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Existence results for a supercritical Neumann problem with a convex–concave non-linearity

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Abstract

We shall consider the following semi-linear problem with a Neumann boundary condition

$$-\Delta u + u = a(|x|)|u|^{p-2}u - b(|x|)|u|^{q-2}u, \quad x \in B_1,$$

where B_1 is the unit ball in $\mathbb{R}^N$, $N \geq 2$, a, b are nonnegative radial functions, and p, q are distinct numbers greater than or equal to 2. We shall assume no growth condition on p and q. Our plan is to use a new variational principle that allows one to deal with problems with supercritical Sobolev non-linearities. Indeed, we first find a critical point of the Euler–Lagrange functional associated with this equation over a suitable closed and convex set. Then we shall use this new variational principle to deduce that the restricted critical point of the Euler–Lagrange functional is an actual critical point.

Keywords Semi-linear elliptic problems · Calculus of variations · Variational principles

Mathematics Subject Classification 35J15 · 58E30

1 Introduction

In this paper, we consider the Neumann problem

$$\begin{cases} -\Delta u + u = a(|x|)|u|^{p-2}u - b(|x|)|u|^{q-2}u, & x \in B_1 \\ u > 0, & x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{1}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 2$ and p, q are distinct numbers greater than or equal to 2. The functions a and b are assumed to be nonnegative and radially

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monotone. We shall assume no growth conditions on p and q, and therefore, this problem can be supercritical.

Our plan is to prove the existence of at least one positive radially non-decreasing solution for problem (1). We would like to remark that there have been several studies on the existence of positive radially non-decreasing solutions for problem (1) when the function b is identically zero, i.e.,

$$\begin{cases} -\Delta u + u = a(|x|)|u|^{p-2}u, & x \in B_1 \\ u > 0, & x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \qquad (2)$$

In most of these studies, when p is supercritical, the authors took advantage of the fact that the function on the right-hand side of equation (2) is an increasingly monotone function in terms of u. However, this property is lost in (1) as the right-hand side of this equation is the difference of two monotone functions in terms of u. Here is one of the main results in this paper.

Theorem 1.1 *Let p and q be two distinct real numbers with $p > q \geq 2$. Assume that functions a and b satisfy the following conditions:*

H1. $a \in L^1(0, 1)$ is non-decreasing and $a(r) > 0$ for a.e. $r \in [0, 1]$.
H2. $b \in L^1(0, 1)$ is nonnegative and non-increasing on $[0, 1]$.

Then problem (1) admits at least one radially non-decreasing positive solution.

We shall show that a similar result as in Theorem 1.1 also holds for $p < q$ (see Theorem 3.7). Here is an example satisfying the assumptions of Theorem 1.1:

$$\begin{cases} -\Delta u + u = |x|^\alpha |u|^{p-2}u - \frac{\mu}{|x|^\beta}|u|^{q-2}u, & x \in B_1 \\ u > 0, & x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial B_1. \end{cases}$$

where $\alpha, \mu \geq 0$ and $0 \leq \beta < N$.

To prove Theorem 1.1, we utilize an abstract variational principle from [26] (see also [27]). To be more specific, let V be a reflexive Banach space, V^* its topological dual and let K be a non-empty convex and weakly closed subset of V. Assume that $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semi-continuous function which is Gâteaux differentiable on K. The Gâteaux derivative of Ψ at each point $u \in K$ will be denoted by $D\Psi(u)$. The restriction of Ψ to K is denoted by Ψ_K and defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \qquad (3)$$

For a given functional $\Phi \in C^1(V, \mathbb{R})$ denote by $D\Phi \in V^*$ its derivative and consider the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K(u) := \Psi_K(u) - \Phi(u).$$

According to Szulkin [30], we have the following definition for critical points of I_K.

Definition 1.2 A point $u_0 \in V$ is said to be a critical point of I_K if $I_K(u_0) \in \mathbb{R}$ and if it satisfies the following inequality

$$\langle D\Phi(u_0), u_0 - v \rangle + \Psi_K(v) - \Psi_K(u_0) \geq 0, \qquad \forall v \in V, \qquad (4)$$

where $\langle ., . \rangle$ is the duality pairing between V and its dual V^*.

We also recall the notion of the point-wise invariance condition from [26].

Definition 1.3 We say that the triple (Ψ, K, Φ) satisfies the point-wise invariance condition at a point $u_0 \in V$ if there exists a convex Gâteaux differentiable function $G : V \to \mathbb{R}$ and a point $v_0 \in K$ such that

$$D\Psi(v_0) + DG(v_0) = D\Phi(u_0) + DG(u_0).$$

We shall now recall the following variational principle recently established in [26] (see also [27]).

Theorem 1.4 *Let V be a reflexive Banach space and K be a convex and weakly closed subset of V. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function which is Gâteaux differentiable on K and let $\Phi \in C^1(V, \mathbb{R})$. Assume that the following two assertions hold:*

(i) *The functional $I_K : V \to \mathbb{R} \cup \{+\infty\}$ defined by $I_K(u) := \Psi_K(u) - \Phi(u)$ has a critical point $u_0 \in V$ as in Definition 1.2, and;*

(ii) *the triple (Ψ, K, Φ) satisfies the point-wise invariance condition at the point u_0.*

Then $u_0 \in K$ is a solution of the equation

$$D\Psi(u) = D\Phi(u). \tag{5}$$

We would like to remark that a particular version of Theorem 1.4 has been successfully applied to the well-known Ambrosetti–Brezis–Cerami concave–convex problem [3] to obtain multiplicity results for supercritical non-linearities (see [24] for more details).

For the convenience of the reader, by choosing appropriate functions Ψ, Φ and a convex set K corresponding to our problem (1), we shall provide a proof to a particular case of Theorem 1.4 applicable to this problem.

We now recall some prior works related to (2). In [6], the authors considered the following problem

$$\begin{cases} -\Delta u + u = |u|^p, \ x \in B_1 \\ u > 0, \qquad\qquad x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, \qquad\qquad x \in \partial B_1, \end{cases} \tag{6}$$

and proved the existence of multiple radial solutions as $p \to +\infty$. They also showed the existence of radial solutions to (6) whose Laplacians weakly converge to measures concentrating at interior spheres, with a simple reflection rule. Also in [28], the authors considered a variant of (2) where $|u|^p$ is replaced with $a(|x|)f(u)$ where a and f are positive and non-decreasing functions. They proved the existence of a positive and radially non-decreasing solution for this problem when there is no growth condition on f. In [7], by using the topological and variational arguments, the authors treated a more general problem without imposing any growth condition on f and proved the existence of positive non-decreasing radial solutions. They have also addressed the existence of a non-constant solution in case the problem admits trivial constant solutions. These results were extended to the p-Laplace version in [13,29]. In recent papers [10,11], the authors studied the p-laplace version when the right-hand side of (6) is replaced with more general non-linearities and the domain is either a ball or an annulus. In particular, a conjecture about the existence of solutions with certain qualitative properties (raised in [7]) was answered in [10]. The methods of [23] were extended to prove results regarding multilayer radials solutions in [6]. The next work related to (6) is [8] where they considered

$$\begin{cases} -\Delta u + u = g(|x|)|u|^{p-1}, \ x \in \Omega \\ u > 0, \qquad\qquad\qquad x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \qquad\qquad\qquad x \in \partial\Omega, \end{cases} \tag{7}$$

where Ω is an annulus in $\mathbb{R}^N$:

$$\Omega := \{x \in \mathbb{R}^N : a < |x| < b\},$$

and $g \in L^1(a, b)$, $g(r) > 0$ a.e. in (a, b) and $p > 2$. They proved that for every p large enough, problem (7) admits at least three distinct redial solutions. Also in [9] the elliptic system of the form

$$\begin{cases} -\Delta u + u = a(|x|)f(u, v), \ x \in B_1 \\ -\Delta v + v = \tilde{a}(|x|)g(u, v), \ x \in B_1 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad\qquad x \in \partial B_1, \end{cases} \tag{8}$$

has been considered, where $a, \tilde{a}$ are assumed to be radially, nonnegative and non-decreasing weights and f, g are non-decreasing in each component. With some assumptions on f and g, they proved the existence of at least one couple of non-decreasing non-trivial solutions for (8).

