Surfaces with a canonical principal direction and prescribed mean curvature

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Abstract

A surface in Euclidean space has a canonical principal direction with respect to a fixed direction $\mathbf{d}$ if its tangent part $\mathbf{d}^\top$ is a principal direction along the surface. In this paper, we classify all such surfaces with prescribed mean curvature given as an affine function of one of the following three functions: the height function, the angle function and the support function.

Keywords Principal direction · Mean curvature · Capillary surface · Self-shrinker · Translating soliton

Mathematics Subject Classification 53B25

1 Introduction and statement of the result

A constant angle surface in the three-dimensional Euclidean space $\mathbb{R}^3$ is an orientable surface whose Gauss map makes a constant angle with a fixed vector direction $\mathbf{d}$ ([1]). In particular, the projection of $\mathbf{d}$ on the tangent plane of the surface is a principal direction and the corresponding principal curvature is 0. Dillen et al. generalized, in [4], the notion of the constant angle surfaces to those surfaces where the tangent projection of $\mathbf{d}$ is a principal direction without necessarily corresponding with zero principal curvature. See also [9] for the case of hypersurfaces.

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Definition 1.1 Let $\mathbf{d} \in \mathbb{R}^3$ be a unitary vector. We say that a surface M in $\mathbb{R}^3$ has a canonical principal direction with respect to $\mathbf{d}$ if the tangent part $\mathbf{d}^\top$ along M is a principal direction.

We abbreviate by saying a CPD surface where the direction $\mathbf{d}$ is understood. The study of CPD surfaces in different ambient spaces has been of interest recently: See [4,5,8,17].

The purpose of this paper is to classify all CPD surfaces whose mean curvature is a linear function of the next three functions on M :

$$\begin{aligned} \text{height function } f(p) &= \langle p, \mathbf{v} \rangle, \\ \text{angle function } f(p) &= \langle N(p), \mathbf{v} \rangle, \\ \text{support function } f(p) &= \langle N(p), p \rangle. \end{aligned}$$

Here, $\mathbf{v} \in \mathbb{R}^3$ is a fixed unit vector and N is the Gauss map of M . The interest for these three types of mean curvatures is the following:

1. Height function. A surface M whose mean curvature H satisfies

$$H(p) = \lambda \langle p, \mathbf{v} \rangle + \mu, \quad \lambda, \mu \in \mathbb{R}, \quad (1)$$

is a model of a liquid drop where the direction of the gravity is indicated by the vector $\mathbf{v}$, the constant λ depends only on the physical properties of the drop and μ is a volume constraint: See [7] as a general reference. A surface whose mean curvature satisfies (1) is called a *capillary surface*.

2. Angle function. A surface M whose mean curvature H satisfies

$$H(p) = \lambda \langle N(p), \mathbf{v} \rangle + \mu, \quad \lambda, \mu \in \mathbb{R}, \quad (2)$$

is called a λ -*translating soliton*. This notion generalizes the translating soliton ($\mu = 0$). Translating solitons appear in the theory of the mean curvature flow being the equation of the limit flow by a proper blowup procedure near type II singular points ([12,13,20]).

3. Support function. A surface M whose mean curvature H satisfies

$$H(p) = \lambda \langle N(p), p \rangle + \mu, \quad \lambda, \mu \in \mathbb{R}, \quad (3)$$

is called a λ -*shrinker surface*. This generalizes the concept of self-expander ($\lambda > 0$, $\mu = 0$) and self-shrinker ($\lambda < 0$, $\mu = 0$). Both types of surfaces appear in the context of self-similar solutions to the mean curvature flow as models of the flow behavior near a singularity, where M does not change the shape but is contracted (self-shrinkers) or dilated (self-expanders) by the flow: See [6,11,13].

We observe that if $\lambda = 0$ in the three above equations, then the mean curvature H is constant on the surface. This particular case was treated previously in [9] (see Proposition 2.2 below).

