On the Fredholm-type theorems and sign properties of solutions for (p, q) -Laplace equations with two parameters

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Abstract

We consider the Dirichlet problem for the nonhomogeneous equation $-\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u + f(x)$ in a bounded domain, where $p \neq q$, and $\alpha, \beta \in \mathbb{R}$ are parameters. We explore assumptions on α and β that guarantee the resolvability of the considered problem. Moreover, we introduce several curves on the (α, β) -plane allocating sets of parameters for which the problem has or does not have positive or sign-changing solutions, provided f is of a constant sign.

Keywords (p, q) -Laplacian · Fredholm alternative · Existence of solutions · Positive solutions · Maximum principle · Linking method

Mathematics Subject Classification 35J62 · 35J20 · 35P30 · 35B50

1 Introduction and preliminaries

Consider the boundary value problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \alpha|u|^{p-2}u + \beta|u|^{q-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (D_{\alpha, \beta, f})$$

where $\Delta_r u := \operatorname{div}(|\nabla u|^{r-2} \nabla u)$ with $r = p$ or $r = q$ defines the r -Laplace operator, $p, q > 1$ and, without loss of generality, $p > q$. Parameters α, β are real numbers, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 -boundary, $N \geq 1$. The source function f belongs to $W^{-1, p'}(\Omega)$, the dual of the Sobolev space $W_0^{1, p} := W_0^{1, p}(\Omega)$, $p' = \frac{p}{p-1}$. The latter space

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is endowed with the norm $\|\nabla(\cdot)\|_p$, where $\|u\|_p := (\int_{\Omega} |u|^p dx)^{1/p}$ defines the norm of $L^p(\Omega)$.

The main “building block” of $(D_{\alpha,\beta,f})$ is the nonlinear eigenvalue problem for the r -Laplacian

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.0)_r$$

We say that λ is an eigenvalue of the r -Laplacian if there exists a nonzero weak solution of $(1.0)_r$ called an eigenfunction associated with λ . Hereinafter, we denote by $\lambda_1(r)$ and φ_r the first eigenvalue and an associated eigenfunction, respectively. It is known (see, e.g., [2,4]) that $\lambda_1(r)$ is simple and isolated, and it can be defined by

$$\lambda_1(r) = \inf \left\{ \frac{\|\nabla u\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r} \setminus \{0\} \right\}. \quad (1.1)$$

Moreover, $\varphi_r \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$ (see Remark 2.4 below), and we can choose φ_r to be strictly positive [27]. By analogy with the linear case $r = 2$, we will denote the set of all eigenvalues of the r -Laplacian as $\sigma(-\Delta_r)$ and the set of all eigenfunctions associated with the parameter $\lambda \in \mathbb{R}$ as $ES(r; \lambda)$. For instance, $ES(r; \lambda_1(r)) \equiv \mathbb{R}\varphi_r$, and $ES(r; \lambda) = \{0\}$ provided $\lambda \notin \sigma(-\Delta_r)$.

If we let $p = q$ and $\alpha = \beta$, then, up to scaling, $(D_{\alpha,\beta,f})$ turns to the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

which had been actively studied in the following two directions:

1. *Existence and multiplicity* of solutions. In fact, if λ is not an eigenvalue of $-\Delta_p$, then the existence is well known, see, e.g., [21, Theorem 3.1, p. 60]. Moreover, in the linear case $p = 2$ the complete information about the existence and multiplicity is provided via the Fredholm alternative. On the other hand, the situation is drastically different in the nonlinear case $p \neq 2$. It was investigated in [12,13,28] (see also references therein), that the nontrivial multiplicity of solutions of (1.2) can occur as for $\lambda = \lambda_1(p)$, and for λ from a punctured neighborhood of $\lambda_1(p)$, depending on assumptions on f . Notice that the investigations of (1.2) were carried out mainly in a neighborhood of $\lambda_1(p)$ due to the lack of description of $\sigma(-\Delta_p)$ at higher eigenvalues.
2. *Sign properties* of solutions. Suppose that $f \in L^\infty(\Omega) \setminus \{0\}$ and $f \geq 0$ a.e. in Ω . Then, the well-known maximum principle states that any solution of (1.2) is positive provided $\lambda < \lambda_1(p)$. Moreover, any solution of (1.2) is either nonnegative or sign-changing for $\lambda > \lambda_1(p)$, see, e.g., [2, Theorem 2.1]. The latter result is strengthened by the anti-maximum principle [11,19]: There exists $\lambda_f > \lambda_1(p)$ such that any solution of (1.2) is negative provided $\lambda \in (\lambda_1(p), \lambda_f)$.

The aim of the present paper is to obtain some basic results on the existence of solutions of $(D_{\alpha,\beta,f})$, as well as on their sign properties. As in the case of the problem (1.2), it is clear that the investigation of $(D_{\alpha,\beta,f})$ should be preceded by the study of the corresponding unperturbed problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \alpha |u|^{p-2} u + \beta |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (D_{\alpha,\beta,0})$$

While the left- and right-hand sides of $(D_{\alpha,\beta,0})$ have the same “homogeneity”, the structure of the solution set of $(D_{\alpha,\beta,0})$ strongly depends on the choice of the parameters α and β , see, e.g., [6–8, 26, 29], where the existence, multiplicity, and behavior of solutions of $(D_{\alpha,\beta,0})$ have been comprehensively studied. In fact, depending on α and β , $(D_{\alpha,\beta,0})$ demonstrates a behavior similar to the one for problems with convex–concave nonlinearities [3] and indefinite nonlinearities [1, 16], but with a more essential inclination to eigenvalue problems.

Let us recall several results from [6, 8] about the existence and nonexistence of positive solutions of $(D_{\alpha,\beta,0})$, some of which will be used in the subsequent sections. In [6], the following family of critical points was introduced:

$$\beta_{ps}(\alpha) := \sup_{u \in \text{int } C_0^1(\bar{\Omega})_+} \inf_{\varphi \in C_0^1(\bar{\Omega})_+ \setminus \{0\}} \mathcal{L}_\alpha(u; \varphi) \quad \text{for } \alpha \geq \lambda_1(p). \quad (1.3)$$

Here, $\mathcal{L}_\alpha(u; \varphi)$ is the so-called *extended functional* (see [17]) defined as

$$\mathcal{L}_\alpha(u; \varphi) := \frac{\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_\Omega |\nabla u|^{q-2} \nabla u \nabla \varphi \, dx - \alpha \int_\Omega |u|^{p-2} u \varphi \, dx}{\int_\Omega |u|^{q-2} u \varphi \, dx},$$

and $\text{int } C_0^1(\bar{\Omega})_+$ is the interior of the positive cone $C_0^1(\bar{\Omega})_+$ of $C_0^1(\bar{\Omega})$ given by

$$\text{int } C_0^1(\bar{\Omega})_+ := \left\{ u \in C_0^1(\bar{\Omega}) : u(x) > 0 \text{ for all } x \in \Omega, \quad \frac{\partial u}{\partial \nu}(x) < 0 \text{ for all } x \in \partial\Omega \right\},$$

where ν is the outward unit normal vector to $\partial\Omega$. It was proved in [6, Theorem 2.2] that $\beta_{ps}(\alpha)$ is the threshold curve on the (α, β) -plane which separates sets of the existence and nonexistence of positive solutions of $(D_{\alpha,\beta,0})$. Namely, if $\alpha > \lambda_1(p)$, then $(D_{\alpha,\beta,0})$ has a positive solution when $\beta < \beta_{ps}(\alpha)$, and $(D_{\alpha,\beta,0})$ does not have positive solutions when $\beta > \beta_{ps}(\alpha)$. Moreover, if $\alpha < \lambda_1(p)$, then $(D_{\alpha,\beta,0})$ possesses a solution if and only if $\beta > \lambda_1(q)$, see [6, Propositions 1 and 2]. Borderline cases were also studied, see, in particular, [6, Proposition 4] and [8, Proposition 3]. As for the properties of the curve $\beta_{ps}(\alpha)$, it is known that $\beta_{ps}(\lambda_1(p)) \geq \beta_*$, $\beta_{ps}(\alpha)$ is nonincreasing, and $\beta_{ps}(\alpha) = \lambda_1(q)$ for all $\alpha \geq \alpha_*$, where

$$\alpha_* := \frac{\|\nabla \varphi_q\|_p^p}{\|\varphi_q\|_p^p}, \quad \beta_* := \frac{\|\nabla \varphi_p\|_q^q}{\|\varphi_p\|_q^q}, \quad (1.4)$$

see [6, Proposition 3] and [8, Proposition 3 (ii)]. Notice that $\alpha_* > \lambda_1(p)$ and $\beta_* > \lambda_1(q)$, since $\lambda_1(p)$ and $\lambda_1(q)$ are simple and have different eigenspaces, see [8, Proposition 13].

We also refer the interested reader to the works [5, 10, 18, 24] for existence results and qualitative properties of solutions of other types of problems driven by the (p, q) -Laplace operator.

2 Main results

In this section, we collect our main results. We group them according to the existence of solutions of $(D_{\alpha,\beta,f})$ and their qualitative properties.

Hereinafter, we will use the notations

$$H_\alpha(u) := \|\nabla u\|_p^p - \alpha \|u\|_p^p \quad \text{and} \quad G_\beta(u) := \|\nabla u\|_q^q - \beta \|u\|_q^q,$$

and

$$\langle f, u \rangle := \int_{\Omega} f u \, dx$$

for the dual pairing between $f \in W^{-1,p'}(\Omega)$ and $u \in W_0^{1,p}$.

2.1 Existence

Theorem 2.1 *Let $f \in W^{-1,p'}(\Omega)$. Assume that $\alpha, \beta \in \mathbb{R}$ are such that one of the following assumptions is satisfied:*

- (i) $G_{\beta}(u) < 0$ for all $u \in ES(p; \alpha) \setminus \{0\}$;
- (ii) $G_{\beta}(u) > 0$ for all $u \in ES(p; \alpha) \setminus \{0\}$.

Then, $(D_{\alpha,\beta,f})$ has at least one solution.

In order to quantify assumptions (i) and (ii) of Theorem 2.1, we introduce the following two families of critical values:

$$\underline{\beta}(\alpha) := \inf \left\{ \frac{\|\nabla u\|_q^q}{\|u\|_q^q} : u \in ES(p; \alpha) \setminus \{0\} \right\}, \quad (2.1)$$

$$\bar{\beta}(\alpha) := \sup \left\{ \frac{\|\nabla u\|_q^q}{\|u\|_q^q} : u \in ES(p; \alpha) \setminus \{0\} \right\}, \quad \alpha \in \mathbb{R}, \quad (2.2)$$

and set $\underline{\beta}(\alpha) = +\infty$ and $\bar{\beta}(\alpha) = -\infty$ for $\alpha \notin \sigma(-\Delta_p)$. Note that $\underline{\beta}(\alpha), \bar{\beta}(\alpha) \in [\lambda_1(q), +\infty)$ provided $\alpha \in \sigma(-\Delta_p)$, see [7, Lemma 3.6], and the simplicity of $\lambda_1(p)$ and $\lambda_1(q)$ yields

$$\underline{\beta}(\lambda_1(p)) = \bar{\beta}(\lambda_1(p)) = \beta_* > \lambda_1(q), \quad (2.3)$$

where β_* is defined in (1.4) and the inequality follows from [8, Proposition 13]. Then, Theorem 2.1 can be reformulated as follows.

