Quasilinear eigenvalue problems with singular weights for the p -Laplacian

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Abstract

In this paper we study quasilinear homogeneous eigenvalue problem with the p -Laplacian involving singular weights. We work on a bounded domain with Lipschitzian boundary and the weights are negative powers of the distance from the boundary. We generalize results concerning the existence and properties of the principal eigenvalue and corresponding eigenfunctions for both quasilinear unweighted case and singular linear case.

Keywords Quasilinear eigenvalue problem · p -Laplacian with singular weights · Principal eigenvalue · Regularity of eigenfunction · Variational characterization

Mathematics Subject Classification 35J62 · 35J92 · 35B45 · 35B50

1 Introduction

In this paper we study the following quasilinear homogeneous eigenvalue problem

$$\begin{cases} -\operatorname{div} \left(\frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \right) + \frac{1}{d^\beta} |u|^{p-2} u = \frac{\lambda}{d^\gamma} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitzian boundary $\partial\Omega$. Here, $d = d(x)$ ($= d(x, \partial\Omega)$) denotes the distance of the point $x \in \Omega$ from the boundary $\partial\Omega$, $1 < p < N$, $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$, $0 \leq \gamma < \alpha + p$ and λ is a real parameter. As usual, we look for values of λ such that there is a nontrivial solution $u \neq 0$ to (1.1).

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Problems of this type generalize both homogeneous nonlinear eigenvalue problems for the p -Laplacian with smooth coefficients and linear eigenvalue problems (for $p = 2$) with singular (unbounded) coefficients.

For the former case, the model problem is

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the standard p -Laplacian as usual and $m \in L^\infty(\Omega)$ is such that $m(x) \geq 0$ (or at least $m^+ \not\equiv 0$). This problem was studied by Anane [1] in smooth domains, proving the existence of the first eigenvalue $\lambda_1 > 0$ having a positive eigenfunction $\varphi_1 \in C^{1,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$ such that $\varphi_1 > 0$ in Ω and $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$ (by the Strong Maximum Principle of Vázquez [38]). Moreover, λ_1 is simple and isolated, see also Díaz and Saa [13]. The simplicity of λ_1 was proved by Lindquist [30] for general domains by using special test functions. Here we use the convexity ideas in Fleckinger et al. [20] in the short formulation for a direct proof in Belloni and Kawohl [2]. (See Drábek and Hernández [15] and Cuesta [5] for previous work.)

In this paper we generalize some results in Díaz and Hernández [11], this time by using variational arguments in weighted Sobolev spaces (with $p \neq 2$). We also improve results in Montenegro and Lorca [33], see the details below.

The paper is organized as follows. In Sect. 2 we state some preliminary results in weighted Sobolev spaces $W_0^{1,p}(\Omega; \frac{1}{d^\sigma})$ which are used all along the paper. Section 3 is devoted to the proof of the existence of the first (principal) eigenvalue having eigenfunctions which do not change sign in Ω . We use variational arguments for a functional associated with (1.1). A fundamental regularity result, namely, that eigenfunctions $u \geq 0$ are bounded, is proved in Sect. 4, together with some additional estimates. We employ a variant of iterative arguments of Nash–Moser type (see Drábek et al. [16], Drábek and Hernández [15] for similar arguments). The authors would like to thank J. I. Díaz for improvement of the original version of Theorem 4.1. Further properties of eigenvalues and eigenfunctions $u \geq 0$ are obtained in Sect. 5. First, we are able to apply Harnack's inequality by Trudinger [37] to show that $u > 0$ on Ω and regularity results by Tolksdorf [36] to show that u is locally $C^{1,\sigma}$ for some $0 < \sigma < 1$. Then following the ideas in Fleckinger et al. [20] we use the Belloni–Kawohl argument (see [2]) to prove that the principal eigenvalue is simple, i.e., that two eigenfunctions are mutually proportional. We also show that eigenfunctions corresponding to other eigenvalues should change sign in Ω and that the first eigenvalue is isolated. For this, we obtain nodal estimates generalizing those in Anane [1], Montenegro and Lorca [33].

2 Preliminaries

For $\varepsilon \in \mathbb{R}$ we let

$$L^p\left(\Omega; \frac{1}{d^\varepsilon}\right) := \left\{ u = u(x), \ x \in \Omega : \int_\Omega \frac{1}{d^\varepsilon} |u(x)|^p \, dx < \infty \right\}$$

to be the weighted Lebesgue space with the norm

$$\|u\|_{p;\varepsilon} = \left(\int_\Omega \frac{1}{d^\varepsilon} |u(x)|^p \, dx \right)^{\frac{1}{p}},$$

and

$$W^{1,p}\left(\Omega; \frac{1}{d^\varepsilon}\right) := \left\{ u = u(x), x \in \Omega : \int_{\Omega} \frac{1}{d^\varepsilon} [|\nabla u(x)|^p + |u(x)|^p] dx < \infty \right\}$$

to be the weighted Sobolev space with the norm

$$\|u\|_{1,p;\varepsilon} = \left(\int_{\Omega} \frac{1}{d^\varepsilon} [|\nabla u(x)|^p + |u(x)|^p] dx \right)^{\frac{1}{p}}.$$

We also define $W_0^{1,p}(\Omega; \frac{1}{d^\varepsilon}) \subset W^{1,p}(\Omega; \frac{1}{d^\varepsilon})$ to be a closure of the set $C_0^\infty(\Omega)$ (smooth functions with compact support in Ω) with respect to the norm $\|\cdot\|_{1,p;\varepsilon}$. By $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ we denote the usual Lebesgue space and Sobolev space of functions having zero trace on $\partial\Omega$, respectively. The corresponding norms will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, respectively. The following embeddings will be used in our proofs.

Proposition 2.1 (Continuous embedding) *Let $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$. Then $W_0^{1,p}(\Omega; \frac{1}{d^\alpha}) \hookrightarrow L^p(\Omega; \frac{1}{d^\beta})$.*

Proposition 2.2 (Continuous embedding) *Let $\alpha \geq 0$, $1 < p < N$. Then*

$$W_0^{1,p}\left(\Omega; \frac{1}{d^\alpha}\right) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad p^* = \frac{Np}{N-p}.$$

Proof of Proposition 2.1 follows from Kufner [23, Sec. 8.8 on p. 57]. The first embedding in Proposition 2.2 holds due to $\alpha \geq 0$ and Ω bounded, the second one is a well known fact (see, e.g., Pick et al. [34]). $\square$

The following result is an extension of Lemma 3.3 in [3].

Proposition 2.3 (Compact embedding) *Let $0 \leq \gamma < \alpha + p$. Then*

$$W_0^{1,p}\left(\Omega; \frac{1}{d^\alpha}\right) \hookrightarrow L^p\left(\Omega; \frac{1}{d^\gamma}\right).$$

Proof For $\sigma > 0$ small enough we define $\Omega_\sigma := \{x \in \Omega : d(x) > \sigma\}$. Consider the commutative diagram:

$$\begin{array}{ccc} W_0^{1,p}(\Omega; \frac{1}{d^\alpha}) & \xrightarrow{I_\sigma} & L^p(\Omega; \frac{1}{d^\gamma}) \\ & \searrow I_\sigma^1 & \nearrow I_\sigma^3 \\ & W^{1,p}(\Omega_\sigma) & \xrightarrow{I_\sigma^2} L^p(\Omega_\sigma) \end{array}$$

