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Gradient behaviour for large solutions to semilinear elliptic problems

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Abstract

Given $p > 1$ and f Lipschitz, under appropriate assumptions on the smoothness of the bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, we give a precise description of the asymptotic behaviour of the gradient of the unique solution of

$$\begin{cases} -\Delta u + |u|^{p-1}u = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

In particular, we show that there exists a corrector function S, finite sum of singular terms, such that

$$z := u - S \in W^{1,\infty}(\Omega).$$

Moreover, we prove that

$$\forall\, \bar{x} \in \partial\Omega \qquad z(\bar{x}) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \frac{z(\bar{x} - \delta\nu(\bar{x}))}{\delta} = 0,$$

where ν is the outward unit normal to $\partial\Omega$.

Keywords Large solutions · Semilinear elliptic equations · Gradient bounds

Mathematics Subject Classification 35J25 · 35J61

1 Introduction

This paper is devoted to the study of semilinear elliptic problems with explosive boundary conditions; more precisely, we are interested in the qualitative behaviour of solutions of

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain of $\mathbb{R}^N$, with $N \geq 1$, $g \in C^1(\mathbb{R})$ is such that

$$g(a) > 0 \quad \text{for some} \quad a \in \mathbb{R} \quad \text{and} \quad g'(s) > 0 \quad \text{for every} \quad s \in \mathbb{R}, \tag{1.2}$$

and f is a Lipschitz continuous function. Here, solutions are meant in the classical sense, i.e. $C^2(\Omega)$ functions which satisfy the differential equation above pointwise and such that

$$\lim_{x \to \partial\Omega} u(x) = +\infty.$$

In the literature, solutions that blow up at the boundary of the domain are known as *large solutions*. Looking naively at (1.1), one naturally wonders under which assumptions on g the existence of a large solution is assured, if the monotonicity assumption on g implies uniqueness of solution and how such a solution behaves near the boundary.

In the seminal works by Keller and Osserman (see [19,28]), it is proved that the necessary and sufficient condition for the existence of a large solution for problem (1.1) is the following:

$$\exists\, t_0 \in [-\infty, +\infty) \ : \ \psi(t) := \int_t^\infty \frac{\mathrm{d}s}{\sqrt{2G(s)}} < \infty \quad \text{for} \quad t > t_0, \quad \text{where} \quad G'(s) = g(s). \tag{1.3}$$

This growth condition at infinity, known as Keller–Osserman condition, arises solving the one-dimensional problem

$$-\phi'' + g(\phi) = 0, \quad s > 0 \quad \text{and} \quad \lim_{s \to 0^+} \phi(s) = +\infty. \tag{1.4}$$

We stress that, in fact, $\phi(s) = \psi^{-1}(s)$ solves problem (1.4). We refer the interested reader to [14] (see also the references cited therein) for existence issues with no monotonicity assumptions on g.

Uniqueness is not a trivial task in the sense that it is not known if the monotonicity of g is a sufficient condition for it; we refer to [26], where it is proved that if g is convex then (1.1) admits a unique large solution, and to [15] (see also [5]), where it is shown that assumptions of the type

$$\frac{g(t)}{t^q} \quad \text{increasing for } t \gg 1 \text{ and some } q > 1$$

imply uniqueness of large solution. It is worthy to mention that the special case $g(s) = |s|^{p-1}s$ with $p > 1$ satisfies the latter condition.

Let us point out now that the function ϕ defined in (1.4) is strongly related to the boundary behaviour of solutions of (1.1). In [4,5], it has been proved that the behaviour of u is, in some sense, *one dimensional* near the boundary, i.e. it holds that

$$\lim_{d(x) \to 0} \frac{\psi(u(x))}{d(x)} = 1 \quad \text{where } d(x) = \mathrm{dist}(x, \partial\Omega).$$

Moreover, if g is such that

$$\liminf_{t \to \infty} \frac{\psi(\beta t)}{\psi(t)} > 1, \quad \forall\, \beta \in (0, 1), \tag{1.5}$$

then

$$\Big| u(x) - \phi(d(x)) \Big| = o(\phi(d(x))) \quad \text{as } d(x) \to 0, \tag{1.6}$$

namely the first-order term in the asymptotic of u near the boundary only depends on the corresponding ODE (1.4) and in particular is not affected by the geometry of the domain. In