Theorem 2.2 *Let $f \in W^{-1,p'}(\Omega)$. Assume that $\alpha, \beta \in \mathbb{R}$ are such that either $\beta < \underline{\beta}(\alpha)$ or $\beta > \bar{\beta}(\alpha)$. Then, $(D_{\alpha,\beta,f})$ has at least one solution.*

The equalities in (2.3) show that if $\alpha = \lambda_1(p)$, then Theorem 2.1 does not provide the existence for $(D_{\alpha,\beta,f})$ only when $\beta = \beta_*$. We have the following result in this case.

Theorem 2.3 *Let $\partial\Omega$ be connected if $N \geq 2$. Assume that $p > 2q$, $\alpha = \lambda_1(p)$ and $\beta = \beta_*$. If $f \in L^2(\Omega)$ is such that $\int_{\Omega} f \varphi_p \, dx = 0$, then $(D_{\alpha,\beta,f})$ has at least one solution.*

Recalling $p > q$, let us remark that the term $-\Delta_q u - \beta|u|^{q-2}u$ in the problem $(D_{\alpha,\beta,f})$ can be considered as a specific case of a lower-order perturbation of the p -Laplacian, and hence some results stated in this section have to be typical for a more general setting, too. In particular, the (p, q) -Laplacian can be seen as a $(p-1)$ -quasihomogeneous operator in the sense of [21, Definition 2.1, p. 58], and hence, [21, Theorem 3.2, p. 73] implies the existence of a solution of $(D_{\alpha,\beta,f})$ whenever $f \in W^{-1,p'}(\Omega)$ and $\alpha \notin \sigma(-\Delta_p)$. Thus, our Theorems 2.1 and 2.3 provide an improvement of this existence result.

2.2 Sign properties

Let us start with the following two remarks about a regularity of solutions of $(D_{\alpha,\beta,f})$ and further properties of nonnegative solutions of $(D_{\alpha,\beta,f})$.

Remark 2.4 Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded, $\{f_n\} \subset L^\infty(\Omega)$ is such that $\{\|f_n\|_\infty\}$ is bounded, and $\{c_n\}$ is nonnegative and bounded. If u_n is a (weak) solution of

$$\begin{cases} -\Delta_p u - c_n \Delta_q u = \alpha_n |u|^{p-2} u + \beta_n |u|^{q-2} u + f_n(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\{u_n\}$ is bounded in $W_0^{1,p}$, then there exist $\gamma \in (0, 1)$ and $M > 0$ independent of $n \in \mathbb{N}$ such that $u_n \in C^{1,\gamma}(\overline{\Omega})$ and $\|u_n\|_{C^{1,\gamma}(\overline{\Omega})} \leq M$ for all n . Indeed, thanks to the boundedness assumptions, Moser's iteration process (see, e.g., [25, Appendix C]) implies that $\{u_n\}$ is bounded in $L^\infty(\Omega)$, and then regularity results [22] and [23, Theorem 1.7] yield the boundedness of $\{u_n\}$ in $C^{1,\gamma}(\overline{\Omega})$. In particular, if $f \in L^\infty(\Omega)$, then any solution of $(D_{\alpha,\beta,f})$ belongs to $C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$. The same regularity holds true for any eigenfunction of the r -Laplacian, $r > 1$.

Remark 2.5 If $f \in L^\infty(\Omega)$, $f \geq 0$ a.e. in Ω , and $u \geq 0$ is a nonzero nonnegative solution of $(D_{\alpha,\beta,f})$, then $u > 0$ in view of the maximum principle [27, Theorem 5.4.1]. Moreover, Remark 2.4 and the boundary point lemma [27, Theorem 5.5.1] yield $u \in \text{int } C_0^1(\overline{\Omega})_+$. Analogously, $\varphi_r \in \text{int } C_0^1(\overline{\Omega})_+$, where φ_r is the first (nonnegative) eigenfunction of the r -Laplacian, $r > 1$.

Assuming $f \in W^{-1,p'}(\Omega)$ and $f \geq 0$ in the weak sense, we introduce the following family of critical values:

$$\beta_f(\alpha) := \inf \{ \Phi_\alpha^+(u) : u \in \mathcal{B}^+(\alpha) \}, \quad \alpha \in \mathbb{R}, \quad (2.4)$$

where

$$\Phi_\alpha^+(u) := \frac{\|\nabla u\|_q^q}{\|u\|_q^q} + \frac{p-1}{p-q} \left(\frac{p-q}{q-1} \right)^{\frac{q-1}{p-1}} \frac{(H_\alpha(u))^{\frac{q-1}{p-1}} \langle f, u \rangle^{\frac{p-q}{p-1}}}{\|u\|_q^q}, \quad (2.5)$$

$$\mathcal{B}^+(\alpha) := \left\{ u \in W_0^{1,p} \setminus \{0\} : u \geq 0 \text{ a.e. in } \Omega \text{ and } H_\alpha(u) \geq 0 \right\}. \quad (2.6)$$

Clearly, $\beta_f(\alpha) \in [\lambda_1(q), +\infty)$ for any $\alpha \in \mathbb{R}$. In Lemma 5.4 below, we study some other properties of β_f . In particular, we show that $\beta_f(\alpha) > \lambda_1(q)$ if and only if $\alpha < \alpha_*$, provided $\langle f, \varphi_q \rangle > 0$, where α_* is defined in (1.4).

Theorem 2.6 Let $f \in W^{-1,p'}(\Omega) \setminus \{0\}$ and $f \geq 0$ in the weak sense. If $\alpha \leq \lambda_1(p)$ and $\beta < \beta_f(\alpha)$, then any solution of $(D_{\alpha,\beta,f})$ is nonnegative. If, in addition, $f \in L^\infty(\Omega)$, then any solution of $(D_{\alpha,\beta,f})$ belongs to $\text{int } C_0^1(\overline{\Omega})_+$.

The result of Theorem 2.6 can be complemented by the following dichotomy.

Proposition 2.7 Let $f \in L^\infty(\Omega) \setminus \{0\}$ and $f \geq 0$ a.e. in Ω . Then, for every $\beta < \beta_f(\lambda_1(p))$ there exists $\delta(\beta) > 0$ such that for all $\alpha \in (\lambda_1(p), \lambda_1(p) + \delta(\beta))$ any solution of $(D_{\alpha,\beta,f})$ is either positive or negative. In particular, $(D_{\alpha,\beta,f})$ has no sign-changing solutions.

By means of the critical curve $\beta_{ps}(\alpha)$ defined in (1.3), we give the following fact.

Proposition 2.8 Let $f \in L^\infty(\Omega) \setminus \{0\}$ and $f \geq 0$ a.e. in Ω . Assume that $\alpha \geq \lambda_1(p)$ and $\beta > \beta_{ps}(\alpha)$. Then, $(D_{\alpha,\beta,f})$ has no nonnegative solutions. That is, any solution of $(D_{\alpha,\beta,f})$ is either nonpositive or sign-changing.

Recall that $\beta_{ps}(\alpha) = \lambda_1(q)$ for all $\alpha \geq \alpha_*$. In this case, Proposition 2.8 can be refined as follows.

3 Proofs. Existence I

In this section, we prove Theorem 2.1. We start by preparing several auxiliary results. We will use the sequence of eigenvalues $\{\lambda_k(p)\} \subset \sigma(-\Delta_p)$ introduced in [14] which can be defined as

$$\lambda_k(p) := \inf_{h \in \mathcal{F}_k(p)} \max_{x \in S^{k-1}} \|\nabla h(x)\|_p^p, \quad (3.1)$$

where S^{k-1} is the unit sphere in $\mathbb{R}^k$, $k \in \mathbb{N}$, and

$$\begin{aligned} \mathcal{F}_k(p) &:= \left\{ h \in C(S^{k-1}, S(p)) : h \text{ is odd} \right\}, \\ S(p) &:= \left\{ u \in W_0^{1,p} : \|u\|_p = 1 \right\}. \end{aligned} \quad (3.2)$$

It is known that $\lambda_k(p) \rightarrow +\infty$ as $k \rightarrow +\infty$, see [14, p. 195]. However, we recall that it is an open problem whether $\{\lambda_k(p)\} = \sigma(-\Delta_p)$, except in the cases $p = 2$ and $N = 1$, where the answer is affirmative.

Along this section, we assume that $f \in W^{-1,p'}(\Omega)$, and we denote by $\|f\|_*$ the norm of f in $W^{-1,p'}(\Omega)$. Recall that weak solutions of $(D_{\alpha,\beta}, f)$ are critical points of the energy functional $E_{\alpha,\beta} \in C^1(W_0^{1,p}, \mathbb{R})$ defined by

$$E_{\alpha,\beta}(u) = \frac{1}{p} H_\alpha(u) + \frac{1}{q} G_\beta(u) - \langle f, u \rangle.$$

To prove Theorem 2.1, we show that $E_{\alpha,\beta}$ has a linking structure provided $\alpha > \lambda_1(p)$ and $\alpha \neq \lambda_k(p)$, $k \in \mathbb{N}$. Then, we can obtain a critical point of $E_{\alpha,\beta}$ whenever $E_{\alpha,\beta}$ satisfies the Palais–Smale condition.

Let us consider the set

$$Y(\lambda) := \left\{ u \in W_0^{1,p} : \|\nabla u\|_p^p \geq \lambda \|u\|_p^p \right\}, \quad \lambda \in \mathbb{R}. \quad (3.3)$$

Hereinafter, S_+^k stands for the closed unit upper hemisphere in $\mathbb{R}^{k+1}$ with the boundary S^{k-1} . We start by formulating the following linking lemma.

Lemma 3.1 ([7, Lemma 3.1]) *Let $k \in \mathbb{N}$. Then, $h(S_+^k) \cap Y(\lambda_{k+1}(p)) \neq \emptyset$ for any $h \in C(S_+^k, W_0^{1,p})$ provided $h|_{S^{k-1}}$ is odd.*

Lemma 3.2 *Let $\alpha, \beta \in \mathbb{R}$. If $\lambda > \max\{\alpha, 0\}$, then $E_{\alpha,\beta}$ is bounded from below and coercive on $Y(\lambda)$.*

Proof Let $u \in Y(\lambda)$. We have

$$E_{\alpha,\beta}(u) \geq \frac{1}{p} \left(1 - \frac{\alpha}{\lambda} \right) \|\nabla u\|_p^p - \frac{\beta}{q(\lambda_1(p))^{q/p}} |\Omega|^{\frac{p-q}{p}} \|\nabla u\|_p^q - \|f\|_* \|\nabla u\|_p,$$

where we used the Hölder and Poincaré inequalities to estimate the term $\|u\|_q^q$ of $E_{\alpha,\beta}(u)$. Since $p > q > 1$ and $\lambda > \max\{\alpha, 0\}$, we easily deduce the desired boundedness from below and coercivity of $E_{\alpha,\beta}$ on $Y(\lambda)$. $\square$

Lemma 3.3 *Let $\alpha, \beta \in \mathbb{R}$ be such that*

$$G_\beta(u) \neq 0 \text{ for any } u \in ES(p; \alpha) \setminus \{0\}. \quad (3.4)$$

Then, $E_{\alpha,\beta}$ satisfies the Palais–Smale condition.