Neumann boundary problems with subcritical non-linearities by means of Sobolev spaces have been studied in several papers, among which we refer to [1,2,4,20,21] and the references therein.

Note that as the right-hand side of equation (2) is an non-decreasing function of u, one can write it as a gradient of a convex function, namely φ. Now if φ^* is the Fenchel dual of φ, then by considering the critical points of the functional

$$I(u) = \varphi^*(Au) - \varphi(u),$$

where $Au = -\Delta u + u$, one can prove the existence of a solution for the equation (See [14,15,25,26]). We shall refer to [14] in particular where this new functional I was used to study problem (2) for non-radial domains.

The paper is organized as follows. In the next section, we shall recall some preliminaries from convex analysis and non-smooth critical point theory. Section 3 is devoted to the proof of our main results. In the last section, we shall also discuss the case of non-constant solutions when the functions a and b are constants.

2 Preliminaries

In this section, we recall some important definitions and results from convex analysis and minimax principles for lower semi-continuous functions.

Let V be a real Banach space and V^* its topological dual and let $\langle ., . \rangle$ be the pairing between V and V^*. The weak topology on V induced by $\langle ., . \rangle$ is denoted by $\sigma(V, V^*)$. A function $\Psi : V \to \mathbb{R}$ is said to be weakly lower semi-continuous if

$$\Psi(u) \leq \liminf_{n \to \infty} \Psi(u_n),$$

for each $u \in V$ and any sequence u_n approaching u in the weak topology $\sigma(V, V^*)$. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The subdifferential $\partial \Psi$ of Ψ is defined to be the following set-valued operator: If $u \in \text{Dom}(\Psi) = \{v \in V; \ \Psi(v) < +\infty\}$, set

$$\partial \Psi(u) = \left\{ u^* \in V^*; \langle u^*, v - u \rangle + \Psi(u) \leq \Psi(v) \text{ for all } v \in V \right\}$$

and if $u \notin \text{Dom}(\Psi)$, set $\partial \Psi(u) = \varnothing$. If Ψ is Gâteaux differentiable at u, denote by $D\Psi(u)$ the derivative of Ψ at u. In this case, $\partial \Psi(u) = \{D\Psi(u)\}$.

We shall now recall some notations and results for the minimax principles for lower semi-continuous functions.

Definition 2.1 Let V be a real Banach space, $\Phi \in C^1(V, \mathbb{R})$ and $\Psi : V \to (-\infty, +\infty]$ be proper (i.e., $\text{Dom}(\Psi) \neq \emptyset$), convex and lower semi-continuous. A point $u \in V$ is said to be a critical point of

$$I := \Psi - \Phi \tag{9}$$

if $u \in \text{Dom}(\Psi)$ and if it satisfies the inequality

$$\langle D\Phi(u), u - v \rangle + \Psi(v) - \Psi(u) \geq 0, \qquad \forall v \in V. \tag{10}$$

Definition 2.2 We say that I satisfies the Palais–Smale compactness condition (PS) if every sequence $\{u_n\}$ such that

- $I[u_n] \to c \in \mathbb{R}$,
- $\langle D\Phi(u_n), u_n - v \rangle + \Psi(v) - \Psi(u_n) \geq -\epsilon_n \|v - u_n\|, \qquad \forall v \in V.$

where $\epsilon_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

The following two theorems are due to Szulkin [30].

Theorem 2.3 (Mountain pass theorem) *Suppose that $I : V \to (-\infty, +\infty]$ is of the form* (9) *and satisfies the Palais–Smale condition and the mountain pass geometry (MPG):*

1. $I(0) = 0.$
2. *there exists $e \in V$ such that $I(e) \leq 0$.*
3. *there exists some ρ such that $0 < \rho < \|e\|$ and for every $u \in V$ with $\|u\| = \rho$ one has $I(u) > 0$.*

Then I has a critical value c which is characterized by

$$c = \inf_{g \in \Gamma} \sup_{t \in [0,1]} I[g(t)],$$

where $\Gamma = \{g \in C([0, 1], V) : g(0) = 0, g(1) = e\}$.

Theorem 2.4 *Suppose that $I : V \to (-\infty, +\infty]$ is of the form* (9) *and satisfies the Palais–Smale condition. If I is bounded from below, then $c = \inf_{u \in V} I(u)$ is a critical value.*

3 Existence results

In this section, we first prove an adapted version of Theorem 1.4 applicable specifically to our problem, and then, we proceed with the proof of our main results.

Throughout this section, we always assume that conditions *H1* and *H2* in Theorem 1.1 hold, and the real numbers p, q are greater than or equal to 2. Let $H^1_{\text{rad}}(B_1)$ be the space of radial function in the Sobolev space $H^1(B_1)$. Consider the Banach space $V = H^1_{\text{rad}}(B_1) \cap L^q_b(B_1) \cap L^p_a(B_1)$ with

$$L^p_a(B_1) := \left\{ u : \int a(|x|)|u|^p dx < \infty \right\}, \quad \& \quad L^p_b(B_1) := \left\{ u : \int b(|x|)|u|^p dx < \infty \right\},$$

equipped with the following norm

$$\|u\|_V = \|u\|_{H^1} + \|u\|_{L^p_a} + \|u\|_{L^q_b},$$

where

$$\|u\|_{L_a^p} = \left(\int_{B_1} a(|x|)|u|^p \mathrm{d}x \right)^{\frac{1}{p}}, \quad \& \quad \|u\|_{L_b^q} = \left(\int_{B_1} b(|x|)|u|^q \mathrm{d}x \right)^{\frac{1}{q}}.$$

Note that the duality pairing between V and its dual V^* is defined by

$$\langle u, u^* \rangle = \int_{\Omega} u(x)u^*(x)\,\mathrm{d}x, \quad \forall u \in V, \forall u^* \in V^*.$$

Our plan is to apply Theorem 1.4 to the Euler–Lagrange functional corresponding to problem (1), i.e.,

$$E(u) := \frac{1}{2} \int_{B_1} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x + \frac{1}{q} \int_{B_1} b(x)u^q \mathrm{d}x - \frac{1}{p} \int_{B_1} a(|x|)|u|^p \mathrm{d}x,$$

over the set

$$K = \left\{ u \in V : u \ge 0, u \text{ is non-decreasing with respect to the radius } r = |x| \right\}. \quad (11)$$

To adapt Theorem 1.4 to our case, define $\Psi : V \to \mathbb{R}$ and $\Phi : V \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{2} \int_{B_1} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x + \frac{1}{q} \int_{B_1} b(|x|)u^q \mathrm{d}x,$$

and

$$\Phi(u) = \frac{1}{p} \int_{B_1} a(|x|)|u|^p \mathrm{d}x.$$

We remark that even though Φ is not even well defined on $H^1(B_1)$, but it is continuously differentiable on the space $V = H^1_{\mathrm{rad}}(B_1) \cap L_b^q(B_1) \cap L_a^p(B_1)$. Finally, let us introduce the functional $E_K(u) : V \to (-\infty, +\infty]$ defined by

$$E_K(u) := \Psi_K(u) - \Phi(u) \quad (12)$$

where

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \quad (13)$$

Note that E_K is indeed the Euler–Lagrange functional corresponding to (1) restricted to K. Here is an adapted version of Theorem 1.4 applicable to our case.