Although in the literature, Eqs. (2) and (3) have been studied when $\mu = 0$, the case $\mu \neq 0$ makes sense in the context of manifolds with density [10,16]. Indeed, consider $\mathbb{R}^3$ with a positive smooth density function e^ϕ , $\phi \in C^\infty(\mathbb{R}^3)$, which serves as a weight for the volume and the surface area. For a variation $\{M_t : t \in (-\epsilon, \epsilon)\}$ of M , let $A_\phi(t)$ and $V_\phi(t)$ be the weighted area and the enclosed weighted volume of M_t , respectively. Then, the first variation of $A_\phi(t)$ and $V_\phi(t)$ is

$$A'_\phi(0) = -2 \int_M H_\phi u \, dA_\phi, \quad V'_\phi(0) = \int_M u \, dA_\phi,$$

where u is the normal component of the variation and $H_\phi = H - \langle \nabla \phi, N \rangle / 2$ is called the weighted mean curvature [19]. Then, it is immediate that M is a critical point of A_ϕ

for a given weighted volume if and only if $H_\phi = \mu$ is a constant function. Equations (2) and (3) appear with suitable choices of the density function ϕ ; namely, for Eq. (2), we take $\phi(p) = 2\lambda\langle p, \mathbf{v} \rangle$, and for Eq. (3), the function if $\phi(p) = \lambda|p|^2$.

Surfaces with a canonical principal direction have suitable parametrization. It was proved in [9] that a CPD surface with respect to $\mathbf{d}$ admits the following parametrization

$$X(s, t) = \gamma(s) + (f(t)\eta(s) + g(t)\mathbf{d}), \quad (4)$$

where $\gamma = \gamma(s)$, $s \in I \subset \mathbb{R}$ is a curve parametrized by the arc length and contained in a plane orthogonal to $\mathbf{d}$, $\eta = \eta(s)$ is a unit vector field along γ and orthogonal to both γ and $\mathbf{d}$. Finally, $f = f(t)$, $g = g(t)$, $t \in J \subset \mathbb{R}$, are smooth functions that satisfy $f'(t)^2 + g'(t)^2 = 1$ for all $t \in J$. The curve γ is named the *directrix* of M , and the curve $(f(t), g(t))$ is the *profile curve* of M . As a consequence, we may take η to be the principal normal vector to γ and thus $\gamma''(s) = \kappa(s)\eta(s)$, where κ is the curvature of γ .

Once introduced the above notation, we give the main result of this paper.

Theorem 1.1 *Let M be a CPD surface parametrized as (4). If M satisfies (1), (2) or (3), then M is one of the following surfaces:*

1. A cylindrical surface where the base curve is $\alpha(t) = f(t)\eta + g(t)\mathbf{d}$, η is a constant vector, the rulings are orthogonal to $\mathbf{d}$ and α satisfies the one-dimensional case of (1), (2) or (3).
2. A surface of revolution whose generating curve is α and the rotational axis is parallel to $\mathbf{d}$. The curve α satisfies a specific second-order ODE depending on condition (1), (2) or (3).
3. A cylindrical surface where the base curve is $\gamma_a(s) = \gamma(s) + a\eta(s)$, $a \in \mathbb{R}$, γ_a satisfies the one-dimensional case of (1), (2) or (3) and the rulings are parallel to $\mathbf{d}$.

By a cylindrical surface, we mean a ruled surface where all the rulings are parallel. Then, the base curve of the surface is contained in a plane orthogonal to the rulings. In Theorem 1.1, items (1) and (3), the base curve of the cylindrical surface satisfies that its curvature is a linear function of the height, angle or support function depending on the case. In such a case, we will say then that the curve satisfies the one-dimensional case of (1), (2) or (3).

The proof of Theorem 1.1 is separated in each one of the three equations: See Sects. 3, 4 and 5, respectively.

2 Preliminaries

In this section, we give an expression of the mean curvature H for a CPD surface and we classify the surfaces with constant mean curvature. Firstly, we need to give some observations about parametrization (4) in the following proposition.

Proposition 2.1 *Let M be a CPD surface parametrized by (4).*

1. We have $1 - f(t)\kappa(s) \neq 0$ for all $s \in I$, $t \in J$.
2. If the function g is constant, then M is a plane orthogonal to $\mathbf{d}$.
3. If γ is a straight line, then M is a cylindrical surface whose rulings are orthogonal to $\mathbf{d}$.
4. If γ is a circle, then M is a surface of revolution whose rotational axis is parallel to $\mathbf{d}$.

