the *horizontal* cohomology in maximum dimension. This result, besides giving an effective algorithm for the computation of the variational cohomology of an invariant Pfaffian system, provides a method for detecting obstructions to the existence of finite or infinitesimal actions leaving a given system invariant.

Keywords Differential systems · Partial differential equations · Pfaffian systems · Standard prolongations · Merihedric prolongations · Lie groups and algebras · Lie groupoids and algebroids · Finite and infinitesimal actions · Variational cohomology · Lie algebra cohomology · Local models · Integration · Foliations

Mathematics Subject Classification Primary 53C05; Secondary 53C15 · 53C17

1 Introduction

We study here a problem that arises naturally in connection with the integration of differential systems invariant under finite or infinitesimal group actions [10] though, of course, it is also present elsewhere. The problem is to know whether a given integrable Pfaffian system $\mathcal{S}$ can be invariant under the infinitesimal action of a given Lie algebra $\mathfrak{g}$ or, more generally, of some Lie algebra $\mathfrak{g}$. Showing the existence of such infinitesimal actions is a rather delicate problem that has to be analyzed in each specific case. On the other hand, showing the nonexistence can be achieved, as is common, by displaying some obstructions via cohomological methods and this is actually our main concern in this paper. Since we want to compare the variational

The original version of this article was revised: Due to open choice cancellation.

✉ A. Kumpera
antoniokumpera@hotmail.com

¹ Universidade Estadual de Campinas (State University of Campinas), Campinas, SP, Brazil

cohomology of $\mathcal{S}$ with the Lie algebra cohomology of $\mathfrak{g}$, we shall only be concerned with transversally free infinitesimal actions as defined in Sect. 6.

The Euler–Lagrange variational complex associated with an integrable Pfaffian system $\mathcal{S}$ is finite. As is usual, we call *horizontal* that part of the complex preceding the Euler operator E and *vertical* that part subsequent to this operator. The horizontal part is a finite augmentation of E and the vertical part a finite resolution. We show (Theorem 1) that if $\mathcal{S}$ is invariant under a transversally free infinitesimal action of the Lie algebra $\mathfrak{g}$, then the above finite resolution is equivalent, in positive dimensions, to the Lie algebra complex of $\mathfrak{g}$ taking values in the horizontal cohomology of maximum dimension. In particular, the resulting cohomology spaces are equal; hence, any discrepancy between the two will single out obstructions to the existence of such an infinitesimal action. The above equivalence also provides an effective method for calculating the vertical variational cohomology when the Pfaffian system is invariant.

Throughout the years, several authors have given distinct, though essentially equivalent, formulations of the variational complex. We adopt here the approach described in [9] since it emphasizes the relationship of this complex with the algebra of generalized symmetries. Inasmuch as the usual *de Rham* complex, on a manifold M , is the differential complex associated with the algebra of all the vector fields on M , the horizontal part of the variational complex is a *de Rham* complex associated with the algebra of all trivial symmetries (total derivatives) and the vertical part is also a *de Rham* complex associated with the algebra of all the generalized symmetries. Our first task, in this paper, consists in writing down explicitly the complex we shall be dealing with, namely, the restriction, to an integrable Pfaffian system, of the general complex defined in [9]. This, unfortunately, is a rather long and boring task; hence, we only state, in Sect. 2, a well-known lemma that provides all the necessary technical information relevant to the restriction procedure and thereafter construct directly, in Sects. 3, 4 and 5, the desired complex. It turns out that the trivial symmetries become simply the vector fields annihilated by $\mathcal{S}$ and the generalized symmetries become the equivalence classes of the infinitesimal automorphisms of $\mathcal{S}$, modulo the trivial symmetries. There is of course nothing new about this restricted complex, just a different makeup. Invariant systems are examined in Sect. 6, a double complex is displayed in Sect. 7, and some examples are placed in evidence in Sect. 8. In the three subsequent sections, we discuss two quite outstanding examples of foliations, namely, foliations having a *compact, simply connected and isolated leaf*. The search for such foliations was suggested by Charles Ehresmann [7] though he did not put much faith in their existence. Reeb's foliation on the 3-sphere was already quite an accomplishment. Both foliations are equivalent and, in fact, are diffeomorphic via a stereographic projection. We insist in constructing both of them since each one has its own beauty and remarkable peculiarities. They are transitively invariant under the action of the unitary quaternions as well as by the special orthogonal group $SO(4, \mathbf{R})$. For these foliations, the Lie algebra cohomology is easily established, whereas a direct computation of the variational cohomology is much more arduous. We end up, in Sect. 12, by showing that the variational complex is also very useful in the study of non-integrable Pfaffian systems and, to do so, we choose the most non-integrable systems of all, namely, the *Flag Systems*, their sole integral manifolds being integral curves. We also show how the variational complex, though hidden in behind the curtains, contributes in proving that all the Flag Systems have isomorphic invariance groups canonically isomorphic to the automorphism group of the *Darboux Flag*, i.e., the Pfaffian system, on the space $\mathbf{R}^3$, generated by the Pfaffian form $dx^2 - x^3 dx^1$. We also show how Élie Cartan took benefit of the variational complex in the study and integration of the second-order *Monge equations* [3]. Finally, we cannot forget mentioning Bott's remarkable paper *On a topological obstruction to integrability* [2] that is, by all means, at the basis of the present work.

For simplicity, we assume that all the data are C^∞ smooth though, in each specific case, C^k smoothness, for some k , would suffice. We also assume that all the manifolds are finite-dimensional, connected and second countable though not necessarily orientable and that all the objects such as functions, vector fields, differential forms, etc., are globally defined on these manifolds unless stated otherwise (e.g., local coordinates, local generators of Pfaffian systems and distributions, etc.).

2 Pfaffian systems and their prolongations

Let $\mathcal{S}$ be a Pfaffian system, for the time being not necessarily integrable, and $\Sigma = \mathcal{S}^\perp$ the corresponding distribution (field of contact elements), both defined on the manifold M ([10], sect. 2). We set $\dim M = n$ and $\text{rank } \mathcal{S} = n - p$; hence, $\dim \Sigma = p$, the last notation indicating the dimension of each contact element Σ_x occasionally referred to as the *rank* of Σ . In terms of partial differential equations, it seems preferable to view the distribution Σ or the Pfaffian system $\mathcal{S}$ as a section of the Grassmannian bundle $\mathbf{G}_1^p M$ of p -dimensional first-order linear contact elements, $p = \dim \Sigma$, or, still better, as the sub-manifold $\mathcal{R}$ of $\mathbf{G}_1^p M$ image of this section ([9], p. 614). Since the dimension p will remain unchanged throughout the present discussion, we abbreviate $\mathbf{G}_k^p M$ by $\mathbf{G}_k$, where k is the order of the contact elements under consideration. Let $\mathcal{R}_1 \subset \mathbf{G}_2$ denote the first prolongation of $\mathcal{R}$.

Lemma 1 *The distribution Σ is integrable (involutive) if and only if $\mathcal{R}_1$ projects onto $\mathcal{R}$.*

A proof of this result can be found in [10], section 13. This proof tells us, in particular, that the fiber $(\mathcal{R}_1)_X$, $X \in \mathcal{R}$, is either empty or else contains a single element, say Y . The first-order linear holonomic contact element $\Sigma_X^{(1)}$ associated with Y is the unique holonomic element tangent to $\mathcal{R}$ at the point X . Since it does not make much sense to define the variational complex for other than *formally integrable* equations (or at least for equations that, after prolongation, become formally integrable at large enough orders), we see that in the present situation it becomes natural to assume that Σ is integrable. This does not mean, however, that variational complexes cannot be associated with non-integrable Pfaffian systems. In this latter context, we ought to specify or determine the dimension $q \leq p$ of the linear contact elements for which a *sufficient number* of integral contact elements do exist (e.g., Pfaffian systems that are *in involution*, in the sense of Cartan, at the dimension q) and consider the variational complex in the realm of the bundles $\mathbf{G}_k^q M$. We shall nevertheless restrict our attention to integrable systems.

When $\mathcal{S}$ (or Σ) is integrable, then $\mathcal{R}_1$ is the set of all second-order contact elements determined by the p -dimensional integral manifolds of $\mathcal{S}$ and the assignment

$$Y \in \mathcal{R}_1 \mapsto \Sigma_X^{(1)} \subset T_X \mathcal{R}, \quad X = \rho_{1,2}(Y),$$

is an integrable distribution $\Sigma^{(1)}$, defined on the manifold $\mathcal{R}$, equivalent to Σ via the diffeomorphism $\beta_1 = \rho_{0,1} : \mathcal{R} \rightarrow M$. In general, the k -th prolongation $\mathcal{R}_k \subset \mathbf{G}_{k+1}$ is the set of all the $(k+1)$ -order contact elements determined by the p -dimensional integral manifolds of $\mathcal{S}$. Furthermore, the assignment

$$Y \in \mathcal{R}_k \mapsto \Sigma_X^{(k)} \subset T_X \mathcal{R}_{k-1}, \quad X = \rho_{k,k+1}(Y),$$

where $\Sigma_X^{(k)}$ indicates the linear holonomic contact element at order k associated with Y , is an integrable distribution $\Sigma^{(k)}$ defined on the manifold $\mathcal{R}_{k-1}$ and equal to the annihilator of the restriction, to $\mathcal{R}_{k-1}$, of the canonical contact structure $\mathcal{S}_k$ of $\mathbf{G}_k$ ($[\Sigma^{(k)}]^\perp = \iota^* \mathcal{S}_k$, $\iota :$

$\mathcal{R}_{k-1} \hookrightarrow \mathbf{G}_k$, cf. [9], sect. 2). For any pair of integers $h \leq k$, the distributions $\Sigma^{(k)}$ and $\Sigma^{(h)}$ are equivalent via the diffeomorphism $\rho_{h,k} : \mathcal{R}_{k-1} \longrightarrow \mathcal{R}_{h-1}$ ($\mathcal{R}_0 = \mathcal{R}$ and $\Sigma^{(0)} = \Sigma$).

3 The variational complex

Pfaffian systems invariant under infinitesimal Lie algebra actions or, equivalently, under local Lie group actions are present in many a context and, in particular, find their place in the integration of invariant differential systems [10]. Motivated by some pending matters in the aforementioned paper, our concern here is to display obstructions to the existence of transversally free infinitesimal actions of a finite-dimensional real Lie algebra $\mathfrak{g}$ leaving invariant a given integrable Pfaffian system $\mathcal{S}$ defined on a manifold M . Exhibiting such obstructions brings some further insight into the global applicability of Lie and Cartan's methods and is achieved by comparing the variational cohomology of $\mathcal{S}$ with the Lie algebra cohomology of $\mathfrak{g}$ with values in an appropriate space.