Theorem 3.1 *Let $V = H^1_{\mathrm{rad}}(B_1) \cap L_b^q(B_1) \cap L_a^p(B_1)$ and let K be the convex and closed subset of V given in (11). Suppose the following two assertions hold:*

(i) *The functional $E_K : V \to \mathbb{R} \cup \{+\infty\}$ defined in (12) has a critical point $\bar{u} \in V$ as in Definition 2.1, and;*
(ii) *there exists $\bar{v} \in K$ with $\frac{\partial \bar{v}}{\partial \nu} = 0$ on the boundary of B_1 such that*

$$-\Delta \bar{v} + \bar{v} + b(x)|\bar{v}|^{q-2}\bar{v} = D\Phi(\bar{u}) = a(|x|)|\bar{u}|^{p-2}\bar{u},$$

in the weak sense, i.e.,

$$\int_{\Omega} \nabla \bar{v}.\nabla \eta \, \mathrm{d}x + \int_{\Omega} \left(\bar{v} + b(x)|\bar{v}|^{q-2}\bar{v}\right)\eta \, \mathrm{d}x = \int_{\Omega} a(|x|)|\bar{u}|^{p-2}\bar{u}\eta \, \mathrm{d}x, \quad \forall \eta \in V.$$

Then $\bar{u} \in K$ is a weak solution of the equation

$$-\Delta u + u = a(|x|)|u|^{p-2}u - b(x)|u|^{q-2}u, \tag{14}$$

with the Neumann boundary condition.

Proof Since $\bar{u}$ is a critical point of E_K, it follows from Definition 2.1 that

$$\langle D\Phi(\bar{u}), \bar{u} - v \rangle + \Psi_K(v) - \Psi_K(\bar{u}) \geq 0, \quad \forall v \in V. \tag{15}$$

On the other hand, by (ii), there exists $\bar{v} \in K$ satisfying

$$\begin{cases} -\Delta\bar{v} + \bar{v} + b(x)|\bar{v}|^{q-2}\bar{v} = a(|x|)|\bar{u}|^{p-2}\bar{u}, & x \in B_1 \\ \frac{\partial\bar{v}}{\partial\nu} = 0, & x \in \partial B_1, \end{cases} \tag{16}$$

in the weak sense. By setting $v = \bar{v}$ in (15), we obtain that

$$\frac{1}{2}\int_{B_1} \left(|\nabla\bar{v}|^2 + \bar{v}^2\right) dx + \frac{1}{q}\int_{B_1} b(x)|\bar{v}|^q dx - \frac{1}{2}\int_{B_1} \left(|\nabla\bar{u}|^2 + \bar{u}^2\right) dx - \frac{1}{q}\int_{B_1} b(x)|\bar{u}|^q dx$$

$$\geq \int_{B_1} a(|x|)|\bar{u}|^{p-2}\bar{u}(\bar{v} - \bar{u}) \, dx$$

$$= \int_{B_1} \nabla\bar{v}.\nabla(\bar{v} - \bar{u}) \, dx + \int_{B_1} \bar{v}(\bar{v} - \bar{u}) \, dx + \int_{B_1} b(x)|\bar{v}|^{q-2}\bar{v}(\bar{v} - \bar{u}) \, dx,$$

where the last line follows from (16). Therefore,

$$\frac{1}{2}\int_{B_1} |\nabla\bar{v} - \nabla\bar{u}|^2 \, dx + \frac{1}{2}\int_{B_1} |\bar{v} - \bar{u}|^2 \, dx$$

$$+ \frac{1}{q}\int_{B_1} b(x)(|\bar{u}|^q - |\bar{v}|^q) \, dx + \int_{B_1} b(x)|\bar{v}|^{q-2}\bar{v}(\bar{v} - \bar{u}) \, dx \leq 0. \tag{17}$$

Note that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = |t|^q/q$ is convex. Thus for all $t_1, t_2 \in \mathbb{R}$ we have that

$$g(t_2) - g(t_1) \geq g'(t_1)(t_2 - t_1) = |t_1|^{q-2}t_1(t_2 - t_1).$$

Substituting $t_2 = \bar{u}$ and $t_1 = \bar{v}$ in the latter inequality implies that

$$\frac{1}{q}|\bar{u}|^q - \frac{1}{q}|\bar{v}|^q \geq |\bar{v}|^{q-2}\bar{v}(\bar{u} - \bar{v}).$$

Multiplying both sides by $b(x)$ and integrating over B_1 yield that

$$\frac{1}{q}\int_{B_1} b(x)|\bar{u}|^q \, dx - \frac{1}{q}\int_{B_1} b(x)|\bar{v}|^q \, dx \geq \int_{B_1} b(x)|\bar{v}|^{q-2}\bar{v}(\bar{u} - \bar{v}) \, dx. \tag{18}$$

Now from (17) and (18), we obtain that

$$\frac{1}{2}\int_{B_1} |\nabla\bar{v} - \nabla\bar{u}|^2 \, dx + \frac{1}{2}\int_{B_1} |\bar{v} - \bar{u}|^2 \, dx \leq 0,$$

which implies that $\bar{u} = \bar{v}$. Taking into account that $\bar{u} = \bar{v}$ in (16), we have that $\bar{u}$ is a weak solution of

$$-\Delta u + u = a(|x|)|u|^{p-2}u - b(x)|u|^{q-2}u,$$

with the Neumann boundary condition. □

It is worth noting that condition $ii)$ in Theorem 3.1 indeed shows that the triple (Ψ, K, Φ) satisfies the point-wise invariance condition at $\bar{u}$ given in Definition 1.3. In fact, it corresponds to the case where $G = 0$. This is why Theorem 3.1 is a very particular case of the general Theorem 1.4.

We shall need some preliminary results before proving our main results. We begin with the following elementary lemma.

Lemma 3.2 *The following assertions hold:*

1. *There exists $C_0 > 0$ such that*

$$\|u\|_{L^\infty(B_1)} \leq C_0 \|u\|_{H^1(B_1)}, \qquad \forall u \in K.$$

2. *There exists a constant $C > 0$ such that*

$$\|u\|_{H^1(B_1)} \leq \|u\|_V \leq C \|u\|_{H^1(B_1)}, \qquad \forall u \in K. \tag{19}$$

Proof Let $0 < r < 1$ and B_r be a ball centered at the origin with radius r. It follows from the continuous embedding of $H^1(B_1 \setminus B_r) \subseteq L^\infty(B_1 \setminus B_r)$ that there exists a constant $C_0 > 0$ such that

$$\|u\|_{L^\infty(B_1)} = \|u\|_{L^\infty(B_1 \setminus B_r)} \leq C_0 \|u\|_{H^1(B_1 \setminus B_r)} \leq C_0 \|u\|_{H^1(B_1)}, \quad (\forall u \in K).$$

This completes the proof of the first part. For the second part, we have that

$$\|u\|_{H^1(B_1)} \leq \|u\|_V = \|u\|_{H^1(B_1)} + \|u\|_{L_a^p(B_1)} + \|u\|_{L_b^q(B_1)}$$
$$\leq \|u\|_{H^1(B_1)} + \|u\|_{L^\infty(B_1)}(\|a\|_{L^1(B_1)} + \|b\|_{L^1(B_1)}) \leq C\|u\|_{H^1(B_1)},$$

where $C = 1 + C_0(\|a\|_{L^1(B_1)} + \|b\|_{L^1(B_1)})$. $\square$

Lemma 3.3 *Let Ω be a bounded open domain in $\mathbb{R}^N$ and $q \geq 2$. Suppose that $f \in L^\infty(\Omega)$, $b \in L^1(\Omega)$ and $u \in H^1(\Omega) \cap L_b^q(\Omega)$ be such that*

$$\int_\Omega \nabla u.\nabla\varphi \mathrm{d}x + \int_\Omega u\varphi \mathrm{d}x + \int_\Omega b(x)|u|^{q-2}u\varphi \mathrm{d}x = \int_\Omega f\varphi \mathrm{d}x, \qquad \forall \varphi \in H^1(\Omega) \cap L_b^q(\Omega). \tag{20}$$

If the functions f and b are nonnegative, then there exists a constant $C > 0$ independent of u such that for a.e. $x \in \Omega$ we have

$$0 \leq u(x) \leq C\|f\|_{L^\infty(\Omega)}. \tag{21}$$

Proof Consider a function $\eta \in C^1(\mathbb{R})$ such that

(i) η is strictly increasing on the interval $[0, \infty)$,
(ii) $\eta(t) = 0, \qquad \forall t \in (-\infty, 0]$.
(iii) both η and its derivative η' are bounded.