Proof 1. Since $\partial_s X = (1 - \kappa(s)f(t))\gamma'(s) \neq 0$ by regularity of M , we conclude that $1 - \kappa(s)f(t) \neq 0$.

2. If $g(t) = a$ for all $s \in I$, then $f(t) = \epsilon t + b$, with $b \in \mathbb{R}$ and $\epsilon \in \{-1, 1\}$. Then, (4) gives

$$X(s, t) = \gamma(s) + (\epsilon t + b)\eta(s) + a\mathbf{d} = (\gamma(s) + a\mathbf{d}) + (\epsilon t + b)\eta(s).$$

It follows that $\gamma(s) + a\mathbf{d}$ is a planar curve and the curves $t \mapsto (\epsilon t + b)\eta(s)$ are segments of straight lines coplanar to γ . This proves that M is a plane orthogonal to $\mathbf{d}$.

3. If γ is a straight line, then η is a unit constant vector and from (4), the surface M is a cylindrical surface whose base curve is $\alpha(t) = f(t)\eta + g(t)\mathbf{d}$ and the rulings are orthogonal to $\mathbf{d}$.
4. After a change of coordinates, we suppose $\gamma(s) = (r \cos(s), r \sin(s), 0)$ and $\mathbf{d} = (0, 0, 1)$. By (4), we find

$$X(s, t) = (r(1 - f(t)) \cos(s), r(1 - f(t)) \sin(s), g(t)),$$

proving that M is a surface of revolution about the $\mathbf{d}$ -axis.

□

We need to have an expression of the mean curvature H of a CPD surface. The following result is proved in [9].

Proposition 2.2 *Let M be a CPD surface. If M is not a plane orthogonal to $\mathbf{d}$, then the principal curvatures κ_1 and κ_2 with respect to the Gauss map $N(s, t) = -g'(t)\eta(s) + f'(t)\mathbf{d}$ are*

$$\kappa_1(X(s, t)) = \frac{f''(t)}{g'(t)}, \quad \kappa_2(X(s, t)) = \frac{g'(t)\kappa(s)}{1 - f(t)\kappa(s)}.$$

The mean curvature H with respect to N is

$$H(X(s, t)) = \frac{1}{2} \left(\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa(s)}{1 - f(t)\kappa(s)} \right). \quad (5)$$

In Theorem 1.1, we exclude the case $\lambda = 0$, that is, that the surface has constant mean curvature. The CPD surfaces with constant mean curvature were obtained in [9, Cor. 14]. By completeness, we give its proof.

Proposition 2.3 *If M is a CPD surface with constant mean curvature H , then M is a plane, a circular cylinder with axis orthogonal to $\mathbf{d}$ or a surface of revolution whose axis is parallel to $\mathbf{d}$.*

Proof It is immediate that a plane orthogonal to $\mathbf{d}$ is a CPD surface with respect to $\mathbf{d}$ and with zero mean curvature. Suppose now that M is not the above surface, in particular, the function $g(t)$ in (4) is not constant by item 2 of Proposition 2.1. By Proposition 2.2, we have

$$\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa(s)}{1 - f(t)\kappa(s)} = 2H,$$

where the right-hand side is a constant. Differentiating with respect to s , we obtain

$$\frac{g'(t)\kappa'(s)}{(1 - f(t)\kappa(s))^2} = 0.$$

Since $g'(t) \neq 0$, we conclude $\kappa'(s) = 0$ for every $s \in I$. This means that γ is a straight line or a circle.

1. Case γ is a straight line. By Proposition 2.1, the surface M is a cylindrical surface whose rulings are parallel to γ . Since H is constant, then M is a plane parallel to γ ($H = 0$) or M is a circular cylinder ($H \neq 0$) whose axis is the straight line, being γ orthogonal to $\mathbf{d}$.
2. Case γ is a circle. By Proposition 2.1 again, M is a surface of revolution whose axis is parallel to $\mathbf{d}$.

