Weakly complete domains in Grauert-type surfaces

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Abstract

The aim of this short note is to investigate the geometry of weakly complete subdomains of Grauert-type surfaces, i.e., open connected sets D , sitting inside a Grauert-type surface X , which admit a smooth plurisubharmonic exhaustion function. We prove that they are either modifications of Stein spaces or Grauert-type surfaces themselves, and we apply these results to the special case of Hopf surfaces.

Keywords Weakly complete · Grauert-type surfaces · Levi problem

Mathematics Subject Classification 32C40 · 32E05 · 32U10

1 Introduction

A weakly complete space is a complex space which admits a plurisubharmonic exhaustion function; if the function is strictly plurisubharmonic outside a compact set, then the space is a modification of a Stein space.

Understanding the geometry of weakly complete spaces can be thought as a generalization of the classical Levi problem; in [4], Ślodkowski, Tomassini and the author classified weakly complete surfaces which admit a real analytic plurisubharmonic exhaustion function. We refer to [3] for a review of some examples and the main steps of the proof; the result, loosely stated, says that such a complex surface is either holomorphically convex or the regular level sets of the exhaustion are Levi-flat hypersurfaces foliated in dense complex curves.

These last instances are called Grauert-type surfaces.

The aim of this short note is to investigate the geometry of weakly complete subdomains of Grauert-type surfaces, i.e., open connected sets D , sitting inside a Grauert-type surface X , which admit a smooth plurisubharmonic exhaustion function.

Instances of this question can be found, for example, in the works of Miebach (see [2]) and Levenberg and Yamaguchi (see [1]), where the problem is raised for Hopf surfaces.

It turns out that a weakly complete domain in a Grauert-type surface can either be a modification of a Stein space or a Grauert-type surface itself; in particular, this implies a kind

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of stability result, namely that, if the ambient space X admits a real analytic plurisubharmonic exhaustion function, then every weakly complete domain $\Omega \subseteq X$ can be endowed with a *real analytic* plurisubharmonic exhaustion function.

The proof we present relies on some observations that hold true not only for Grauert-type surfaces, but also for holomorphically convex ones; therefore, we present our results for any weakly complete surface with a real analytic plurisubharmonic exhaustion function.

2 The kernel of a weakly complete space

Let X be a weakly complete complex surface, endowed with a real analytic plurisubharmonic exhaustion function $\alpha : X \rightarrow \mathbb{R}$; suppose also that $D \subseteq X$ is a weakly complete domain, with an exhaustion function $\phi : D \rightarrow \mathbb{R}$ which is plurisubharmonic and smooth.

Following [6], we denote by Σ_X^1 and Σ_D^1 their kernels; namely, Σ_X^1 is the set of points of X where every plurisubharmonic exhaustion function fails to be strictly plurisubharmonic.

By [6, Lemma 3.1], we can suppose that both α and ϕ are minimal for X and D , respectively, i.e., that Σ_X^1 is exactly the set of points where α is not strictly plurisubharmonic and Σ_D^1 is exactly the set of points where ϕ is not strictly plurisubharmonic. Moreover, by [4, Proposition 3.1], we can assume that both α and ϕ are nonnegative.

The following lemma collects some properties of kernels and minimal functions that we will need later on.

Lemma 2.1 *Let X , D , α , ϕ be as above. Then*

- (1) $\Sigma_D^1 \subseteq \Sigma_X^1$
- (2) for every $p \in \Sigma_D^1$ which is a regular point for ϕ , $d\alpha(p) = \lambda_p d\phi(p)$
- (3) for every $p \in \Sigma_D^1$, $d\phi(p)$ vanishes on the kernel of $\partial\bar{\partial}\phi(p)$.

Proof Take a point $p \in D$; if $p \notin \Sigma_X^1$, then the Levi form of α is positive definite at p , so $\alpha + \phi$ is a plurisubharmonic exhaustion function for D which is strictly plurisubharmonic at p ; hence, $p \notin \Sigma_D^1$. This proves (1).