We first show that $u(x) \geq 0$ for a.e. $x \in \Omega$. Set $\varphi = -\eta(-u)$. Note that $\varphi \in H^1(\Omega) \cap L_b^q(\Omega)$, so by (20) we have

$$\int_\Omega |\nabla u|^2 \eta'(-u)\mathrm{d}x - \int_\Omega u\eta(-u)\mathrm{d}x - \int_\Omega b(x)|u|^{q-2}u\eta(-u)\mathrm{d}x = -\int_\Omega f\eta(-u)\mathrm{d}x. \tag{22}$$

Note that $f \geq 0$ and $\eta(-u) \geq 0$. Thus from (22) one can deduce that

$$-\int_\Omega (u + b(x)|u|^{q-2}u)\eta(-u)\mathrm{d}x \leq 0.$$

On the other hand, for all $t \in \mathbb{R}$ and $x \in \Omega$, we have $(t + b(x)|t|^{q-2}t)\eta(t) \geq 0$. Therefore,

$$-(u + b(x)|u|^{q-2}u)\eta(-u) = 0, \quad a.e. \quad x \in \Omega.$$

So by the definition of η, we get $u(x) \geq 0$ for a.e. $x \in \Omega$.

We now prove the second inequality in (21). Let $b_0 = \inf_{x \in \Omega} b(x)$ and note that $b_0 \geq 0$. Set $M = \|f\|_{L^\infty(\Omega)}$. Since the map $t \to t + b_0|t|^{q-2}t - M$ is strictly increasing, there exists $C > 0$ such that $t + b_0|t|^{q-2}t - M \leq 0$ if and only if $t \leq CM$. Set $\varphi = \eta(u - CM)$. It follows from (iii) that $\varphi \in H^1(\Omega) \cap L_b^q(\Omega)$, so by (20) we have

$$\int_\Omega |\nabla u|^2 \eta'(u - CM)dx + \int_\Omega u\eta(u - CM)dx + \int_\Omega b(x)|u|^{q-2}u\eta(u - CM)dx$$
$$= \int_\Omega f\eta(u - CM)dx,$$

from which one has that

$$\int_\Omega |\nabla u|^2 \eta'(u - CM)dx + \int_\Omega (u - M)\eta(u - CM)dx + \int_\Omega b(x)|u|^{q-2}u\eta(u - CM)dx$$
$$= \int_\Omega (f - M)\eta(u - CM)dx. \tag{23}$$

Note that $f - M \leq 0$ and $\eta(u - CM) \geq 0$. Thus from (23) one can deduce that

$$\int_\Omega (u + b(x)|u|^{q-2}u - M)\eta(u - CM)dx \leq 0.$$

Since u is nonnegative, it follows that

$$\int_\Omega (u + b_0|u|^{q-2}u - M)\eta(u - CM)dx \leq \int_\Omega (u + g(x)|u|^{q-2}u - M)\eta(u - CM)dx \leq 0.$$

On the other hand, by the properties of the function η, we have

$$(t + b_0|t|^{q-2}t - M)\eta(t - CM) \geq 0, \quad \forall t \in \mathbb{R}.$$

Therefore,

$$\left(u + b_0|u|^{q-2}u - M\right)\eta(u - CM) = 0, \quad a.e. \quad x \in \Omega.$$

Thus again by using the definition of η we obtain that $u(x) \leq CM$ for a.e. $x \in \Omega$. $\quad\square$

Lemma 3.4 *The functional E_K defined in* (12) *satisfies the (PS) compactness condition if either of the following assertions hold:*

1. $p > q$;
2. *If $p < q$ then $(a^q/b^p)^{\frac{1}{q-p}} \in L^1(0, 1)$.*

Proof Suppose that $\{u_n\}$ is a sequence in K such that $E(u_n) \to c \in \mathbb{R}$, $\epsilon_n \to 0$ and

$$\Psi_K(v) - \Psi_K(u_n) + \langle D\Phi(u_n), u_n - v\rangle \geq -\epsilon_n\|v - u_n\|_V, \quad \forall v \in V. \tag{24}$$

We must show that $\{u_n\}$ has a convergent subsequence in V. Firstly, we prove that $\{u_n\}$ is bounded in V. To do this, we will consider two cases $p > q$ and $p < q$ separately.

Case 1, $p > q$:

Note that since $E(u_n) \to c$, then for large values of n we have

$$\frac{1}{2}\|u_n\|_{H^1}^2 + \frac{1}{q}\|u_n\|_{L_b^q}^q - \frac{1}{p}\int_{B_1} a(|x|)|u_n|^p dx \leq c + 1. \tag{25}$$

Now consider the function $g(r) = r^q - p(r-1) - 1$ on the interval $(1, +\infty)$. One can see easily that if we set $r^* = (\frac{p}{q})^{\frac{q}{q-1}}$, then for every $r \in (1, r^*)$ we have $g(r) < 0$. Choose such a number r. Thus, we have $r > 1$ and $r^q - 1 < p(r-1)$. Note that

$$\langle D\varphi(u_n), u_n \rangle = \int_{B_1} a(|x|)u_n(x)^p \mathrm{d}x.$$

Thus, by setting $v = ru_n$ in (24) we get

$$\frac{(1-r^2)}{2}\|u_n\|_{H^1}^2 + \frac{(1-r^q)}{q}\|u_n\|_{L_b^q}^q + (r-1)\int_{B_1} a(|x|)|u_n|^p \mathrm{d}x \le \epsilon_n(r-1)\|u_n\|_V \le C\|u_n\|_V. \tag{26}$$

Remember that $r^q - 1 < p(r-1)$, so take $\alpha > 0$ such that

$$\frac{1}{p(r-1)} < \alpha < \frac{1}{r^q - 1}.$$

Multiply (26) by α and sum it up with (25) to get

$$\frac{1+\alpha(1-r^2)}{2}\|u_n\|_{H^1}^2 + \frac{1+\alpha(1-r^q)}{q}\|u_n\|_{L_b^q}^q$$
$$+ \left[\alpha(r-1) - \frac{1}{p}\right]\int_{B_1} a(|x|)|u_n|^p \mathrm{d}x \le c + 1 + \alpha C\|u_n\|_V.$$

Now from the choice of α, the fact that $r > 1$ and $q \ge 2$, all the coefficients in the left-hand side of the latter inequality are positive, thus for suitable constants $C_1, C_2, C_3, C_4 > 0$ we have that

$$\|u_n\|_{H^1}^2 + C_1\|u_n\|_{L_b^q}^q + C_2\int_{B_1} a(|x|)|u_n|^p \mathrm{d}x \le C_3 + C_4\|u_n\|_V.$$

As a consequence, by the second part of Lemma 3.2, one obtains that

$$\|u_n\|_{H^1}^2 \le C_3 + C_4\|u_n\|_V \le C_3 + C_4 C\|u_n\|_{H^1}$$

Therefore, $\{u_n\}$ is bounded in H^1. This completes the boundedness proof for the case $p > q$.
Case 2, $p < q$:
Let $u \in K$. By the Hölder inequality one has that

$$\int_{B_1} a(|x|)|u|^p \mathrm{d}x \le \|a(b)^{\frac{-p}{q}}\|_{L^{\frac{q}{q-p}}} \left(\int_{B_1} b(|x|)|u|^q \mathrm{d}x\right)^{\frac{p}{q}}. \tag{27}$$