Here $I_\sigma^1 u(x) = u(x)$, $x \in \Omega_\sigma$ is the “restriction” (bounded and linear map) from $W_0^{1,p}(\Omega; \frac{1}{d^\alpha})$ into $W^{1,p}(\Omega_\sigma)$; $I_\sigma^2 u(x) = u(x)$, $x \in \Omega_\sigma$ is “embedding” (compact linear map) from $W^{1,p}(\Omega_\sigma)$ into $L^p(\Omega_\sigma)$; $I_\sigma^3 u(x) = u(x)$, $x \in \Omega_\sigma$ and $I_\sigma^3 u(x) = 0$, $x \in \Omega \setminus \Omega_\sigma$ is “zero extension” (bounded linear map) from $L^p(\Omega_\sigma)$ onto $L^p(\Omega; \frac{1}{d^\gamma})$. Then for any $\sigma > 0$, $I_\sigma : W_0^{1,p}(\Omega; \frac{1}{d^\alpha}) \rightarrow L^p(\Omega; \frac{1}{d^\gamma})$ is a compact linear map. Denote by

$$I : W_0^{1,p}\left(\Omega; \frac{1}{d^\alpha}\right) \hookrightarrow L^p\left(\Omega; \frac{1}{d^\gamma}\right)$$

the embedding of $W_0^{1,p}(\Omega; \frac{1}{d^\alpha})$ into $L^p(\Omega; \frac{1}{d^\gamma})$. Then for $u \in W_0^{1,p}(\Omega; \frac{1}{d^\alpha})$ there exists $c > 0$ (due to Proposition 2.1) such that

$$\begin{aligned} \|(I_\sigma - I)u\|_{p;\gamma}^p &= \int_{\Omega \setminus \Omega_\sigma} \frac{1}{d^\gamma} |u(x)|^p dx = \int_{\Omega \setminus \Omega_\sigma} \frac{1}{d^\gamma} d^{\alpha+p} \frac{|u(x)|^p}{d^{\alpha+p}} dx \\ &\leq \max_{x \in \Omega \setminus \Omega_\sigma} d^{\alpha+p-\gamma}(x) \|u\|_{p;\alpha+p}^p \leq c \max_{x \in \Omega \setminus \Omega_\sigma} d^{\alpha+p-\gamma}(x) \|u\|_{1,p;\alpha}^p. \end{aligned}$$

The assertion now follows from the fact that $\max_{x \in \Omega \setminus \Omega_\sigma} d^{\alpha+p-\gamma}(x) \rightarrow 0$ as $\sigma \rightarrow 0$. $\square$

In what follows, we denote $X := W_0^{1,p}(\Omega; \frac{1}{d^\alpha})$ and introduce an equivalent norm

$$\|u\|_X := \left(\int_{\Omega} \frac{1}{d^\alpha} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

Then $(X, \|\cdot\|_X)$ is a uniformly convex (and hence reflexive) Banach space.

3 Existence of principal eigenfunctions

In this section we study the eigenvalue problem (1.1). Any weak solution $u \neq 0$ of (1.1) is called an *eigenfunction* corresponding to an *eigenvalue* λ . More precisely, u is an eigenfunction and λ is an eigenvalue, if $u \in X$ satisfies the integral identity

$$\int_{\Omega} \frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} \frac{1}{d^\beta} |u|^{p-2} u \varphi dx = \lambda \int_{\Omega} \frac{1}{d^\gamma} |u|^{p-2} u \varphi dx \quad (3.1)$$

for any test function $\varphi \in X$. We have the following existence result.

Theorem 3.1 *Let $1 < p < N$, $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$ and $0 \leq \gamma < \alpha + p$. Then there exists a unique eigenvalue $\lambda_1 > 0$ of (1.1) with associated eigenfunction which does not change sign in Ω . The eigenvalue λ_1 is the least eigenvalue of (1.1).*

Proof Define

$$R(u) := \frac{\|u\|_X^p + \|u\|_{p;\beta}^p}{\|u\|_{p;\gamma}^p}, \quad u \in X \setminus \{0\}.$$

By Proposition 2.3 there exists a constant $C_1 > 0$ such that for all $u \in X$,

$$\|u\|_{p;\gamma} \leq C_1 \|u\|_X.$$

Then for $u \in X$, $u \neq 0$, we have

$$R(u) \geq \frac{\|u\|_X^p}{\|u\|_{p;\gamma}^p} \geq \frac{1}{C_1^p} > 0.$$

Let us define a manifold in X as follows

$$\mathcal{S} := \{u \in X : \|u\|_{p;\gamma} = 1\}.$$

Since $1 < p < \infty$, the norm in the dual space $(L^p(\Omega; \frac{1}{d^\gamma}))^*$ is uniformly convex. Therefore, according to Fabian et al. [19], Theorem 9.10 (ii), the norm $\|\cdot\|_{p;\gamma}$ is uniformly Fréchet

differentiable off the origin and by [19], Fact 9.7 (iii) it is also of class C^1 on $L^p(\Omega; \frac{1}{d^\gamma}) \setminus \{0\}$. Hence, $\mathcal{S}$ is a C^1 -manifold modeled on X due to our Proposition 2.1.

Apparently, R is bounded from below on $\mathcal{S}$ by a constant $\frac{1}{C_1^p} > 0$. Assume that $\{u_n\} \subset \mathcal{S}$ is a minimizing sequence for $R|_{\mathcal{S}}$, i.e.,

$$\lim_{n \rightarrow \infty} R(u_n) = \inf_{u \in \mathcal{S}} R(u). \quad (3.2)$$

Then $\{u_n\}$ is a bounded sequence in $X \hookrightarrow L^p(\Omega; \frac{1}{d^\beta})$ and since $X, L^p(\Omega; \frac{1}{d^\beta})$ are uniformly convex Banach spaces, $X \hookrightarrow \hookrightarrow L^p(\Omega; \frac{1}{d^\gamma})$, there exists $u \in X$ such that, up to a subsequence we have

$$u_n \rightharpoonup u \text{ weakly in } X, \quad (3.3)$$

$$u_n \rightharpoonup u \text{ weakly in } L^p\left(\Omega; \frac{1}{d^\beta}\right), \quad (3.4)$$

$$u_n \rightarrow u \text{ strongly in } L^p\left(\Omega; \frac{1}{d^\gamma}\right), \quad (3.5)$$

$$u_n \rightarrow u \text{ a.e. in } \Omega. \quad (3.6)$$

We deduce $u \neq 0$, $\|u\|_{p;\gamma} = 1$. By the lower semicontinuity of the norm in X and $L^p(\Omega; \frac{1}{d^\beta})$, we have

$$\liminf_{n \rightarrow \infty} \|u_n\|_X^p \geq \|u\|_X^p, \quad \liminf_{n \rightarrow \infty} \|u_n\|_{p;\beta}^p \geq \|u\|_{p;\beta}^p. \quad (3.7)$$

But (3.2) and (3.7) yield

$$\lim_{n \rightarrow \infty} \|u_n\|_X = \|u\|_X, \quad \lim_{n \rightarrow \infty} \|u_n\|_{p;\beta} = \|u\|_{p;\beta}. \quad (3.8)$$

Uniform convexity of X and $L^p(\Omega; \frac{1}{d^\beta})$ together with (3.3), (3.4) and (3.8) yield

$$u_n \rightarrow u \text{ strongly in } X \text{ and } u_n \rightarrow u \text{ strongly in } L^p\left(\Omega; \frac{1}{d^\beta}\right). \quad (3.9)$$

Then (3.9) combined with (3.5) imply

$$R(u) = \inf_{v \in \mathcal{S}} R(v).$$

Since $R|_{\mathcal{S}}$ is a C^1 -functional on C^1 -manifold $\mathcal{S}$, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that for any $\varphi \in X$ we have

$$p \int_{\Omega} \frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + p \int_{\Omega} \frac{1}{d^\beta} |u|^{p-2} u \varphi \, dx = \mu p \int_{\Omega} \frac{1}{d^\gamma} |u|^{p-2} u \varphi \, dx.$$

The special choice $\varphi = u$ leads to

$$\|u\|_X^p + \|u\|_{p;\beta}^p = \mu \|u\|_{p;\gamma}^p,$$

which is equivalent to

$$\mu = R(u) = \inf_{v \in \mathcal{S}} R(v).$$

We call $\mu = \lambda_1 > 0$ the principal eigenvalue of (1.1) and $u \in X$ is the corresponding principal eigenfunction. Since for $u \in X$ we have $|u| \in X$ (cf. Gilbarg and Trudinger [22], Section 7.4, Lemma 7.6) and $R(u) = R(|u|)$, we may assume that $u \geq 0$ a.e. in Ω .