Proof Let $\{u_n\} \subset W_0^{1,p}$ be a Palais–Smale sequence for $E_{\alpha,\beta}$. In view of the (S_+) -property of the operator $-\Delta_p - \Delta_q$ (see, e.g., [7, Remark 3.5]), it is sufficient to show the boundedness of $\{u_n\}$ in order to establish the desired Palais–Smale condition for $E_{\alpha,\beta}$. Suppose, by contradiction, that $\|\nabla u_n\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$, up to a subsequence. Then, arguing in the same way as in [7, Lemma 3.3], we see that the sequence of normalized functions $v_n := u_n / \|\nabla u_n\|_p$ converges strongly in $W_0^{1,p}$ (up to a subsequence) to some $v_0 \in ES(p; \alpha) \setminus \{0\}$. Hence, we get a contradiction whenever $\alpha \notin \sigma(-\Delta_p)$. On the other hand, if $\alpha \in \sigma(-\Delta_p)$, then

$$o(1) = \frac{1}{\|\nabla u_n\|_p^q} \left(p E_{\alpha,\beta}(u_n) - \langle E'_{\alpha,\beta}(u_n), u_n \rangle \right) = \left(\frac{p}{q} - 1 \right) G_\beta(v_n) - \frac{p-1}{\|\nabla u_n\|_p^{q-1}} \langle f, v_n \rangle$$

as $n \rightarrow +\infty$. This yields $G_\beta(v_0) = 0$, which contradicts (3.4). $\square$

We are now in a position to prove Theorem 2.1. The proof will be split into three cases, each of which is considered in a separate subsection.

3.1 Case $\alpha \notin \{\lambda_k(p) : k \in \mathbb{N}\}$

First, we briefly handle the case $\alpha < \lambda_1(p)$. Since $Y(\lambda_1(p)) = W_0^{1,p}$, where $Y(\lambda_1(p))$ is given by (3.3), Lemma 3.2 implies that $E_{\alpha,\beta}$ is bounded from below and coercive on $W_0^{1,p}$. Therefore, there exists a global minimizer of $E_{\alpha,\beta}$ which is a solution of $(D_{\alpha,\beta,f})$.

Assume now that $\alpha > \lambda_1(p)$ and $\alpha \notin \{\lambda_k(p) : k \in \mathbb{N}\}$. (Recall that the case $\alpha \in \sigma(-\Delta_p)$ is still possible.) Since $\lambda_k(p) \rightarrow +\infty$ as $k \rightarrow +\infty$, we can find $k \in \mathbb{N}$ such that $\lambda_k(p) < \alpha < \lambda_{k+1}(p)$. Define

$$\begin{aligned} \gamma &:= \inf \{ E_{\alpha,\beta}(u) : u \in Y(\lambda_{k+1}(p)) \} > -\infty, \\ \Lambda &:= \left\{ h \in C(S_+^k, W_0^{1,p}) : \max_{x \in S^{k-1}} E_{\alpha,\beta}(h(x)) \leq \gamma - 1 \text{ and } h|_{S^{k-1}} \text{ is odd} \right\}, \\ c &:= \inf_{h \in \Lambda} \max_{x \in S_+^k} E_{\alpha,\beta}(h(x)), \end{aligned}$$

where γ is well defined by Lemma 3.2. Note that if $\Lambda \neq \emptyset$, then Lemma 3.1 implies $c \geq \gamma$. Moreover, under assumption (i) or (ii) of the theorem, $E_{\alpha,\beta}$ satisfies the Palais–Smale condition by Lemma 3.3. Therefore, by the standard arguments via the deformation lemma we see that c is a critical value of $E_{\alpha,\beta}$ provided $\Lambda \neq \emptyset$. Thus, let us construct an admissible map to show that $\Lambda \neq \emptyset$.

Let us choose some $0 < \varepsilon < (\alpha - \lambda_k(p))/2$. Then, due to the definition of $\lambda_k(p)$, there exists $h_0 \in \mathcal{F}_k(p)$ such that

$$\max_{x \in S^{k-1}} \|\nabla h_0(x)\|_p^p < \lambda_k(p) + \varepsilon.$$

This implies

$$\max_{x \in S^{k-1}} H_\alpha(h_0(x)) < \lambda_k(p) + \varepsilon - \alpha < -\varepsilon. \quad (3.5)$$

Therefore, recalling that $p > q > 1$, we can find a sufficiently large $T_0 > 0$ such that

$$\max_{x \in S^{k-1}} E_{\alpha,\beta}(Th_0(x)) < -\frac{T^p \varepsilon}{p} + \frac{T^q}{q} \max_{x \in S^{k-1}} G_\beta(h_0(x)) + T \|f\|_* (\lambda_k(p) + \varepsilon)^{1/p} \leq \gamma - 1 \quad (3.6)$$

for all $T \geq T_0$. Using Dugundji's extension theorem [15], we may assume that $h_0 \in C(S_+^k, W_0^{1,p})$, and hence $T_0 h_0 \in \Lambda$. Thus, $\Lambda \neq \emptyset$, which implies that c is a critical value of $E_{\alpha,\beta}$. $\square$

3.2 Case $\alpha \in \{\lambda_k(p) : k \in \mathbb{N}\}$ under assumption (i)

We may assume that $\lambda_k(p) = \alpha < \lambda_{k+1}(p)$ for some $k \in \mathbb{N}$. Let $\{\alpha_n\} \subset \mathbb{R}$ be a decreasing sequence satisfying

$$\lambda_k(p) = \alpha < \alpha_n < \lambda_{k+1}(p) \quad \text{for all } n \in \mathbb{N}, \text{ and } \lim_{n \rightarrow +\infty} \alpha_n = \alpha.$$

Similar to Sect. 3.1 above, consider for each $n \in \mathbb{N}$,

$$\begin{aligned} \gamma_n &:= \inf\{E_{\alpha_n,\beta}(u) : u \in Y(\lambda_{k+1}(p))\} > -\infty, \\ \Lambda_n &:= \left\{ h \in C(S_+^k, W_0^{1,p}) : \max_{x \in S^{k-1}} E_{\alpha_n,\beta}(h(x)) \leq \gamma_n - 1 \text{ and } h|_{S^{k-1}} \text{ is odd} \right\}, \\ c_n &:= \inf_{h \in \Lambda_n} \max_{x \in S_+^k} E_{\alpha_n,\beta}(h(x)). \end{aligned} \quad (3.7)$$

Arguing as in Sect. 3.1, we get $\Lambda_n \neq \emptyset$, and hence, Lemma 3.1 yields $c_n \geq \gamma_n$. Moreover, since $\{\alpha_n\}$ is a decreasing sequence, we have $H_{\alpha_n}(u) \geq H_{\alpha_1}(u)$ for any $n \in \mathbb{N}$ and $u \in W_0^{1,p}$, whence

$$c_n \geq \gamma_n \geq \inf\{E_{\alpha_1,\beta}(u) : u \in Y(\lambda_{k+1}(p))\} > -\infty, \quad (3.8)$$

where the last inequality follows from Lemma 3.2. That is, $\{c_n\}$ is bounded from below.

Now, we claim that for any $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $u_n^\varepsilon \in W_0^{1,p}$ such that

$$|E_{\alpha_n,\beta}(u_n^\varepsilon) - c_n| < \varepsilon \quad \text{and} \quad \|E'_{\alpha_n,\beta}(u_n^\varepsilon)\|_* < \varepsilon. \quad (3.9)$$

Suppose, by contradiction, that there exist some $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$u \in E_{\alpha_n,\beta}^{-1}([c_n - \varepsilon, c_n + \varepsilon]) \quad \text{implies} \quad \|E'_{\alpha_n,\beta}(u)\|_* \geq \varepsilon.$$

(Clearly, the set $E_{\alpha_n,\beta}^{-1}([c_n - \varepsilon, c_n + \varepsilon])$ is nonempty.) Let us fix some $\varepsilon' \in (0, \min\{\varepsilon, c_n - (\gamma_n - 1)\})$. Then, using the deformation lemma (see, e.g., [9, Theorem 3.4 of Chapter I]), we can find a map $\eta \in C(W_0^{1,p}, W_0^{1,p})$ which satisfies the following two assertions:

- (a) $\eta(u) = u$ for any u such that $u \notin E_{\alpha_n,\beta}^{-1}([c_n - \varepsilon', c_n + \varepsilon'])$;
- (b) $E_{\alpha_n,\beta}(\eta(u)) \leq c_n - \varepsilon'/2$ whenever $E_{\alpha_n,\beta}(u) \leq c_n + \varepsilon'/2$.

By the definition (3.7) of c_n , there exists $h \in \Lambda_n$ such that $\max_{x \in S_+^k} E_{\alpha_n,\beta}(h(x)) \leq c_n + \varepsilon'/2$.

Since $h \in \Lambda_n$, we have $\max_{x \in S^{k-1}} E_{\alpha_n,\beta}(h(x)) \leq \gamma_n - 1$. Thus, in view of the inequality $\gamma_n - 1 < c_n - \varepsilon'$, Assertion (a) implies that $\eta(h(x)) = h(x)$ for all $x \in S^{k-1}$, and hence $\eta \circ h \in \Lambda_n$. On the other hand, Assertion (b) yields $\max_{x \in S_+^k} E_{\alpha_n,\beta}(\eta(h(x))) \leq c_n - \varepsilon'/2$, which

contradicts the definition of c_n . Consequently, the existence of $\{u_n^\varepsilon\} \subset W_0^{1,p}$ satisfying (3.9) is shown.

In view of (3.9), we can find for each $n \in \mathbb{N}$ a function $u_n \in W_0^{1,p}$ such that

$$|E_{\alpha_n,\beta}(u_n) - c_n| < \frac{1}{n} \quad \text{and} \quad \|E'_{\alpha_n,\beta}(u_n)\|_* < \frac{1}{n}. \quad (3.10)$$

If $\|\nabla u_n\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$, up to a subsequence, then by the same arguments as in the proof of [7, Lemma 3.3] the second inequality in (3.10) implies that the sequence of normalized functions $v_n := u_n / \|\nabla u_n\|_p$ has a subsequence strongly convergent in $W_0^{1,p}$ to some $v_0 \in ES(p; \alpha) \setminus \{0\}$. Moreover, passing to the limit along this subsequence in

$$\begin{aligned} \frac{p(c_n - 1)}{\|\nabla u_n\|_p^q} - \frac{1}{n\|\nabla u_n\|_p^{q-1}} &\leq \frac{1}{\|\nabla u_n\|_p^q} \left(pE_{\alpha_n, \beta}(u_n) - \langle E'_{\alpha_n, \beta}(u_n), u_n \rangle \right) \\ &= \left(\frac{p}{q} - 1 \right) G_\beta(v_n) - \frac{p-1}{\|\nabla u_n\|_p^{q-1}} \langle f, v_n \rangle \end{aligned}$$

and recalling that $\{c_n\}$ is bounded from below [see (3.8)], we get $G_\beta(v_0) \geq 0$. However, this contradicts assumption (i) of the theorem. Therefore, $\{u_n\}$ is bounded in $W_0^{1,p}$ and, consequently, $\{c_n\}$ is also bounded. Furthermore, noting that

$$\|E'_{\alpha, \beta}(u_n)\|_* \leq \|E'_{\alpha, \beta}(u_n) - E'_{\alpha_n, \beta}(u_n)\|_* + \|E'_{\alpha_n, \beta}(u_n)\|_* \leq \frac{\alpha_n - \alpha}{\lambda_1(p)^{1/p}} \|u_n\|_p^{p-1} + \frac{1}{n},$$

we obtain from (3.10) that $\{u_n\}$ is a (bounded) Palais–Smale sequence for $E_{\alpha, \beta}$. Consequently, $E_{\alpha, \beta}$ possesses a critical point since it satisfies the Palais–Smale condition by Lemma 3.3.