Let $\mathcal{A}$ be the algebra of all the infinitesimal automorphisms (symmetries) of $\mathcal{S}$ and $\mathcal{B}$ the set of all the vector fields annihilated by the local sections of $\mathcal{S}$, i.e., the sections of Σ . Then, $\mathcal{B}$, composed by the trivial symmetries, is an ideal in $\mathcal{A}$ and one shows, based on Lemma 8 of [10], that the algebra $\mathcal{S}(\mathcal{R})$ of generalized symmetries of the first-order partial differential equation $\mathcal{R}$ associated with $\mathcal{S}$ ($\mathcal{S}(D)$ in the notations of [9], sect. 9) identifies with $\mathcal{A}/\mathcal{B}$. Further, $\mathcal{A}$ is a module over the ring $\mathcal{I}$ of all the first integrals of $\mathcal{S}$ and $\mathcal{B}$ a module over the ring $\mathcal{F}$ of all the functions on M ; hence, $\mathcal{S}(\mathcal{R})$ is a module over $\mathcal{I}$. We adopt here the approach to the variational complex as described in [9] for it emphasizes its relationship with the algebra $\mathcal{S}(\mathcal{R})$ and seems best suited for our purpose. In the sequel, we describe what the general (full) horizontal and vertical complexes as well as the Euler operator, considered in [9], become upon restriction to the equation $\mathcal{R}$.

4 The horizontal operator

We now construct directly the *horizontal* part of the variational complex associated with an integrable Pfaffian system $\mathcal{S}$, namely, that part preceding the Euler operator. Let us denote, as above, by $\mathcal{B}$ the algebra of all the (globally defined) vector fields η tangent to the distribution $\Sigma = \mathcal{S}^\perp$, i.e., $\eta \in \Gamma(\Sigma)$, and by $\mathcal{F}$ the ring of all the (globally defined) C^∞ functions on the underlying manifold M . The dual space $\mathcal{H} = \mathcal{B}^*$, with respect to the $\mathcal{F}$ -module structure, is equal to the set of all the global sections of the dual bundle $\Sigma^* \simeq T^*M/\mathcal{S}$ and, correspondingly, $\wedge \mathcal{H} \simeq \Gamma(\wedge \Sigma^*)$. The differential

$$d_H : \wedge^s \mathcal{H} \longrightarrow \wedge^{s+1} \mathcal{H}$$

is defined by the usual formula:

$$\begin{aligned} d_H \mu(\eta_1, \dots, \eta_{s+1}) &= \sum_i (-1)^{i+1} \theta(\eta_i) \mu(\eta_1, \dots, \widehat{\eta}_i, \dots, \eta_{s+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \mu([\eta_i, \eta_j], \eta_1, \dots, \widehat{\eta}_i, \dots, \widehat{\eta}_j, \dots, \eta_{s+1}), \quad (1) \end{aligned}$$

where the η_i are vector fields tangent to Σ , $\theta(\eta_i)$ is the usual Lie derivative and $[\eta_i, \eta_j]$ is the usual Lie bracket. Let $(U; x^1, \dots, x^p, y^1, \dots, y^q)$, $q = n - p$, be a foliated chart of Σ for which the integral manifolds, in U , are given by the equations $y^\lambda = c^\lambda$, the c^λ being

constants. Then, an element μ of $\wedge^s \mathcal{H}$ has the local expression

$$\mu = \sum a_{i_1 \dots i_s} dx^{i_1} \wedge \dots \wedge dx^{i_s},$$

the coefficients $a_{i_1 \dots i_s}$ being C^∞ functions defined on U , and

$$\begin{aligned} d_H \mu &= \sum (da_{i_1 \dots i_s} | \Sigma) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s} \\ &= \sum \frac{\partial a_{i_1 \dots i_s}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}, \end{aligned} \quad (2)$$

where $da_{i_1 \dots i_s} | \Sigma$ (resp. dx^i) stands for the restriction of this differential to the integral manifolds of Σ or, still simpler, its restriction to the contact elements Σ_x , $x \in M$.

Let us now extend the differential (1) by adding, in the cochains, a term that corresponds, in [9], to the module $\mathcal{C}$ of all the contact 1-forms. Here we consider $\mathcal{C} = \Gamma(\mathcal{S})$ to be the module of all the global sections of $\mathcal{S}$, take the cochain space $\Phi^{r,s} = (\wedge^r \mathcal{C}) \otimes_{\mathcal{F}} (\wedge^s \mathcal{H})$ and consider its elements as horizontal forms with values in $\wedge^r \mathcal{C}$. The extended differential

$$d_H : \wedge^r \mathcal{C} \otimes \wedge^s \mathcal{H} \longrightarrow \wedge^r \mathcal{C} \otimes \wedge^{s+1} \mathcal{H}$$

is then defined by

$$\begin{aligned} d_H(\omega \otimes \mu)(\eta_1, \dots, \eta_{s+1}) &= \sum_i (-1)^{i+1} \theta(\eta_i) [\mu(\eta_1, \dots, \widehat{\eta}_i, \dots, \eta_{s+1}) \omega] \\ &\quad + \sum_{i < j} (-1)^{i+j} \mu([\eta_i, \eta_j], \eta_1, \dots, \widehat{\eta}_i, \dots, \widehat{\eta}_j, \dots, \eta_{s+1}) \omega, \end{aligned} \quad (3)$$

where $\eta_i \in \Gamma(\Sigma)$ and $\theta(\eta_i)$ is the Lie derivative. The second term, on the right-hand side, belongs of course to $\wedge^r \mathcal{C}$, the same being true for the first term since $\mathcal{S}$ is integrable and consequently $\theta(\eta_i) \mathcal{C} \subset \mathcal{C}$. Let $(U; x^1, \dots, x^p, y^1, \dots, y^q)$ be a foliated chart for the distribution Σ in which the integral manifolds are given by the slices $y^\lambda = c^\lambda$. Then, a typical element of $\Phi^{r,s}$ is locally a sum of terms

$$\mu = a dy^{j_1} \wedge \dots \wedge dy^{j_r} \otimes dx^{i_1} \wedge \dots \wedge dx^{i_s},$$

and

$$d_H \mu = \sum \frac{\partial a}{\partial x^i} dy^{j_1} \wedge \dots \wedge dy^{j_r} \otimes dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}, \quad (4)$$

where dx^i stands for the restriction $dx^i | \Sigma$. The last formula as well as the formula (2), though helpful in theoretical considerations, is most often useless in practice since they require the local integration of Σ . We can nevertheless remedy this situation as follows. Let us consider any coordinate system $(U; x^i, y^j)$ with the sole requirement that the family $\{dx^i | \Sigma\}$ be free at every point of U , thus providing a field of co-frames for $\Sigma^* | U$. Next, we consider the local basis $\{\eta_i\}$ of Σ defined by the conditions $\langle \eta_i, dx^j \rangle = \delta_i^j$. Since Σ is integrable and since each η_i projects onto $\partial/\partial x^i$, it follows that $[\eta_i, \eta_j] = 0$. A typical element of $\Phi^{r,s}$ can now be written locally as a sum of terms

$$\eta = a \omega^{j_1} \wedge \dots \wedge \omega^{j_r} \otimes dx^{i_1} \wedge \dots \wedge dx^{i_s},$$

where dx^i stands for $dx^i | \Sigma$ and the family $\{\omega^\lambda = dy^\lambda - \sum Y_i^\lambda dx^i\}$ is a local basis of $\mathcal{S}$. The formula (3) then reduces to

$$d_H \eta = \sum_i [\theta(\eta_i)(a \omega^{j_1} \wedge \dots \wedge \omega^{j_r})] \otimes dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}, \quad (5)$$

and a similar formula can also replace (2), the derivatives $\partial a_{\dots}/\partial x^i$ being replaced by $\theta(\eta_i)(a_{\dots})$.

Let $\mathcal{I}^r = \mathcal{I}^r(S)$ denote the module of all the globally defined *invariant forms* ω of degree r with respect to the Pfaffian system $\mathcal{S}$, namely, those forms satisfying the condition ([10], sect. 4):

$$\theta(\eta)\omega = 0, \quad \forall \eta \in \Gamma(\mathcal{S}^\perp).$$

Then, $\mathcal{I}^0 = \mathcal{I}$ is the ring of all the global first integrals of $\mathcal{S}$, $\oplus_r \mathcal{I}^r$ is a graded $\mathcal{I}$ -subalgebra of $\mathcal{A}$ and the formula (3) shows that the *short* sequence

$$0 \longrightarrow \mathcal{I}^r \longrightarrow \Phi^{r,0} \xrightarrow{d_H} \Phi^{r,1} \quad (6)$$

is exact for each integer r . We next show that the *long* sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \Phi^{0,0} \xrightarrow{d_H} \Phi^{0,1} \xrightarrow{d_H} \dots \xrightarrow{d_H} \Phi^{0,p} \longrightarrow 0 \quad (7)$$

is locally exact. In fact, let $(U; x^i, y^\lambda)$ be a foliated chart for Σ . Then the formula (2) defines, for each fixed set of values $y^\lambda = c^\lambda$, the differential of the *de Rham* complex on the corresponding slice, whereupon results the local exactness of (7) since the usual homotopy operators can be written incorporating the parameters y^λ . Let us finally show that

$$\Phi^{r,p-1} \xrightarrow{d_H} \Phi^{r,p} \longrightarrow 0 \quad (8)$$

is locally exact. A typical element of $\Phi^{r,p}$ is, locally, a sum

$$\omega = \sum a_{j_1 \dots j_r} dy^{j_1} \wedge \dots \wedge dy^{j_r} \otimes dx^{i_1} \wedge \dots \wedge dx^{i_p};$$

hence, upon integrating for example along x^1 , we obtain the element

$$\Omega = \sum A_{j_1 \dots j_r} dy^{j_1} \wedge \dots \wedge dy^{j_r} \otimes dx^{i_2} \wedge \dots \wedge dx^{i_p}, \quad \frac{\partial A_{[j]}}{\partial x^1} = a_{[j]},$$

such that $d_H \Omega = \omega$.

5 The vertical operator

Let us next construct the *vertical* part of the Euler–Lagrange complex, namely, that part subsequent to the Euler operator. We recall that $\mathcal{A}$ denotes the algebra of all the infinitesimal automorphisms of S and $\mathcal{B}$ the ideal of those vector fields that are tangent to $\Sigma = \mathcal{S}^\perp$. The system $\mathcal{S}$ being integrable, any vector field tangent to Σ is an infinitesimal automorphism. Based on Lemma 1, it can be shown, as observed earlier, that the algebra $\mathcal{S}(\mathcal{R})$ of the generalized symmetries of the equation $\mathcal{R}$ associated with $\mathcal{S}$ identifies with $\mathcal{A}/\mathcal{B}$. The following remarks will be used later:

- (a) $\mathcal{A}$ is a module over the ring $\mathcal{I}$ and $\mathcal{B}$ a module over $\mathcal{F}$; hence, $\mathcal{S}(\mathcal{R})$ is a module over $\mathcal{I}$.
- (b) When $\xi \in \mathcal{A}$ is tangent to an integral leaf of $\mathcal{S}$ at some point x_0 , then it is also tangent to that leaf at all the other points.