Let w be any other eigenfunction of (1.1) with associated eigenvalue λ . Then choosing $\varphi = w$ in (3.1) we arrive at

$$\lambda = R(w) \geq \inf_{v \in \mathcal{S}} R(v) = \lambda_1.$$

In other words, λ_1 is the first (the least, the principal) eigenvalue of (1.1). $\square$

Remark 3.1 The problem (1.2) for $\partial\Omega$ piecewise C^1 was studied by Montenegro and Lorca [33] using the version of Hardy–Sobolev inequality in [26] for $m = m(x)$ such that $m(\cdot)d^\tau(\cdot) \in L^a(\Omega)$ ($a > 1$), $0 < \tau < 1$ and p satisfying either

(A)

$$\frac{p}{1-\tau} \leq a \leq \frac{Np}{N-\tau p} \quad \text{if } N > \tau p$$

or

(B)

$$p < \frac{N}{1-\tau} < a.$$

Notice that for $m(x) = d^{-\gamma}(x)$, $\gamma > 0$, $m(x)d^\tau(x) = d^{\tau-\gamma}(x)$ and the integrability condition $a(\tau - \gamma) + 1 > 0$ is obviously satisfied for any $a > 1$ if $\tau - \gamma > 0$ (which holds only if $0 < \gamma < 1$) or for $1 < a < \frac{1}{\gamma-\tau}$ if $\gamma - \tau > 0$. In this case $\gamma - \tau < 1$ which gives the limitation $\gamma < 2$ for γ .

If $N > p$, then $N > p > p(1-\tau)$ for any $\tau \in (0, 1)$ and condition (A) is *never* satisfied since it reads $\frac{1}{1-\tau} \leq \frac{N}{N-\tau p}$ or, equivalently, $p \geq N$, a contradiction.

Concerning condition (B), if $\tau < \gamma$, a should satisfy $p < \frac{N}{1-\tau} < a < \frac{1}{\gamma-\tau}$ which is equivalent to

$$\gamma < \frac{\tau(N-1)+1}{N} < 1 \quad \text{for all } \tau \in (0, 1)$$

giving again the limitation $\gamma < 1$ for γ . Finally, if now $\tau > \gamma$, then condition (B) can be satisfied.

All this shows limitations on γ (for $\alpha = 0$), whereas in this paper we have (for $\alpha = 0$) the limitation $\gamma < p$. Already in [3, 11] were $\beta < 2$ and $\gamma < 2$ allowed.

Remark 3.2 The results in [1] were generalized in [5]. On the other hand, the existence and multiplicity of principal eigenvalues in [21] were extended in [6] to the p -Laplacian, but this time by using variational methods in a more or less classical way. For the limit case $\gamma = \frac{N}{2}$ in the above paper, see [31]. See also [25, 35] for related work. Results for nonhomogeneous operators were obtained by Drábek [14] and other boundary conditions were treated in [28]. For applications of related ideas to Fučík problems, see [29], and for higher eigenvalues see [6, 33].

Remark 3.3 Problem (1.1) with $\alpha = 0$, $p = 2$ and unbounded coefficients in the spaces $L^r(\Omega)$ with $r > \frac{N}{2}$ (or $r > N$) were studied in [21] by using an alternative approach reducing (1.1) to an equivalent fixed point problem for the (unique) principal eigenvalue $r(\mu)$ of an associated eigenvalue problem where all coefficients are positive. This problem was posed in $L^p(\Omega)$, in general for $p \neq 2$, and then a generalized version of the Krein–Rutman theorem for cones with empty interior (as, e.g., $L^p(\Omega)$, $1 < p < \infty$) in [7] was a basic auxiliary tool. This gives a way to extend and unify previous work concerning the

existence and multiplicity of principal eigenvalues. That a similar approach could be tempted also for the p -Laplacian is supported by the considerations in [32] concerning the plausibility of such a theorem of Krein–Rutman type for the p -Laplacian. A nice survey paper, including original contributions, about all these results is due to Chang [4].

We have studied problem (1.1) with the sign plus on the left-hand side. Now, we will show that the case with the minus sign can be treated by reducing it to the precedent case using some fixed point argument.

More precisely, we deal next with the eigenvalue problem

$$\begin{cases} -\operatorname{div} \left(\frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \right) - \frac{1}{d^\beta} |u|^{p-2} u = \frac{\lambda}{d^\gamma} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.10)$$

where $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$, $0 \leq \gamma < \alpha + p$.

Note that principle eigenvalue are not necessarily positive in this case. We look first for a positive least eigenvalue. For $\lambda \geq 0$ fixed we consider the associated eigenvalue problem

$$\begin{cases} -\operatorname{div} \left(\frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \right) = \mu \left(\frac{1}{d^\beta} |u|^{p-2} u + \frac{\lambda}{d^\gamma} |u|^{p-2} u \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.11)$$

where μ is the eigenvalue parameter. Now, we can argue as in the proof of Theorem 3.1 but this time working on a manifold associated to the “bigger” $L^p(\Omega; b)$ space, namely, in

$$\mathcal{S}^* := \left\{ u \in X : \int_{\Omega} \left(\frac{|u|^p}{d^\beta} + \frac{|u|^p}{d^\gamma} \right) dx = 1 \right\}.$$

Notice that it is easy to see that $\beta > \gamma$ implies $L^p(\Omega; \frac{1}{d^\beta}) \subset L^p(\Omega; \frac{1}{d^\gamma})$ and viceversa. This gives us the first positive eigenvalue to (3.11) with the corresponding variational characterization

$$\mu(\lambda) := \inf_{w \in \mathcal{S}^*} \frac{\int_{\Omega} \frac{|\nabla w|^p}{d^\alpha} dx}{\int_{\Omega} \frac{|w|^p}{d^\beta} dx + \lambda \int_{\Omega} \frac{|w|^p}{d^\gamma} dx}.$$

It is straightforward to prove that $\mu(\lambda)$ is a continuous and monotone (decreasing) function of λ . Moreover, we have

$$\mu(\lambda) \leq \frac{1}{\lambda} \inf_{w \in \mathcal{S}^*} \frac{\int_{\Omega} \frac{|\nabla w|^p}{d^\alpha} dx}{\int_{\Omega} \frac{|w|^p}{d^\gamma} dx},$$

which implies $\mu(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Hence, the existence of a positive eigenvalue to (3.11) is equivalent to the existence of $\bar{\lambda} > 0$ such that $\mu(\bar{\lambda}) = 1$, which in turn is equivalent to

$$\mu(0) = \inf_{w \in \mathcal{S}^*} \frac{\int_{\Omega} \frac{|\nabla w|^p}{d^\alpha} dx}{\int_{\Omega} \frac{|w|^p}{d^\beta} dx} > 1.$$

Next we treat the case $\lambda \leq 0$ by making the change of variable $\lambda \rightarrow -\lambda$. Then the associated eigenvalue problem takes the form

$$\begin{cases} -\operatorname{div} \left(\frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \right) + \frac{\lambda |u|^{p-2} u}{d^\gamma} = v \frac{|u|^{p-2} u}{d^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.12)$$

We look for $\bar{\lambda} > 0$ such that $v(\bar{\lambda}) = 1$, where $v(\lambda) > 0$ is the first eigenvalue to (3.12) provided by Theorem 3.1 again. The variational characterization for $v(\lambda)$ is given by

$$v(\lambda) := \inf_{w \in \mathcal{S}^*} \frac{\int_{\Omega} \frac{|\nabla w|^p}{d^\alpha} dx + \lambda \int_{\Omega} \frac{|w|^p}{d^\gamma} dx}{\int_{\Omega} \frac{|w|^p}{d^\beta} dx},$$

where $v(\lambda)$ is again continuous and increasing in λ with $v(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$. Hence, there exists $\bar{\lambda} > 0$ such that $v(\bar{\lambda}) = 1$ if and only if

$$v(0) = \inf_{w \in \mathcal{S}^*} \frac{\int_{\Omega} \frac{|\nabla w|^p}{d^\alpha} dx}{\int_{\Omega} \frac{|w|^p}{d^\beta} dx} < 1.$$

Notice that $v(0) = \mu(0)$.