Now, observe that, if ψ is a nonnegative plurisubharmonic function and q is a regular point for ψ , then

$$\partial\bar{\partial}\psi^2 = 2(\partial\psi \wedge \bar{\partial}\psi + \psi\partial\bar{\partial}\psi)$$

is again positive semidefinite and the kernel of the associated linear map is given by the intersection of the kernel of $\partial\bar{\partial}\phi$ and the kernel of $\partial\psi$ (or, as ψ is real valued, of $d\psi$). Therefore, if $p \in \Sigma_D^1 \subseteq \Sigma_X^1$, then both ϕ and ϕ^2 must both have degenerate Levi forms at p ; hence, $d\phi$ has to vanish on the kernel of $\partial\bar{\partial}\phi$. So (3) is proved.

Now, consider $\phi^2 + \alpha^2$. In order for its Levi form to be degenerate at p , the kernels of $d\phi$ and $d\alpha$ and the kernels of $\partial\bar{\partial}\phi$ and $\partial\bar{\partial}\alpha$ must all coincide; hence, (2) follows. $\square$

From the Main Theorem in [4], in particular it follows that, for every $p \in \Sigma_X^1$, regular point for α , there exist an irreducible complex curve F_p and an injective holomorphic map $i_p : F_p \rightarrow X$ such that $p \in C_p = i(F_p) \subseteq \Sigma_X^1 \cap \{\alpha = \alpha(p)\}$. Moreover, $T_p C_p$, if defined, is the kernel of $\partial\bar{\partial}\alpha(p)$.

Proposition 2.2 *Consider $p \in \Sigma_D^1$ and suppose that $\phi(p)$ is a regular value for ϕ . Then $C_p \subseteq \Sigma_D^1$.*

Proof By [6, Lemma 4.1], $\Sigma_D^1 \cap \{\phi = \phi(p)\}$ is foliated in complex curves; let S_p be the leaf passing through p . By Lemma 2.1-(2), as ϕ is constant along S_p , so it is α . Therefore, the immersed complex curve S_p sits in $\Sigma_D^1 \cap \{\phi = \phi(p)\}$ as well as in $\Sigma_X^1 \cap \{\alpha = \alpha(p)\}$. But the latter, by the Main Theorem in [4], is either union of a finite number of irreducible compact curves or it is a real three-dimensional space foliated with complex leaves, which are then the only complex curves in it. Therefore, $S_p = C_p$. $\square$

Putting these observations together with the geometric structure of X given by the Main Theorem in [4], we obtain the result we claimed.

Theorem 2.3 *Let X be a weakly complete complex surface endowed with a real analytic plurisubharmonic exhaustion function $\alpha : X \rightarrow \mathbb{R}$; consider $D \subseteq X$ a domain with a C^∞ plurisubharmonic exhaustion function $\phi : D \rightarrow \mathbb{R}$. Then, one of the following is true:*

- (1) D is a modification of a Stein space
- (2) both D and X are union of complex curves
- (3) both D and X are Grauert-type surfaces.

Proof We separate two cases.

First case

If we find a point $p \in \Sigma_D^1$ and a minimal function ψ for D such that the level set

$$Z_\psi(p) = \{q \in D : \psi(q) = \psi(p)\}$$

is regular for ψ , then, by Proposition 2.2, $C_p \subseteq \Sigma_D^1 \subseteq D$.

Obviously, also the closure of C_p will be contained in D ; in the case of Grauert-type surfaces, this implies that D contains the connected component of the level set of α which contains p .

Now, if X is fibered in compact curves, we have a (holomorphic) proper map with connected fibers $F : X \rightarrow T$, where T is an open connected Riemann surface. If X is a surface of Grauert type with $M = \min \alpha$ of real dimension 2, then there is a (pluriharmonic) proper map [4, Main Theorem] with connected fibers [5, Theorem 3.1] $F : X \setminus M \rightarrow T$, where T is the real line. If M has real dimension 3, up to passing to a double cover X^* , we find a (pluriharmonic) proper map with connected fibers $F : X^* \rightarrow T$, where T is an open connected subset of the real line.

In all these three scenarios, we have a proper map with connected fibers $F : X^* \rightarrow T$, where X^* is either X , $X \setminus M$ or a double cover of X and T is either an open connected Riemann surface or an open connected subset of the real line.

We define

$$\Omega = \{x \in T : F^{-1}(x) \subseteq D\}$$

and, by the first part of this proof, we know that $\Omega \neq \emptyset$; we also set $V = F(D)$. We know that V is open and connected.