We now define $\Xi^r = \Phi^{r,s}/d_H \Phi^{r,s-1}$, denote by $\mathbf{q}_r : \Phi^{r,s} \longrightarrow \Xi^r$ the quotient map and observe that Ξ^r is an $\mathcal{I}$ -module since $d_H \mathcal{I} = 0$. An element $\omega \otimes \mu \in (\wedge^r \mathcal{C}) \otimes (\wedge^s \mathcal{H})$ can also be considered as an $\mathcal{I}$ -multilinear form on $\mathcal{S}(\mathcal{R})$ with values in $\wedge^s \mathcal{H}$ by setting

$$(\omega \otimes \mu)([\xi_1], \dots, [\xi_r]) = \omega(\xi_1, \dots, \xi_r)\mu, \quad \xi_i \in \mathcal{A},$$

where each $[\xi_i]$ denotes the class of $\xi_i \bmod \mathcal{B}$. In fact, when $\eta \in \mathcal{B}$, then $(\omega \otimes \mu)(\dots, \eta, \dots) = 0$, and consequently, $(\omega \otimes \mu)([\xi_1], \dots, [\xi_r])$ is well defined on $\mathcal{S}(\mathcal{R})$. Furthermore ([9], sect. 11), since

$$[d_H(\omega \otimes \mu)](\xi_1, \dots, \xi_r) = d_H[(\omega \otimes \mu)(\xi_1, \dots, \xi_r)],$$

the form $d_H(\omega \otimes \mu)$, $\omega \otimes \mu \in (\wedge^r \mathcal{C}) \otimes (\wedge^s \mathcal{H})$, considered as a multilinear form on $\mathcal{S}(\mathcal{R})$, takes values that vanish under the projection $\mathbf{q}_0 : \Phi^{0,s} \longrightarrow \Xi^0$. Hence, to any element $\tau \in \Xi^r$, we can associate an $\mathcal{I}$ -multilinear form $[\tau]$, defined on $\mathcal{S}(\mathcal{R})$ and taking values in Ξ^0 , as follows. We take $\Omega = \sum \omega_i \otimes \mu_i \in \Phi^{r,s}$ such that $\mathbf{q}_r(\Omega) = \tau$ and set

$$[\tau](\xi_1, \dots, \xi_r) = \mathbf{q}_0 \Omega([\xi_1], \dots, [\xi_r]).$$

The mapping $\tau \mapsto [\tau]$ being injective ([9], sect.11), we are led to consider the defining formula

$$\begin{aligned} d_V(\mathbf{q}_r(\omega \otimes \mu))([\xi_1], \dots, [\xi_{r+1}]) \\ = \mathbf{q}_0 \left\{ \sum_i (-1)^{i+1} \theta([\xi_i])[\omega([\xi_1], \dots, \widehat{[\xi_i]}, \dots, [\xi_{r+1}])\mu] \right\} \\ + \mathbf{q}_0 \left\{ \sum_{i < j} (-1)^{i+j} \omega([\xi_i], [\xi_j], [\xi_1], \dots, \widehat{[\xi_i]}, \dots, \widehat{[\xi_j]}, \dots, [\xi_{r+1}])\mu \right\}, \quad (9) \end{aligned}$$

where $\omega \otimes \mu \in \Phi^{r,s}$ and $\xi_i \in \mathcal{A}$. The second term on the right-hand side clearly belongs to Ξ^0 . As for the first term, let us write

$$\mu_i = \omega([\xi_1], \dots, \widehat{[\xi_i]}, \dots, [\xi_{r+1}])\mu \in \Phi^{0,s}$$

and let us assume that some $\xi_i = \eta \in \mathcal{B}$. If $j \neq i$, then $\mu_i = 0$ and if $j = i$ then, since $d_H \mu_i = 0$,

$$\theta(\eta)\mu_i = i(\eta)d_H \mu_i + d_H i(\eta)\mu_i = d_H i(\eta)\mu_i,$$

and consequently, $\mathbf{q}_0 \theta(\eta)\mu_i = 0$. In any case, (9) provides a well defined multilinear form on $\mathcal{S}(\mathcal{R})$ taking values in Ξ^0 and it can be shown (e.g., in coordinates) that the multilinear form $d_V(\mathbf{q}_r(\omega \otimes \mu))$ is the image of an element $\tau \in \Xi^{r+1}$.

The Euler–Lagrange variational complex associated with the integrable Pfaffian system $\mathcal{S}$ is the finite sequence ($p = \text{rank } \Sigma$, $q = \text{rank } \mathcal{S}$)

$$\begin{aligned} 0 \longrightarrow \mathcal{I} \longrightarrow \Phi^{0,0} \xrightarrow{d_H} \Phi^{0,1} \xrightarrow{d_H} \dots \xrightarrow{d_H} \Phi^{0,p-1} \xrightarrow{d_H} \\ \xrightarrow{d_H} \Phi^{0,p} \xrightarrow{E} \Xi^1 \xrightarrow{d_V} \dots \xrightarrow{d_V} \Xi^{q-1} \xrightarrow{d_V} \Xi^q \longrightarrow 0, \quad (10) \end{aligned}$$

where E , the Euler operator, is the composite

$$\Phi^{0,p} \xrightarrow{\mathbf{q}_0} \Xi^0 \xrightarrow{d_V} \Xi^1$$

and $q = \text{rank } \mathcal{S}$. This complex is locally exact and reduces, locally, to (7) since Ξ^r vanishes on account of (8).

6 Invariant Pfaffian systems

Let $\mathcal{S}$ be an integrable Pfaffian system invariant under the infinitesimal action $\Phi : g \longrightarrow \chi(M)$ of a finite-dimensional real Lie algebra g ([10], sect. 3). For every

$v \in g$, $\Phi(v) \in \mathcal{A}$; hence, the action induces a Lie algebra morphism $\Phi : g \rightarrow \mathcal{S}(\mathcal{R})$. If we further assume that the action Φ is transversally free ([10], sect. 4), i.e., if

(i) $\dim g = \dim \Phi(g)_x$, $\forall x \in M$, and

(ii) $T_x M = \Sigma_x \oplus \Phi(g)_x$, $\forall x \in M$,

then the above morphism is injective and the following result holds:

Lemma 2 *The algebra $\mathcal{S}(\mathcal{R})$ is generated by $\Phi(g)$ over the ring $\mathcal{I}$ of first integrals of $\mathcal{S}$, any $\mathbf{R}$ -basis of $\Phi(g)$ is an $\mathcal{I}$ -basis of $\mathcal{S}(\mathcal{R})$, and the $\mathcal{I}$ -dual $\mathcal{S}(\mathcal{R})^*$ identifies with $\mathcal{I}^1$.*

Let us now recall some properties of the invariant forms associated with $\mathcal{S}$. By definition ([10], sect. 4), the exterior form ω is an invariant form of $\mathcal{S}$ if $\theta(\eta)\omega = 0$ for all the vector fields $\eta \in \Gamma(\Sigma)$, $\Sigma = \mathcal{S}^\perp$. It follows that $\theta(f\eta)\omega = 0$ for any function f ; hence, ω is invariant if and only if $i(\eta)\omega = i(\eta)d\omega = 0$, for all $\eta \in \Gamma(\Sigma)$. We infer that a necessary and sufficient condition for ω to be an invariant form is that it be expressible, locally, in terms of the first integrals of $\mathcal{S}$ and their differentials. When ω is invariant, then so are the forms $f\omega$, $d\omega$ and $\theta(\xi)\omega$, where f is a first integral and ξ an infinitesimal automorphism of the system $\mathcal{S}$; hence, the set of all the invariant forms is a differential algebra, over the ring $\mathcal{I}$, invariant under the infinitesimal action Φ via the Lie derivative. Let $\{v_i\}$ be a basis of g . Then the linear forms $\omega^i \in \mathcal{C}$ defined by the conditions $\langle \Phi(v_i), \omega^j \rangle = \delta_i^j$ are a global basis of invariant forms of $\mathcal{S}$, a so-called Cartan basis ([10], sect. 6, 8), and

$$d\omega^i = \sum_{j < k} c_{jk}^i \omega^j \wedge \omega^k, \quad (11)$$

where $\{c_{jk}^i\}$ is the set of structure constants of g with respect to the above chosen basis. The real subspace $\Omega \subset \Gamma(T^*M)$ generated by the forms ω^i only depends on Φ and acts as an $\mathbf{R}$ -dual to the space $h = \Phi(g)$. Let us denote by $\mathcal{F}$ the ring of C^∞ functions on M and by $\wedge\Omega$ the exterior algebra of Ω over the field $\mathbf{R}$. Since the set $\{\omega^i\}$ is a global basis of the Pfaffian system $\mathcal{S}$, it follows that $\mathcal{C} \simeq \Omega \otimes_{\mathbf{R}} \mathcal{F}$ and, more generally, that

$$\Phi^{r,s} = (\wedge^r \mathcal{C}) \otimes_{\mathcal{F}} (\wedge^s \mathcal{H}) \simeq (\wedge^r \Omega) \otimes_{\mathbf{R}} (\wedge^s \mathcal{H}) \simeq (\wedge^r \mathcal{S}(\mathcal{R})^*) \otimes_{\mathcal{I}} (\wedge^s \mathcal{H}). \quad (12)$$

Furthermore, since $d\omega(\xi_1, \xi_2) = -\omega([\xi_1, \xi_2])$, for any $\omega \in \Omega$ and $\xi_i \in \Phi(g)$, it also follows that the formula (9) reduces, whenever $\omega \in \wedge^r \Omega$ and $\xi_i \in \Phi(g) \subset \mathcal{S}(\mathcal{R})$, to the expression

$$[d\omega \otimes \mathbf{q}_0\mu + (-1)^{\deg \omega} \omega \wedge d_V(\mathbf{q}_0\mu)](\xi_1, \dots, \xi_{r+1}), \quad (13)$$

where $\mathbf{q}_0\mu(\xi)$ is the Lie derivative $\mathbf{q}_0(\theta(\xi)\mu)$. We shall see later that the above formula still holds for any invariant form ω since $d_H\omega = 0$ implies that $d_V\omega = d\omega$.