3.3 Case $\alpha \in \{\lambda_k(p) : k \in \mathbb{N}\}$ under assumption (ii)

We may assume that $\lambda_k(p) < \alpha = \lambda_{k+1}(p)$ for some $k \in \mathbb{N}$. Let $\{\alpha_n\}$ be an increasing sequence satisfying

$$\lambda_k(p) < \alpha_n < \alpha = \lambda_{k+1}(p) \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \rightarrow +\infty} \alpha_n = \alpha.$$

Let us define γ_n , Λ_n , and c_n as in Sect. 3.2 above. By the same arguments as in Sect. 3.2, for any $n \in \mathbb{N}$ we can find $u_n \in W_0^{1,p}$ such that

$$|E_{\alpha_n, \beta}(u_n) - c_n| < \frac{1}{n} \quad \text{and} \quad \|E'_{\alpha_n, \beta}(u_n)\|_* < \frac{1}{n}.$$

We claim that $\{c_n\}$ is bounded from above. If the claim is true, then, as in Sect. 3.2, the inequality

$$\begin{aligned} \frac{p(c_n + 1)}{\|\nabla u_n\|_p^q} + \frac{1}{n\|\nabla u_n\|_p^{q-1}} &\geq \frac{1}{\|\nabla u_n\|_p^q} \left(pE_{\alpha_n, \beta}(u_n) - \langle E'_{\alpha_n, \beta}(u_n), u_n \rangle \right) \\ &= \left(\frac{p}{q} - 1 \right) G_\beta(v_n) - \frac{p-1}{\|\nabla u_n\|_p^{q-1}} \langle f, v_n \rangle, \end{aligned}$$

in combination with assumption (ii) of the theorem, implies the boundedness of $\{u_n\}$ in $W_0^{1,p}$, which yields the existence of a critical point of $E_{\alpha, \beta}$.

Let us prove that $\{c_n\}$ is bounded from above. Note that $\Lambda_n \neq \emptyset$ for any $n \in \mathbb{N}$ by the same arguments as in Sect. 3.1. In particular, for $n = 1$ we can find a map $h_1 \in C(S_+^k, W_0^{1,p})$ and a sufficiently large $T_1 > 0$ such that $h_1|_{S^{k-1}}$ is odd, and

$$\max_{x \in S^{k-1}} E_{\alpha_1, \beta}(Th_1(x)) \leq \gamma_1 - 1 \quad \text{for any } T \geq T_1, \quad (3.11)$$

$$\max_{x \in S^{k-1}} E_{\alpha_1, \beta}(Th_1(x)) \rightarrow -\infty \quad \text{as } T \rightarrow +\infty, \quad (3.12)$$

see (3.6). Moreover, recalling that $\{\alpha_n\}$ is increasing, we have $H_{\alpha_n}(u) \leq H_{\alpha_1}(u)$ and hence $E_{\alpha_n, \beta}(u) \leq E_{\alpha_1, \beta}(u)$ for all $n \in \mathbb{N}$ and $u \in W_0^{1,p}$. Therefore, in view of (3.12), for any $n \in \mathbb{N}$ there exists a sufficiently large $T_n > 0$ such that

$$\max_{x \in S^{k-1}} E_{\alpha_n, \beta}(T_n h_1(x)) \leq \max_{x \in S^{k-1}} E_{\alpha_1, \beta}(T_n h_1(x)) \leq \gamma_n - 1. \quad (3.13)$$

We may assume that $\{T_n\}$ is increasing.

Define a map $h_n \in C(S_+^k, W_0^{1,p})$ for $n \geq 2$ by

$$h_n(x) := \begin{cases} (2x_{k+1}T_1 + T_n(1 - 2x_{k+1})) h_1\left(\frac{x'}{\sqrt{1-x_{k+1}^2}}, 0\right) & \text{if } 0 \leq x_{k+1} \leq \frac{1}{2}, \\ T_1 h_1\left(\frac{2}{\sqrt{3}}x', \frac{2}{\sqrt{3}}\sqrt{x_{k+1}^2 - \frac{1}{4}}\right) & \text{if } \frac{1}{2} \leq x_{k+1} \leq 1, \end{cases}$$

where $x = (x_1, \dots, x_{k+1}) \in S_+^k$ and $x' = (x_1, \dots, x_k)$. Considering $x_{k+1} = 0$, we deduce from (3.13) that $h_n \in \Lambda_n$. Moreover, if $0 \leq x_{k+1} \leq \frac{1}{2}$, then

$$E_{\alpha_n, \beta}(h_n(x)) \leq E_{\alpha_1, \beta}(h_n(x)) \leq \max_{T_1 \leq T \leq T_n} \max_{x \in S^{k-1}} E_{\alpha_1, \beta}(Th_1(x)) \leq \gamma_1 - 1$$

by (3.11), and if $\frac{1}{2} \leq x_{k+1} \leq 1$, then

$$E_{\alpha_n, \beta}(h_n(x)) \leq \max_{x \in S_+^k} E_{\alpha_n, \beta}(T_1 h_1(x)) \leq \max_{x \in S_+^k} E_{\alpha_1, \beta}(T_1 h_1(x)) < +\infty.$$

Therefore, since $c_n \leq \max_{x \in S_+^k} E_{\alpha_n, \beta}(h_n(x))$ for each $n \in \mathbb{N}$, we conclude that $\{c_n\}$ is bounded from above, which finishes the proof of Theorem 2.1.

4 Proofs. Existence II

In this section, we prove Theorem 2.3. First, we provide the following auxiliary result. Let us decompose any $u \in L^2(\Omega)$ as $u = \gamma_u \varphi_p + u^\perp$, where φ_p is the first eigenfunction of the p -Laplacian,

$$\gamma_u := \|\varphi_p\|_2^{-2} \int_{\Omega} u \varphi_p \, dx \quad \text{and} \quad \int_{\Omega} u^\perp \varphi_p \, dx = 0. \quad (4.1)$$

It is clear that $u^\perp = u - \gamma_u \varphi_p \in W_0^{1,p}$ provided $u \in W_0^{1,p}$.

Lemma 4.1 *Let $p \geq 2$. Assume that $\beta \in (\lambda_1(q), \beta_*]$. Then, there exists a constant $C > 0$ such that for any $u \in W_0^{1,p}$ satisfying $G_\beta(u) < 0$ there holds*

$$|G_\beta(u)| \leq C |\gamma_u|^{q-1} \left(\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u^\perp|^2 \, dx \right)^{\frac{1}{2}} + C \int_{\Omega} |\nabla u^\perp|^q \, dx. \quad (4.2)$$

If, in addition, $p \geq 2q$ and $\gamma_u \neq 0$, then the following inequality is satisfied:

$$\begin{aligned} |G_\beta(u)| &\leq C |\gamma_u|^{-\frac{p-2q}{2}} \left(|\gamma_u|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u^\perp|^2 \, dx + \int_{\Omega} |\nabla u^\perp|^p \, dx \right)^{\frac{1}{2}} \\ &\quad + C \int_{\Omega} |u^\perp|^q \, dx. \end{aligned} \quad (4.3)$$

Proof Let us fix any $u = \gamma_u \varphi_p + u^\perp \in W_0^{1,p}$. Using the mean value theorem, we can find $\varepsilon \in (0, 1)$ such that

$$\begin{aligned} 0 > G_\beta(u) &= |\gamma_u|^q G_\beta(\varphi_p) + \langle G'_\beta(\gamma_u \varphi_p + \varepsilon u^\perp), u^\perp \rangle \geq \langle G'_\beta(\gamma_u \varphi_p + \varepsilon u^\perp), u^\perp \rangle \\ &\geq -q \int_\Omega |\nabla(\gamma_u \varphi_p + \varepsilon u^\perp)|^{q-1} |\nabla u^\perp| \, dx - \beta q \int_\Omega |\gamma_u \varphi_p + \varepsilon u^\perp|^{q-1} |u^\perp| \, dx, \end{aligned} \quad (4.4)$$

where $G_\beta(\varphi_p) \geq 0$ follows from the assumption $\beta \in (\lambda_1(q), \beta_*]$.

During the proof, we will denote by $C > 0$ various constants independent of u . To estimate (4.4), we develop an approach from the proof of [8, Proposition 11]. Let us start with the first summand in (4.4). Since $\varepsilon \in (0, 1)$, we have

$$\int_\Omega |\nabla(\gamma_u \varphi_p + \varepsilon u^\perp)|^{q-1} |\nabla u^\perp| \, dx \leq \int_\Omega \left(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp| \right)^{q-1} |\nabla u^\perp| \, dx.$$

Then, using Hölder's inequality, we get

$$\begin{aligned} &\int_\Omega \left(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp| \right)^{q-1} |\nabla u^\perp| \, dx \\ &\leq C |\gamma_u|^{q-1} \int_\Omega |\nabla \varphi_p|^{q-1} |\nabla u^\perp| \, dx + C \int_\Omega |\nabla u^\perp|^q \, dx \\ &= C |\gamma_u|^{q-1} \int_\Omega |\nabla \varphi_p|^{\frac{p-2}{2}} |\nabla u^\perp| \cdot |\nabla \varphi_p|^{\frac{2q-p}{2}} \, dx + C \int_\Omega |\nabla u^\perp|^q \, dx \\ &\leq C |\gamma_u|^{q-1} \left(\int_\Omega |\nabla \varphi_p|^{p-2} |\nabla u^\perp|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \frac{dx}{|\nabla \varphi_p|^{p-2q}} \right)^{\frac{1}{2}} + C \int_\Omega |\nabla u^\perp|^q \, dx. \end{aligned} \quad (4.5)$$

Note that $\int_\Omega |\nabla \varphi_p|^{p-2} |\nabla u^\perp|^2 \, dx < +\infty$, since $\varphi_p \in C^1(\overline{\Omega})$ (see Remark 2.4) and $p \geq 2$. Moreover, $\int_\Omega \frac{dx}{|\nabla \varphi_p|^{p-2q}} < +\infty$, too. Indeed, if $p \leq 2q$, then this boundedness easily follows from the regularity $\varphi_p \in C^1(\overline{\Omega})$, while in the case $p > 2q$ the desired boundedness was discussed in [8, p. 1234].