Consider a sequence $\{x_n\} \subset \Omega$, converging to a point $x \in V$. If the fibers of F are irreducible compact complex curves, then obviously any plurisubharmonic function is constant on them. If the fibers of F are smooth Levi-flat hypersurfaces with dense complex leaves, then again any plurisubharmonic function is constant on them (see the end of the proof of [4, Corollary 3.8]); α being real analytic and proper, its critical values cannot accumulate, so, up to passing to a subsequence, we can suppose that $F^{-1}(x_n)$ is always contained in a smooth Levi-flat hypersurface. So, in both cases, as $F^{-1}(x_n)$ is contained in D , then ϕ is constant on it, for every n .

For every $p \in F^{-1}(x) \cap D$ (which is nonempty), there is a sequence of points p_n such that $p_n \in F^{-1}(x_n)$, $p_n \rightarrow p$. This implies that ϕ is constant on $F^{-1}(x) \cap D$, but then $F^{-1}(x) \subset D$, because ϕ is an exhaustion function and the fibers of F are compact and connected. Hence, Ω is closed in V .

On the other hand, if $x \in \Omega$, then ϕ is constant on $F^{-1}(x)$, which is then a compact subset of D . By an easy topological argument, there exists a neighborhood U of $x \in T$ such that $F^{-1}(U) \subseteq D$, i.e., $U \subseteq \Omega$, i.e., Ω is open in V .

Therefore, Ω is both open and closed in V , which is connected; so $\Omega = V$, i.e., $D = F^{-1}(V)$.

Second case

Suppose that for every point $p \in \Sigma_D^1$, the level set $Z_\phi(p)$ is singular. By Sard's theorem, there exists a sequence $\{c_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ of regular values of ϕ such that $c_n \rightarrow +\infty$; hence, there exists a sequence of regular level sets $Z_\phi(p_n)$ with $p_n \rightarrow bD$.

By hypothesis, none of these level sets intersects Σ_D^1 , so, ϕ being minimal, all these levels are strongly pseudoconvex. By a standard argument, this implies that D is a modification of a Stein space. $\square$

Remark 2.1 All the contents of this section, apart from the second case in the previous theorem, hold when ϕ is supposed to be only C^2 ; however, in order to apply Sard's theorem, we need ϕ to be at least C^4 .

3 Applications

One special case of weakly complete surfaces with a real analytic plurisubharmonic exhaustion function is Hopf surfaces with one compact curve removed (see [4, Section 2] for a worked example). Applying Theorem 2.3 to this particular case, we obtain a different proof for the Main Theorem in [2] and Theorem 1.1. in [1], under the hypothesis of the existence of a smooth plurisubharmonic exhaustion.

More generally, suppose X is the total space of a topologically trivial line bundle over a compact Riemann surface M ; if the class in $H^1(M, S^1)$ associated with the bundle is unipotent, then X is fibered in compact curves; otherwise, it is a Grauert-type surface (see [4, Section 2] and [5, Section 4] for related discussions and examples).

In both cases, we have a minimal plurisubharmonic function whose only critical level is the minimum level, coinciding with M ; so, any open set containing M will also contain a regular level set of the exhaustion function.

Combining these observations with Theorem 2.3, we obtain the following

Corollary 3.1 *Suppose X is the total space of a topologically trivial line bundle ξ over a compact Riemann surface M and let $D \subset X$ be a weakly complete domain with a smooth plurisubharmonic exhaustion. Then one of the following happens:*

- (1) D is Stein
- (2) both D and X are fibered in complex curves
- (3) both D and X are Grauert-type surfaces.

Proof From Theorem 2.3 we know that, if cases (2) and (3) don't apply, then D is a modification of a Stein space. However, if D were not Stein, then it would contain some compact curves. If X is a surface of Grauert type, the only compact curve in it is M , which doesn't have any Stein neighborhood; on the other hand, if X is fibered in complex curves, as soon

as D contains one of them, it is a union of curves, repeating the argument in the end of the first case in the proof of Theorem 2.3.

Therefore, D is Stein. $\square$

Remark 3.1 The proof we have just given applies whenever X is a weakly complete surface which admits a real analytic plurisubharmonic exhaustion function whose only critical level set is the minimum level set.

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