Thus

$$E_K(u) \ge \frac{1}{2}\int_{B_1} (|\nabla u|^2 + u^2)\,\mathrm{d}x + \frac{1}{q}\int_{B_1} b(|x|)|u|^q \mathrm{d}x$$
$$- \frac{1}{p}\|a(b)^{\frac{-p}{q}}\|_{L^{\frac{q}{q-p}}} \left(\int_{B_1} b(|x|)|u|^q \mathrm{d}x\right)^{\frac{p}{q}}$$
$$= \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{q}\|u\|_{L_b^q}^q - \frac{1}{p}\|a(b)^{\frac{-p}{q}}\|_{L^{\frac{q}{q-p}}}\|u\|_{L_b^q}^p. \tag{28}$$

Since $p < q$ and $\|a(b)^{\frac{-p}{q}}\|_{L^{\frac{q}{q-p}}} < \infty$, the latter inequality implies that E_K is bounded from below and coercive on V taking into consideration the second part of Lemma 3.2. This indeed implies that the sequence $\{u_n\}$ must also be bounded in V for the case $p < q$.

Therefore, for both cases $p > q$ and $p < q$ we have the boundedness of $\{u_n\}$ in V. Using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $\bar{u} \in V$ such that $u_n \rightharpoonup \bar{u}$ weakly in H^1, $u_n \to \bar{u}$ strongly in L^2 and $u_n \to \bar{u}$ a.e.. We also have the weak convergence of u_n to $\bar{u}$ in both L_b^q and L_a^p. Also according to Lemma 3.2 from boundedness of $\{u_n\} \subset K$ in H^1 one can deduce that $\{u_n\}$ is bounded in L^∞. Note that every u_n is radial, so $\bar{u}$ is radial too, and moreover, $\bar{u} \in K$.

It follows from the properties of weak convergence that

$$\|\bar{u}\|_{H^1} \le \liminf_{n\to\infty} \|u_n\|_{H^1},$$
$$\|\bar{u}\|_{L_b^q} \le \liminf_{n\to\infty} \|u_n\|_{L_b^q}. \tag{29}$$

Now in (24) set $v = \bar{u}$:

$$\frac{1}{2}\left(\|\bar{u}\|_{H^1}^2 - \|u_n\|_{H^1}^2\right) + \frac{1}{q}(\|\bar{u}\|_{L_b^q}^q - \|u_n\|_{L_b^q}^q) + \int_{B_1} a(|x|)|u_n|^{p-2}u_n(u_n - \bar{u})\mathrm{d}x \ge -\epsilon_n\|u_n - \bar{u}\|_V. \tag{30}$$

Since $\|u_n\|_{L^\infty}$ is bounded, one has that

$$a(|x|)|u_n|^{p-1}|\bar{u}(x) - u_n(x)| \le a(|x|)\|u_n\|_{L^\infty}^{p-1}(\|u_n\|_{L^\infty} + \|\bar{u}\|_{L^\infty}) \le \tilde{C}a(|x|),$$

and

$$a(|x|)|u_n|^p \le a(|x|)\|u_n\|_{L^\infty}^p \le \tilde{C}a(|x|),$$

for some constant $\tilde{C} > 0$. Since $a \in L^1$, it follows from dominated convergence theorem that

$$\lim_{n\to\infty}\int_{B_1} a(|x|)|u_n|^{p-2}u_n(\bar{u} - u_n)\mathrm{d}x = 0, \quad \& \quad \lim_{n\to\infty}\int_{B_1} a(|x|)|u_n|^p\,\mathrm{d}x = \int_{B_1} a(|x|)|u|^p\,\mathrm{d}x. \tag{31}$$

Therefore, it follows from (30) and (31) that

$$\frac{1}{2}\left(\limsup_{n\to\infty}\|u_n\|_{H^1}^2 - \|\bar{u}\|_{H^1}^2\right) + \frac{1}{q}\left(\limsup_{n\to\infty}\|u_n\|_{L_b^q}^q - \|\bar{u}\|_{L_b^q}^q\right) \le 0.$$

Hence, by taking into account the inequalities in (29) we obtain that

$$\|\bar{u}\|_{H^1}^2 = \lim_{n\to\infty}\|u_n\|_{H^1}, \quad \& \quad \lim_{n\to\infty}\|u_n\|_{L_b^q}^q = \|\bar{u}\|_{L_b^q}^q. \tag{32}$$

The latter together with (31) yield that

$$u_n \to \bar{u} \quad \text{strongly in} \quad V$$

as desired. $\qquad\square$

To prove the second condition in Theorem 3.1, we shall need the following result for which a proof already exists in [15]. However, for the convenience of the reader we shall also provide a sketch for the proof in the present paper.

Lemma 3.5 *Let $h \in L^1(0, 1)$ be a nonnegative monotone function. Then there exists a sequence of smooth monotone functions $\{h_m\}$ with the property that $0 \le h_m \le h$ and $h_m \to h$ strongly in $L^1(0, 1)$.*

Proof We do the proof for the case where the function h is non-decreasingly monotone. The same argument works if h is non-increasingly monotone. For large integers m we define q_m on $[0, \infty)$ via $q_m(r) = \min\{h(r), m\}$ and so note for each m that q_m is non-decreasing on $(0, 1)$. Now extend $q_m(r)$ to $q_m(1)$ for $r > 1$ and $q_m = 0$ for $r < 0$. Let $0 \le \eta$ be smooth with $\eta = 0$ on $(-\infty, -1) \cup (0, \infty)$ and $\eta > 0$ on $(-1, 0)$. We also assume that $\int_{-1}^0 \eta(\tau) d\tau = 1$. For $\epsilon > 0$, define $\eta_\epsilon(r) := \frac{1}{\epsilon} \eta(\frac{r}{\epsilon})$ and

$$q_m^\epsilon(r) := \int_{-\epsilon}^0 \eta_\epsilon(\tau) q_m(r + \tau) d\tau,$$

note that this is just the usual mollification except the support of η is adjusted slightly. Since q_m is non-decreasing, we see that for each fixed small $\epsilon > 0$ that q_m^ϵ is non-decreasing in r. Then note that we have

$$0 \le q_m^\epsilon(r) = \int_{-\epsilon}^0 \eta_\epsilon(\tau) q_m(r + \tau) d\tau \le q_m(r) \int_{-\epsilon}^0 \eta_\epsilon(\tau) d\tau = q_m(r) \le h(r).$$

We now let $\epsilon_m \searrow 0$ and we set $h_m(r) := q_m^{\epsilon_m}$. So we have $0 \le h_m(r) \le h(r)$ for all m. Also $r \mapsto h_m(r)$ is non-decreasing in r. One can now show that $h_m \to h$ strongly in $L^1(0, 1)$. □

Lemma 3.6 *Suppose $\bar{u} \in K$. Then there exists $v \in K$ satisfying*

$$\begin{cases} -\Delta v + v + b(x)|v|^{q-2}v = a(|x|)|\bar{u}|^{p-2}\bar{u}, & x \in B_1 \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{33}$$

in the weak sense.