Actually, we have proved the following theorem.

Theorem 3.2 *Problem (3.10) with $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$, and $0 \leq \gamma < \alpha + p$ has a first positive (resp. negative) eigenvalue $\lambda_1 > 0$ (resp. $\lambda_1 < 0$) if and only if $\mu(0) > 1$ (resp. $v(0) < 1$).*

4 L^∞ -estimates

Theorem 4.1 *Let $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$, $0 \leq \gamma < \alpha + p$ and $\gamma < \beta$. Let $u \in X$, $u \geq 0$, be the principal eigenfunction of (1.1) associated with $\lambda_1 > 0$. Then $\|u\|_\infty < +\infty$ and $u \in L^p \left(\Omega; \frac{1}{d^\beta} \right)$ for any $1 < p \leq +\infty$.*

Proof For $M > 0$ we set $v_M(x) := \min\{u(x), M\}$ and for $\kappa > 0$ we choose $\varphi(x) = v_M^{\kappa p+1}(x) \in L^\infty(\Omega)$ as a test function in (3.1):

$$(\kappa p + 1) \int_{\Omega} \frac{1}{d^\alpha} v_M^{\kappa p} |\nabla v_M|^p dx + \int_{\Omega} \frac{1}{d^\beta} u^{p-1} v_M^{\kappa p+1} dx \leq \lambda_1 \int_{\Omega} \frac{1}{d^\gamma} u^{p-1} v_M^{\kappa p+1} dx. \quad (4.1)$$

From (4.1) we deduce by Hölder's inequality,

$$\begin{aligned} \int_{\Omega} \frac{1}{d^\beta} u^{p-1} v_M^{\kappa p+1} dx &\leq (\kappa p + 1) \int_{\Omega} \frac{1}{d^\alpha} v_M^{\kappa p} |\nabla v_M|^p dx + \int_{\Omega} \frac{1}{d^\beta} u^{p-1} v_M^{\kappa p+1} dx \\ &\leq \lambda_1 \int_{\Omega} \frac{1}{d^\gamma} (u^{p-1} v_M^{\kappa p+1})^{\frac{\gamma}{\beta}} (u^{p-1} v_M^{\kappa p+1})^{\frac{\beta-\gamma}{\beta}} dx \\ &\leq \lambda_1 \left(\int_{\Omega} \frac{1}{d^\beta} u^{p-1} v_M^{\kappa p+1} dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} u^{p-1} v_M^{\kappa p+1} dx \right)^{\frac{\beta-\gamma}{\beta}}. \end{aligned}$$

From here we get

$$\int_{\Omega} \frac{1}{d^\beta} u^{p-1} v_M^{\kappa p+1} dx \leq \lambda_1^{\frac{\beta}{\beta-\gamma}} \int_{\Omega} u^{p-1} v_M^{\kappa p+1} dx. \quad (4.2)$$

Lemma 4.1 *Let $\kappa > 0$ be such that $\|u\|_{(\kappa+1)p} < +\infty$. Then*

$$\begin{aligned} \lim_{M \rightarrow \infty} \int_{\Omega} \frac{1}{d^{\beta}} v_M^{(\kappa+1)p} dx &= \lim_{M \rightarrow \infty} \int_{\Omega} \frac{1}{d^{\beta}} u^{p-1} v_M^{\kappa p+1} dx = \int_{\Omega} \frac{1}{d^{\beta}} u^{(\kappa+1)p} dx \\ &\leq \lambda_1^{\frac{\beta}{\beta-\gamma}} \int_{\Omega} u^{(\kappa+1)p} dx. \end{aligned} \quad (4.3)$$

Proof For $M \rightarrow \infty$ we have $v_M(x) \nearrow u(x)$ monotonically for a.e. $x \in \Omega$. It follows from (4.2) that

$$\begin{aligned} \int_{\Omega} \frac{1}{d^{\beta}} v_M^{(\kappa+1)p} dx &\leq \int_{\Omega} \frac{1}{d^{\beta}} u^{p-1} v_M^{\kappa p+1} dx \\ &\leq \lambda_1^{\frac{\beta}{\beta-\gamma}} \int_{\Omega} u^{p-1} v_M^{\kappa p+1} dx \leq \lambda_1^{\frac{\beta}{\beta-\gamma}} \int_{\Omega} u^{(\kappa+1)p} dx. \end{aligned} \quad (4.4)$$

Then (4.3) follows from (4.4), $\|u\|_{(\kappa+1)p} < +\infty$ and the monotone convergence theorem. $\square$

Now, let us estimate separately the left-hand side (LHS) and the right-hand side (RHS) of (4.1):

$$\begin{aligned} \text{LHS} &\geq \frac{\kappa p + 1}{(\kappa + 1)^p} \int_{\Omega} \frac{1}{d^{\alpha}} |\nabla v_M^{\kappa+1}|^p dx \geq C_1 \frac{\kappa p + 1}{(\kappa + 1)^p} \int_{\Omega} |\nabla v_M^{\kappa+1}|^p dx \\ &= C_1 \frac{\kappa p + 1}{(\kappa + 1)^p} \left(\int_{\Omega} |\nabla v_M^{\kappa+1}|^p dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} |\nabla v_M^{\kappa+1}|^p dx \right)^{\frac{\beta-\gamma}{\beta}} \\ &\geq C_2 \frac{\kappa p + 1}{(\kappa + 1)^p} \left(\int_{\Omega} \frac{1}{d^{\beta}} v_M^{(\kappa+1)p} dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} v_M^{(\kappa+1)p^*} dx \right)^{\frac{p}{p^*} \cdot \frac{\beta-\gamma}{\beta}}. \end{aligned} \quad (4.5)$$

Here, $C_1 \leq \frac{1}{d^{\alpha}(x)}$, $x \in \Omega$, and $C_2 > 0$ is the constant from embeddings in Propositions 2.1 and 2.2;

$$\begin{aligned} \text{RHS} &\leq \lambda_1 \left(\int_{\Omega} \frac{1}{d^{\beta}} u^{p-1} v_M^{\kappa p+1} dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} u^{p-1} v_M^{\kappa p+1} dx \right)^{\frac{\beta-\gamma}{\beta}} \\ &\leq \lambda_1 \left(\int_{\Omega} \frac{1}{d^{\beta}} u^{(\kappa+1)p} dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} u^{(\kappa+1)p} dx \right)^{\frac{\beta-\gamma}{\beta}}. \end{aligned} \quad (4.6)$$