[7], the authors improve (1.6); assuming in addition to (1.5) that

$$\frac{G(s)}{s^2} \text{ is strictly increasing for large s} \quad \text{and} \quad \limsup_{\beta\to 1, s\to 0} \frac{\phi'(\beta s)}{\phi'(s)} < \infty,$$

they prove that

$$|u(x) - \phi(d(x))| \le c\phi(d(x))d(x) \quad \text{as } d(x) \to 0,$$

where the positive constant c depends on the mean curvature of the boundary of Ω. After this first clue, the influence of the geometry of $\partial\Omega$ in the expansion of u has been studied in [21,29] under different assumptions on g. The most general result in this direction has been proved in [8]; in order to state it, we need to define

$$J(s) := \frac{N-1}{2}\int_0^s \Gamma(\phi(t))\mathrm{d}t, \quad \text{where} \quad \Gamma(t) := \frac{\int_0^t \sqrt{2G(s)}\mathrm{d}s}{G(t)}$$

and to assume that

$$\lim_{\delta\to 0} \frac{B(\phi(\delta(1+o(1))))}{B(\phi(\delta))} = 1 \quad \text{and} \quad \limsup_{t\to\infty} B(t)\Gamma(t) < \infty, \tag{1.7}$$

where

$$B(s) := \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{2G(s)} = \frac{g(s)}{\sqrt{2G(s)}}.$$

Assuming (1.7), together with (1.3) and (1.5), it follows that

$$\Big| u(x) - \phi\big[d(x) - H(x)J(d(x))\big] \Big| \le \phi(d(x)o(d(x)) \quad \text{as} \quad d(x) \to 0, \tag{1.8}$$

where H is a smooth function whose restriction to $\partial\Omega$ coincides with the mean curvature of the domain; moreover, it is worth stressing that (1.7) implies

$$J(d(x)) = O(d^2(x)).$$

The relation above, together with (1.8), tells us that the second-order contribution to the explosion of u is affected by the geometry of the domain through the mean curvature of $\partial\Omega$. More recently in [12] (see also [10]), by means of an interesting application of the contraction theorem, all the singular terms of the asymptotic of u have been implicitly calculated in the special case $\Omega = B$.

For power-type nonlinearities, it is also possible to obtain the first asymptotic of the gradient of the solution by means of scaling arguments. In particular, in [4,6] (see also [30]) it is proved that if

$$\lim_{s\to\infty} \frac{g(s)}{s^p} = 1 \quad \text{for some} \quad p > 1,$$

it holds true that

$$\left| \frac{\partial u(x)}{\partial \nu} - \frac{\partial \phi(d(x))}{\partial \nu} \right| + \left| \frac{\partial u(x)}{\partial \tau} \right| \le o(\phi'(d(x))) \quad \text{as} \quad d(x) \to 0, \tag{1.9}$$

where ν is the unit normal to $\partial\Omega$ (recall that $\nu(\bar{x}) = -\nabla d(\bar{x})$ for $\bar{x} \in \partial\Omega$) and $\tau \in \mathbb{S}^{N-1}$ is such that $\tau(\bar{x}) \cdot \nu(\bar{x}) = 0$ for every $\bar{x} \in \partial\Omega$. However, a general result for the second-order term in the expansion of ∇u in the same spirit of (1.8) is not available in the literature (see anyway [3] for a partial result in convex domains).