□

3 CPD capillary surfaces

In this section, we prove Theorem 1.1 for capillary surfaces. By the expression of H in (5), Eq. (1) is written as

$$\frac{1}{2} \left(\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa(s)}{1 - f(t)\kappa(s)} \right) = \lambda (\langle \gamma(s), \mathbf{v} \rangle + (f(t)\langle \eta(s), \mathbf{v} \rangle + g(t)\langle \mathbf{d}, \mathbf{v} \rangle)) + \mu,$$

for all $s \in I, t \in J$. We differentiate with respect to s , obtaining

$$\frac{g'(t)\kappa'(s)}{(1 - f(t)\kappa(s))^2} = 2\lambda(1 - f(t)\kappa(s))\langle \gamma'(s), \mathbf{v} \rangle, \quad (6)$$

where we have used $\gamma''(s) = \kappa(s)\eta(s)$. We discuss according to κ .

1. Case $\kappa'(s) = 0$ for every $s \in I$. Then, the curvature function κ is constant. Moreover, Eq. (6) implies $\langle \gamma'(s), \mathbf{v} \rangle = 0$, that is, $\langle \gamma(s), \mathbf{v} \rangle$ is a constant function. From Proposition 2.1, we know that M is a cylindrical surface whose rulings are orthogonal to $\mathbf{d}$ or M is a surface of revolution whose axis is parallel to $\mathbf{d}$.

(a) Case $\kappa = 0$. Then, η is a constant vector and Eq. (1) becomes

$$\frac{1}{2} \frac{f''(t)}{g'(t)} = \lambda \langle f(t)\eta + g(t)\mathbf{d}, \mathbf{v} \rangle + \lambda \langle \gamma(s), \mathbf{v} \rangle + \mu.$$

Define $\alpha(t) = f(t)\eta + g(t)\mathbf{d}$ and $\tilde{\mu} = \lambda \langle \gamma(s), \mathbf{v} \rangle + \mu$. Let us observe that $\tilde{\mu}$ is constant. On the other hand, α is parametrized by the arc length and its curvature $\tilde{\kappa}$ satisfies

$$\frac{\tilde{\kappa}(t)}{2} = \lambda \langle \alpha(t), \mathbf{v} \rangle + \tilde{\mu}.$$

Consequently, the planar curve α satisfies the one-dimensional case of Eq. (1) as a linear combination of its height function.

- (b) Case $\kappa \neq 0$. We know that M is a rotational surface whose axis is parallel to $\mathbf{d}$. Since $\langle \gamma'(s), \mathbf{v} \rangle = 0$, a differentiation with respect to s yields $\kappa(s)\langle \eta(s), \mathbf{v} \rangle = 0$. We deduce that $\langle \eta(s), \mathbf{v} \rangle = 0$, and it follows that $\mathbf{v}$ is orthogonal to both γ' and η . This says that $\mathbf{d}$ is parallel to $\mathbf{v}$. By Eq. (1), the generating curve $\alpha(t) = f(t)\eta + g(t)\mathbf{d}$ satisfies the ODE

$$\frac{1}{2} \left(\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa}{1 - f(t)\kappa} \right) = \epsilon \lambda g(t) + b$$

where $\epsilon = \langle \mathbf{d}, \mathbf{v} \rangle = \pm 1$ and $b = \lambda \langle \gamma(s), \mathbf{v} \rangle + \mu$ is a constant.

2. Case $\kappa'(s_0) \neq 0$ at some $s_0 \in I$. In particular, $\kappa'(s) \neq 0$ in a subinterval I' of I around $s = s_0$. By simplicity, we assume $I' = I$. We now write (6) as

$$2 \frac{\lambda \langle \gamma'(s), \mathbf{v} \rangle}{\kappa'(s)} = \frac{g'(t)}{(1 - f(t)\kappa(s))^3}.$$

By differentiating with respect to t , we have

$$g''(t)(1 - f(t)\kappa(s)) + 3g'(t)f'(t)\kappa(s) = 0. \quad (7)$$

We distinguish two possibilities:

- (a) Subcase $g''(t) \neq 0$ in some $t_0 \in J$. Then, $g''(t) \neq 0$ in an interval $K \subset J$ around $t = t_0$. In $I \times K$, we can write (7) as

$$\frac{1}{\kappa(s)} = f(t) - 3 \frac{g'(t)f'(t)}{g''(t)}.$$

Since the left-hand side depends only on s and the right-hand side on t , then they are constant functions; in particular, $1/\kappa(s)$ is constant, a contradiction because $\kappa'(s) \neq 0$.