But (4.5) and (4.6) yield

$$\begin{aligned} &\left(\int_{\Omega} \frac{1}{d^{\beta}} v_M^{(\kappa+1)p} dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} v_M^{(\kappa+1)p^*} dx \right)^{\frac{p}{p^*} \cdot \frac{\beta-\gamma}{\beta}} \\ &\leq C_3 \frac{(\kappa + 1)^p}{\kappa p + 1} \left(\int_{\Omega} \frac{1}{d^{\beta}} u^{(\kappa+1)p} dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} u^{(\kappa+1)p} dx \right)^{\frac{\beta-\gamma}{\beta}} \end{aligned}$$

where $C_3 = C_3(\lambda_1, \alpha, \gamma, p, \Omega) > 0$. This is equivalent to

$$\begin{aligned} & \left(\int_{\Omega} \frac{1}{d^{\beta}} v_M^{(\kappa+1)p} dx \right)^{\frac{\gamma}{(\kappa+1)p(\beta-\gamma)}} \left(\int_{\Omega} v_M^{(\kappa+1)p^*} dx \right)^{\frac{1}{(\kappa+1)p^*}} \\ & \leq \frac{\beta}{C_3^{(\kappa+1)p(\beta-\gamma)}} \left[\frac{(\kappa+1)^p}{\kappa p + 1} \right]^{\frac{\beta}{(\kappa+1)p(\beta-\gamma)}} \left(\int_{\Omega} \frac{1}{d^{\beta}} u^{(\kappa+1)p} dx \right)^{\frac{\gamma}{(\kappa+1)p(\beta-\gamma)}} \\ & \quad \times \left(\int_{\Omega} u^{(\kappa+1)p} dx \right)^{\frac{1}{(\kappa+1)p}}. \end{aligned} \quad (4.7)$$

We set $\kappa_1 := \frac{p^*}{p} - 1$, i.e., $(\kappa_1 + 1)p = p^*$. Since $u \in L^{p^*}(\Omega)$ by Proposition 2.2, the last integral in (4.7) is finite and we can apply Lemma 4.1 with $\kappa = \kappa_1$. Passing to the limit for $M \rightarrow \infty$ in (4.7) with $\kappa = \kappa_1$, we conclude

$$\begin{aligned} & \left(\int_{\Omega} \frac{1}{d^{\beta}} u^{p^*} dx \right)^{\frac{\gamma}{(\kappa_1+1)p(\beta-\gamma)}} \|u\|_{(\kappa_1+1)p^*} \\ & \leq C_3^{\frac{\beta}{(\kappa_1+1)p(\beta-\gamma)}} \left[\frac{(\kappa_1+1)^p}{\kappa_1 p + 1} \right]^{\frac{\beta}{(\kappa_1+1)p(\beta-\gamma)}} \left(\int_{\Omega} \frac{1}{d^{\beta}} u^* dx \right)^{\frac{\gamma}{(\kappa_1+1)p(\beta-\gamma)}} \|u\|_{p^*}, \end{aligned}$$

i.e.,

$$\|u\|_{(\kappa_1+1)p^*} \leq C_3^{\frac{\beta}{(\kappa_1+1)p(\beta-\gamma)}} \left[\frac{(\kappa_1+1)^p}{\kappa_1 p + 1} \right]^{\frac{\beta}{(\kappa_1+1)p(\beta-\gamma)}} \|u\|_{p^*}. \quad (4.8)$$

In the next step we set $\kappa_2 := (\frac{p^*}{p})^2 - 1$, i.e., $(\kappa_2 + 1)p = (\kappa_1 + 1)p^*$. Due to (4.8) we can apply Lemma 4.1 with $\kappa = \kappa_2$ and pass to the limit for $M \rightarrow \infty$ in (4.7) with $\kappa = \kappa_2$. We deduce

$$\|u\|_{(\kappa_2+1)p^*} \leq C_3^{\frac{\beta}{(\kappa_2+1)p(\beta-\gamma)}} \left[\frac{(\kappa_2+1)^p}{\kappa_2 p + 1} \right]^{\frac{\beta}{(\kappa_2+1)p(\beta-\gamma)}} \|u\|_{(\kappa_1+1)p^*}. \quad (4.9)$$

By induction, for $\kappa_n := (\frac{p^*}{p})^n - 1$, i.e., $(\kappa_n + 1)p = (\kappa_{n-1} + 1)p^*$, we get

$$\|u\|_{(\kappa_n+1)p^*} \leq C_3^{\frac{\beta}{(\kappa_n+1)p(\beta-\gamma)}} \left[\frac{(\kappa_n+1)^p}{\kappa_n p + 1} \right]^{\frac{\beta}{(\kappa_n+1)p(\beta-\gamma)}} \|u\|_{(\kappa_{n-1}+1)p^*}. \quad (4.10)$$

From (4.8)–(4.10) we obtain

$$\|u\|_{(\kappa_n+1)p^*} \leq C_3^{\frac{\beta}{p(\beta-\gamma)}} \sum_{k=1}^n \frac{1}{\kappa_k + 1} \prod_{k=1}^n \left[\left(\frac{(\kappa_n+1)^{\frac{\beta}{\beta-\gamma}}}{(\kappa_n p + 1)^{\frac{\beta}{p(\beta-\gamma)}}} \right)^{\frac{1}{\sqrt{\kappa_k+1}}} \right]^{\frac{1}{\sqrt{\kappa_k+1}}} \|u\|_{p^*}. \quad (4.11)$$

We consider the function

$$f(y) := \left(\frac{(y+1)^{\frac{\beta}{\beta-\gamma}}}{(yp+1)^{\frac{\beta}{p(\beta-\gamma)}}} \right)^{\frac{1}{\sqrt{y+1}}}, \quad y \in (0, +\infty).$$

Since f is continuous and $\lim_{y \rightarrow +\infty} f(y) = 1$, there exists a constant $C_4 > 1$ such that for any $k = 1, 2, \dots, n$, we have

$$\left(\frac{(\kappa_k + 1)^{\frac{\beta}{\beta-\gamma}}}{(\kappa_k p + 1)^{\frac{\beta}{p(\beta-\gamma)}}} \right)^{\frac{1}{\sqrt{\kappa_k + 1}}} \leq C_4. \quad (4.12)$$

Hence, from (4.11), (4.12) we get

$$\|u\|_{(\kappa_n+1)p^*} \leq C_3^{\frac{\beta}{p(\beta-\gamma)}} \sum_{k=1}^n \frac{1}{\kappa_k+1} C_4^{\sum_{k=1}^n \frac{1}{\sqrt{\kappa_k+1}}} \|u\|_{p^*}. \quad (4.13)$$

But $\frac{1}{\kappa_k+1} = (\frac{p}{p^*})^k$, $\frac{1}{\sqrt{\kappa_k+1}} = (\sqrt{\frac{p}{p^*}})^k$ with $\frac{p}{p^*} < \sqrt{\frac{p}{p^*}} < 1$. That is,

$$\sum_{k=1}^{\infty} \frac{1}{\kappa_k + 1} < +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{\kappa_k + 1}} < +\infty.$$

It follows from here and (4.13) that there exists a constant $C_5 > 0$, independent of $n \in \mathbb{N}$, such that

$$\|u\|_{(\kappa_n+1)p^*} \leq C_5 \|u\|_{p^*}, \quad n = 1, 2, \dots \quad (4.14)$$

But $\lim_{n \rightarrow +\infty} (\kappa_n + 1)p^* = +\infty$, (4.14) and Fatou's lemma imply

$$\|u\|_{\infty} \leq C_5 \|u\|_{p^*} < +\infty$$

(see [16, p. 116]) for a precise proof of this fact).