Proof By Lemma 3.5, there exists a sequence $\{a_m\}$ (resp. $\{b_m\}$) of smooth functions such that $0 \le a_m \le a$ (resp. $0 \le b_m \le b$) and each a_m is non-decreasing (resp. each b_m is non-increasing) on $(0, 1)$ and $a_m \to a$ in $L^1(0, 1)$ (resp. $b_m \to b$ in $L^1(0, 1)$). We shall consider the following problem.

$$\begin{cases} -\Delta v + v + b_m|v|^{q-2}v = a_m(|x|)|\bar{u}|^{p-2}\bar{u}, & x \in B_1 \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{34}$$

Set $f_m(x) = a_m(|x|)|\bar{u}|^{p-2}\bar{u}$. Note that $f_m \in H^1$ as a_m is smooth and $\bar{u} \in K$. Here we have used the fact that $K \subset L^\infty$ by virtue of Lemma 3.2. Now define

$$J(v) := \frac{1}{2} \int_{B_1} \left(|\nabla v|^2 + |v|^2 \right) dx + \frac{1}{q} \int_{B_1} b_m |v|^q dx - \int_{B_1} f_m(x) v dx,$$

on the space H^1_{rad}. Observe that J is convex and lower semi-continuous. Also

$$\lim_{\|v\|_{H^1} \to \infty} J(v) = \infty,$$

therefore J takes its minimum at some $v_m \in H^1_{\text{rad}}$. It is also easily seen that $\frac{\partial v_m}{\partial \nu} = 0$. We want to show that $v_m \in K$. Note first that $v_m \in H^1(B_1) \cap L^q_{b_m}$ satisfies the following equation

$$\int_{B_1} \nabla v_m . \nabla \varphi dx + \int_{B_1} v_m \varphi dx + \int_{B_1} b_m |v_m|^{q-2} v_m \varphi dx$$

$$= \int_{B_1} f_m \varphi dx, \quad \forall \varphi \in H^1(B_1) \cap L^q_{b_m}(B_1). \tag{35}$$

Therefore, by Lemma 3.3 together with the fact that $f_m \geq 0$ we obtain that $v_m \geq 0$, a.e. in B_1.

Now we need to prove some regularity for v_m. Note that a_m is smooth and $\bar{u} \in K$, so by Lemma 3.2 one has that $f_m \in L^\infty(B_1)$. Thus from Lemma 3.3, we can deduce that $v_m \in L^\infty(B_1)$. Now $v_m \in H^1(B_1)$ is a weak solution of the following equation

$$-\Delta v_m + v_m = f_m - b_m |v_m|^{q-2} v_m := g_m,$$

and observe that $g_m \in L^2$ since $v_m \in L^\infty$ and b_m is smooth. Also $\nabla g_m \in L^2(B_1)$ as the functions $a_m, b_m, v_m \in H^1$ and $v_m \in L^\infty$. Thus $g_m \in H^1(B_1)$. Now by standard regularity theory one can deduce that $v_m \in H^3(B_1)$.

It remains to show that the radial function v_m is non-decreasing in $r = |x|$. For $0 < r < 1$, set $w_m(x) := (v_m)_r(|x|)$, the derivative of v_m with respect to $r = |x|$. Thus, w_m satisfies

$$\begin{cases} -\Delta w_m + (\frac{N-1}{|x|^2} + 1 + (q-1)b_m |v_m|^{q-2}) w_m = Q_m(x), & x \in B_1 \backslash \{0\} \\ w_m = 0, & x \in \partial B_1, \end{cases} \tag{36}$$

where

$$Q_m(x) := a_m'(r)\bar{u}(r)^{p-1} + a_m(r)(p-1)\bar{u}(r)^{p-2}\bar{u}'(r) - b_m'|v_m|^{q-2}v_m,$$

on $(0, 1)$. Since a_m and $\bar{u}$ are nonnegative and non-decreasing, the first two terms in Q_m are nonnegative. Also v_m is nonnegative and b_m is non-increasing which makes the last term in Q_m nonnegative. Thus, $Q_m \geq 0$ on $(0, 1)$. Note that $w_m \in H^1_{\text{rad}}(B_1)$ and has enough regularity to extend the solution of (36) to the full ball B_1. If we define the linear operator L as

$$L(w_m) := -\Delta w_m + \left(\frac{N-1}{|x|^2} + 1 + (q-1)b_m |v_m|^{q-2}\right) w_m,$$

then $L(w_m) \geq 0$. We now show that $w_m \geq 0$ in B_1. Note that $w_m = w_m^+ - w_m^-$ where $w_m^+ = \max\{0, w_m(x)\}$ and $w_m^- = \max\{0, -w_m(x)\}$. Multiplying the inequality $L(w_m) \geq 0$ by $-w_m^-$ and integrating over B_1 implies that

$$\int_{B_1} \left(|\nabla w_m^-|^2 + |w_m^-|^2\right) dx \leq 0,$$

from which we obtain that $w_m^- = 0$. Thus, $w_m \geq 0$ and consequently v_m is non-decreasing in r. Therefore, $v_m \in K$.

We now show that $\{v_m\}$ is bounded in V. It follows from (35) with $\varphi = v_m$ that

$$\int_{B_1} \left(|\nabla v_m|^2 + |v_m|^2 + b_m |v_m|^q\right) dx = \int_{B_1} f_m v_m dx. \tag{37}$$

On the other hand,

$$\int_{B_1} f_m v_m dx = \int_{B_1} a_m(|x|)|\bar{u}|^{p-2}\bar{u}v_m \, dx \leq \int_{B_1} a(|x|)|\bar{u}|^{p-2}\bar{u}v_m \, dx$$
$$\leq \|\bar{u}\|_{L^\infty}^{p-1} \|a\|_{L^1} \|v_m\|_{L^\infty} \leq C_1 \|v_m\|_{H^1}, \tag{38}$$

where C_1 is a constant independent of m. Here we have used Lemma 3.2 together with the fact that $a \in L^1(0, 1)$ and $\bar{u}, v_m \in K$. It now follows from (37) and (38) that

$$\|v_m\|_{H^1}^2 + \int_{B_1} b_m |v_m|^q \, dx \leq C_1 \|v_m\|_{H^1}, \tag{39}$$

from which the boundedness of $\|v_m\|_{H^1}^2$ and $\int_{B_1} b_m |v_m|^q \, dx$ follows. By Lemma 3.2, we obtain that $\|v_m\|_{L^\infty}$ is also bounded. By passing to a subsequence, we can assume there is $0 \le v \in H^1_{\mathrm{rad}}(B_1)$ such that $v_m \to v$ a.e. and $v_m \rightharpoonup v$ weakly in $H^1(B_1)$. Also v is non-decreasing in r on $(0, 1)$ since each v_m has this property. Thus, by Lemma 3.2, $v \in L^\infty$ from which we obtain particularly that $v \in K$. It remains to show that v satisfies Eq. (33). Suppose $\eta \in H^1(B_1) \cap L^\infty(B_1)$, then from (35) we get

$$\int_{B_1} \left(\nabla v_m \nabla \eta + v_m \eta + |v_m|^{q-2} v_m \eta \right) dx = \int_{B_1} a_m |\bar{u}|^{p-2} \bar{u} \eta \, dx. \tag{40}$$

We show that

$$\int_{B_1} b_m |v_m|^{q-2} v_m \eta \, dx \to \int_{B_1} b |v|^{q-2} v \eta \, dx. \tag{41}$$

It follows from the boundedness of $\|v_m\|_{L^\infty}$ that

$$\left| b_m |v_m|^{q-2} v_m \eta \right| \le b \|v_m\|_{L^\infty}^{q-1} \|\eta\|_{L^\infty} \le C_2 b,$$

for some positive constant C_2. Thus, the relation (41) follows from the dominated convergence theorem. Therefore, by passing to the limit in (40) we get that

$$\int_{B_1} \left(\nabla v \nabla \eta + v \eta + b(x) |v|^{q-2} v \eta \right) dx = \int_{B_1} a |\bar{u}|^{p-2} \bar{u} \eta \, dx.$$

By density, the latter holds for all $\eta \in H^1(B_1) \cap L^q_b(B_1)$. This means that v is a solution of (33) in the weak sense. □

Proof of Theorem 1.1 First, we shall use Theorem 2.3 to prove the existence of a non-trivial critical point for E_K. Since by Lemma 3.4 the functional E_K satisfies the (PS) compactness condition, we just need to verify the mountain pass geometry of the functional E_K.