We next consider the elements $\tau \in \Xi^r$ as $\mathcal{I}$ -multilinear (and skew-symmetric) forms $[\tau]$ defined on $\mathcal{S}(\mathcal{R})$ and taking values in Ξ^0 (cf. sect. 4). Each form $[\tau]$ restricts to an $\mathbf{R}$ -multilinear form $\varrho = [\tau]_{\mathbf{R}}$ defined on the real subspace $h \subset \mathcal{S}(\mathcal{R})$, and conversely, on account of Lemma 2, each $\mathbf{R}$ -multilinear form ϱ defined on h and taking values in $\tau \in \Xi^0$ extends, by $\mathcal{I}$ -multilinearity, to a form $[\tau]$ defined on $\mathcal{S}(\mathcal{R})$ and verifying $[\tau]_{\mathbf{R}} = \varrho$. Furthermore, since in the realm of real vector spaces the subspace $d_H(\Phi^{0,s-1})$ admits a complement in $\Phi^{0,s}$, the form ϱ lifts to a real form $\tilde{\varrho}$ defined on h and taking values in $\Phi^{0,s} = \wedge^s \mathcal{H}$. Finally, making use of $\mathcal{F}$ -multilinearity on h , $\tilde{\varrho}$ extends to an $\mathcal{F}$ -multilinear form $\tilde{\varrho}$ defined on $\chi(M)$ and taking values in $\wedge^s \mathcal{H}$ by requiring that $i(\eta)\tilde{\varrho} = 0$ for all the vector fields $\eta \in \Gamma(\Sigma)$. We thus obtain an element $\tilde{\varrho} \in \Phi^{r,s}$ such that $[\tilde{\varrho}]_{\mathbf{R}} = \varrho$ and, consequently, the assignment $\tau \mapsto [\tau]$ of Sect. 4 becomes bijective when $\mathcal{S}$ is invariant under a transversally free infinitesimal action. Identifying g with h and thereafter Ω with g^* , the expression (13)

or, equivalently, the formula (9) shows that the cochain complex defining the cohomology of g with values in Ξ^0 and relative to the representation $\rho(v)(\mathbf{q}_0\mu) = \mathbf{q}_0(\theta(\xi)\mu)$, with $v \in g$, $\mu \in \Phi^{0,s}$ and $\xi = \Phi(v)$, is equal to

$$\Xi^0 \xrightarrow{d_V} \Xi^1 \xrightarrow{d_V} \Xi^2 \xrightarrow{d_V} \dots \quad (14)$$

Since $\mathbf{q}_0$ is surjective, we can rewrite the above complex by

$$\Phi^{0,p} \xrightarrow{E} \Xi^1 \xrightarrow{d_V} \Xi^2 \xrightarrow{d_V} \dots, \quad (15)$$

without affecting the cohomology groups in positive dimensions, the latter being the vertical part of the Euler–Lagrange complex associated with $\mathcal{S}$, namely, the finite resolution of E .

Theorem 1 *Let $\mathcal{S}$ be an integrable Pfaffian system invariant under a transversally free infinitesimal action of the Lie algebra g . Then the finite resolution of the Euler operator E is equal, in positive dimensions, to the cochain complex of the Lie algebra g with values in Ξ^0 .*

Since $E = d_V \circ \mathbf{q}_0$, we also infer that the space of cocycles in $\Phi^{0,p}$ (i.e., $\ker E$) is equal to the inverse image, by $\mathbf{q}_0$, of the 0-dimensional cohomology of g namely, the inverse image of the subspace composed by the g -invariant elements of Ξ^0 ($\rho(v)(\mathbf{q}_0\mu) = 0$, $\forall v \in g$). Given an integrable Pfaffian system $\mathcal{S}$, we can now confront the variational cohomology with the cohomology of g taking values in Ξ^0 and eventually detect obstructions to the existence of a transversally free infinitesimal action of g leaving the system $\mathcal{S}$ invariant. When $\dim_{\mathbf{R}} \Xi^0 < \infty$, which is very seldom the case, then we usually have more information on the cohomology of g . For instance, when g is semi-simple, its cohomology vanishes in dimensions one and two (Whitehead's lemmas), and consequently, the same must hold for the variational cohomology.

The above discussion leads us finally to define a *twisted complex* very convenient in calculations. Identifying the algebra g with its image $h = \Phi(g)$, the elements of g^* can be considered as those belonging to a subset of the invariant forms considered previously and we define the complex $(\wedge g^* \otimes_{\mathbf{R}} \Xi^0, \partial)$ by setting:

$$\partial(\omega \otimes \mathbf{q}_0\mu) = d\omega \otimes \mathbf{q}_0\mu + (-1)^{\deg \omega} \omega \wedge d_V \mathbf{q}_0\mu, \quad (16)$$

$d\omega$ being the differential of the invariant form ω and $(d_V \mathbf{q}_0\mu)(\xi)$ the Lie derivative $\mathbf{q}_0(\theta(\xi)\mu)$. The inclusion $g \hookrightarrow \mathcal{S}(\mathcal{R})$ determines, by transposition, a natural transformation (cochain complex morphism) from the Euler–Lagrange complex toward the twisted chain complex associated with g and this, in turn, induces morphisms on the corresponding cohomology groups. If, furthermore, the structure constants of g are known, then the first term on the right-hand side of (16) is easily calculable. We also consider the trivial extension $(\wedge g^* \otimes_{\mathbf{R}} \Xi^0, d \otimes Id)$ of the standard cochain complex associated with g . The corresponding cohomology is clearly isomorphic to the cohomology of g , and it becomes very useful to compare, via homological methods, the twisted complex with the standard one (e.g., homotopical deformation of $d \otimes Id$ into ∂).

7 A double complex

Usually, the Euler–Lagrange complex is defined in terms of a double complex. We shall also define, in this section, a double complex not for the sake of redefining what has already

been defined earlier but in view of obtaining further invariants for the Pfaffian system $\mathcal{S}$. We begin by stating an almost obvious result that will, however, be needed in the sequel. Two first-order distributions (field of contact elements) Σ_V and Σ_H , defined on a manifold M , will be said to be *complementary* when $T_x M = (\Sigma_V)_x \oplus (\Sigma_H)_x$ for any $x \in M$, which means that the tangent bundle TM is the direct sum of the two distributions. The reader will find a specific example (of almost product structures) in [12], Part A, Example (6), pg. 13. Let us denote by V and H the corresponding projections on each factor and by d_V and d_H the associated Frölicher–Nijenhuis differentials of type d ([6]). Recalling some definitions, given a vector-valued 1-form $u : TM \rightarrow TM$ defined on M (e.g., the projections V and H) and a scalar differential form ω of degree r , we shall denote by $i_u \omega$ the scalar form (indicated in the previous reference by $u \overline{\wedge} \omega$) defined as follows:

$$i_u \omega(\xi_1, \dots, \xi_r) = \sum_j \omega(\xi_1, \dots, \xi_{j-1}, u(\xi_j), \xi_{j+1}, \dots, \xi_r). \quad (17)$$

The map $i_u : \wedge TM^* \rightarrow \wedge TM^*$ becomes an interior derivation of degree zero, namely, it verifies the identity

$$i_u(\omega \wedge \mu) = (i_u \omega) \wedge \mu + \omega \wedge i_u \mu \quad (18)$$

and *interior* means that i_u vanishes on functions. The associated Frölicher–Nijenhuis derivation $d_u = [i_u, d] = i_u \circ d - d \circ i_u$ is of degree one and

$$d_u(\omega \wedge \mu) = (d_u \omega) \wedge \mu + (-1)^{\deg \omega} \omega \wedge d_u \mu. \quad (19)$$

Moreover, the map $u \mapsto d_u$ is $\mathbf{R}$ -linear, $d_{Id} = d$ and $[d_u, d] = d_u \circ d + d \circ d_u = 0$, i.e., d_u is a derivation of type d . In particular, $d_H + d_V = d$ since $H + V = Id$. We now cite the desired result.

Theorem 2 *The distributions Σ_V and Σ_H are simultaneously integrable if and only if $d_V^2 = 0$ or, equivalently, if and only if $d_H^2 = 0$, where Σ_V and Σ_H are arbitrary complementary distributions and V, H the corresponding projections.*

Let us now assume that both distributions are integrable. Then

$$0 = [d_{V+H}, d_{V+H}] = [d_V, d_V] + [d_H, d_H] + [d_V, d_H] + [d_H, d_V] \quad (20)$$

and we infer that $0 = [d_V, d_H] + [d_H, d_V] = 2(d_V d_H + d_H d_V)$, where after $d_V d_H = -d_H d_V$. Let $\Psi^{r,s}$ denote the module of all the differential forms of type (r, s) with respect to the splitting $TM = \Sigma_V \oplus \Sigma_H$. We shall say that a vector field ξ is vertical (resp. horizontal) whenever it is tangent to Σ_V (resp. Σ_H) and observe that a differential form ω of degree $r + s$ belongs to $\Psi^{r,s}$ if and only if $\omega(\xi_1, \dots, \xi_{r+s}) = 0$ whenever more than r entries are vertical or more than s entries are horizontal. Forms of type $(r, 0)$ are those forms that vanish whenever some entry is horizontal and, similarly, for the forms of type $(0, s)$, this property translating formally by $i(\xi)\omega = 0$, $\xi \in \Gamma(\Sigma_H)$, and correspondingly for the type $(0, s)$ forms. Since all the modules of differential forms here considered are projective of finite type, we infer that $\Psi^{r,s} = \Psi^{r,0} \wedge \Psi^{0,s}$, $\Psi^{r,0} = \wedge^r \Psi^{1,0}$, $\Psi^{0,s} = \wedge^s \Psi^{0,1}$, $\Psi^{1,0} = \mathcal{C}$ and $\Psi^{0,1}$ is the module of all the forms vanishing on Σ_V . We now show that $\Psi^{0,1} \simeq \mathcal{H}$. In fact, any $\omega \in \Psi^{0,1}$ induces, by restriction, an element of $\mathcal{H}$ and, conversely, any element of $\mathcal{H}$ extend (lift) to an element of $\Psi^{0,1}$ by requiring that it vanishes on the distribution Σ_V . More generally, the elements of $\Psi^{0,s}$ are in one-to-one correspondence with the elements of $\wedge^s \mathcal{H}$, the mapping $\Psi^{0,s} \rightarrow \wedge^s \mathcal{H}$ being simply the restriction of the forms to Σ_H , with the inverse defined by $\mu \mapsto \mu \circ \wedge^r H$ (read *the projection H*). We can also identify $\Psi^{1,0} = \mathcal{C}$ and,