- (b) Subcase $g''(t) = 0$ for every t . From (7), we have $f'(t)\kappa(s) = 0$ for every $s \in I$, $t \in J$. Since $\kappa'(s) \neq 0$, then $\kappa(s) \neq 0$ for some $s \in I$, and thus $f'(t) = 0$ for every $t \in J$. Hence, $g'(t) = \epsilon \in \{-1, 1\}$, f is a constant function $f(t) = a$, $a \in \mathbb{R}$, and $g(t) = \epsilon t + b$, $b \in \mathbb{R}$. Parametrization (4) of M becomes

$$X(s, t) = \gamma(s) + a\eta(s) + (\epsilon t + b)\mathbf{d}.$$

This proves that M is a cylindrical surface whose base curve is the planar curve $\gamma_a(s) = \gamma(s) + a\eta(s)$ and the rulings are parallel to $\mathbf{d}$. Under this situation, Eq. (1) becomes

$$\frac{\epsilon\kappa(s)}{2(1 - a\kappa(s))} = \lambda(\langle \gamma_a(s), \mathbf{v} \rangle + (\epsilon t + b)\langle \mathbf{d}, \mathbf{v} \rangle) + \mu.$$

The derivative with respect to t gives $\lambda\epsilon\langle \mathbf{d}, \mathbf{v} \rangle = 0$, and we deduce that $\mathbf{d}$ is orthogonal to $\mathbf{v}$. Eq. (1) simplifies into

$$\frac{\kappa(s)}{1 - a\kappa(s)} = 2\epsilon\lambda\langle \gamma_a(s), \mathbf{v} \rangle + 2\epsilon\mu. \quad (8)$$

Finally, a simple computation gives that the curvature of γ_a is $\kappa_{\gamma_a} = \frac{\epsilon\kappa(s)}{1 - a\kappa(s)}$.

Thus, from (8), we deduce that γ_a satisfies the one-dimensional case of (1).

We point out that the one-dimensional case of capillary Eq. (1) and the rotational capillary surfaces are well known. For the first one, we refer [18, pp. 1130–40], also [14]. A detailed description of the rotational capillary surfaces lies in [7].

4 CPD λ -translating solitons

We now give the proof of Theorem 1.1 for CPD λ -translating solitons. In order to compute (2), we need to know $\langle N(p), \mathbf{v} \rangle$. By Proposition 2.2, we find

$$\langle N(s, t), \mathbf{v} \rangle = -g'(t)\langle \eta(s), \mathbf{v} \rangle + f'(t)\langle \mathbf{d}, \mathbf{v} \rangle.$$

By combining (2) and (5), we obtain

$$\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa(s)}{1 - f(t)\kappa(s)} = -2\lambda g'(t)\langle \eta(s), \mathbf{v} \rangle + 2\lambda f'(t)\langle \mathbf{d}, \mathbf{v} \rangle + 2\mu,$$

for all $s \in I$, $t \in J$. By taking the derivative with respect to s , and using that $g'(t) \neq 0$, we get

$$\frac{\kappa'(s)}{(1 - f(t)\kappa(s))^2} = -2\lambda \langle \eta'(s), \mathbf{v} \rangle = 2\lambda \kappa(s) \langle \gamma'(s), \mathbf{v} \rangle.$$

We differentiate this expression with respect to t , obtaining now

$$\frac{2\kappa(s)\kappa'(s)f'(t)}{(1 - f(t)\kappa(s))^3} = 0.$$

Then, $(\kappa^2(s))' f'(t) = 0$. We discuss two cases.

1. Suppose there exists $t_0 \in J$ such that $f'(t_0) \neq 0$. Then, κ is a constant function. In such a case, by Proposition 2.1 we conclude that M is either a cylindrical surface whose rulings are orthogonal to $\mathbf{d}$ or M is a surface of revolution with axis parallel to $\mathbf{d}$.