On the other hand, if $p \geq 2q$ and $\gamma_u \neq 0$, then, recalling that $\varepsilon \in (0, 1)$ and using Hölder's inequality again, we can estimate the first summand in (4.4) as follows:

$$\begin{aligned} &\int_\Omega |\nabla(\gamma_u \varphi_p + \varepsilon u^\perp)|^{q-1} |\nabla u^\perp| \, dx \leq \int_\Omega \left(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp| \right)^{q-1} |\nabla u^\perp| \, dx \\ &= \int_\Omega \left(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp| \right)^{\frac{p-2}{2}} |\nabla u^\perp| \cdot \left(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp| \right)^{\frac{2q-p}{2}} \, dx \\ &\leq \left(\int_\Omega \left(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp| \right)^{p-2} |\nabla u^\perp|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \frac{dx}{(|\gamma_u| |\nabla \varphi_p| + |\nabla u^\perp|)^{p-2q}} \right)^{\frac{1}{2}} \\ &\leq C \left(|\gamma_u|^{p-2} \int_\Omega |\nabla \varphi_p|^{p-2} |\nabla u^\perp|^2 \, dx + \int_\Omega |\nabla u^\perp|^p \, dx \right)^{\frac{1}{2}} \frac{1}{|\gamma_u|^{\frac{p-2q}{2}}} \left(\int_\Omega \frac{dx}{|\nabla \varphi_p|^{p-2q}} \right)^{\frac{1}{2}}. \end{aligned} \quad (4.6)$$

Let us now estimate the second term in (4.4). Since $\varepsilon \in (0, 1)$ and $\|\varphi_p\|_\infty < +\infty$ (see Remark 2.4), we have

$$\int_{\Omega} |\gamma_u \varphi_p + \varepsilon u^\perp|^{q-1} |u^\perp| \, dx \leq C |\gamma_u|^{q-1} \int_{\Omega} |u^\perp| \, dx + C \int_{\Omega} |u^\perp|^q \, dx.$$

Thus, using Hölder's inequality and an embedding result of [20, Lemma 4.2], we get

$$\begin{aligned} \int_{\Omega} |\gamma_u \varphi_p + \varepsilon u^\perp|^{q-1} |u^\perp| \, dx &\leq C |\gamma_u|^{-\frac{p-2q}{2}} \left(|\gamma_u|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u^\perp|^2 \, dx \right)^{\frac{1}{2}} \\ &\quad + C \int_{\Omega} |u^\perp|^q \, dx. \end{aligned} \quad (4.7)$$

Finally, combining (4.7) with (4.6), we obtain (4.3). To obtain (4.2), we apply the Sobolev embedding theorem to estimate the last summand in (4.7), and then combine (4.7) with (4.5). $\square$

Proof of Theorem 2.3 We will show that, under the imposed assumptions, $E_{\lambda_1(p), \beta_*}$ attains a global minimum. If $f \equiv 0$, then the existence of a global minimizer is given by [8, Theorem 2.6 (ii)]. Thus, we may assume that $f \not\equiv 0$. Let $\{u_n\} \subset W_0^{1,p}$ be a minimizing sequence for $E_{\lambda_1(p), \beta_*}$. It is not hard to see that each $E_{\lambda_1(p), \beta_*}(u_n) < 0$. Indeed, fixing an arbitrary $u \in W_0^{1,p}$ satisfying $\langle f, u \rangle > 0$, and recalling that $p > q > 1$, we can find a sufficiently small $t > 0$ such that

$$E_{\lambda_1(p), \beta_*}(tu) = \frac{t^p}{p} H_{\lambda_1(p)}(u) + \frac{t^q}{q} G_{\beta_*}(u) - t \langle f, u \rangle < 0.$$

Let us prove that $\{u_n\}$ is bounded in $W_0^{1,p}$. Suppose, by contradiction, that $\|\nabla u_n\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$, up to a subsequence. Making the $L^2(\Omega)$ -decomposition $u_n = \gamma_n \varphi_p + u_n^\perp$ [see (4.1)], we conclude that $|\gamma_n| \rightarrow +\infty$ or $\|\nabla u_n^\perp\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$. To reach a contradiction, let us estimate $E_{\lambda_1(p), \beta_*}(u_n)$ from below. First, to estimate $H_{\lambda_1(p)}(u_n)$, we use the improved Poincaré inequality obtained in [20], which states that

$$H_{\lambda_1(p)}(u_n) \geq C |\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 \, dx + C \|\nabla u_n^\perp\|_p^p. \quad (4.8)$$

Hereinafter, $C > 0$ is a constant independent of $n \in \mathbb{N}$. Second, to estimate $\langle f, u_n \rangle$, we recall that $f \in L^2(\Omega)$, $\langle f, \varphi_p \rangle = 0$, and $p > 2q > 2$, which yields

$$|\langle f, u_n \rangle| = \left| \int_{\Omega} f u_n \, dx \right| = \left| \int_{\Omega} f u_n^\perp \, dx \right| \leq \|f\|_2 \|u_n^\perp\|_2 \leq C \|f\|_2 \|\nabla u_n^\perp\|_p. \quad (4.9)$$

Finally, in order to apply Lemma 4.1 for estimating $G_{\beta_*}(u_n)$, let us show that we may assume $G_{\beta_*}(u_n) < 0$ for all $n \in \mathbb{N}$. Indeed, if $G_{\beta_*}(u_n) \geq 0$ for all n , up to a subsequence, then, using the estimates (4.8) and (4.9), we get

$$E_{\lambda_1(p), \beta_*}(u_n) \geq C |\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 \, dx + C \|\nabla u_n^\perp\|_p^p - C \|f\|_2 \|\nabla u_n^\perp\|_p. \quad (4.10)$$

If $\|\nabla u_n^\perp\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$, then $E_{\lambda_1(p), \beta_*}(u_n) \rightarrow +\infty$ regardless the behavior of $\{\gamma_n\}$, which contradicts the minimization property of $\{u_n\}$. Hence, $\{u_n^\perp\}$ is bounded in $W_0^{1,p}$. Therefore, since $\|\nabla u_n\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$, we have $|\gamma_n| \rightarrow +\infty$. We see from (4.10) that if $|\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 \, dx \rightarrow +\infty$ as $n \rightarrow +\infty$, up to a subsequence, then $E_{\lambda_1(p), \beta_*}(u_n) \rightarrow +\infty$, which is again impossible. This implies

$\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \rightarrow 0$. In view of the embedding result [20, Lemma 4.2], we get $\|u_n^\perp\|_2 \rightarrow 0$, and hence, (4.9) yields $\langle f, u_n \rangle \rightarrow 0$. Thus, we conclude from (4.10) and the behavior of $\langle f, u_n \rangle$ that $E_{\lambda_1(p), \beta_*}(u_n) \geq 0 + o(1)$ as $n \rightarrow +\infty$, which contradicts the minimization property of $\{u_n\}$ and the fact that $E_{\lambda_1(p), \beta_*}(u_n) < 0$ for each $n \in \mathbb{N}$. Thus, $G_{\beta_*}(u_n) < 0$ for all $n \in \mathbb{N}$.

Substituting now (4.8), (4.9), and the estimate (4.2) for $G_{\beta_*}(u_n)$ (combined with Hölder's inequality) into $E_{\lambda_1(p), \beta_*}(u_n)$, we get

$$\begin{aligned} E_{\lambda_1(p), \beta_*}(u_n) &\geq C|\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx + C\|\nabla u_n^\perp\|_p^p \\ &\quad - C|\gamma_n|^{q-1} \left(\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \right)^{\frac{1}{2}} - C\|\nabla u_n^\perp\|_p^q - C\|f\|_2 \|\nabla u_n^\perp\|_p. \end{aligned} \quad (4.11)$$

Let us consider the following three possible cases.

1. $\|\nabla u_n^\perp\|_p \rightarrow +\infty$ and $|\gamma_n| \rightarrow +\infty$ as $n \rightarrow +\infty$. Using Young's inequality, we have

$$\begin{aligned} |\gamma_n|^{q-1} \left(\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \right)^{\frac{1}{2}} &= \varepsilon |\gamma_n|^{\frac{p-2}{2}} \left(\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \right)^{\frac{1}{2}} \cdot \frac{1}{\varepsilon} |\gamma_n|^{\frac{2q-p}{2}} \\ &\leq \varepsilon^2 |\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx + \frac{1}{\varepsilon^2} |\gamma_n|^{2q-p} \end{aligned} \quad (4.12)$$

for any $\varepsilon > 0$. Substituting (4.12) into (4.11), we obtain

$$\begin{aligned} E_{\lambda_1(p), \beta_*}(u_n) &\geq (C - C\varepsilon^2) |\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx + C\|\nabla u_n^\perp\|_p^p \\ &\quad - \frac{C}{\varepsilon^2} |\gamma_n|^{2q-p} - C\|\nabla u_n^\perp\|_p^q - C\|f\|_2 \|\nabla u_n^\perp\|_p. \end{aligned} \quad (4.13)$$

Taking $\varepsilon > 0$ small enough and recalling that $p > 2q$, we easily conclude that $E_{\lambda_1(p), \beta_*}(u_n) \rightarrow +\infty$ as $n \rightarrow +\infty$, which is impossible since $\{u_n\}$ is a minimizing sequence.

2. $\|\nabla u_n^\perp\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\{\gamma_n\}$ is bounded. Since $\varphi_p \in C^1(\overline{\Omega})$ (see Remark 2.4) and $p > 2$, we have

$$|\gamma_n|^{q-1} \left(\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \right)^{\frac{1}{2}} \leq C|\gamma_n|^{q-1} \|\nabla u_n^\perp\|_2 \leq C|\gamma_n|^{q-1} \|\nabla u_n^\perp\|_p,$$

and hence, recalling that $p > q > 1$, we obtain from (4.11) that

$$E_{\lambda_1(p), \beta_*}(u_n) \geq C\|\nabla u_n^\perp\|_p^p - C|\gamma_n|^{q-1} \|\nabla u_n^\perp\|_p - C\|\nabla u_n^\perp\|_p^q - C\|f\|_2 \|\nabla u_n^\perp\|_p \rightarrow +\infty$$

as $n \rightarrow +\infty$, a contradiction.

3. $\{\|\nabla u_n^\perp\|_p\}$ is bounded and $|\gamma_n| \rightarrow +\infty$ as $n \rightarrow +\infty$. In this case, $\{u_n^\perp\}$ converges weakly in $W_0^{1,p}$ and strongly in $L^r(\Omega)$, $r \in (1, p^*)$, to some $u_0^\perp \in W_0^{1,p}$, up to a subsequence. Recalling that $p > 2q$, we see from the estimate (4.13) that if $|\gamma_n|^{p-2} \int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \rightarrow +\infty$ as $n \rightarrow +\infty$, up to a subsequence, then $E_{\lambda_1(p), \beta_*}(u_n) \rightarrow +\infty$, which is impossible. This implies $\int_{\Omega} |\nabla \varphi_p|^{p-2} |\nabla u_n^\perp|^2 dx \rightarrow 0$. In view of the embedding result [20, Lemma 4.2], we get $\|u_n^\perp\|_2 \rightarrow 0$, which yields $u_0^\perp \equiv 0$ and hence $\|u_n^\perp\|_q \rightarrow 0$ due to the strong convergence of $\{u_n^\perp\}$ in $L^r(\Omega)$, $r \in (1, p^*)$. Consequently, we see from (4.3) and (4.9) that

$$|G_{\beta_*}(u_n)| \rightarrow 0 \quad \text{and} \quad |\langle f, u_n \rangle| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Thus,

$$E_{\lambda_1(p), \beta_*}(u_n) = H_{\lambda_1(p)}(u_n) + o(1) \geq o(1) \quad \text{as } n \rightarrow +\infty,$$

which contradicts the facts that each $E_{\lambda_1(p), \beta_*}(u_n) < 0$ and $\{u_n\}$ is a minimizing sequence for $E_{\lambda_1(p), \beta_*}$.

Therefore, we conclude that $\{u_n\}$ is bounded in $W_0^{1,p}$, and hence $\inf_{W_0^{1,p}} E_{\lambda_1(p), \beta_*} = E_{\lambda_1(p), \beta_*}(u_n) + o(1) > -\infty$. Moreover, since $E_{\lambda_1(p), \beta_*}$ is weakly lower-semicontinuous, we see that, up to a subsequence, $\{u_n\}$ converges strongly in $W_0^{1,p}$ to a global minimizer u of $E_{\lambda_1(p), \beta_*}$, and hence, u is a critical point of $E_{\lambda_1(p), \beta_*}$. $\square$

5 Proofs. Sign properties

In this section, we prove the results stated in Sect. 2.2. We start with the following auxiliary lemma which will be employed several times.