more generally, $\Psi^{r,0} = \wedge^r \mathcal{C}$ with the modules $\Gamma(\Sigma_V^*)$ and $\Gamma(\wedge^r \Sigma_V^*)$, respectively. Given any two complementary integrable distributions, they both play technically similar roles and can be, for whatever purpose, interchanged. Nevertheless, in our earlier approach to the Euler–Lagrange complex, privilege was accorded to the initially given distribution Σ . We now identify $\Phi^{r,s}$ with $\Psi^{r,s}$ by firstly identifying $\wedge^s \mathcal{H}$ with $\Phi^{0,s}$ and secondly by identifying the elements $\omega \otimes \mu \in \Phi^{r,s}$ with $\omega \wedge \mu \in \Psi^{r,s}$. Under this identification, the operator d_H , defined in (3), identifies with the Frölicher–Nijenhuis operator $(-1)^r d_H$ and, furthermore, we can now proceed to write down the desired double complex ($p = \dim \Sigma_H$, $q = \dim \Sigma_V$ and d_H is equal to (3) or to $(-1)^r d_H$ in the Frölicher–Nijenhuis sense). Obviously, we shall choose $\Sigma_H = \Sigma$, but the first component Σ_V has no privileged choices, the resulting double complexes being equivalent for any two different choices. Nevertheless, in order to obtain the largest benefit with respect to the additional invariants, we shall choose $\Sigma_V = \Phi(g)$, *i.e.*, $(\Sigma_V)_x = (\Phi(g))_x$, for all $x \in M$, since the Lie algebra action provides considerable additional information. Let us write:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \mathcal{J}^0(\mathcal{S}) & \xrightarrow{d} & \mathcal{J}^1(\mathcal{S}) & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{J}^{p-1}(\mathcal{S}) & \xrightarrow{d} & \mathcal{J}^p(\mathcal{S}) \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow \mathcal{I}^0(\mathcal{S}) \rightarrow \Phi^{0,0} & \xrightarrow{d_H} & \Phi^{0,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Phi^{0,p-1} & \xrightarrow{d_H} & \Phi^{0,p} & \xrightarrow{q_0} & \Xi^0 \\
 & d \downarrow & d_V \downarrow & d_V \downarrow & & d_V \downarrow & d_V \downarrow & d_V \downarrow & d_V \downarrow & d_V \downarrow \\
 0 \rightarrow \mathcal{I}^1(\mathcal{S}) \rightarrow \Phi^{1,0} & \xrightarrow{d_H} & \Phi^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Phi^{1,p-1} & \xrightarrow{d_H} & \Phi^{1,p} & \xrightarrow{q_1} & \Xi^1 \\
 & d \downarrow & d_V \downarrow & d_V \downarrow & & d_V \downarrow & d_V \downarrow & d_V \downarrow & d_V \downarrow & d_V \downarrow \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 0 \rightarrow \mathcal{I}^r(\mathcal{S}) \rightarrow \Phi^{r,0} & \xrightarrow{d_H} & \Phi^{r,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Phi^{r,p-1} & \xrightarrow{d_H} & \Phi^{r,p} & \xrightarrow{q_r} & \Xi^r \\
 & d \downarrow & d_V \downarrow & d_V \downarrow & & d_V \downarrow & d_V \downarrow & d_V \downarrow & d_V \downarrow & d_V \downarrow \\
 0 \rightarrow \mathcal{I}^q(\mathcal{S}) \rightarrow \Phi^{q,0} & \xrightarrow{d_H} & \Phi^{q,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Phi^{q,p-1} & \xrightarrow{d_H} & \Phi^{q,p} & \xrightarrow{q_q} & \Xi^q
 \end{array}$$

where $q_\ell = \mathbf{q}_\ell$, $\forall \ell$, in the previous notation and, on the one before the last line, r stands for $q-1$ (due to the lack of space). The operators $d_V : \Xi^\ell \longrightarrow \Xi^{\ell+1}$ are those defined by the formula (9) and the relation $d_H \circ d_V = -d_V \circ d_H$, in the Frölicher–Nijenhuis sense, now yields

$$(-1)^{r+1} d_H \circ d_V = (-1)^r d_V \circ d_H;$$

hence, all the operators in the above diagram commute. We already know that the lines of this diagram are exact at $\Phi^{\ell,0}$, $\mathcal{I}^\ell(\mathcal{S})$ being the module of the invariant forms of degree ℓ with respect to the system $\mathcal{S}$. Let us now compute the kernel of $\Phi^{0,\ell} \longrightarrow \Phi^{1,\ell}$. We first observe that a form $\omega \in \Psi^{r,s}$ is equal to zero if and only if $\omega(\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s) = 0$ for all the entries r of which are vertical and s are horizontal. Furthermore, since vertical vector fields are spanned by $\Phi(g)$, it will suffice to consider only the elements $\xi_i \in \Phi(g)$. With this in mind and recalling that $[\xi, \eta] \in \Gamma(\Sigma_H)$ whenever $\xi \in \Phi(g)$ and $\eta \in \Gamma(\Sigma_H)$, a straightforward calculation leads to the formula

$$d_V \mu(\xi, \eta_1, \dots, \eta_\ell) = [\theta(\xi)\mu](\eta_1, \dots, \eta_\ell),$$

where $\mu \in \Psi^{0,\ell}$, $\xi \in \Phi(g)$ and $\eta_i \in \Gamma(\Sigma_H)$. Consequently, $d_V \mu = 0$ if and only if $\theta(\xi)\mu = 0$ for all $\xi \in \Phi(g)$. The space $\mathcal{J}^\ell(S)$, kernel of $d_V : \Phi^{0,\ell} \rightarrow \Phi^{1,\ell}$, is equal to the $\mathcal{I}(S)$ -module of the horizontal forms of degree ℓ invariant under the infinitesimal action Φ . Since the relation $[\xi, \eta] \in \Gamma(\Sigma_H)$ continues to hold for η horizontal and ξ an arbitrary infinitesimal automorphism of Σ_H tangent to Σ_V , all the elements of $\mathcal{J}^\ell(S)$ are invariant under such automorphisms. Still better, the relation

$$\theta(f\xi)\mu = f\theta(\xi)\mu + df \wedge i(\xi)\mu$$

shows that all the elements of $\mathcal{J}^\ell(S)$ are invariant by any vertical vector field (tangent to Σ_V). It should, however, be remarked that the Lie derivative $\theta(\xi)\mu$ only has a sense, along infinitesimal automorphisms ξ of Σ_H , when $\Phi^{0,\ell}$ is considered in its original definition as $\wedge^\ell \mathcal{H}$. A similar calculation will also show, once more, that the kernel of $d_H : \Phi^{\ell,0} \rightarrow \Phi^{\ell,1}$ is the set of all the forms of degree ℓ invariant with respect to Σ_H . Since $d = d_V + (-1)^r d_H$ (d_H in the sense of the previous diagram) and since $d_H(\mathcal{I}^\ell(S)) = 0$, we obtain the complex

$$0 \rightarrow \mathbf{R} \rightarrow \mathcal{I}^0(S) \xrightarrow{d} \mathcal{I}^1(S) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{I}^q(S) \rightarrow 0 \quad (21)$$

whose cohomology is the invariant cohomology of $\Sigma (= \Sigma_H)$. It only depends on Σ since the cochains are the invariant forms. When Σ is simple (*i.e.*, when M admits a differentiable quotient structure, modulo the integral leaves of Σ), then (21) is equivalent to the *de Rham* complex of the quotient manifold M/Σ . We can thus read, on (21), the non-quotientability of Σ or, at least, discard prospective candidates to quotients by scanning the cohomological invariants (*e.g.*, Betti and Lefschetz numbers). Since in our context of a transversally free g -action we are provided with a global basis of invariant 1-forms (*cf.* Sect. 6), the cohomology of (21) can, in principle, be calculated as soon as we know the constants of structure of g . Similarly, we can also display a complex

$$0 \rightarrow \mathbf{R} \rightarrow \mathcal{J}^0(S) \xrightarrow{d} \mathcal{J}^1(S) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{J}^p(S) \rightarrow 0 \quad (22)$$

whose cochains are the g -invariant horizontal forms. Its cohomology is the equivariant cohomology of (Σ, Φ) . Our earlier remark shows as well that (22) is also the complex of g -invariant forms with respect to Σ_V . When the integral foliation of the distribution Σ_V admits a differentiable quotient, this complex is equivalent to the *de Rham* complex of the quotient manifold M/Σ_V . It should further be emphasized that neither (21) nor (22) does rely upon the construction of the double complex exhibited on the previous diagram.

There are other invariants of Σ that can be detected on the double complex. For instance, we can detect the *relatively invariant* forms of Σ (ω is said to be relatively invariant when its differential $d\omega$ is invariant). Assuming that ω is of degree ℓ , let us write

$$\omega = \omega^{\ell,0} + \omega^{\ell-1,1} + \omega^{\ell-2,2} + \dots + \omega^{1,\ell-1} + \omega^{0,\ell},$$

where $\omega^{r,s}$, $r+s=\ell$, denotes the component of type (r,s) of ω and, using d_H in the sense of the double complex, we can also write

$$\begin{aligned} d\omega &= d_V \omega^{\ell,0} + (-1)^\ell d_H \omega^{\ell,0} + d_V \omega^{\ell-1,1} + (-1)^{\ell-1} d_H \omega^{\ell-1,1} + \\ &\quad + d_V \omega^{\ell-2,2} + d_V \omega^{\ell-2,2} + (-1)^{\ell-2} d_H \omega^{\ell-2,2} + \dots + \\ &\quad + d_V \omega^{1,\ell-1} - d_H \omega^{1,\ell-1} + d_V \omega^{0,\ell} + d_H \omega^{0,\ell}. \end{aligned}$$

If $d\omega$ is invariant, then it belongs to $\mathcal{I}^{\ell+1}(\mathcal{S}) \subset \Phi^{\ell+1,0}$. Writing out, symbolically, the types in the expression of $d\omega$, we find the following display:

$$d\omega = (\ell + 1, 0) + (\ell, 1) + (\ell, 1) + (\ell - 1, 2) + (\ell - 1, 2) + (\ell - 2, 3) + (\ell - 2, 3) + \cdots + (2, \ell - 1) + (1, \ell) + (1, \ell) + (0, \ell + 1);$$

hence, in order that $d\omega$ be of type $(\ell + 1, 0)$, it is necessary and sufficient that all the remaining terms cancel out; namely, the following equalities must be verified:

$$\begin{aligned} (-1)^\ell d_H \omega^{\ell,0} + d_V \omega^{\ell-1,1} &= 0 & (-1)^{\ell-1} d_H \omega^{\ell-1,1} + d_V \omega^{\ell-2,2} &= 0 \\ \cdots & & -d_H \omega^{1,\ell-1} + d_V \omega^{0,\ell} &= 0 & d_H \omega^{0,\ell} &= 0. \end{aligned} \quad (23)$$

Let us now assume that the above conditions do hold. Then, $d\omega$ is of type $(\ell + 1, 0)$ and

$$d_H d\omega = d_H d_V \omega^{\ell,0} = -d_V d_H \omega^{\ell,0} = d_V ((-1)^\ell d_V \omega^{\ell-1,1}) = 0$$

on account of the first relation in (23); hence, $d\omega$ is invariant. The conditions (23) can be retraced in the *spectral sequence* associated with the double complex and with respect to the *vertical* filtration. At present, we shall not inquire any further in these matters since they escape our main concern, namely, the Euler–Lagrange complex that appears at the upper right corner of the double complex, the composite $d_V \circ q_0 = q_1 \circ d_V$ being the Euler operator. In our notations, $\mathcal{J}^p(\mathcal{S})$ is the set of null Lagrangians. In particular, when $\omega = \omega^{\ell,0}$, then (23) reduces to $d_H \omega^{\ell,0} = 0$ and $d\omega$ is invariant if and only if ω is invariant.