(a) Case $\kappa = 0$. Now η is a constant vector. Equation (2) is now

$$\frac{f''(t)}{g'(t)} = -2\lambda g'(t)\langle \eta(s), \mathbf{v} \rangle + 2\lambda f'(t)\langle \mathbf{d}, \mathbf{v} \rangle + 2\mu. \quad (9)$$

Define the planar curve $\alpha(t) = f(t)\eta + g(t)\mathbf{d}$ is a planar curve. A computation of its principal normal vector N_α gives $N_\alpha(s) = -g'(t)\eta + f'(t)\mathbf{d}$ and its curvature is $\kappa_\alpha(t) = \frac{f''(t)}{g'(t)}$. Thus, (9) is written as

$$\kappa_\alpha(t) = 2\lambda \langle N_\alpha(t), \mathbf{v} \rangle + 2\mu,$$

proving that α satisfies the one-dimensional case of (2).

- (b) Case $\kappa \neq 0$. Then, $\langle \gamma'(s), \mathbf{v} \rangle = 0$, that is, $\langle \gamma(s), \mathbf{v} \rangle$ is a constant function. Moreover, taking another derivative with respect to s , we find $\kappa \langle \eta(s), \mathbf{v} \rangle = 0$, so $\mathbf{v}$ is orthogonal to both $\gamma'(s)$ and $\eta(s)$ which proves that $\mathbf{v}$ is parallel to $\mathbf{d}$. Now Eq. (2) becomes

$$\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa}{1 - f(t)\kappa} = 2\epsilon \lambda f'(t) + 2\mu,$$

with $\epsilon = \langle \mathbf{d}, \mathbf{v} \rangle = \pm 1$.

2. Case $f'(t) = 0$ for every $t \in J$. Then, f is a constant function $f(t) = a$, $a \in \mathbb{R}$. From $f'(t)^2 + g'(t)^2 = 1$, we deduce that $g(t) = \epsilon t + b$, $\epsilon \in \{-1, 1\}$, $b \in \mathbb{R}$. The parametrization X in (4) is now

$$X(s, t) = (\gamma(s) + a\eta(s)) + (\epsilon t + b)\mathbf{d}.$$

This proves that M is a cylindrical surface over the planar curve $\gamma_a(s) = \gamma(s) + a\eta(s)$ and the rulings are parallel to $\mathbf{d}$. Now Eq. (2) becomes

$$\frac{\kappa(s)}{1 - a\kappa(s)} = -2\lambda \langle \eta(s), \mathbf{v} \rangle + 2\mu\epsilon. \quad (10)$$

Again the left-hand side of (10) is the curvature κ_{γ_a} of γ_a , and consequently (10) asserts that γ_a satisfies the one-dimensional case of (2).

We point out that the rotational surfaces that are λ -translating solitons are classified in [3] if $\mu = 0$ and in [15] when $\mu \neq 0$.

5 CPD λ -shrinker surfaces

We now give the proof of Theorem 1.1 for CPD λ -shrinker surface. From the expression of N in Proposition 2.2, we have

$$\begin{aligned}\langle N(p), p \rangle &= \langle -g'(t)\eta(s) + f'(t)d, \gamma(s) + (f(t)\eta(s) + g(t)d) \rangle \\ &= -g'(t)\langle \eta(s), \gamma(s) \rangle - f(t)g'(t) + f'(t)g(t).\end{aligned}$$

By the expression of H in (5), Eq. (3) is written as

$$\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa(s)}{1 - f(t)\kappa(s)} = -2\lambda g'(t)\langle \eta(s), \gamma(s) \rangle - 2\lambda f(t)g'(t) + 2\lambda f'(t)g(t) + 2\mu,$$

for all $s \in I$, $t \in J$. Since $g'(t) \neq 0$, we divide the above expression by $g'(t)$, obtaining

$$\frac{f''(t)}{g'(t)^2} + \frac{\kappa(s)}{1 - f(t)\kappa(s)} = -2\lambda\langle \eta(s), \gamma(s) \rangle - 2\lambda f(t) + 2\lambda \frac{f'(t)g(t)}{g'(t)} + 2\frac{\mu}{g'(t)}.$$

We observe that the right-hand side in the above equation is a sum of a function depending only on s and other function depending only on t . Thus, if we differentiate with respect to s and then with respect to t , we obtain 0. Therefore, looking at the left-hand side, we conclude

$$0 = \frac{2\kappa(s)\kappa'(s)f(t)}{(1 - f(t)\kappa(s))^3} = \frac{(\kappa^2(s))'f(t)}{(1 - f(t)\kappa(s))^3}.$$

Then, $(\kappa^2(s))'f(t) = 0$. Now the discussion follows the next two cases.