Our aim is to complete the picture of the asymptotic behaviour of the gradient of solutions of problem (1.10) in the case $g(s) = |s|^{p-1}s$, with $p > 1$ and Ω smooth enough. Thus, the problem we deal with is

$$\begin{cases} -\Delta u + |u|^{p-1}u = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $f \in W^{1,\infty}(\Omega)$. It is easy to verify that with such a choice of g, assumptions (1.3)–(1.7) are satisfied. It is also worth to recall that in this case problem (1.10) has a unique large solution and that the function ϕ defined in (1.4) has the following explicit form:

$$\phi(s) = \frac{\sigma_0}{s^\alpha} \quad \text{with} \quad \alpha = \frac{2}{p-1} \quad \text{and} \quad \sigma_0 = [\alpha(\alpha+1)]^{\frac{1}{p-1}}. \tag{1.11}$$

The result that we present in this paper will describe not only the second-order behaviour of the gradient of the large solution of (1.10), but also the complete asymptotic expansion of all the singular terms of u and ∇u, for every arbitrary sufficiently smooth domain and every $p > 1$. As a by-product of this expansion, we will be able to provide the expected second-order asymptotic for the normal and tangential components of ∇u with respect to $\partial\Omega$. Indeed, we will prove

$$\begin{aligned} &\lim_{x\to\bar{x}} \left[d^\alpha(x) \frac{\partial u(x)}{\partial \nu} - \alpha\sigma_0 d(x) \right] = c(\alpha, N) H(\bar{x}) \\ &\lim_{x\to\bar{x}} d^\alpha(x) \frac{\partial u(x)}{\partial \tau} = 0 \end{aligned} \quad \text{uniformly with respect to } \bar{x} \in \partial\Omega, \tag{1.12}$$

where $c(\alpha, N)$ is a precise constant that depends only on α and N (see Corollary 1.4 for more details). More in general, we will be able to prove (see Theorem 1.3 for the precise statement) that there exists a unique explicit function S, sum of $[\alpha] + 1$ singular terms where α is as in (1.11), such that

$$z := u - S \in W^{1,\infty}(\Omega).$$

Let us say that the formula above expresses the leitmotiv of the paper, that is, try to find an explicit simple corrector function that describes the explosive behaviour of u.

Moreover, using a scaling argument and the previously obtained information on z, in Theorem 2.9 we prove that the function z satisfies the following boundary conditions:

$$z(\bar{x}) = 0 \quad \text{and} \quad \lim_{\delta\to 0} \frac{z(\bar{x} - \delta\nu(\bar{x}))}{\delta} = 0 \quad \forall\, \bar{x} \in \partial\Omega.$$

The latter condition is a weak form of expressing the fact that the normal derivative of z is zero at the boundary of Ω.

Finally, we consider a more general class of nonlinearities that will be easily treated with an extension of our method.

Before stating precisely our main results, we need to give some notation.

1.1 Notation

We shall often work in tubular neighbourhoods of $\partial\Omega$ of the type

$$\Omega_\delta = \{y \in \Omega : \text{dist}(u, \partial\Omega) < \delta\}, \; \delta > 0.$$

We recall that Ω is always at least of class C^2. Hence, the function $\mathrm{dist}(\cdot, \partial\Omega)$ distance from the boundary is well defined and twice differentiable near $\partial\Omega$. More precisely, the following theorem, proved in [20], gives the relation between the regularity of the boundary and the regularity of the distance function.

Theorem 1.1 (Theorem 3 in [20]) *Let Ω be a domain of class C^γ with $\gamma \geq 2$. Then,*

$$\exists\bar{\delta} > 0 \text{ such that } dist(\cdot, \partial\Omega) \in C^\gamma(\overline{\Omega_{\bar{\delta}}}). \tag{1.13}$$

Thanks to the previous theorem, we can define the following smooth versions of the distance function.

Definition 1.2 Let Ω be a domain of class C^γ with $\gamma \geq 2$, and let $\bar{\delta} > 0$ be given by (1.13). Then, we define the regularized distance as a function $d \in C^\gamma(\Omega)$ such that $d(x) = \mathrm{dist}(x, \partial\Omega)$ for every x that belongs to $\Omega_{\bar{\delta}}$. We moreover denote $d_n(x) := d(x) + \frac{1}{n}$.

It is worthy to stress that $d_n(\cdot)$ inherits from $\mathrm{dist}(\cdot, \partial\Omega)$ the following important properties

$$|\nabla d_n(x)|^2 = 1 \;\; x \in \Omega_{\bar{\delta}}, \quad \nabla d_n(\bar{x}) = -\nu(\bar{x}) \quad \text{and} \quad \Delta d_n(\bar{x}) = -(N-1)H(\bar{x}) \;\; \bar{x} \in \partial\Omega,$$

where ν is the outward normal to $\partial\Omega$ and $H(\bar{x})$ is the mean curvature at $\bar{x} \in \partial\Omega$.