Let us now return to the end of Sect. 6. We consider a Pfaffian system $\mathcal{S}$ defined on the manifold M and invariant under the transversally free infinitesimal action $\Phi : g \rightarrow \chi(M)$, as well as a second system $\tilde{\mathcal{S}}$, defined on $M \times N$, constructed via a basis of invariant forms of $\mathcal{S}$ and a second infinitesimal action $\Psi : g \rightarrow \chi(N)$. In order to avoid any confusion, we denote by $\bar{\mathcal{S}}$ the system, on $M \times N$, shifted of $\mathcal{S}$, namely, the system generated by $p_M^* \mathcal{S} + p_N^* T^* N$, where p_M and p_N are the projections onto the respective factors. This system is invariant under the *horizontal* action $\bar{\Phi} : g \rightarrow \chi(M \times N)$ defined by $\bar{\Phi}(v) = (\Phi(v), 0)$. We also consider the *vertical* action $\tilde{\Phi} : g \rightarrow \chi(M \times N)$ given by $\tilde{\Phi}(v) = (0, \Psi(v))$. Finally, we set $\bar{\Sigma} = \bar{\mathcal{S}}^\perp$ and $\tilde{\Sigma} = \tilde{\mathcal{S}}^\perp$. Then, $\bar{\Sigma} \subset \tilde{\Sigma}$, $\tilde{\Sigma} \subset \bar{\Sigma}$ and the inclusion $\iota : \bar{\Sigma} \hookrightarrow \tilde{\Sigma}$ transposes to $\iota^* : \Gamma(\wedge \tilde{\Sigma}^*) \rightarrow \Gamma(\wedge \bar{\Sigma}^*)$. Since vector fields tangent to $\bar{\Sigma}$ are also tangent to $\tilde{\Sigma}$, we infer, from the formula (1), that $d_H \circ \iota^* = \iota^* \circ d_H$; hence, ι^* is a natural transformation of cochain complexes. The same, however, is not true for the variational parts of the Euler–Lagrange complexes associated with $\bar{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ since $\text{rank } \bar{\Sigma} \neq \text{rank } \tilde{\Sigma}$ (unless $\Sigma = TM$). Define $\bar{\Phi} \oplus \tilde{\Phi} : g \oplus g \rightarrow \chi(M \times N)$ by setting, in the obvious way, $\bar{\Phi} \oplus \tilde{\Phi}(v, w) = \bar{\Phi}(v) + \tilde{\Phi}(w)$. In studying this product infinitesimal action that is transversally free with respect to the sum of the aforementioned distributions, we shall be able to find new invariants for the system $\mathcal{S}$. Nevertheless, we shall not pursue any further this line of investigation since, for the time being, we do not foresee any interesting applications for these new invariants.

8 Examples

Throughout this section, we replace integrable Pfaffian systems by the corresponding integral foliations since the latter are much more visible. Though all the foliations considered in the sequel are entirely naïve, the resulting homological calculations are not always so.

Example 1 *The torus.*

Let $\mathcal{F}$ be the foliation on the 2-dimensional torus T whose leaves are the cosets of a 1-dimensional subgroup H . Then $\mathcal{F}$ is invariant under the infinitesimal action generated by any element of the Lie algebra of T ($\equiv \mathbf{R}^2$), this action being transversally free when the chosen element does not belong to the Lie algebra $\mathfrak{h}$ of H . The variational cohomology at Ξ^1 is equal to $\mathbf{R}$ and, using Green's formula, it can be shown that the cohomology class of an element $[\omega] \in \Xi^1$ identifies with the real number $\int_T \omega$. When the slope of the element, in $\mathfrak{h}$, is rational, the calculations are very simple and both spaces Ξ^0 and Ξ^1 identify with the set of all the global first integrals of $\mathcal{F}$. However, when this slope is irrational, the global first integrals of $\mathcal{F}$ reduce to the constants (the curves become everywhere dense) and it becomes more involved to describe the spaces Ξ^0 and Ξ^1 as well as to calculate directly the cohomology. This is an example where the advantages of the Lie algebra cohomology calculations become apparent.

Example 2 *The Möbius strip.*

Let $\mathcal{F}$ be the foliation on the Möbius strip $\mathcal{M}$ whose leaves are the “double” circles, except for the central circle (under the usual identification $(1, y) \equiv (-1, -y)$, where $\mathcal{F}$ is the foliation induced by the segments parallel to the x -axis). Both spaces Ξ^0 and Ξ^1 identify again with the set of all the global first integrals of $\mathcal{F}$ that, in turn, identifies with the set of all the even functions defined on the interval $] -1, 1[$. The variational cohomology at Ξ^1 vanishes and, whatever the representation $\rho : g = \mathbf{R} \rightarrow \text{Der } \Xi^0$, the cohomology of g , in dimension one, is non-trivial (the derivative of an even function usually ceases to be even). The above disagreement shows that $\mathcal{F}$ cannot be invariant by any transversally free infinitesimal action, a fact that is geometrically obvious since such an action would provide an orientation to $\mathcal{M}$.

Example 3 *Foliation with a compact attractor.*

We consider, on the infinite cylinder $\mathcal{C} = \mathbf{R} \times S^1$ with the coordinates (t, θ) , the foliation $\mathcal{F}$ obtained by integrating the vector field

$$\eta = t \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}. \quad (24)$$

The nature of the spaces Ξ^0 and Ξ^1 is rather involved, but a straightforward calculation shows that the cohomology at Ξ^1 is null. On the other hand, whatever the representations of the Lie algebra $g = \mathbf{R}$ into $\text{Der } \Xi^0$, the Lie algebra cohomology, in dimension one, cannot vanish. Consequently, the foliation $\mathcal{F}$ does not admit any 1-dimensional transversally free infinitesimal action that leaves it invariant. This fact is also geometrically obvious since the local 1-parameter group (ϕ_u) generated by any such infinitesimal action would be defined, for small u , on a whole neighborhood of the limit circle $\{0\} \times S^1$ and would transform this circle into open compact subsets of the neighboring leaves, this being of course excluded. An entirely similar situation arises in the double solid torus (two solid tori glued together by their boundaries) upon taking the Reeb foliation inside each of the tori. The common boundary torus is the unique compact leaf.

Example 4 *Spheres and rays.*

On the space $M = \mathbf{R}^{p+1} - \{0\}$, let $\mathcal{F}_1$ be the foliation whose leaves are the spheres centered at the origin and $\mathcal{F}_2$ the foliation whose leaves are the rays issued from the origin. We first consider the *sphere* foliation $\mathcal{F}_1$ and calculate Ξ^0 , one possible argument being as follows: Each element $\mu \in \Phi^{0,q}$ identifies canonically with a differentiable 1-parameter family $(\bar{\mu}_\rho)$ of differentiable q -forms defined on the unit sphere S^p and, under this identification, the differential $d_H \mu \in \Phi^{0,q+1}$ also identifies with $d\bar{\mu}_\rho$. We next take a differentiable

1-parameter family $(\bar{\mu}_t)$, $t > 0$, of p -forms on S^p . Then, upon choosing a fixed volume form Ω on this unit sphere (e.g., the volume form associated with the induced Euclidean metric), we can determine, by integration, a differentiable function $\varphi : \mathbf{R}_+ \rightarrow \mathbf{R}$ such that $\bar{\mu}_t - \varphi(t)\Omega$ is, for each value of t , a coboundary. Restating Lemma 4.2 of [15] in its stronger version (as is proved in the subsequent two pages of the book), we can use it to establish a stronger 1-parameter version of Lemma 4.2 and prove, in the aforementioned context, that there exists a differentiable 1-parameter family $\bar{\eta}_t$ of $(p-1)$ -forms defined on S^p such that $\bar{\mu}_t - \varphi(t)\Omega = d\bar{\eta}_t$. Returning to $\Phi^{0,p}$ and taking the form $\tilde{\Omega} = r^*\Omega$, $r : X \mapsto (1/\|X\|)X$, defined on M , we conclude that each $\mu \in \Phi^{0,p}$ determines a differentiable function φ such that $\mu - \varphi\tilde{\Omega} = d_H\eta$, where $\eta \in \Phi^{0,p-1}$ and consequently that Ξ^0 is equal to the set of all the real-valued differentiable functions defined on $\mathbf{R}_+$, i.e., equal to the set of all the global first integrals of $\mathcal{F}_1$.

Let us now calculate Ξ^1 . Observing that $d\rho$ is a global generator of the Pfaffian system that annihilates $\mathcal{F}_1$, any element $\eta_1 \in \Phi^{1,p}$ writes as $\eta_1 = \eta \wedge d\rho$, with $\eta \in \Phi^{0,q}$, and $d_H\eta_1 = (d_H\eta) \wedge d\rho$ since $d_H(d\rho) = 0$. Consequently, the element $\mu_1 = \mu \wedge d\rho \in \Phi^{1,p}$ is equal to $d_H\eta_1$, with $\eta_1 \in \Phi^{1,p-1}$, if and only if $\mu = d_H\eta$; hence, the present calculation reduces to the previous one and Ξ^1 is again equal to the set of all the global first integrals of $\mathcal{F}_1$.

It now becomes easy to show that the variational cohomology at Ξ^1 is null. The foliation $\mathcal{F}_1$ is, of course, invariant under many 1-dimensional transversally free infinitesimal actions and the vanishing of the Lie algebra cohomology in dimension one can easily be verified.

We next consider the *radial* foliation $\mathcal{F}_2$. Here we can proceed locally, on open sets saturated by rays, and integration along these rays will show that $\Xi^q = 0$ for $0 \leq q \leq p$. The variational and the Lie algebra cohomologies vanish, their comparison *not* revealing the following rather important geometrical facts:

- (a) When p is even, there cannot exist a transversally free infinitesimal action leaving the radial foliation $\mathcal{F}_2$ invariant. In fact, since the tangent spaces to $\mathcal{F}_1$ and $\mathcal{F}_2$ are complementary, any such infinitesimal action would project onto the spheres producing an infinitesimal action operating tangentially to the spheres and, in restriction to these spheres, would be free. However, even-dimensional spheres do not admit nowhere vanishing vector fields.
- (b) When p is odd, such transversally free infinitesimal actions do exist only for $p = 1, 3$. Their nonexistence for $p = 7$ is essentially a consequence of the fact that S^7 is not a group manifold and, for all the other values of p , that the corresponding spheres are not parallelizable.

We can enhance the variational cohomology by adding non-trivial cocycles to the space M . For example, let us take for the manifold M the portion of $\mathbf{R}^{p+1} - \{0\}$ in between the spheres $S^p(1)$ and $S^p(2)$, (the numbers indicating the radii) and identify these two spheres by the radial map. Then, $\mathcal{F}_1$ induces a foliation $\bar{\mathcal{F}}_1$ in spheres, $\mathcal{F}_2$ a foliation $\bar{\mathcal{F}}_2$ in circles (in fact, $M = S^p \times S^1$) and we can show, for the foliation $\bar{\mathcal{F}}_2$, that Ξ^0 is equal to the set of all the differentiable functions defined on the sphere $S^p(1)$ or, equivalently, to the set of all the global first integrals of $\bar{\mathcal{F}}_2$. Furthermore, Ξ^r is equal to the product of $\binom{p}{r}$ copies of Ξ^0 and $\Xi^p = \Xi^0$. As for the variational cohomology, we can apply again the 1-parameter version of Lemma 4.2 and conclude that it vanishes at Ξ^{r+1} whenever $r+1 < p$, it being equal to $\mathbf{R}$ at Ξ^p . Stokes' formula will then show that the cohomology class of an element $[\omega] \in \Xi^p$ identifies with the real number $\int_M \omega$.