Lemma 5.1 *Let $\alpha > \lambda_1(p)$. Assume that $u \in W_0^{1,p}$ satisfies $H_\alpha(u) < 0$. Then, there exists a sequence $\{u_n\} \subset W_0^{1,p}$ such that $H_\alpha(u_n) = 0$ for all $n \in \mathbb{N}$, and $u_n \rightarrow u$ weakly in $W_0^{1,p}$ and strongly in $W_0^{1,q}$ as $n \rightarrow +\infty$. Moreover, if $u \geq 0$ a.e. in Ω , then $\{u_n\}$ can be chosen such that $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$.*

Proof The claims can be obtained arguing in much the same way as in the proof of [8, Theorem 2.5 (ii)] by considering u instead of φ_p . $\square$

Now, we provide several properties of the critical value $\beta_f(\alpha)$ defined by (2.4). Recall that the functional Φ_α^+ and the set $\mathcal{B}^+(\alpha)$ are defined by (2.5) and (2.6), respectively.

Proposition 5.2 *Let $f \in W^{-1,q'}(\Omega) \setminus \{0\}$ and $f \geq 0$ in the weak sense. Let $\text{supp } f$ be the support of the distribution f . Then, the following assertions are satisfied:*

- (i) Φ_α^+ is 0-homogeneous and $\mathcal{B}^+(\alpha) \neq \emptyset$ for all $\alpha \in \mathbb{R}$;
- (ii) if $\alpha < \alpha_*$, then $\beta_f(\alpha) \geq \lambda_1(q)$, and $\beta_f(\alpha) > \lambda_1(q)$ if and only if $\langle f, \varphi_q \rangle > 0$;
- (iii) if $\alpha \geq \alpha_*$, then $\beta_f(\alpha) = \lambda_1(q)$;
- (iv) if $\alpha \geq \lambda_1(p)$, then $\beta_f(\alpha) \leq \beta_*$;
- (v) β_f is nonincreasing and left-continuous;
- (vi) if there is a nonempty open set $\tilde{\Omega} \subset \Omega \setminus \text{supp } f$, then $\sup\{\beta_f(\alpha) : \alpha \leq \lambda_1(p)\} < +\infty$.

If, in addition, $\langle f, v \rangle > 0$ for any $v \in W_0^{1,q} \setminus \{0\}$ with $v \geq 0$ a.e. in Ω , then the following assertions are satisfied:

- (vii) β_f is continuous in $(-\infty, \lambda_1(p))$;
- (viii) $\beta_f(\alpha) \rightarrow +\infty$ as $\alpha \rightarrow -\infty$.

Proof Assertion (i). The 0-homogeneity of Φ_α^+ is trivial. To obtain $\mathcal{B}^+(\alpha) \neq \emptyset$, we consider an eigenfunction u of the p -Laplacian associated with some eigenvalue $\lambda_k(p) > \max\{\alpha, \lambda_1(p)\}$. Then, $u^+ \not\equiv 0$ and $H_\alpha(u^+) = (\lambda_k(p) - \alpha)\|u^+\|_p^p > 0$, which yields $u^+ \in \mathcal{B}^+(\alpha)$.

Assertion (ii). The inequality $\beta_f(\alpha) \geq \lambda_1(q)$ is trivial. Moreover, if $\langle f, \varphi_q \rangle = 0$, then $\beta_f(\alpha) = \lambda_1(q)$ by the definition (2.4) of $\beta_f(\alpha)$ since $\varphi_q \in \mathcal{B}^+(\alpha)$. Let $\langle f, \varphi_q \rangle > 0$ and suppose, by contradiction, that there is some $\alpha < \alpha_*$ such that $\beta_f(\alpha) = \lambda_1(q)$. Then, in

view of Assertion (i), we can find a minimizing sequence $\{u_n\} \subset \mathcal{B}^+(\alpha)$ for $\beta_f(\alpha)$ such that $\|u_n\|_q = 1$ for all $n \in \mathbb{N}$,

$$\|\nabla u_n\|_q^q \rightarrow \lambda_1(q) \quad \text{and} \quad (H_\alpha(u_n))^{\frac{q-1}{p-1}} \langle f, u_n \rangle^{\frac{p-q}{p-1}} \rightarrow 0 \quad (5.1)$$

as $n \rightarrow +\infty$. Passing to a subsequence, we see that $u_n \rightarrow \varphi_q$ strongly in $W_0^{1,q}$ and $L^q(\Omega)$, where we assumed $\|\varphi_q\|_q = 1$. Therefore,

$$\lim_{n \rightarrow +\infty} \langle f, u_n \rangle = \langle f, \varphi_q \rangle > 0,$$

and hence, the second convergence in (5.1) implies $\lim_{n \rightarrow +\infty} H_\alpha(u_n) = 0$. Let us show that

$\{u_n\}$ is bounded in $W_0^{1,p}$. Indeed, if we suppose, by contradiction, that $\|\nabla u_n\|_p \rightarrow +\infty$ as $n \rightarrow +\infty$, up to a subsequence, then $\lim_{n \rightarrow +\infty} H_\alpha(u_n) = 0$ implies $\|\nabla u_n\|_p^p \leq (\alpha + 1)\|u_n\|_p^p$ for all sufficiently large $n \in \mathbb{N}$. Thus, according to [29, Lemma 9], there exists a constant $C = C(\alpha) > 0$ independent of n such that

$$\|\nabla u_n\|_p \leq C\|u_n\|_q = C, \quad (5.2)$$

which is a contradiction, and hence, $\{u_n\}$ is bounded in $W_0^{1,p}$. Recalling that $u_n \rightarrow \varphi_q$ strongly in $W_0^{1,q}$, we conclude from (5.2) that $u_n \rightarrow \varphi_q$ weakly in $W_0^{1,p}$, up to a subsequence, whence we get

$$H_\alpha(\varphi_q) \leq \liminf_{n \rightarrow +\infty} H_\alpha(u_n) = 0. \quad (5.3)$$

However, by the definition (1.4) of α_* , (5.3) contradicts the assumption $\alpha < \alpha_*$.

Assertion (iii). Note that $\beta_f(\alpha_*) = \lambda_1(q)$ since $\varphi_q \in \mathcal{B}^+(\alpha_*)$. Thus, let us assume $\alpha > \alpha_*$. Then, $H_\alpha(\varphi_q) < 0$, and hence, applying Lemma 5.1 to φ_q , we can find a sequence $\{u_n\} \subset W_0^{1,p} \setminus \{0\}$ such that $u_n \geq 0$ a.e. in Ω and $H_\alpha(u_n) = 0$ for any $n \in \mathbb{N}$, and $\frac{\|\nabla u_n\|_q^q}{\|u_n\|_q^q} \rightarrow \lambda_1(q)$ as $n \rightarrow +\infty$. That is, $\{u_n\} \subset \mathcal{B}^+(\alpha)$, and the assertion follows.

Before proving Assertion (iv), let us establish Assertion (v). We start with the monotonicity of β_f . Suppose, by contradiction, that there exist α_1 and α_2 such that $\alpha_1 < \alpha_2$ and $\beta_f(\alpha_1) < \beta_f(\alpha_2)$. That is, we can find $u \in \mathcal{B}^+(\alpha_1)$ such that $\Phi_{\alpha_1}^+(u) < \beta_f(\alpha_2)$. If $H_{\alpha_2}(u) \geq 0$, then $u \in \mathcal{B}^+(\alpha_2)$. Moreover, $\alpha_1 < \alpha_2$ implies $H_{\alpha_1}(u) > H_{\alpha_2}(u)$, and hence, $\Phi_{\alpha_1}^+(u) \geq \Phi_{\alpha_2}^+(u)$, which contradicts the definition of $\beta_f(\alpha_2)$. Therefore, $H_{\alpha_2}(u) < 0$. Applying Lemma 5.1, we can find a sequence $\{u_n\} \subset W_0^{1,p}$ such that $u_n \geq 0$ a.e. in Ω ,

$$H_{\alpha_2}(u_n) = 0 \quad \text{and} \quad \Phi_{\alpha_2}^+(u_n) = \frac{\|\nabla u_n\|_q^q}{\|u_n\|_q^q} \rightarrow \frac{\|\nabla u\|_q^q}{\|u\|_q^q} \leq \Phi_{\alpha_1}^+(u) \quad \text{as } n \rightarrow +\infty. \quad (5.4)$$

We see that $\{u_n\} \subset \mathcal{B}^+(\alpha_2)$, and hence, (5.4) leads to a contradiction since $\Phi_{\alpha_1}^+(u) < \beta_f(\alpha_2)$.

Now we prove the left-continuity of β_f . Let us fix some $\alpha \in \mathbb{R}$ and consider a sequence $\{\alpha_n\}$ such that $\alpha_n < \alpha$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow +\infty} \alpha_n = \alpha$. Since we know that $\liminf_{n \rightarrow +\infty} \beta_f(\alpha_n) \geq \beta_f(\alpha)$ by the monotonicity obtained above, let us show that $\limsup_{n \rightarrow +\infty} \beta_f(\alpha_n) \leq \beta_f(\alpha)$. Arguing by contradiction, we suppose that, up to a subsequence,

$$\delta := \lim_{n \rightarrow +\infty} \beta_f(\alpha_n) > \beta_f(\alpha).$$

Thus, we can find $u \in \mathcal{B}^+(\alpha)$ satisfying $\Phi_\alpha^+(u) < \delta$. Since each $\alpha_n < \alpha$, we have $u \in \mathcal{B}^+(\alpha_n)$. Therefore, recalling that $\alpha_n \rightarrow \alpha$ as $n \rightarrow +\infty$, we get a contradiction by

$$\delta = \lim_{n \rightarrow +\infty} \beta_f(\alpha_n) \leq \lim_{n \rightarrow +\infty} \Phi_{\alpha_n}^+(u) = \Phi_\alpha^+(u) < \delta.$$

Assertion (iv) easily follows from Assertion (v) by noting that $\beta_f(\lambda_1(p)) \leq \beta_*$ in view of $\varphi_p \in \mathcal{B}^+(\lambda_1(p))$.

Assertion (vi). Taking a nonnegative function $u \in C_0^\infty(\Omega) \setminus \{0\}$ such that $\text{supp } u \subset \tilde{\Omega}$, we see that $H_\alpha(u) \geq 0$ for all $\alpha \leq \lambda_1(p)$ and $\langle f, u \rangle = 0$, which yields the desired bound:

$$\beta_f(\alpha) \leq \Phi_\alpha^+(u) = \frac{\|\nabla u\|_q^q}{\|u\|_q^q} \quad \text{for any } \alpha \leq \lambda_1(p).$$

Assertion (vii). Let $\alpha < \lambda_1(p)$ and let $\{\alpha_n\}$ be an arbitrary sequence convergent to α . Since we already know that β_f is nonincreasing and left-continuous by Assertion (v), it is sufficient to assume that $\alpha < \alpha_n < \lambda_1(p)$ and to show that $\liminf_{n \rightarrow +\infty} \beta_f(\alpha_n) \geq \beta_f(\alpha)$. Suppose, by contradiction, that, up to a subsequence, $\lim_{n \rightarrow +\infty} \beta_f(\alpha_n) < \beta_f(\alpha)$. Thus, for any sufficiently large $n \in \mathbb{N}$ we can choose $u_n \in \mathcal{B}^+(\alpha_n)$ such that

$$\beta_f(\alpha_n) \leq \Phi_{\alpha_n}^+(u_n) \leq \sup_m \Phi_{\alpha_m}^+(u_m) < \beta_f(\alpha), \quad (5.5)$$

and we may assume $\|u_n\|_q = 1$ for all n . The latter inequality in (5.5) implies the existence of $C > 0$ such that

$$\|\nabla u_n\|_q^q \leq C \quad \text{and} \quad (H_{\alpha_n}(u_n))^{\frac{q-1}{p-1}} \langle f, u_n \rangle^{\frac{p-q}{p-1}} \leq C \quad (5.6)$$

for all n . In view of the first bound in (5.6) and the choice $\|u_n\|_q = 1$, we see that $\{u_n\}$ converges to some nonnegative function $u_0 \in W_0^{1,q} \setminus \{0\}$ weakly in $W_0^{1,q}$ and strongly in $L^q(\Omega)$, up to a subsequence. By our assumption, we get $\lim_{n \rightarrow +\infty} \langle f, u_n \rangle = \langle f, u_0 \rangle > 0$, which implies the uniform boundedness of $H_{\alpha_n}(u_n)$ for all n . Hence, by the same argument as in Assertion (ii) [see (5.2)], we conclude that $\{u_n\}$ is bounded in $W_0^{1,p}$, and thereby it converges to u_0 weakly in $W_0^{1,p}$, up to a subsequence. Moreover, $H_\alpha(u_0) \geq 0$ by $\alpha \leq \lambda_1(p)$. Thus, we see that $u_0 \in \mathcal{B}^+(\alpha)$ and we deduce from (5.5) that

$$\Phi_\alpha^+(u_0) \leq \liminf_{n \rightarrow +\infty} \Phi_{\alpha_n}^+(u_n) < \beta_f(\alpha),$$

which contradicts the definition of $\beta_f(\alpha)$.