It follows from Lemma 4.1 that

$$\|u\|_{(\kappa+1)p;\beta} \leq \lambda_1^{\frac{\beta}{\beta-\gamma} \cdot \frac{1}{(\kappa+1)p}} \|u\|_{(\kappa+1)p}.$$

Hence, there exists a constant $C_6 > 0$ such that for $n \in \mathbb{N}$,

$$\|u\|_{(\kappa_n+1)p;\beta} \leq \lambda^{\frac{\beta}{\beta-\gamma} \cdot \frac{1}{(\kappa_n+1)p}} \|u\|_{(\kappa_n+1)p} \leq C_6 \|u\|_{p^*}$$

and so, by the same argument as above we have

$$\lim_{p \rightarrow +\infty} \|u\|_{p;\beta} = \|u\|_{\infty;\beta} := \sup_{x \in \Omega} \text{ess } u(x) \leq C_6 \|u\|_{p^*} < +\infty.$$

The proof is complete. $\square$

Remark 4.1 In particular, the eigenvalue problem

$$\begin{cases} -\Delta_p u + \frac{1}{d^p} |u|^{p-2} u = \frac{\lambda}{d^\gamma} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 \leq \gamma < p$ has the first eigenvalue $\lambda_1 > 0$ and corresponding eigenfunction $u \in W_0^{1,p}(\Omega)$, $u \geq 0$ in Ω , satisfying $u \in L^\infty(\Omega)$, $\frac{1}{d^p} u \in L^\infty(\Omega)$ and $\frac{1}{d^\gamma} u \in L^\infty(\Omega)$.

Remark 4.2 As it was noticed in the Introduction, the boundary behavior and regularity of positive eigenfunctions in the spaces $C^{1,\delta}(\overline{\Omega})$ ($0 < \delta < 1$) was studied in [23] for $p = 2$, $\alpha = 0$, $0 \leq \beta$, $\gamma < 2$. Some partial results were obtained in [3] and then extended in [11] by applying the results in [8,9]. These results show that in the critical case positive eigenfunctions are “flat” in the sense that they are positive on Ω with zero boundary derivative. The above

estimates in Theorem 4.1 imply that if $1 < \gamma < \beta < \alpha + p$, these eigenfunctions are also “flat”. In particular, they violate the Hopf and/or Vázquez maximum principle. Another result in this direction was obtained recently in [17] in the radial case by using the results of [18]. All these questions will be studied in the forthcoming work [10].

5 Further properties of the principal eigenvalue and eigenfunction

Since $C_c^\infty(\Omega)$ is a dense set in $X = W_0^{1,p}(\Omega; \frac{1}{d^\alpha})$ (see [27, Section 7.2]), it is enough to consider only smooth φ with a compact support in Ω as a test function in (3.1). In particular, any weak solution of (3.1) is also weak solution in the sense of [36,37]. Hence, we have the following qualitative properties of the principal eigenfunction.

Theorem 5.1 *Let $u \in X$, $u \geq 0$ a.e. in Ω , be the principal eigenfunction of (1.1) associated with $\lambda_1 > 0$. Then $u > 0$ in Ω . Assume, moreover, that $\partial\Omega$ is of class C^1 . Then for any compact set $K \subset\subset \Omega$ there exists $\delta = \delta(K) \in (0, 1)$ such that $u \in C^{1,\delta}(K)$.*

Proof Let $K \subset\subset \Omega$ be a compact set. Then there exist constants $0 < k_1(K) < k_2(K)$ such that

$$k_1(K) \leq \frac{1}{d^\alpha(x)} \leq k_2(K), \quad k_1(K) \leq \frac{1}{d^\beta(x)} \leq k_2(K), \quad k_1(K) \leq \frac{1}{d^\gamma(x)} \leq k_2(K) \quad (5.1)$$

for all $x \in K$. Then it follows from (5.1), [37, Theorem 1], Theorem 1 and our Theorem 4.1 that $u > 0$ everywhere in K . Since K is chosen arbitrarily, $u > 0$ everywhere in Ω , too.

In case $\partial\Omega$ is of class C^1 , $\frac{1}{d^\alpha}$, $\frac{1}{d^{\alpha+p}}$ and $\frac{1}{d^\gamma}$ are C^1 functions in every compact set $K \subset\subset \Omega$ and satisfy (5.1) (see [22, Lemma 14.16]). The existence of $\delta \in (0, 1)$ such that $u \in C^{1,\delta}(K)$ follows from [36, Theorem 1] combined with Theorem 4.1. $\square$

Besides regularity and positivity of the first eigenfunction we have also the following simplicity result for λ_1 .

Theorem 5.2 *The principal eigenvalue is simple, that is, the corresponding principal eigenfunction u is unique up to a multiple by a nonzero real number (and hence does not change sign in Ω).*

Proof The following argument is taken from [2], see also [20]. Let u and v be eigenfunctions associated with λ_1 , normalized by $\|u\|_{p;\gamma} = \|v\|_{p;\gamma} = 1$. It follows from the proof of Theorem 3.1 that both u and v minimize $R(u) = \|u\|_X^p + \|u\|_{p;\beta}^p$. By Theorem 5.1 we may assume that $u > 0$ and $v > 0$ in Ω . We construct a function $w := (\frac{u^p + v^p}{2})^{\frac{1}{p}}$. Then, clearly, we have

$$\int_{\Omega} \frac{1}{d^\gamma} w^p \, dx = \frac{1}{2} \left(\int_{\Omega} \frac{1}{d^\gamma} u^p \, dx + \int_{\Omega} \frac{1}{d^\gamma} v^p \, dx \right) = 1,$$

i.e., $w \in \mathcal{S}$. We also have (pointwise in Ω):

$$\begin{aligned}
 |\nabla w|^p &= \left(\frac{u^p + v^p}{2} \right)^{1-p} \left| \frac{1}{2} (u^{p-1} \nabla u + v^{p-1} \nabla v) \right|^p \\
 &= \frac{u^p + v^p}{2} \left| \frac{1}{2} \left(\frac{u^p}{\frac{u^p + v^p}{2}} \cdot \frac{\nabla u}{u} + \frac{v^p}{\frac{u^p + v^p}{2}} \cdot \frac{\nabla v}{v} \right) \right|^p \\
 &= \frac{u^p + v^p}{2} \left| \frac{u^p}{u^p + v^p} \cdot \frac{\nabla u}{u} + \left(1 - \frac{u^p}{u^p + v^p} \right) \frac{\nabla v}{v} \right|^p \\
 &\leq \frac{u^p + v^p}{2} \left(\frac{u^p}{u^p + v^p} \left| \frac{\nabla u}{u} \right|^p + \left(1 - \frac{u^p}{u^p + v^p} \right) \left| \frac{\nabla v}{v} \right|^p \right) \\
 &= \frac{1}{2} \left(u^p \left| \frac{\nabla u}{u} \right|^p + v^p \left| \frac{\nabla v}{v} \right|^p \right) = \frac{1}{2} (|\nabla u|^p + |\nabla v|^p). \tag{5.2}
 \end{aligned}$$

This implies

$$\int_{\Omega} \frac{1}{d^\alpha} |\nabla w|^p \, dx \leq \frac{1}{2} \left(\int_{\Omega} \frac{1}{d^\alpha} |\nabla u|^p \, dx + \int_{\Omega} \frac{1}{d^\alpha} |\nabla v|^p \, dx \right). \tag{5.3}$$

Since both u and v minimize R on $\mathcal{S}$, equality must hold in (5.3). In particular, it follows from here, that equality must hold also in (5.2) almost everywhere in Ω . The strict convexity of the function $t \mapsto |t|^p$, $p > 1$, then implies that $\frac{\nabla u}{u} = \frac{\nabla v}{v}$ almost everywhere in Ω . But this yields $\nabla(\frac{u}{v}) = 0$ almost everywhere in Ω . Since Ω is a connected set and $u, v \in C^1(\Omega)$, there exists a constant $C > 0$ such that $u = Cv$ holds everywhere in Ω . But $u, v \in \mathcal{S}$ immediately implies that $C = 1$. $\square$

The eigenfunctions associated with an eigenvalue different from λ_1 must change its sign in Ω . Namely, we have