Returning to the geometrical facts described earlier, we can retrace (a) by looking at the variational cohomology. In fact, since any global vector field on an even-dimensional sphere

has a singularity, whatever the representation ρ of a Lie algebra $\mathfrak{g}$ into $\text{Der } \Xi^0 \simeq \chi(S^p)$, the corresponding Lie algebra cohomology cannot vanish in dimension one. As for the property (b), it requires a deeper analysis that seems out of reach in the present setting. Nevertheless, it can be shown that transversally free abelian infinitesimal actions leaving $\mathcal{F}_2$ invariant cannot exist since the corresponding Lie algebra cohomologies, with values in Ξ^0 , would vanish in dimension p thus contradicting the variational cohomology.

Example 5 *Exterior differential systems.*

Exterior differential systems are a generalization of Pfaffian systems and allow the presence of exterior differential forms of arbitrary degree. Almost all that has been discussed in the present paper can be extended to these systems as soon as integrability is replaced by *involutiveness*, within the appropriate definitions. These systems were introduced by Élie Cartan in the beginning of the twentieth century and afterward much exploited by him in view of solving fundamental geometrical problems. ([4], [5], [8], see also [13]). We shall not discuss here this extended context since much has to be re-evaluated including the initial notion of generalized symmetry. Moreover, it might be considerably beneficial to replace Lie group actions by Lie groupoid actions since the latter will embrace many more situations.

9 A foliation with a compact, simply connected and isolated leaf

In the early 1950s, Charles Ehresmann was asking himself whether the leaves sufficiently close to a compact and simply connected leaf, of a given foliation, enjoyed necessarily the same property.

Let F_0 be a compact and simply connected leaf of a foliated manifold. Is any leaf F passing by a point sufficiently close to F_0 homeomorphic to F_0 ? ([7], Problem 73).

The Reeb foliation, defined on the sphere S^3 and also arising from a problem posed by Ehresmann, exhibits a compact isolated leaf that, however, is not simply connected. We presently construct a foliation having a leaf as imagined by Ehresmann, that is to say, a compact, simply connected and *isolated* leaf, all the other nearby leaves being non-compact. The argument, indeed very simple, is based upon the consideration of a duly modified *loxodromy curve* (Fig. 1), also known as the Napier navigational curve. It verifies some remarkable properties inasmuch in geometry as in nature and one can even say that the loxodromy curve is the transposition (via the inverse of the stereographic projection), to a 2-sphere, of the Bernoulli *Spira Mirabilis* (Fig. 2) ([1]). Once established the construction principle of such a foliation, nothing will hinder the construction of many other. Our example is displayed on the Euclidean unit 3-sphere deprived of its two poles (along the first canonical axis of $\mathbf{R}^4$), the isolated simply connected leaf being a 2-sphere. It should be remarked, however, that, in general, this isolated leaf is not necessarily unique though it is unique in our example.

10 The modified loxodromy

For the definition of the loxodromic curve as well as a nice image of it, we refer the reader to Wikipedia,¹ the angle β indicated there, on the image, being as well the one used in what follows. However, we adopt a slightly different parametrization and convene that each loxodromy curve initiates, for $t = 0$, at a point located on the *Tropic of Cancer*, namely, at

¹ < <https://upload.wikimedia.org/wikipedia/commons/d/d6/Loxodrome.png> >.

23°26'12.8" latitude North². We shall now modify this curve in view of obtaining one that accumulates on the equator. For this, let us consider the angle β as a function of the real variable t and assume that:

- (a) $\beta(t)$ converges *fast enough* to $\pi/2$, when $t \rightarrow -\infty$, so as for the curve to remain in the northern hemisphere and accumulate onto the equator.
- (b) $\beta(t)$ converges *fast enough* to 0, when $t \rightarrow +\infty$, so as for the curve to stop winding endlessly around the North Pole. This last requirement is not by necessity but merely by convenience.

We might, for example, choose the function $\beta(t) = (1 - \operatorname{arctg}(t))\pi/4$, where arctg denotes the inverse of the *tangent* function.

Next, we simply project (reflect) the above curve onto the southern hemisphere thus obtaining an entirely similar curve that accumulates, from below, onto the equator.

In order to make our discussion simpler, let us take the Euclidean coordinates (x, y, z, w) in $\mathbf{R}^4$ and fix the x -axis denoting it, for simplicity, by ℓ . This being so, any 3-plane (3-dimensional linear subspace), containing the axis ℓ , cuts the 3-sphere along a unit 2-sphere. We can now *draw*, on each of these 2-spheres, the two modified loxodromic curves and obtain the following 4-dimensional picture:

Choosing any fixed (modified loxodromic) curve on the initially given 2-sphere, we can reproduce its equivalent (correspondent) curve on each one of the 2-sphere obtained by cutting the 3-sphere with a plane as described above, *i.e.*, with a 3-plane containing the axis ℓ . The union of all these curves is a non-singular differentiable surface (its dimension being equal to 2), and two such surfaces are entirely disjoint (no common points). However, the remarkable feature of these surfaces is that not only they are open (non-compact) on account of the loxodromy curves accumulating onto the equators of the 2-spheres, but, more important, these surfaces accumulate onto the 2-sphere contained in the 3-sphere and defined by the equation $x = 0$.

In conclusion, we obtain an example of a smooth 2-dimensional foliation on a 3-sphere, devoid of its two poles ($x = 1$ and $x = -1$), with an isolated compact and simply connected leaf (the 2-sphere defined by the equation $x = 0$). Needless to say that these exceptional leaves need not, in general, be unique, let us also observe that such an example can be rendered analytic by simply choosing β to be a real analytic function.

The above discussion provides a pictorial account of our foliation, and we now need to give the proofs. For this, we shall appeal to the *quaternionic structure* defined on $\mathbf{R}^4$. To begin with, let us denote, as is usual, by $1, i, j, k$ the real and the three purely imaginary units and identify them, respectively, with the canonical unit vectors, *i.e.*, $1 \equiv (1, 0, 0, 0)$, $i \equiv (0, 1, 0, 0)$, etc. Next, we remark the following properties, where $\mathbf{W}$ denotes a generic hyperplane in $\mathbf{R}^4$ (a 3-dimensional subspace), $[\dots]$ denotes the subspace generated by some vectors, and the *orthogonality* is taken with respect to the standard inner product of the numerical space.

- (c) The axis ℓ belongs to $\mathbf{W}$ if and only if $[1]^\perp \cap \mathbf{W}$ is two-dimensional. Moreover, any two-dimensional subspace contained in $[1]^\perp$ generates a hyperplane containing the vector 1 , since it will suffice to sum both spaces. Consequently, we derive that the set of all the hyperplanes that contain the vector 1 can be identified with the two-dimensional real projective space associated with $[1]^\perp$ by simply associating to each hyperplane, contained in $[1]^\perp$, its orthogonal complement in $[1]^\perp$.

Let us now specify the following hyperplanes:

$$\mathbf{W}_1 = [1, i, j], \quad \mathbf{W}_2 = [1, i, k], \quad \mathbf{W}_3 = [1, j, k], \quad \mathbf{W}_4 = [i, j, k],$$

² We could, of course, initiate these curves on any other *parallel* of the northern hemisphere.

and try to obtain all the previously considered hyperplanes by means of *left multiplication*, with elements of $\mathbf{R}^4$, of a given fixed hyperplane. Given a quaternion q and a hyperplane $\mathbf{W}$, we set

$$q \cdot \mathbf{W} = \{qw \mid w \in \mathbf{W}\}$$

and observe that q can be replaced by the unit quaternion $\frac{1}{\|q\|}q$ without affecting the result; hence, we shall restrict our elements q to the unit sphere S^3 . We next observe that

- (d) $q \cdot \mathbf{W} = r \cdot \mathbf{W}$ if and only if $q^{-1}r \cdot \mathbf{W} = \mathbf{W}$, i.e., if and only if $q^{-1}r$ (or, equivalently, $r^{-1}q$) leaves invariant the hyperplane $\mathbf{W}$.

Let us finally chose and fix a definite hyperplane $\mathbf{W}$, for instance $\mathbf{W}_1$. It should be emphasized that this is a choice and not a canonical procedure. Now, if $h \cdot \mathbf{W}_1 = \mathbf{W}_1$, then $h1, hi, hj \in \mathbf{W}_1$, and consequently, $h \in \mathbf{W}_1, h \in \mathbf{W}_1 \cdot i$ and $h \in \mathbf{W}_1 \cdot j$; hence, ultimately, $h \in \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3 = [1]$. Since we have opted for unit quaternions, there are only two choices possible, namely, $h = 1$ and $h = -1$. In choosing the first option, we can now claim the following statement.

Lemma 3 *Under the above choices, there exists a unique element q transforming the hyperplane $\mathbf{W}_1$ into a give hyperplane $\mathbf{W}$.*

Furthermore, since $i \cdot \mathbf{W}_1 = \mathbf{W}_2$ and $j \cdot \mathbf{W}_1 = \mathbf{W}_3$, we infer that any hyperplane $\mathbf{W}$ containing the unit 1 can be reached from $\mathbf{W}_1$ with a suitable *left multiplication* by a unit quaternion.

One last remark is due. If two distinct hyperplanes $q \cdot \mathbf{W}_1$ and $r \cdot \mathbf{W}_1$, containing ℓ , have a common point not belonging to $[\ell]$, then their intersection is 2-dimensional and

$$q \cdot \mathbf{W}_1 \cap r \cdot \mathbf{W}_1 \cap [\ell]^\perp$$

is a 1-dimensional line that cuts the two 2-spheres, one on each hyperplane, along two points belonging to a common diameter; hence, the two spheres coincide and there is no obstruction to the continuation of the process. Furthermore, if $q \cdot w_1 = r \cdot w_2$ with $w_1, w_2 \in \mathbf{W}_1$, then $q^{-1}rw_2 = w_1$, i.e., $q^{-1}r$ transforms an element of $\mathbf{W}_1$ into an element of $\mathbf{W}_1$ and this context holds for all the points belonging to the 2-dimensional space $q \cdot \mathbf{W}_1 \cap r \cdot \mathbf{W}_1$ with the additional merit that it also holds for all the points of $[\ell]$. Arguing as above, we conclude as well that $q^{-1}r \in [\ell]$, where after we shall chose the value $+1$ to ensure that the two transformed *modified loxodromy curves* do coincide.

We now define 2-dimensional open (non-compact) sub-manifolds of S^3 by dragging the initial (modified) loxodromy curves. For this, let us denote by S_1^2 the 2-sphere obtained as the intersection of $\mathbf{W}_1$ with S^3 and let us choose a given (modified) loxodromic curve γ upon this sphere. Given a unit quaternion q and under the above mentioned “uniqueness” choices, we can *draw* the loxodromic curve $q \cdot \gamma$ upon the sphere $(q \cdot \mathbf{W}_1) \cap S^3$. The union, for all q , of these loxodromic curves will yield a 2-dimensional open surface in S^3 , and the set of all these surfaces together with the 2-dimensional sphere defined by the equation $x = 0$ will finally provide a differentiable foliation of S^3 , minus its two poles, where the open leaves accumulate on the compact and simply connected leave represented by a 2-sphere.