Finally, unless otherwise specified, we indicate with C a constant that depends only on the data of the problem and that can vary line to line also in the proof on the same theorem.

1.2 Main results

The ansatz that guides our approach is that it is possible to give an explicit description of the explosive behaviour of the large solution u and of its gradient ∇u by means of a finite sum of singular terms. Inspired by (1.6), (1.8) and (1.9), we conjecture that

$$u(x) \sim \sigma_0 d^{-\alpha} + \sigma_1 d^{-\alpha+1} + \sigma_2 d^{-\alpha+2} + \cdots,$$

where σ_k with $k = 0, 1, \ldots$ are smooth functions, and define the following *regularized* function

$$z := u - \left(\sigma_0 d^{-\alpha} + \sigma_1 d^{-\alpha+1} + \sigma_2 d^{-\alpha+2} + \cdots\right). \tag{1.14}$$

Hence, the first question we want to answer is:

Can we find σ_k with $k = 0, 1, \ldots$ such that z and $|\nabla z|$ belong to $L^\infty(\Omega)$?

Of course, the functions $\sigma_1, \ldots, \sigma_k$ shall take into account several characteristics of the problem, among others the geometry of the domain. Notice moreover that the definition (1.14) suggests that we need $[\alpha] + 2$ terms for having $z \in W^{1,\infty}(\Omega)$. Indeed, we have the following result.

Theorem 1.3 *Let us assume $p > 1$ and fix $\alpha := \frac{2}{p-1}$. Let Ω be a bounded domain of class $C^{[\alpha]+5}$ with $[\alpha]$ the integer part of α, let f belong to $W^{1,\infty}(\Omega)$, and let u be the unique large* *solution of* (1.10). *Let us define the following functions*

$$\begin{aligned}
\sigma_0 :=&[\alpha(\alpha+1)]^{\frac{1}{p-1}}\\
\sigma_1(x) :=& -\frac{1}{2}\frac{\alpha\sigma_0}{1+2\alpha}\Delta d(x) = \sigma_0\frac{\alpha(N-1)H(x)}{2(1+2\alpha)}\\
\sigma_k(x) :=& \frac{(\alpha+1-k)[\sigma_{k-1}(x)\Delta d(x) + 2\nabla\sigma_{k-1}(x)\nabla d(x)] + \Delta\sigma_{k-2}(x)}{(k-\alpha)(k-\alpha-1)-(2+\alpha)(\alpha+1)}\\
&+\frac{\sigma_0^p}{(k-\alpha)(k-\alpha-1)-(2+\alpha)(\alpha+1)}\\
&\sum_{j=2}^{k}\left[\binom{p}{j}\sigma_0^{-j}\sum_{i_1+\cdots+i_j=k}\sigma_{i_1}(x)\cdots\sigma_{i_j}(x)\right]\\
&\mathit{for}\, k = 2\cdots[\alpha]+1 \text{ and } i_1,\ldots,i_j \text{ positive integers.}
\end{aligned}\tag{1.15}$$

Then, $\sigma_k \in C(\overline{\Omega})^{[\alpha]+5-k}$ *with* $k = 0,\ldots,[\alpha]+1$, *and the function* $S \in C^4(\Omega)$, *defined as*

$$S(x) = \sum_{k=0}^{[\alpha]+1}\sigma_k(x)\, d^{k-\alpha}(x), \tag{1.16}$$

is such that

$$z(x) := u(x) - S(x) \in W^{1,\infty}(\Omega).$$

Moreover, it also holds true

$$\forall\, \bar{x} \in \partial\Omega \qquad z(\bar{x}) = 0 \quad \mathit{and} \quad \lim_{\delta\to 0}\frac{z(\bar{x}-\delta\nu(\bar{x}))}{\delta} = 0\,. \tag{1.17}$$