Theorem 5.3 *Let $0 \leq \gamma < 1$ and $v \in X$ be an eigenfunction associated with an eigenvalue $\lambda \neq \lambda_1$. Then $v^+ \not\equiv 0$ and $v^- \not\equiv 0$, where v^+ and v^- denote the positive and negative part of v , respectively.*

Proof It follows from the proof of Theorem 3.1 that $\lambda > \lambda_1$. Assume by contradiction that $v \geq 0$, $v \not\equiv 0$ in Ω . Since u and v are eigenfunctions associated with λ_1 and λ , respectively, we have

$$\int_{\Omega} \frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} \frac{1}{d^\beta} |u|^{p-2} u \varphi \, dx = \lambda_1 \int_{\Omega} \frac{1}{d^\gamma} |u|^{p-2} u \varphi \, dx, \tag{5.4}$$

$$\int_{\Omega} \frac{1}{d^\alpha} |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dx + \int_{\Omega} \frac{1}{d^\beta} |v|^{p-2} v \psi \, dx = \lambda \int_{\Omega} \frac{1}{d^\gamma} |v|^{p-2} v \psi \, dx \tag{5.5}$$

for all $\varphi, \psi \in X$. For $\varepsilon > 0$ we set

$$\begin{aligned}
 u_\varepsilon &:= u + \varepsilon \quad \text{and} \quad v_\varepsilon := v + \varepsilon, \\
 \varphi &:= \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}} \quad \text{and} \quad \psi := \frac{v_\varepsilon^p - u_\varepsilon^p}{v_\varepsilon^{p-1}}.
 \end{aligned}$$

The assumption $v \geq 0$ allows to apply Theorem 4.1 to v . We conclude $v \in L^\infty(\Omega)$. It then follows that $\varphi, \psi \in X$. Adding (5.4) and (5.5) with φ and ψ chosen above, and performing

the estimates as in [16, p. 120], we arrive at

$$\begin{aligned} & \int_{\Omega} \frac{1}{d^{\gamma}} \left[\lambda \left(\frac{v}{v_{\varepsilon}} \right)^{p-1} - \lambda_1 \left(\frac{u}{u_{\varepsilon}} \right)^{p-1} \right] (v_{\varepsilon}^p - u_{\varepsilon}^p) dx \\ & \geq \int_{\Omega} \frac{1}{d^{\beta}} \left[\left(\frac{v}{v_{\varepsilon}} \right)^{p-1} - \left(\frac{u}{u_{\varepsilon}} \right)^{p-1} \right] (v_{\varepsilon}^p - u_{\varepsilon}^p) dx \geq 0. \end{aligned} \quad (5.6)$$

The last inequality is due to the fact that

$$f(x, y, \varepsilon) = \left[\left(\frac{x}{x + \varepsilon} \right)^{p-1} - \left(\frac{y}{y + \varepsilon} \right)^{p-1} \right] ((x + \varepsilon)^p - (y + \varepsilon)^p) \geq 0$$

for $x \geq 0$, $y \geq 0$, $\varepsilon > 0$. It follows from $0 \leq \gamma < 1$ that $\int_{\Omega} \frac{1}{d^{\gamma(x)}} dx < +\infty$. This fact together with $u \in L^{\infty}(\Omega)$, $v \in L^{\infty}(\Omega)$ imply that we can apply Lebesgue's theorem and pass to the limit for $\varepsilon \rightarrow 0_+$ in (5.6). We derive

$$(\lambda - \lambda_1) \int_{\Omega} \frac{1}{d^{\gamma}} (v^p - u^p) dx \geq 0. \quad (5.7)$$

Since both $u > 0$ and $v \geq 0$ are unique up to a multiple by a positive constant, we can normalize them in such a way that (5.7) gives a contradiction. $\square$

As a consequence of Theorem 5.3 we get the isolatedness of λ_1 .

Theorem 5.4 *Let $0 \leq \gamma < 1$. Then the first eigenvalue λ_1 is isolated.*

Proof Let v be an eigenfunction associated with an eigenvalue $\lambda > \lambda_1$. Then v changes sign in Ω (see Theorem 5.3). Let us denote

$$\Omega^{-} := \{x \in \Omega : v(x) < 0\} (\neq \emptyset).$$

Let us choose $\psi = v^{-}$ as a test function in (5.5). We obtain

$$\begin{aligned} \|v^{-}\|_X^p & \leq \int_{\Omega} \frac{1}{d^{\alpha}} |\nabla v^{-}|^p dx + \int_{\Omega} \frac{1}{d^{\beta}} |v^{-}|^p dx = \lambda \int_{\Omega} \frac{1}{d^{\gamma}} |v^{-}|^p dx \\ & = \lambda \int_{\Omega} \frac{1}{d^{\gamma}} (|v^{-}|^p)^{\frac{\gamma}{\beta}} (|v^{-}|^p)^{1-\frac{\gamma}{\beta}} dx \leq \lambda \left(\int_{\Omega} \frac{1}{d^{\beta}} |v^{-}|^p dx \right)^{\frac{\gamma}{\beta}} \left(\int_{\Omega} |v^{-}|^p dx \right)^{1-\frac{\gamma}{\beta}} \\ & \leq \lambda C_7 \|v^{-}\|_X^{\frac{p\gamma}{\beta}} \left(\int_{\Omega} |v^{-}|^p dx \right)^{1-\frac{\gamma}{\beta}} \\ & \leq \lambda C_8 \|v^{-}\|_X^{\frac{p\gamma}{\beta}} \left(\int_{\Omega} |v^{-}|^{p^*} dx \right)^{\frac{p}{p^*} \left(1 - \frac{\gamma}{\beta} \right)} (\text{meas } \Omega^{-})^{\frac{p^*-p}{p^*} \left(1 - \frac{\beta}{\gamma} \right)}. \end{aligned}$$

From here we deduce

$$\|v^{-}\|_X \leq \lambda^{\frac{1}{p}} C_8^{\frac{1}{p}} \|v^{-}\|_{p^*}^{\frac{p^*-p}{pp^*}} \leq \lambda^{\frac{1}{p}} C_9 \|v^{-}\|_X (\text{meas } \Omega^{-})^{\frac{p^*-p}{pp^*}},$$

i.e., there is a constant $C_{10} > 0$ such that

$$\text{meas } \Omega^{-} \geq C_{10} > 0. \quad (5.8)$$

Here C_{10} does not depend on λ for a given bounded interval. Let us assume by contradiction that there are eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of (1.1), such that $\lambda_n \searrow \lambda_1$. Let u_n be an eigenfunction

associated with λ_n which is normalized by $\|u_n\|_X = 1$. Without loss of generality we may assume that there exists $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly both in X and $L^p(\Omega; \frac{1}{d^\beta})$ and $u_n \rightarrow u$ strongly in $L^p(\Omega; \frac{1}{d^\gamma})$. It follows from the continuity of Nemytskij's operator between $L^p(\Omega; \frac{1}{d^\gamma})$ and $L^{p'}(\Omega; \frac{1}{d^\gamma})$, $p' = \frac{p}{p-1}$ (see [16, Theorem 1.1]) that

$$\lambda_n |u_n|^{p-2} u_n \rightarrow \lambda |u|^{p-2} u \text{ strongly in } L^{p'}\left(\Omega; \frac{1}{d^\gamma}\right).$$

Now, let us take $v = u_n$ in (5.5), $\varphi = \psi = (u_n - u)$ and subtract (5.4) from (5.5). We get

$$\begin{aligned} & \int_{\Omega} \frac{1}{d^\alpha} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot (\nabla u_n - \nabla u) \, dx \\ & + \int_{\Omega} \frac{1}{d^\beta} (|u_n|^{p-2} u_n - |u|^{p-2} u) (u_n - u) \, dx \\ & = \int_{\Omega} \frac{1}{d^\gamma} (\lambda_n |u_n|^{p-2} u_n - \lambda |u|^{p-2} u) (u_n - u) \, dx. \end{aligned} \quad (5.9)$$