Assertion (viii). Suppose, by contradiction, that $C := \sup\{\beta_f(\alpha) : \alpha \leq \lambda_1(p)\} < +\infty$. The monotonicity of β_f [see Assertion (v)] implies the existence of a sequence $\{\alpha_n\}$ such that $\alpha_n \rightarrow -\infty$ and $\beta_f(\alpha_n) \rightarrow C$ as $n \rightarrow +\infty$. Therefore, by the definition of $\beta_f(\alpha_n)$, we can find a sequence $\{u_n\} \subset \mathcal{B}^+(\alpha_n)$ such that $\|u_n\|_q = 1$ for each $n \in \mathbb{N}$ and $\Phi_{\alpha_n}^+(u_n) \leq C + 1$. Arguing as in Assertion (vii) [see (5.6)], we obtain that $u_n \rightarrow u_0 \not\equiv 0$ weakly in $W_0^{1,q}$ and strongly in $L^q(\Omega)$, up to a subsequence. Moreover, $H_{\alpha_n}(u_n)$ is uniformly bounded for all n . Since $\alpha_n \rightarrow -\infty$ as $n \rightarrow +\infty$, we conclude that $\{u_n\}$ is bounded in $W_0^{1,p}$ and $\|u_n\|_p \rightarrow 0$. However, this contradicts $u_0 \not\equiv 0$. $\square$

The following result is crucial for the proof of Theorem 2.6 and Proposition 2.10 (ii).

Proposition 5.3 *Let $f \in W^{-1,p'}(\Omega)$ and $f \geq 0$ in the weak sense. Assume that $\alpha \in \mathbb{R}$ and $\beta < \beta_f(\alpha)$. If $u \in W_0^{1,p}$ is such that $u \geq 0$ a.e. in Ω and $H_\alpha(u) > 0$, then*

$$H_\alpha(u) + G_\beta(u) + \langle f, u \rangle > 0. \quad (5.7)$$

Proof If $u \in W_0^{1,p}$ is such that $u \geq 0$ a.e. in Ω and $H_\alpha(u) > 0$, then $\langle f, u \rangle \geq 0$. Evidently, if $G_\beta(u) \geq 0$, then (5.7) is satisfied. Thus, let us assume that $G_\beta(u) < 0$. Note that $u \in \mathcal{B}^+(\alpha)$,

Fig. 2 Schematic behavior of $Q_\beta(t)$ and $Q_{\tilde{\beta}}(t)$

i.e., u is admissible for the minimization problem (2.4) of $\beta_f(\alpha)$. Consequently, if $\langle f, u \rangle = 0$, then $\beta < \beta_f(\alpha)$ implies $G_\beta(u) > 0$, which is impossible. Therefore, $\langle f, u \rangle > 0$.

Consider the function

$$Q_\beta(t) := t^p H_\alpha(u) + t^q G_\beta(u) + t \langle f, u \rangle, \quad t \geq 0.$$

Since $p > q > 1$, we see that $t^p H_\alpha(u) > 0$ is the leading term at $t \rightarrow +\infty$, $t \langle f, u \rangle > 0$ is the leading term at $t \rightarrow 0$, and $t^q G_\beta(u) < 0$ has an impact in a middle range of t , see Fig. 2.

If $Q_\beta(1) > 0$, then (5.7) is satisfied. Thus, let us suppose that $Q_\beta(1) \leq 0$. In view of the behavior of $Q_\beta(t)$, we can find $t_0, t_1 > 0$ such that $t_0 < 1 < t_1$, $Q_\beta(t_0) > 0$, and $Q_\beta(t_1) > 0$. Let us define the value

$$q(b) := \min_{t_0 \leq t \leq t_1} Q_b(t),$$

and let $t_b \in (t_0, t_1)$ be a corresponding minimizer. We see that $q(\beta) = Q_\beta(t_\beta) \leq 0$ and $Q'_\beta(t_\beta) = 0$. Note that $Q_b(t)$ is strictly decreasing with respect to b for any fixed $t > 0$. Moreover, if $\hat{\beta} < \beta$ is such that $G_{\hat{\beta}}(u) \geq 0$, then $Q_{\hat{\beta}}(t) \geq q(\hat{\beta}) \geq c > 0$ for some constant c and all $t \geq t_0$. Therefore, noting that $q(\cdot)$ is continuous, we obtain the existence of $\tilde{\beta} \in (\hat{\beta}, \beta]$ such that

$$Q_{\tilde{\beta}}(t_{\tilde{\beta}}) = 0 \quad \text{and} \quad Q'_{\tilde{\beta}}(t_{\tilde{\beta}}) = 0. \quad (5.8)$$

Let us denote for simplicity $t = t_{\tilde{\beta}}$, and rewrite (5.8) as

$$\begin{cases} t^p H_\alpha(u) + t^q G_{\tilde{\beta}}(u) + t \langle f, u \rangle = 0, \\ pt^p H_\alpha(u) + qt^q G_{\tilde{\beta}}(u) + t \langle f, u \rangle = 0. \end{cases}$$

Solving this system with respect to $G_{\tilde{\beta}}(u)$ and t , we obtain

$$t = \left(\frac{q-1}{p-q} \right)^{\frac{1}{p-1}} \left(\frac{\langle f, u \rangle}{H_\alpha(u)} \right)^{\frac{1}{p-1}}$$

and

$$G_{\tilde{\beta}}(u) = -\frac{p-1}{p-q} \left(\frac{p-q}{q-1} \right)^{\frac{q-1}{p-1}} (H_\alpha(u))^{\frac{q-1}{p-1}} \langle f, u \rangle^{\frac{p-q}{p-1}}. \quad (5.9)$$

Expressing now $\tilde{\beta}$ from $G_{\tilde{\beta}}(u)$, we get

$$\tilde{\beta} = \frac{\|\nabla u\|_q^q}{\|u\|_q^q} + \frac{p-1}{p-q} \left(\frac{p-q}{q-1} \right)^{\frac{q-1}{p-1}} \frac{(H_\alpha(u))^{\frac{q-1}{p-1}} \langle f, u \rangle^{\frac{p-q}{p-1}}}{\|u\|_q^q}. \quad (5.10)$$

However, this is impossible since $\tilde{\beta} \leq \beta < \beta_f(\alpha)$, and u is an admissible function for the minimization formulation of $\beta_f(\alpha)$. $\square$

Proof of Theorem 2.6 Suppose that there is a solution u of $(D_{\alpha,\beta,f})$ with $\alpha \leq \lambda_1(p)$ and $\beta < \beta_f(\alpha)$ such that $u = u^+ + u^-$ and $u^- \not\equiv 0$. Then, $-u^- \geq 0$ and we have

$$\left\langle E'_{\alpha,\beta}(u), u^- \right\rangle = H_\alpha(-u^-) + G_\beta(-u^-) + \langle f, -u^- \rangle = 0. \quad (5.11)$$

Thus, if $\alpha < \lambda_1(p)$, then $H_\alpha(-u^-) > 0$, and hence, we get a contradiction to Proposition 5.3. In the case $\alpha = \lambda_1(p)$, it is enough to show that $u^- \notin \mathbb{R}\varphi_p$. Suppose, by contradiction and without loss of generality, that $-u^- = \varphi_p$. Then, we get

$$\left\langle E'_{\alpha,\beta}(u), u^- \right\rangle = G_\beta(\varphi_p) + \langle f, \varphi_p \rangle = 0,$$

and hence, $G_\beta(\varphi_p) \leq 0$. On the other hand, since $\beta < \beta_f(\lambda_1(p)) \leq \beta_*$ by Lemma 5.2 (iv), we have $G_\beta(\varphi_p) > 0$, which leads to a contradiction. Therefore, any solution of $(D_{\alpha,\beta,f})$ is nonnegative. Moreover, if $f \in L^\infty(\Omega) \setminus \{0\}$ and $f \geq 0$ a.e. in Ω , then we have $u \in \text{int } C_0^1(\overline{\Omega})_+$ (see Remark 2.5), that is, u is positive. $\square$

Proof of Proposition 2.7 Let us fix some $\beta < \beta_f(\lambda_1(p))$ and consider an arbitrary sequence $\{\alpha_n\}$ such that each $\alpha_n > \lambda_1(p)$ and $\alpha_n \rightarrow \lambda_1(p)$ as $n \rightarrow +\infty$. Let u_n be a solution of $(D_{\alpha,\beta,f})$ with $\alpha = \alpha_n$. Assume first that $\{u_n\}$ is bounded in $W_0^{1,p}$. Then, we deduce from Remark 2.4 and the Arzelà–Ascoli theorem that $u_n \rightarrow u$ in $C^1(\overline{\Omega})$, up to a subsequence, where u is a solution of $(D_{\alpha,\beta,f})$. Noting that $u \in \text{int } C_0^1(\overline{\Omega})_+$ by Theorem 2.6 and Remark 2.5, we conclude that $u_n \in \text{int } C_0^1(\overline{\Omega})_+$ for all sufficiently large $n \in \mathbb{N}$.

Assume now that $\{u_n\}$ is unbounded in $W_0^{1,p}$. Considering the normalized sequence $v_n := u_n / \|\nabla u_n\|_p$, we can argue as in [7, Lemma 3.3] to obtain that $\{v_n\}$ converges strongly in $W_0^{1,p}$ to φ_p or $-\varphi_p$, up to a subsequence. Applying again Remark 2.4 and the Arzelà–Ascoli theorem, we deduce that either $v_n \rightarrow \varphi_p$ or $v_n \rightarrow -\varphi_p$ in $C^1(\overline{\Omega})$, up to a subsequence. Since $\varphi_p \in \text{int } C_0^1(\overline{\Omega})_+$ (see Remark 2.5), we get either $v_n \in \text{int } C_0^1(\overline{\Omega})_+$ or $-v_n \in \text{int } C_0^1(\overline{\Omega})_+$ for all sufficiently large $n \in \mathbb{N}$.