Remark 1.4 Let us stress that the higher the value of α (i.e. the closer p is to 1), the higher the number of singular terms is and the higher the regularity of Ω has to be.
Moreover, if we split the above estimate along normal and tangential directions we get a very precise estimate of all the singular terms in the expansion of the gradient. More specifically, we have that

$$\lim_{d(x)\to 0}\frac{\partial u}{\partial\nu} - \sum_{k=0}^{[\alpha]+1}(\alpha-k)\sigma_k(x)\, d^{k-\alpha-1}(x) + \frac{\partial\sigma_k(x)}{\partial\nu}\, d^{k-\alpha}(x) = 0 \tag{1.18}$$

while

$$\left|\frac{\partial u}{\partial\tau} - \sum_{k=0}^{[\alpha]+1}\frac{\partial\sigma_k(x)}{\partial\tau}\, d^{k-\alpha}(x)\right| \in L^\infty(\Omega) \tag{1.19}$$

$\forall\tau \in \mathbb{S}^{N-1}$ such that $\tau\cdot\nu = 0$.

From (1.18) and (1.19), we easily obtain the second-order asymptotic of the gradient (1.12) mentioned in Introduction.

The core of Theorem 1.3 is a Bernstein-type estimate for $|\nabla z|$. This type of technique, already used in the framework of large solutions for quasilinear problem in [22] (see also [23]), has been originally developed in [24,25], and it allows to obtain $L^\infty(\Omega)$-estimates for solutions of a vast class of boundary value problems. Of course, we do not know a priori the boundary condition (if any) satisfied by $u - S$; thus, it is not possible to obtain Bernstein estimates directly for z and $|\nabla z|$. We overcome this obstacle arguing by approximation and considering the following regularized corrector function

$$S_n(x) = \sum_{k=0}^{[\alpha]+1} \sigma_k(x) d_n^{k-\alpha}(x), \qquad d_n = d(x) + \frac{1}{n}, \tag{1.20}$$

where $\sigma_0, \ldots, \sigma_{[\alpha]+1}$ are the functions defined in (1.15), and the following approximated problem

$$\begin{cases} -\Delta u_n + |u_n|^{p-1} u_n = f, & \text{in } \Omega \\ \frac{\partial u_n}{\partial \nu} = \frac{\partial S_n}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Moreover, we define $z_n(x) := u_n(x) - S_n(x)$ that solves

$$\begin{cases} -\Delta z_n + |z_n + S_n|^{p-1}(z_n + S_n) - |S_n|^{p-1} S_n = \widetilde{f}_n & \text{in } \Omega \\ \dfrac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.22}$$

where

$$\widetilde{f}_n = f + \Delta S_n - |S_n|^{p-1} S_n. \tag{1.23}$$

Let us stress that the choice of the Neumann boundary condition in (1.21) and in turn in (1.22) is not the only possible, but it is the most convenient for our scope; indeed, Neumann problems are particularly suited for the implementation of the previously mentioned Bernstein estimates. Observe at this point that u_n converges, at least in $C^2_{\text{loc}}(\Omega)$ (see Proposition 2.6) to the unique large solution to (1.10), and this in turn implies that $z_n \to z := u - S$ in $C^2_{\text{loc}}(\Omega)$ where $S = \lim_{n\to\infty} S_n$. Hence, once a uniform estimate (with respect to n) in $W^{1,\infty}(\Omega)$ is obtained for the solution z_n of (1.22), it can be inherited by z as n diverges.

The proof of Theorem 1.3 is divided into the following main steps:

- we prove that there exists a constant $\bar{C} = \bar{C}(\sigma_0, \ldots, \sigma_{[\alpha]+1}, f)$ such that $d_n|\widetilde{f}_n| + d_n^2|\nabla \widetilde{f}_n| \le \bar{C} d^{1+[\alpha]-\alpha}$ for every $n \in \mathbb{N}$;
- we show that for every $n \in \mathbb{N}$ problem (1.21) admits a solution u_n and we describe the first-order behaviour of u_n near the boundary;
- through a Bernstein-type estimate, we show that there exists a positive constant $B = B(\sigma_0, \ldots, \sigma_{[\alpha]+1}, f)$ such that $\|z_n\|_{W^{1,\infty}(\Omega)} \le B$ for every $n \in \mathbb{N}$. This implies that $\|z\|_{W^{1,\infty}(\Omega)} \le B$.