It is clear that $E_K(0) = 0$. Take $e \in K$. It follows that

$$E_K(te) = \frac{t^2}{2} \int_{B_1} \left(|\nabla e|^2 + |e|^2 \right) dx + \frac{t^q}{q} \int_{B_1} b(|x|) |e|^q dx - \frac{t^p}{p} \int_{B_1} a(|x|) |e|^p dx$$

Now, since $p > q \ge 2$, thus for t sufficiently large $E_K(te)$ is negative.

We now prove condition (3) of (MPG). Take $u \in K$ with $\|u\|_V = \rho > 0$. We have

$$E_K(u) = \frac{1}{2} \|u\|_{H^1}^2 + \frac{1}{q} \|u\|_{L^q_b}^q - \frac{1}{p} \int_{B_1} a(|x|) |u|^p dx.$$

Note that from Lemma 3.2, there exist positive constant C such that for every $u \in K$ one has

$$\|u\|_{H^1} \le \|u\|_V \le C \|u\|_{H^1}. \tag{42}$$

We also have that

$$\int_{B_1} a(|x|) |u|^p dx \le \|u\|_V^p.$$

Therefore,

$$\begin{aligned}
E_K(u) &\ge \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \int_{B_1} a(|x|) |u|^p dx \ge \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \|u\|_V^p \\
&\ge \frac{1}{2C^2} \|u\|_V^2 - \frac{1}{p} \|u\|_V^p = \frac{1}{2C^2} \rho^2 - \frac{1}{p} \rho^p > 0,
\end{aligned}$$

provided $\rho > 0$ is small enough, since $p > 2$. If $u \notin K$, then clearly $E_K(u) > 0$. Thus, MPG holds for the functional E_K. Therefore, the functional E_K has a non-trivial critical point $\bar{u} \in K$. It also follows from Lemma 3.6 that there exists $v \in K$ which satisfies the equation in the weak sense

$$\begin{cases} -\Delta v + v + b(x)|v|^{q-2}v = a(|x|)|\bar{u}|^{p-2}\bar{u}, & x \in B_1 \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial B_1, \end{cases}$$

It now follows from Theorem 3.1 that $\bar{u}$ is indeed a solution of (1) with $E_K(\bar{u}) > 0$. $\qquad \square$

In the following theorem, we show that a similar result to Theorem 1.1 also holds for the case where $p < q$.

Theorem 3.7 *Let p and q be two real numbers $2 \leq p < q$. Assume that the functions a and b satisfy the following conditions:*

H1. $a \in L^1(0, 1)$ is non-decreasing and $a(r) > 0$ for a.e. $r \in [0, 1]$.
H2. $b \in L^1(0, 1)$ is nonnegative and non-increasing on $[0, 1]$.
H3. $(a^q/b^p)^{\frac{1}{q-p}} \in L^1(0, 1)$ and there exists $e \in K$ such that $E_K(e) < 0$.

Then problem (1) admits at least one radially non-decreasing positive solution.

Proof Set $\mu = \inf_V E(u)$. It follows from (28) that E_K is bounded from below. By Lemma 3.4, the functional E_K satisfies the (PS) compactness condition. Thus, it follows from Theorem 2.4 that $\min_{u \in V} E_K(u)$ is attained at some $\bar{u} \in K$ which is indeed a critical point of E_K.

It also follows from Lemma 3.6 that there exists $v \in K$ which satisfies the equation in the weak sense

$$\begin{cases} -\Delta v + v + b(x)|v|^{q-2}v = a(|x|)|\bar{u}|^{p-2}\bar{u}, & x \in B_1 \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial B_1, \end{cases}$$

Thus, by Theorem 3.1, $\bar{u}$ is a solution of (1). Note also that

$$E_K(\bar{u}) = \min_{u \in V} E_K(u) \leq E_K(e) < 0,$$

and therefore, $\bar{u}$ is non-trivial. $\qquad \square$

4 Non-constant solutions

In this section, we consider the problem

$$\begin{cases} -\Delta u + u = a|u|^{p-2}u - b|u|^{q-2}u, & x \in B_1, \\ u > 0, & x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{43}$$

where $a > 0$ and $b > 0$ are two constants. In Theorem 1.1, we have proved that this problem has a nonzero solution when $p > q \geq 2$. However, any nonzero constant function u satisfying

$$Q(u) := a|u|^{p-2}u - b|u|^{q-2}u - u = 0,$$

is also a nonzero solution. In the following theorem, by adapting an argument from [5,7], we show that for certain values of p and q the solution obtained in Theorem 1.1 is indeed

non-constant. We also refer to the interesting paper [10] where more general non-linearities are considered and a generalization to higher eigenvalues is provided.

We need the following lemma which states some simple properties of the eigenfunction associated with λ_2, the second radial eigenvalue of $-\Delta + 1$ in the unit ball with Neumann boundary conditions.

Lemma 4.1 *Let v be an eigenfunction associated with λ_2, that is, a non-trivial solution of*

$$
\begin{cases}
-\Delta v + v = \lambda_2 v, \ x \in B_1, \\
\frac{\partial v}{\partial \nu} = 0, \qquad\quad x \in \partial B_1.
\end{cases}
\tag{44}
$$

Then $\lambda_2 > 1$, v is radial and unique up to a multiplicative factor and we can choose it increasing. Moreover, $\int_{B_1} v \mathrm{d}x = 0$.

We refer to ([7], Lemma 4.8) for the proof of Lemma 4.1. Here is our result for the existence of non-constant solutions.

Theorem 4.2 *Suppose that $2 < q < p$. Also assume $a \equiv 1$, $b > 0$ and $b(q - 2) < p - 2$. Then there exists at least one non-constant non-decreasing radial solution of (1).*

Proof First we show that the function Q has a unique nonzero root. Set $h(t) := t^{p-2} - bt^{q-2} - 1$. Note that $h(1) = -b < 0$ and $\lim_{t \to +\infty} h(t) = +\infty$. Thus, $h(t) = 0$ has a root in the interval $(1, +\infty)$. Also $h'(t) = t^{q-3}[(p - 2)t^{p-q} - b(q - 2)] > 0$ in the interval $(1, +\infty)$ as $b(q - 2) < p - 2$. Thus, h is strictly increasing in the interval $(1, +\infty)$ which means that h has a unique root in this interval. On the other hand, in the interval $[0, 1]$ we have $h(t) < 0$. Therefore, problem (43) has a unique constant solution. Denote this solution by $\bar{u}$ and note that $\bar{u} \in (1, +\infty)$.

Now we show that the solution obtained in Theorem 1.1 is different from $\bar{u}$. Recall that problem (1) has a positive solution u with $E(u) = E_K(u) = c > 0$ where the critical value c is characterized by

$$
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_K[\gamma(t)],
$$

where

$$
\Gamma = \{\gamma \in C([0, 1], V) : \gamma(0) = 0 \neq \gamma(1), E_K(\gamma(1)) \leq 0\}.
$$

We shall show that $E(u) = c < E(\bar{u})$. It then implies that u is non-constant. Let v be as in Lemma 4.1, and $s > 0$ be such that $s < \|\bar{u}\|_\infty / \|v\|_\infty$. Thus, $\bar{u} + sv \in K$. For $r \in \mathbb{R}$ we have

$$
E[(\bar{u} + sv)r] = \frac{r^2}{2} \int_{B_1} (\bar{u} + s\sqrt{\lambda_2}v)^2 \mathrm{d}x + \frac{r^q}{q} \int_{B_1} b(\bar{u} + sv)^q \mathrm{d}x - \frac{r^p}{p} \int_{B_1} (\bar{u} + sv)^p \mathrm{d}x.
$$