1. Suppose there exists $t_0 \in J$ such that $f'(t_0) \neq 0$. Then, κ is a constant function. In such a case, by Proposition 2.1, we conclude that M is either a cylindrical surface whose rulings are orthogonal to $\mathbf{d}$ or M is a surface of revolution with axis parallel to $\mathbf{d}$. By the above equations, κ constant implies that the function $\langle \gamma(s), \eta(s) \rangle$ is constant.

- (a) Case $\kappa = 0$. In particular, γ' and η are constant coplanar vectors and mutually orthogonal. Define the curve $\alpha(t) = f(t)\eta + g(t)\mathbf{d}$. The principal normal vector of α is $N_\alpha(t) = -g'(t)\eta + f'(t)\mathbf{d}$. Equation (3) is written as

$$\frac{f''(t)}{g'(t)} = 2\lambda\langle N_\alpha(t), \gamma(s) + \alpha(t) \rangle + 2\mu.$$

Let us observe that $\langle N_\alpha(t), \gamma(s) \rangle$ does not depend on s and $\kappa_\alpha(t) = f''(t)/g'(t)$ is the curvature of α . Consequently, α satisfies the one-dimensional case of (3).

- (b) Case $\kappa \neq 0$. Then, γ is a circle. Furthermore, we have

$$-\kappa\langle \gamma(s), \gamma'(s) \rangle = \langle \gamma(s), \eta'(s) \rangle = \langle \gamma(s), \eta(s) \rangle' = 0,$$

which implies that $\langle \gamma(s), \gamma'(s) \rangle = 0$. This shows that γ is a circle with center in the origin. Taking another derivative, we find

$$\langle \gamma(s), \gamma'(s) \rangle' = \langle \gamma(s), \gamma''(s) \rangle + \langle \gamma'(s), \gamma'(s) \rangle = \kappa\langle \gamma(s), \eta(s) \rangle + 1 = 0.$$

Therefore, $\langle \gamma(s), \eta(s) \rangle = -1/\kappa$. Thus, M is a rotational surface whose axis of revolution passes through the origin. Moreover, the generating curve satisfies

$$\frac{f''(t)}{g'(t)} + \frac{g'(t)\kappa}{1 - f(t)\kappa} = (2/\kappa)\lambda g'(t) - 2\lambda f(t)g'(t) + 2\lambda f'(t)g(t) + 2\mu.$$

2. Case $f'(t) = 0$ for every $t \in J$. Then, f is a constant function $f(t) = a$, $a \in \mathbb{R}$. From $f'(t)^2 + g'(t)^2 = 1$, we deduce that $g(t) = \epsilon t + b$, $\epsilon \in \{-1, 1\}$, $b \in \mathbb{R}$. From (4), the parametrization X is written as

$$X(s, t) = (\gamma(s) + a\eta(s)) + (\epsilon t + b)\mathbf{d}.$$

This proves that M is a cylindrical surface over the planar curve $\gamma_a = \gamma(s) + a\eta(s)$ and the rulings are parallel to $\mathbf{d}$. Finally, Eq. (3) becomes

$$\frac{\kappa(s)}{1 - a\kappa(s)} = -2\lambda\langle\eta(s), \gamma(s)\rangle - 2\lambda a + 2\epsilon\mu = 2\lambda\langle\eta(s), \gamma_a(s)\rangle + 2\epsilon\mu.$$

Because the curvature of γ_a is $\kappa_{\gamma_a} = \kappa(s)/(1 - a\kappa(s))$, then γ_a satisfies the one-dimensional case of (3).

The cylindrical shrinker surfaces are classified in [2].

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