Hence, Theorem 1.3 tells us that $z \in C^2(\Omega)$ satisfies

$$\begin{cases} -\Delta z + |z + S|^{p-1}(z + S) - |S|^{p-1} S = \widetilde{f} & \text{in } \Omega, \\ z \in W^{1,\infty}(\Omega), \end{cases}$$

where

$$\widetilde{f}(x) = f(x) + \Delta S(x) - |S(x)|^{p-1} S(x) \tag{1.24}$$

and $\widetilde{f} = \lim_{n\to\infty} \widetilde{f}_n$. Note that so far we do not have any information on the boundary behaviour of z, apart from the fact that is globally Lipschitz continuous. Thus, it is natural to wonder if z satisfies some boundary condition; and indeed using scaling arguments, in the same spirit of [4,30], we prove (1.17).

Let us now consider a class of nonlinearities for which Theorem 2.9 can be generalized with minor modifications. Let us thus focus on the following problem

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = r(x,u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.25}$$

where $p > 1$, $h \in C^4(\bar{\Omega})$ is such that for $0 < A < B$

$$A \le h(x) \le B \quad \forall\, x \in \bar{\Omega}, \tag{1.26}$$

and $r \in C^1(\bar{\Omega} \times \mathbb{R})$ satisfies

$$r(x,s)s \ge 0 \quad \text{and} \quad \frac{\partial}{\partial s}\left(h(x)|s|^{p-1}s - r(x,s)\right) \ge 0 \quad \forall\,(x,s) \in \Omega \times \mathbb{R}. \tag{1.27}$$

In Theorem 2.7 of [5], it is proved that for any bounded domain Ω of class C^2, under the assumptions (1.26) and (1.27), problem (1.25) admits a positive large solution; moreover, every large solution u of (1.25) has the following asymptotic behaviour near $\partial\Omega$

$$\lim_{d(x)\to 0} \frac{u(x)}{\sigma_0\left(\sqrt{h(x)}d(x)\right)^{-\alpha}} = 1. \tag{1.28}$$

Now, we make additional growth assumptions on the function $r(x,s)$ in order to be able to implement the Bernstein technique as in Theorem 1.3. We require that

$$\begin{cases} \sup\limits_{0<s<1} |r(x,s^{-\alpha})|s \in L^\infty(\Omega), \\ \sup\limits_{0<s<1} |r_x(x,s^{-\alpha})|s^2 \in L^\infty(\Omega), \\ \sup\limits_{0<s<1} |r_s(x,s^{-\alpha})|s^2 = o(1), & \text{as } d(x) \to 0, \\ \sup\limits_{0<s<1} |r_s(x,s^{-\alpha})|s^{-\alpha+1} \in L^\infty(\Omega), \end{cases} \tag{1.29}$$

where $r_x := \nabla_x r$ and $r_s := \frac{\partial r}{\partial s}$. As a first consequence of (1.29), we deduce that for $1 < q < p$ the function $g(x,s) := h(x)|s|^{p-1}s - r(x,s)$ satisfies

$$\frac{g(x,s)}{s^q} \quad \text{is increasing for large values of } s. \tag{1.30}$$

Indeed,

$$\frac{d}{ds}\frac{g(x,s)}{s^q} = \frac{(p-q)s^{p-1} - r_s(x,s) + r(x,s)s^{-1}}{s^q} > 0 \quad \text{for large value of } s, \quad \forall x \in \Omega.$$

Thus, using (1.28) and (1.30) we can take advantage of (the proof of) Theorem 2 of [15] to infer that problem (1.25) admits a unique large solution.

We stress here that we obtain the asymptotic expansion of large solutions and their gradient via an approximation procedure; thus, in the absence of a uniqueness result, our method gives a description only of the large solution obtained by the approximating scheme, i.e. the minimal large solution.

We can state our last result.