Note that $q < p$, so there exists $r > 1$ such $E[(\bar{u} + sv)r] \leq 0$. Set $\gamma_s(t) = t(\bar{u} + sv)r$. Note that $\gamma_s \in \Gamma$. We shall show that there exists $s > 0$ such that for every $t \in [0, 1]$ one has $E(\gamma_s(t)) < E(\bar{u})$. Therefore,

$$
c \leq \max_{t \in [0,1]} E(\gamma_s(t)) < E(\bar{u}).
$$

Define $\xi : \mathbb{R}^2 \to \mathbb{R}$ by

$$
\xi(s, t) := E'(t(\bar{u} + sv)r)(\bar{u} + sv)r.
$$

Note that ξ is C^1 and $\xi(0, 1/r) = 0$. Also

$$
\begin{aligned}
\frac{d}{dt}\big|_{(0,1/r)}\xi(s, t) &= E''(\bar{u})(r\bar{u}, r\bar{u}) \\
&= r^2 \int_{B_1} \left(1 + b(q-1)\bar{u}^{q-2} - (p-1)\bar{u}^{p-2}\right)\bar{u}^2 dx \\
&= r^2 \int_{B_1} \bar{u}^p - b\bar{u}^q + b(q-1)\bar{u}^q - (p-1)\bar{u}^p dx \\
&= r^2 \int_{B_1} b(q-2)\bar{u}^q - (p-2)\bar{u}^p dx < 0,
\end{aligned}
$$

where we have used the fact that $Q(\bar{u}) = 0$, $\bar{u} \geq 1$ and $b(q-2) < p - 2$. It follows from the implicit function theorem that there exist $\epsilon_1, \epsilon_2 > 0$ and a C^1 function $g : (-\epsilon_1, \epsilon_1) \to (1/r - \epsilon_2, 1/r + \epsilon_2)$ such that $g(0) = 1/r$ and for $(s, t) \in (-\epsilon_1, \epsilon_1) \times (1/r - \epsilon_2, 1/r + \epsilon_2)$ one has $\xi(s, t) = 0$ iff $t = g(s)$. Note also that

$$
\begin{aligned}
\frac{d}{ds}\big|_{(0,1/r)}\xi(s, t) &= E''(\bar{u})(\bar{u}, rv) + E'(\bar{u})rv \\
&= r \int_{B_1} \left(1 + b(q-1)\bar{u}^{q-2} - (p-1)\bar{u}^{p-2}\right)\bar{u}v dx \\
&= r \left(1 + b(q-1)\bar{u}^{q-2} - (p-1)\bar{u}^{p-2}\right)\bar{u} \int_{B_1} v dx = 0
\end{aligned}
$$

where we have used the fact that $E'(\bar{u}) = 0$ and $\int_{B_1} v dx = 0$. Therefore, $g'(0) = 0$. Now we claim that

$$
E\big(g(s)(\bar{u} + sv)r\big) < E(\bar{u}), \qquad \forall s \in (-\epsilon_1, \epsilon_1). \tag{45}
$$

For this, note that since $g'(0) = 0$, then for every $s \in (-\epsilon_1, \epsilon_1)$ one has $g(s) = 1/r + o(s)$. Thus

$$
g(s)(\bar{u} + sv)r - \bar{u} = s\big(1/r + o(s)\big)rv + \bar{u}\big(rg(s) - 1\big) = sv + o(s).
$$

Hence, by using the fact that $E'(\bar{u}) = 0$ one can deduce that

$$
\begin{aligned}
E\big(g(s)(\bar{u} + sv)r\big) - E(\bar{u}) &= \frac{1}{2}E''(\bar{u})\big(g(s)(\bar{u} + sv)r - \bar{u}, g(s)(\bar{u} + sv)r - \bar{u}\big) + o(s^2) \\
&= \frac{1}{2}E''(\bar{u})\big(sv + o(s), sv + o(s)\big) + o(s^2) \\
&= \frac{s^2}{2}E''(\bar{u})(v, v) + o(s^2).
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
E\big(g(s)(\bar{u} + sv)r\big) &- E(\bar{u}) \\
&= \frac{s^2}{2} \int_{B_1} |\nabla v|^2 + |v|^2 + b(q-1)\bar{u}^{q-2}v^2 - (p-1)\bar{u}^{p-2}v^2 \, dx + o(s^2) \\
&= \frac{s^2}{2} \int_{B_1} \lambda_2 v^2 + b(q-1)\bar{u}^{q-2}v^2 - (p-1)\bar{u}^{p-2}v^2 \, dx + o(s^2) \\
&= \frac{s^2}{2\bar{u}^2} \int_{B_1} \big(\lambda_2\bar{u}^2 + b(q-1)\bar{u}^q - (p-1)\bar{u}^p\big)v^2 \, dx + o(s^2) \\
&= \frac{s^2}{2\bar{u}^2} \int_{B_1} \big(\lambda_2\bar{u}^p - \lambda_2 b\bar{u}^q + b(q-1)\bar{u}^q - (p-1)\bar{u}^p\big)v^2 \, dx + o(s^2),
\end{aligned}
$$

where we have used the identity $Q(\bar{u}) = 0$ in the last line. Therefore,

$$
\begin{aligned}
E\big(g(s)(\bar{u} + sv)r\big) - E(\bar{u}) &= \frac{s^2}{2\bar{u}^2} \int_{B_1} \big(b(q-1-\lambda_2)\bar{u}^q - (p-1-\lambda_2)\bar{u}^p\big)v^2 \, dx \\
&\quad + o(s^2) < 0,
\end{aligned}
$$

where the last inequity follows from the fact that

$$
b \leq \frac{p-2}{q-2} < \frac{p-1-\lambda_2}{q-1-\lambda_2},
$$

considering $\lambda_2 > 1$. Now we claim that the function $t \mapsto E\big(\gamma_0(t)\big)$ has a unique maximum point at $t = 1/r$. For this, observe that

$$
\begin{aligned}
\frac{d}{dt}E\big(\gamma_0(t)\big) = \frac{d}{dt}E(t\bar{u}r) &= E'(t\bar{u}r)r\bar{u} \\
&= |B_1|\big(t\bar{u}^2r^2 + bt^{q-1}\bar{u}^q r^q - t^{p-1}\bar{u}^p r^p\big)\bar{u} = -|B_1|t\bar{u}^2r^2 h(t\bar{u}r), \quad (46)
\end{aligned}
$$

where $|B_1|$ is te measure of the ball B_1. Thus $\frac{d}{dt}E\big(\gamma_0(t)\big) > 0$ if $t < 1/r$ and $\frac{d}{dt}E\big(\gamma_0(t)\big) < 0$ if $t > 1/r$. Thus $E\big(\gamma_0(t)\big)$ has a unique maximum point at $t = 1/r$.

By continuity of the function $(t, s) \to E(\gamma_s(t))$ we can choose $0 < s_0 < \epsilon_1$ small enough such that the maximum of the function $t \to E(\gamma_{s_0}(t))$ lies in the interval $(1/r - \epsilon_2, 1/r + \epsilon_2)$. Assume that $E(\gamma_{s_0}(.))$ takes its maximum at the point t_0. Then

$$
0 = \frac{d}{dt}E\big(\gamma_{s_0}(t)\big)|_{t=t_0} = E'\big(t_0(\bar{u} + s_0v)r\big)(\bar{u} + s_0v)r,
$$

and therefore $t_0 = g(s_0)$. So by (45) one obtains that

$$
E\big(t_0(\bar{u} + s_0v)r\big) < E(\bar{u}).
$$

Thus,

$$
E\big(t(\bar{u} + s_0v)r\big) < E(\bar{u}), \qquad \forall t \in [0, 1].
$$

In other words, $\max_{t \in [0,1]} E\big(\gamma_{s_0}(t)\big) < E(\bar{u})$. This indeed shows that

$$
c \leq \max_{t \in [0,1]} E\big(\gamma_{s_0}(t)\big) < E(\bar{u}),
$$

as desired.   □

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