Finally, recalling that the sequence $\{\alpha_n\}$ has been chosen in an arbitrary way, we finish the proof. $\square$

Proof of Proposition 2.8 Let $\alpha \geq \lambda_1(p)$ and $\beta > \beta_{ps}(\alpha)$. Suppose, by contradiction, that $(D_{\alpha,\beta,f})$ has a nonnegative solution u for some $f \in L^\infty(\Omega)$ satisfying $f \geq 0$ and $f \not\equiv 0$. Then, $u \in \text{int } C_0^1(\overline{\Omega})_+$ (see Remark 2.5), and hence, u is a super-solution of $(D_{\alpha,\beta,0})$. Noting that $\beta > \beta_{ps}(\alpha)$ implies $\beta > \lambda_1(q)$ (see [6, Proposition 3 (ii)]) and applying the sub-super-solution method [6, Lemma 6], we obtain a positive solution of $(D_{\alpha,\beta,0})$. However, it contradicts [6, Theorem 2.2] which asserts that $(D_{\alpha,\beta,0})$ has no positive solutions. $\square$

Proof of Proposition 2.9 Suppose, by contradiction, that there exists $\alpha > \alpha_*$ such that for any $n \in \mathbb{N}$ there exists $\beta_n > \lambda_1(q) - \frac{1}{n}$ and a nonnegative solution u_n of $(D_{\alpha,\beta,f})$ with $\beta = \beta_n$.

Since $u_n \in \text{int } C_0^1(\overline{\Omega})_+$ for each $n \in \mathbb{N}$ (see Remark 2.5), we can choose φ_q^p/u_n^{p-1} as a test function for $(D_{\alpha,\beta,f})$. Then, we obtain

$$\begin{aligned} & \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \left(\frac{\varphi_q^p}{u_n^{p-1}} \right) dx + \int_{\Omega} |\nabla u_n|^{q-2} \nabla u_n \nabla \left(\frac{\varphi_q^p}{u_n^{p-1}} \right) dx \\ &= \alpha \int_{\Omega} \varphi_q^p dx + \beta_n \int_{\Omega} \varphi_q^p u_n^{q-p} dx + \int_{\Omega} f \frac{\varphi_q^p}{u_n^{p-1}} dx. \end{aligned}$$

The classical Picone identity (see [2, Theorem 1.1]) implies

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \left(\frac{\varphi_q^p}{u_n^{p-1}} \right) dx \leq \int_{\Omega} |\nabla \varphi_q|^p dx = \alpha_* \int_{\Omega} \varphi_q^p dx$$

and the generalized Picone's identity from [16, Lemma 1] yields

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^{q-2} \nabla u_n \nabla \left(\frac{\varphi_q^p}{u_n^{p-1}} \right) dx &\leq \int_{\Omega} |\nabla \varphi_q|^{q-2} \nabla \varphi_q \nabla \left(\frac{\varphi_q^{p-q+1}}{u_n^{p-q}} \right) dx \\ &= \lambda_1(q) \int_{\Omega} \varphi_q^p u_n^{q-p} dx. \end{aligned}$$

Consequently, we get

$$(\alpha - \alpha_*) \|\varphi_q\|_p^p + (\beta_n - \lambda_1(q)) \int_{\Omega} \varphi_q^p u_n^{q-p} dx + \int_{\Omega} f \frac{\varphi_q^p}{u_n^{p-1}} dx \leq 0 \quad (5.12)$$

for each $n \in \mathbb{N}$. Note that the first and third terms in (5.12) are positive, and the first term does not depend on n . Thus, if $\beta_n \geq \lambda_1(q)$ for some $n \in \mathbb{N}$, we get a contradiction. Therefore, we may assume that $\beta_n \uparrow \lambda_1(q)$, and hence, (5.12) implies

$$\int_{\Omega} \varphi_q^p u_n^{q-p} dx \rightarrow +\infty \quad \text{as } n \rightarrow +\infty. \quad (5.13)$$

Let us show that this is impossible. Since $\alpha_* > \lambda_1(p)$ and each $u_n > 0$, we can argue as in [7, Lemma 3.3] to prove that $\{\|\nabla u_n\|_p\}$ is bounded. Hence, by Remark 2.4 and the Arzelà–Ascoli theorem we obtain that $u_n \rightarrow u$ in $C^1(\overline{\Omega})$, up to a subsequence, where $u \in \text{int } C_0^1(\overline{\Omega})_+$ is a solution of $(D_{\alpha,\beta,f})$. Since φ_q, u , and each u_n satisfy the boundary point lemma (see, e.g., [27, Theorem 5.5.1]), the $C^1(\overline{\Omega})$ -convergence implies that we can find $c_1, c_2 > 0$ such that

$$c_1 \text{dist}(x, \partial\Omega) < \varphi_q(x), u(x), u_n(x) < c_2 \text{dist}(x, \partial\Omega)$$

for any $x \in \Omega$ and all sufficiently large $n \in \mathbb{N}$. Thus, we get

$$\int_{\Omega} \varphi_q^p u_n^{q-p} dx \leq c_2^p c_1^{q-p} \int_{\Omega} (\text{dist}(x, \partial\Omega))^q dx < +\infty,$$

which contradicts (5.13). $\square$

Let us turn to the proof of Proposition 2.10. We start by showing some basic properties of the critical value $\beta^f(\alpha)$ defined by (2.7). Recall that the functional Φ_{α}^- and the set $\mathcal{B}^-(\alpha)$ are defined by (2.8) and (2.9), respectively.

Lemma 5.4 *Let $f \in W^{-1,p'}(\Omega) \setminus \{0\}$ and $f \geq 0$ in the weak sense. Then, the following assertions are satisfied:*

- (i) Φ_{α}^{-} is 0-homogeneous and $\mathcal{B}^{-}(\alpha) \neq \emptyset$ for any $\alpha \geq \lambda_1(p)$;
- (ii) $\beta^f(\alpha) < +\infty$ for any $\alpha \geq \lambda_1(p)$;
- (iii) β^f is nondecreasing in $[\lambda_1(p), +\infty)$;
- (iv) $\beta^f(\lambda_1(p)) = \beta_*$ and $\beta^f(\alpha) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$.

Proof Assertion (i) is trivial. Let us prove Assertion (ii). Note that

$$\mathcal{B}^{-}(\alpha) \subset X(\alpha) := \left\{ u \in W_0^{1,p} : \|\nabla u\|_p^p \leq \alpha \|u\|_p^p \right\}$$

for all $\alpha \geq \lambda_1(p)$, and hence, [29, Lemma 9] implies the existence of a constant $C = C(\alpha) > 0$ such that

$$\|\nabla u\|_p \leq C \|u\|_q \quad \text{for any } u \in \mathcal{B}^{-}(\alpha).$$

Applying Hölder's inequality, we get $\|\nabla u\|_q \leq |\Omega|^{\frac{p-q}{pq}} \|\nabla u\|_p$, which yields the desired boundedness:

$$\beta^f(\alpha) \leq \Phi_{\alpha}^{-}(u) \leq \frac{\|\nabla u\|_q^q}{\|u\|_q^q} \leq |\Omega|^{\frac{p-q}{p}} C^q < +\infty \quad \text{for any } u \in \mathcal{B}^{-}(\alpha).$$

Assertion (iii). We argue similarly to the proof of Proposition 5.2 (v). Suppose, by contradiction, that there exist $\alpha_1, \alpha_2 \geq \lambda_1(p)$ such that $\alpha_1 < \alpha_2$ and $\beta^f(\alpha_1) > \beta^f(\alpha_2)$. That is, we can find $u \in \mathcal{B}^{-}(\alpha_1)$ such that $\Phi_{\alpha_1}^{-}(u) > \beta^f(\alpha_2)$. Since $H_{\alpha_1}(u) \leq 0$, we have $H_{\alpha_2}(u) < H_{\alpha_1}(u) \leq 0$. Applying Lemma 5.1, we can find a sequence $\{u_n\} \subset W_0^{1,p}$ such that $u_n \geq 0$ a.e. in Ω ,

$$H_{\alpha_2}(u_n) = 0 \quad \text{and} \quad \Phi_{\alpha_2}^{-}(u_n) = \frac{\|\nabla u_n\|_q^q}{\|u_n\|_q^q} \rightarrow \frac{\|\nabla u\|_q^q}{\|u\|_q^q} \geq \Phi_{\alpha_1}^{-}(u) \quad \text{as } n \rightarrow +\infty. \quad (5.14)$$

We see that $\{u_n\} \subset \mathcal{B}^{+}(\alpha_2)$, and hence, (5.14) leads to a contradiction since $\Phi_{\alpha_1}^{-}(u) > \beta^f(\alpha_2)$.

Assertion (iv). The equality $\beta^f(\lambda_1(p)) = \beta_*$ is trivial. To show that $\beta^f(\alpha) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$, let us assume, without loss of generality, that $0 \in \Omega$, and let us fix a ball $B \subset \Omega$ such that $0 \in B$. Consider any nonnegative $u \in C_0^{\infty}(\Omega)$ and $\alpha \geq \lambda_1(p)$ such that $\text{supp } u \subset B$ and $\|\nabla u\|_p^p = \alpha \|u\|_p^p$, i.e., $H_{\alpha}(u) = 0$. Now, we define a function u_n by $u_n(x) = u(nx)$ for each $n \in \mathbb{N}$. Since $\text{supp } u \subset B$, we get $u_n \in C_0^{\infty}(\Omega)$. Moreover, it is not hard to obtain that

$$\|\nabla u_n\|_p^p = \alpha n^p \|u_n\|_p^p \quad \text{and} \quad \Phi_{\alpha n^p}^{-}(u_n) = \frac{\|\nabla u_n\|_q^q}{\|u_n\|_q^q} = n^q \frac{\|\nabla u\|_q^q}{\|u\|_q^q}. \quad (5.15)$$

Thus, we see from (5.15) that $u_n \in \mathcal{B}^{-}(\alpha n^p)$ and $\beta^f(\alpha n^p) \geq \Phi_{\alpha n^p}^{-}(u_n) \rightarrow +\infty$ as $n \rightarrow +\infty$. $\square$

The following result can be obtained in much the same way as Proposition 5.3.

Proposition 5.5 *Let $f \in W^{-1,p'}(\Omega)$ and $f \geq 0$ in the weak sense. Assume that $\alpha > \lambda_1(p)$ and $\beta > \beta^f(\alpha)$. If $u \in W_0^{1,p}$ is such that $u \geq 0$ a.e. in Ω and $H_{\alpha}(u) < 0$, then*

$$H_{\alpha}(u) + G_{\beta}(u) - \langle f, u \rangle < 0. \quad (5.16)$$

Proof of Proposition 2.10 (i) Let $\beta < \beta^f(\alpha)$ and $u^{-} \not\equiv 0$. Suppose, by contradiction, that $H_{\alpha}(u^{-}) \geq 0$. As in the proof of Theorem 2.6, $-u^{-}$ satisfies the equality (5.11). Therefore, if $H_{\alpha}(u^{-}) > 0$, then we get a contradiction to Proposition 5.3 applied to $-u^{-}$. If $H_{\alpha}(u^{-}) = 0$,

then $-u^- \in \mathcal{B}^+(\alpha)$, and hence, $\beta < \beta_f(\alpha)$ implies $G_\beta(-u^-) > 0$, which contradicts (5.11). (ii) Let $\beta > \beta_f(\alpha)$ and $u^+ \neq 0$. Suppose, by contradiction, that $H_\alpha(u^+) \leq 0$. Since u is a solution of $(D_{\alpha,\beta,f})$, we have

$$\langle E'_{\alpha,\beta}(u), u^+ \rangle = H_\alpha(u^+) + G_\beta(u^+) - \langle f, u^+ \rangle = 0. \quad (5.17)$$

If $H_\alpha(u^+) < 0$, then we get a contradiction to Proposition 5.5. If $H_\alpha(u^+) = 0$, then $u^+ \in \mathcal{B}^-(\alpha)$, and hence, $\beta > \beta_f(\alpha)$ implies $G_\beta(u^+) < 0$, which contradicts (5.17). The proof is complete. $\square$

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