Theorem 1.5 *Let us assume $p > 1$, fix $\alpha := \frac{2}{p-1}$, and let Ω be a bounded domain of class $C^{[\alpha]+5}$. Assume moreover that (1.26), (1.27) and (1.29) hold true. Then, there exist functions*

$\sigma_{h,k} = \sigma_{h,k}(p, \Omega, h)$ (see (2.21) for the precise definition) with $\sigma_{h,k} \in C(\overline{\Omega})^{[\alpha]+5-k}$ $k = 0, \dots, [\alpha]+1$, such that, defining S_h as

$$S_h(x) = \sum_{k=0}^{[\alpha]+1} \sigma_{h,k}(x) \left(\sqrt{h(x)}d(x)\right)^{k-\alpha}, \tag{1.31}$$

it results $u - S_h \in W^{1,\infty}(\Omega)$ and $z(\bar{x}) := u(\bar{x}) - S_h(\bar{x}) = 0$ for every $\bar{x} \in \partial\Omega$. If moreover we assume

$$\sup_{0<s<1} |r(x, s^{-\alpha})| = o(1) \quad \text{as } d(x) \to 0 \tag{1.32}$$

we also have that

$$\forall\, \bar{x} \in \partial\Omega \qquad \lim_{\delta \to 0} \frac{z(\bar{x} - \delta\nu(\bar{x}))}{\delta} = 0\,.$$

Let us stress that the functions $\sigma_{h,k}$ do not depend on the function r, due to assumptions (1.29). Indeed, these growth conditions imply that the contribution of the perturbation $r(x, s)$ does not affect the asymptotic behaviour prescribed by $h(x)|s|^{p-1}s$.
On the other hand, the presence of the weight h requires some modifications in the definition of the corrector function S_h. This in turn yields to even more involved formulas for $\sigma_{h,0}, \dots, \sigma_{h,[\alpha]+1}$ than (1.15). Notice that

$$\sigma_{h,0} = [\alpha(\alpha+1)]^{\frac{1}{p-1}} \equiv \sigma_0,$$
$$\sigma_{h,1}(x) = \alpha\sigma_0 h^{-1}(x) \frac{\alpha h^{-\frac{1}{2}}(x)\nabla h(x)\nabla d_n(x) + h^{\frac{1}{2}}(x)(N-1)H(x)}{2(1+2\alpha)},$$

namely the first-order behaviour does not see the influence of the weight that comes into play from the second one onward. As a last comment to Theorem 1.5, notice that, in order to recover the Neumann boundary conditions for z, the additional growth assumption (1.32) is needed (see Remark 2.2).

Unfortunately, we are not able to treat problem (1.1) with g that satisfies just (1.2) and (1.3). The main obstacles in considering a general $g(s)$ [that satisfies anyway (1.2) and (1.3)] are, on the one side, that the simple structure of the corrector function S is lost and, on the other, that it becomes much harder to manipulate terms as $g(z+S) - g(S)$.

2 Gradient bound

2.1 The choice of S_n

In this first section, we determine the regularity of the functions σ_k, $k = 0, \dots, [\alpha]+1$, defined in (1.15) and we show that $\widetilde{f}_n$, defined in (1.23), is such that

$$\exists\, \bar{C} = \bar{C}(\sigma_0, \dots, \sigma_{[\alpha]+1}, f) : d_n|\widetilde{f}_n| + d_n^2|\nabla \widetilde{f}_n| \le \bar{C} d_n^{1+[\alpha]-\alpha}.$$

We prove a slightly more general result that emphasizes the relationship between the number of elements of S_n and the required regularity of Ω.

Proposition 2.1 *Let us take a natural number $M \in [0, [\alpha]+1]$, Ω a bounded open domain of class C^{M+4}, σ_k as in (1.15) with $k = 0, \dots, M$ and S_n as in (1.20). Then, we have that*

$\sigma_k \in C(\bar{\Omega})^{M+3-k}$ with $k = 0, \ldots, M$ and that there exists $n_0 = n_0(\sigma_0, \ldots, \sigma_M)$ such that for every $n > n_0$

$$|(\Delta S_n - |S_n|^{p-1} S_n) d_n| + |\nabla(\Delta S_n - |S_n|^{p-1} S_n) d_n^2| \leq C d_n^{M-\alpha} \quad \text{in } \Omega, \tag{2.1}$$

where $C = C(N, \alpha, \partial\Omega)$.