The above given example has a unique isolated compact leaf, but it is very easy, adapting a similar technique, to construct a foliation on S^3 that will have several such leaves, even a countable infinity³. For this, it suffices to draw, on the 2-sphere, several parallels and, in between two consecutive ones, draw the necessary modified loxodromy curves by an appropriate choice of the initial point and the function β .

³ A continuous infinity being impossible.

Fig. 1 Napier navigational curve

We finally observe that the existence of such foliations inasmuch as those of the Reeb type has unimaginable cosmological implications. In fact, the displacement of mass in the expansion of the Universe is most probably subject to flow in accordance with a global foliation and the big black hole⁴ is, most probably, contained in one of these isolated compact leaves. We can also state the following result that is obvious by construction (Fig. 1).

Theorem 3 *The above foliation is transitively invariant under the group of unitary quaternions.*

11 The spira mirabilis

Let us initially take the unit 2-sphere S^2 in the Euclidean space $\mathbf{R}^3$ and denote by ρ the stereographic projection along the rays initiating at the point $(0, 0, 1)$ (the north pole) and projecting onto the (x, y) -plane ($z = 0$). Then, each *parallel* (a circle on the sphere that is parallel to the (x, y) -plane) projects onto a circle of this plane that is centered at the origin, the equator being the only invariant circle. Inasmuch, the meridians (the intersections of the sphere with a plane containing the z -axis) project onto the rays issued from the origin. Observing how *angles do project*, we shall conclude that the *loxodromy curve* projects onto another very famous curve, namely, the *Spira Mirabilis* (Fig. 2). This curve, also known as the Bernoulli spiral or the logarithmic spiral, has been defined in the seventeenth century by Jacob Bernoulli (1654-1705) who named it *Spira Mirabilis* ([1]). His concern was mainly to search for an analytical description of several natural phenomena, and this curve reveals itself, besides its usefulness in geometrical problems and in physics, to be quite extraordinary

⁴ The dark *fixed point* in our *space-time Universe*.

in this respect. The seeds of a sunflower conglomerate according to the design of the *Spira Mirabilis* as can be seen in *van Gogh's* paintings and many species of snails inasmuch as shellfish do present their shells in the format of a *Spira* like, for instance, the *Nautilus shell*. Let us also point out that tornadoes and cyclones do present this spiraling form and needless to say that this does not happen by mere chance or casualty but has a very deep biological meaning as well as functional significance.

The simplest definition of this spiral is given in polar coordinates (r, θ) where $0 \leq r \leq \infty$ and $-\infty \leq \theta \leq \infty$. It goes without saying that the variation imposed above to the angle θ makes (r, θ) cease being a coordinate system in the strict sense of the word. However, there is a big advantage to make such. We set, by definition,

$$\ln(r/R) = \theta \cot g(\alpha) \quad (25)$$

or equivalently,

$$r(\theta) = R e^{\theta \cot g(\alpha)}, \quad (26)$$

R being the radius of the spiral, namely, the distance, in the Euclidean plane, of the origin to the point with coordinates $(r(\theta), \theta)$ and α the inclination angle of the tangent line to the spiral with the line issued from the origin and passing through the point under consideration. Let us further specify that the angle θ is measured from the x -axis in the anticlockwise mode, whereas the negative angles are taken in the clockwise manner. The fundamental property of the *Spira Mirabilis* that further serves to define it in a purely geometrical manner is expressed by the *constancy* of the angle α . The more this angle (taken from the exterior) is small, the faster this spiral flees to infinity. We further observe that the origin 0 is an accumulation point of the spiral when θ tends to $-\infty$. For the sake of our present discussion, we shall simplify the above expressions and only consider the spiral defined by:

$$\ln(r) = \theta \cot g(\pi/4). \quad (27)$$

with $R = 1$. The choice of the value $\pi/4$ for the angle α has to do, of course, with the choice of the angle β for the *loxodromy curve*. Let us further remark that the *modified loxodromy curve*, considered above, will be transformed by the stereographic projection, onto a curve that we shall call the *modified Spira Mirabilis*. In defining the *modified loxodromy curve*, we did not care much to specify or render explicit the dependence of the angle β upon the variable t since it was quite irrelevant. However, we shall in the present case give a more precise definition of the *modified Spira* since it will bring a much clearer geometrical insight into the subject. It goes without saying that any *modified Spira* (consider, for instance, the curves defined in the Example 3) is entirely equivalent, for the purpose of our argument, to the *modified Spira* that we shall define hereafter. It should also be noted that any *modified Spira* provides a corresponding *modified loxodromy curve*. Now, the curve we want to consider is defined, in polar coordinates, by the equation:

$$r(\theta) = e^{\theta \cot g(\pi/4)} + 1, \quad (28)$$

tends to infinity when θ does so and accumulates, from the outside, onto the unit circle centered at the origin when θ tends to $-\infty$. Further, we consider a second *modified Spira Mirabilis* defined by the equation:

$$r(\theta) = 1/(e^{\theta \cot g(\pi/4)} + 1). \quad (29)$$

which approaches the origin 0 when θ tends to infinity and accumulates, from the inside, onto the unit circle when θ tends to $-\infty$. For the sake of distinguishing these two curves,

Fig. 2 Spira Mirabilis

we shall call the first one the *exterior Spira*, the second one being the *interior Spira*, with respect to the unit circle.

Let us now return to the full *loxodromy example* described above and consider, in the Euclidean space $\mathbf{R}^4$, the 3-sphere S^3 with its *North pole* at the point $+1$. We also consider the stereographic projection, still denoted by ρ , defined by the rays issued from the above considered *North pole* and that maps the 3-sphere, devoid of its North and South poles ($\{+1, -1\}$), onto the Euclidean 3-space $\mathbf{R}^3 = [i, j, k]$ deprived of its origin. Given a 3-dimensional subspace $\mathbf{W}$ that contains the line ℓ (the axis generated by the element $+1$), the projection of $\mathbf{W} \cap S^3$, under the stereographic mapping, is equal to the intersection of a 2-plane, deprived of its origin, contained in the 3-space $\mathbf{W}_4 = [i, j, k]$ and the image of a 2-sphere. Examining the relationship between the left multiple, by a unit quaternion, of an element in the sphere S^3 and its stereographic image, we shall be able to *drag around*, in the 3-space $\mathbf{W}_4$, the *modified Spiras* defined on the initial 2-plane and obtain a foliation where the open leaves accumulate on a compact and simply connected leaf defined as the stereographic image of a 2-sphere.

12 Flag systems and Monge equations

Let S be a Flag system defined on the manifold M , i.e., a Pfaffian system whose derived systems S_μ diminish successively their ranks by one unit, at each step, and terminate by the null system. We also denote by $\bar{S}_\mu$ the same system S_μ that, however, has been relocated to a more suitable manifold so as to eliminate the nonvanishing *Cartan Class* ([11]). Though hidden in behind the curtains, the variational complex is the main ingredient in proving the following results:

Theorem 4 *Let $v < \mu$. Then, every i.a. ξ of $\bar{S}_\mu$ prolongs (lifts) uniquely to an i.a. denoted $p_{\mu-v}\xi$ of $\bar{S}_v$ that projects onto ξ . Composition of two such prolongations is again a prolongation at the appropriate level and each prolongation map $p_{\mu-v} : \mathcal{L}(\bar{S}_\mu) \longrightarrow \mathcal{L}(\bar{S}_v)$ is bijective, its inverse being the projection (extended to vector fields). Moreover, p_κ is a Lie*

functor preserving all the operations including brackets and multiplication by functions and commutes with equivalences.

Corollary 1 *Let $\mathcal{S}$ be a flag system of length ℓ . Each i.a. of $\mathcal{S}$ projects onto an i.a. of $\overline{\mathcal{S}}_v$, $1 \leq v \leq \ell$ and, conversely, every i.a. of $\overline{\mathcal{S}}_v$, $v < \ell$, prolongs (lifts) to an i.a. of $\mathcal{S}$. This prolongation–projection procedure commutes and establishes thereafter, for all v , $0 < v < \ell$, a bijective Lie functor with source $\mathcal{L}(\overline{\mathcal{S}}_v)$ and target $\mathcal{L}(\mathcal{S})$.*

Corollary 2 *The pseudo-group of all the local (and global) automorphisms of an arbitrary Flag System is canonically isomorphic to the corresponding pseudo-group of the Darboux Flag.*

A proof of these results can be found in [11].

As for the integration of the second-order *Monge Equations* and the role played by the variational complex, the argumentation is far too long and we refer the reader to Élie Cartan's beautiful *Mémoire* [3] (see also [11]).

Acknowledgements Our acknowledgements are due to Ercüment Ortaçgil for his precious help and encouragement and also for suggesting us the use of quaternions. Initially, the author tried to apply Élie Cartan's beautiful technique of the moving frame, but the quaternions simplified and rendered more concise our argumentation. It was also Prof. Ortaçgil that indicated to us the so practical text [14].

References

1. Bernoulli, J.: *Ars Conjectandi—Tractatus de Seriebus Infinitis*. Thurnisiorum, Basilea (1713)
2. Bott, R.: On a topological obstruction to integrability. *Global analysis*. In: *Proceedings of Symposium in Pure Mathematics*, Berkeley, California, vol. 16, pp. 127–131 (1970)
3. Cartan, E.: Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup.* **27**, 109–192 (1910)
4. Cartan, E.: *Les systèmes différentiels extérieurs et leurs applications géométriques*. Hermann, Paris (1945)
5. Cartan, E.: *Oeuvres Complètes*, Part II. Gauthier-Villars, Paris (1953)
6. Frölicher, A., Nijenhuis, A.: Theory of vector-valued differential forms, Part I. *Indag. Math.* **18**, 338–359 (1956)
7. Granas, A. (ed.): *Toruńska Księga Problemów Matematycznych*. Polska Akad. Nauk (2016)
8. Kumpera, A.: *Exterior Differential Systems*. Princeton University, Princeton (1962)
9. Kumpera, A.: Les symétries généralisées et le complexe d'Euler–Lagrange. *J. Fac. Sci. Univ. Tokyo* **38**, 589–622 (1991)
10. Kumpera, A.: On the Lie and Cartan theory of invariant differential systems. *J. Math. Sci. Univ. Tokyo* **6**, 229–314 (1999)
11. Kumpera, A.: Automorphisms of Flag Systems. arXiv, to be published in *JDEQ*, 1411.0951:25 (2015)
12. Kumpera, A., Spencer, D.: *Lie Equations*, Vol. 1: General Theory. Princeton University Press, Princeton (1972)
13. Lie, S.: Verwertung des Gruppenbegriffes für Differentialgleichungen, I. *Ber. Ges. Leipz. Math. Phys.* **15**, 261–322 (1895)
14. Meister, L., Schaeben, H.: A concise quaternion geometry of rotations. *Math. Methods Appl. Sci.* **28**, 101–126 (2005)
15. Sternberg, S.: *Lectures on Differential Geometry*. Prentice-Hall, Inc., Upper Saddle (1964)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.