We estimate the left-hand side of (5.9) using Hölder's inequality from below:

$$\begin{aligned} \text{LHS} & \geq \int_{\Omega} \frac{1}{d^\alpha} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot (\nabla u_n - \nabla u) \, dx \\ & \geq \left[\left(\int_{\Omega} \frac{1}{d^\alpha} |\nabla u_n|^p \, dx \right)^{\frac{1}{p'}} - \left(\int_{\Omega} \frac{1}{d^\alpha} |\nabla u|^p \, dx \right)^{\frac{1}{p'}} \right] \\ & \quad \times \left[\left(\int_{\Omega} \frac{1}{d^\alpha} |\nabla u_n|^p \, dx \right)^{\frac{1}{p}} - \left(\int_{\Omega} \frac{1}{d^\alpha} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \right] \\ & = (\|u_n\|_X^{p-1} - \|u\|_X^{p-1}) (\|u_n\|_X - \|u\|_X) \geq 0. \end{aligned} \quad (5.10)$$

On the other hand, using Hölder's inequality, we estimate the right-hand side of (5.9) from above:

$$\begin{aligned} \text{RHS} & = \int_{\Omega} \frac{1}{d^{\frac{\gamma}{p'}}} (\lambda_n |u_n|^{p-2} u_n - \lambda |u|^{p-2} u) \frac{1}{d^{\frac{\gamma}{p}}} (u_n - u) \, dx \\ & \leq \left(\int_{\Omega} \frac{1}{d^\gamma} |\lambda_n |u_n|^{p-2} u_n - \lambda |u|^{p-2} u|^{p'} \, dx \right)^{\frac{1}{p'}} \left(\int_{\Omega} \frac{1}{d^\gamma} |u_n - u|^p \, dx \right)^{\frac{1}{p}} \rightarrow 0 \end{aligned} \quad (5.11)$$

as $n \rightarrow \infty$ by the convergences in $L^{p'}(\Omega; \frac{1}{d^\gamma})$ and $L^p(\Omega; \frac{1}{d^\gamma})$ mentioned above. But then (5.9)–(5.11) imply that also $\text{LHS} \rightarrow 0$ as $n \rightarrow \infty$. In particular, $\|u_n\|_X \rightarrow \|u\|_X$. Since $u_n \rightharpoonup u$ in X and X is a uniformly convex Banach space, we get $u_n \rightarrow u$ in X . This implies $u \neq 0$, and hence u is a normalized eigenfunction associated with λ_1 . Without loss of generality we assume that $u > 0$ in Ω . Then the strong convergence $u_n \rightarrow u$ in $L^p(\Omega; \frac{1}{d^\gamma})$ and Egorov's theorem imply that u_n converges uniformly to u except on a subset of Ω with arbitrarily small measure. On the other hand, every set $\Omega_n^- := \{x \in \Omega : u_n(x) < 0\}$ must satisfy (5.8) with $C_{10} > 0$ independent of n , a contradiction with $u > 0$ in Ω . This proves that λ_1 is isolated from the right. The isolatedness from the left follows from definition of λ_1 to be the least eigenvalue of (1.1). $\square$

6 Final remarks

Problem (1.2), but this time with unbounded $m(x)$ was studied by Montenegro and Lorca [33] under various sets of assumptions on $m(x)$ by using Hardy–Sobolev inequality allowing to use variational arguments and getting similar results.

If $\alpha = 0$ and $p = 2$, the corresponding linear eigenvalue problem

$$\begin{cases} -\Delta u + \frac{1}{d^\beta} u = \frac{\lambda}{d^\gamma} u & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \quad (6.1)$$

was studied by Bertsch and Rostamian [3]. The motivation for this work was to obtain linearized stability results for stationary positive solutions $\bar{u} > 0$ (satisfying $\frac{\partial \bar{u}}{\partial n} < 0$ on $\partial\Omega$ as well) of the quasilinear degenerate parabolic problems

$$\begin{cases} \beta(u)_t - \Delta u = f(u) & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega \end{cases}$$

of porous media type with nonlinear diffusion. A model example is $\beta(u) = u^{\frac{1}{m}}$, $f(u) = u^{\frac{p}{m}}$ with $1 \leq p < m$. They proved existence and other qualitative properties of the first eigenvalue λ_1 using the variational arguments in the framework of weighted Sobolev spaces $H_0^1(\Omega, b)$ for some singular weight $b(x)$. The case $\beta < \gamma < 2$ was covered and some partial results were obtained also for the critical cases $\beta = 2$ and $\gamma = 2$. They showed that $\lambda_1 > 0$ (resp. $\lambda_1 < 0$) implies asymptotic stability (resp. instability) in the sense of Lyapunov.

Problems of the same type as (6.1) arise later, when dealing with linearized stability for stationary positive solutions $\bar{u} > 0$ (again with $\frac{\partial \bar{u}}{\partial n} < 0$ on $\partial\Omega$) of semilinear singular parabolic problems

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{in } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where $f: (0, \infty) \rightarrow \mathbb{R}^+$ with $\lim_{u \searrow 0} f(u) = +\infty$. A model example is $f(u) = \frac{1}{u^\alpha}$ with $\alpha > 0$. The associated linearized problems (6.1) were studied in Hernández et al. [23] for much more general differential operators, not necessarily in divergence form, working in $C_0^1(\overline{\Omega})$ and some $C_0^{1,\gamma}(\overline{\Omega})$ ($0 < \gamma < 1$) spaces. They used functional analytic methods, in particular the Krein–Rutman theorem (the Strong Maximum Principle was extended to such situation in Hernández et al. [23]). It was then proved that linearized stability implies the Lyapunov stability. Applications were given in [23, 24]. In the model example, only the case $0 < \alpha < 1$ was covered.

In a recent work Díaz and Hernández [11] extend and unify the results from [3] (and in some sense also from [23]), working again in weighted Sobolev spaces but following an approach different from [3]. In particular, this allows to deal also with the case $\alpha \geq 1$ in the model example. The interesting problem of the boundary behavior of the eigenfunction $\varphi_1 > 0$ (associated with $\lambda_1 > 0$) for $\beta = 2$, $\gamma = 0$ was dealt with in Díaz [8, 9], exhibiting the difference between $\beta = 2$ and $\beta < 2$. For stability results obtained by different methods for ground states of some non-Lipschitz nonlinearities, see [12].

The optimal regularity for positive eigenfunctions was obtained in [23] in the framework of $C^{1,\delta}(\overline{\Omega})$ spaces ($0 < \delta < 1$) for $p = 2$, $\alpha = 0$ when $0 \leq \beta, \gamma < 2$, excluding the critical

cases $\beta = 2$ and/or $\gamma = 2$. Some partial regularity results were obtained in [3] starting from the work in the Sobolev spaces $W_0^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega; b)$ but both best regularity and boundary behavior in the critical case were not considered. Some results in this direction were given in [11] by using the results by Díaz in [8,9] showing that positive eigenfunctions are “flat” in this case. Something similar can be done for the p -Laplacian but we do not intend to address it here. We also do not touch here the links with the linearization of the p -Laplacian.

A different type of linear eigenvalue problem with unbounded weights,

$$\begin{cases} -\Delta u + a(x)u = \lambda m(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

was studied in Fleckinger et al. [21] (where also references to previous work can be found), this time a, m satisfy integrability conditions like $a, m \in L^\gamma(\Omega)$ with $\gamma > \frac{N}{2}$. An alternative version of the Krein–Rutman theorem (see Daners and Koch-Medina [7]) was a useful tool in this study.

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