Proof Note at first that the positive root of $(k-\alpha)(k-\alpha-1) - (2+\alpha)(\alpha+1) = 0$ (seen as an equation in the variable k) is bigger then $[\alpha]+1$: indeed denoting by k_i $i = 1, 2$ the two roots, we have that

$$k_1 < 0 < k_2 = \frac{2\alpha + 1 + \sqrt{(2\alpha+1)^2 + 2(\alpha+1)}}{2} \quad \text{and} \quad k_2 > 2\alpha + 1 > [\alpha] + 1 \quad \alpha > 0.$$

Thus, the denominator in (1.15) is always different from zero. As far as the regularity of the terms involved in (2.1) is concerned, Theorem 1.1 assures us that $d_n \in C^{M+4}(\Omega)$ (see also [16,20]); moreover, as it is clear from the formulas in (1.15), the evaluation of σ_k involves derivatives of d_n of order $k+1$. Hence, the regularity of σ_k is $M+4-(k+1) = M+3-k$, i.e. $\sigma_k \in C^{M+3-k}(\bar{\Omega})$ with $k = 1, \ldots, M$.

Let us show now that such a choice of σ_k implies that (2.1) holds true. Thanks to the proved regularity property, we are allowed to compute both the gradient and the Laplacian of $S_n(x)$. Recalling that by definition $\nabla d_n(x) = \nabla d(x)$ and $\Delta d_n(x) = \Delta d(x)$, we have that

$$\begin{aligned}\Delta S_n(x) = \sum_{k=0}^{M} \Big[&(k-\alpha)(k-\alpha-1)\sigma_k d_n^{k-\alpha-2}(x) |\nabla d(x)|^2 \\ &+ (k-\alpha)\sigma_k(x) d_n^{k-\alpha-1}(x) \Delta d(x) \\ &+ 2(k-\alpha) d_n^{k-\alpha-1}(x) \nabla\sigma_k(x) \nabla d(x) + \Delta\sigma_k(x) d_n^{k-\alpha}(x) \Big].\end{aligned}$$

Ordering the previous expression according to the power of the distance function and working in $\Omega_{\bar{\delta}}$, in order to use that $|\nabla d|^2 = 1$, we obtain

$$\begin{aligned}\Delta S_n(x) = \;& \alpha(\alpha+1)\sigma_0 d_n^{-\alpha-2}(x) + \big[\alpha(\alpha-1)\sigma_1(x) - \alpha\sigma_0 \Delta d(x)\big] d_n^{-\alpha-1}(x) \\ &+ \sum_{k=2}^{M} \Big\{ (k-\alpha)(k-\alpha-1)\sigma_k(x) + (k-\alpha-1)\big[\sigma_{k-1}(x)\Delta d(x) \\ &+ 2\nabla\sigma_{k-1}(x)\nabla d(x)\big] + \Delta\sigma_{k-2}(x) \Big\} d_n^{k-\alpha-2}(x) \\ &+ r(x) d_n^{M-\alpha-1}(x) \quad \text{in } \Omega_{\bar{\delta}},\end{aligned}$$

where $r = r(\sigma_{M-1}, \sigma_M) \in C^1(\bar{\Omega})$.

Now, let us focus on the nonlinear term $|S_n|^{p-1} S_n$. Since any σ_k is bounded, there exists $\delta_0 = \delta_0(\sigma_0, \ldots, \sigma_M)$, with $\delta_0 < \bar{\delta}$, $n_0 = n_0(\delta_0)$ and a function $R = R(\sigma_0, \ldots, \sigma_M) \in C^1(\bar{\Omega})$ such that

$$|S_n|^{p-1}S_n = \sigma_0^p d_n^{-\alpha-2}\left(\sum_{k=0}^{M}\frac{\sigma_k}{\sigma_0}d_n^k\right)^p = \sigma_0^p d_n^{-\alpha-2} + p\sigma_0^{p-1}\sigma_1 d_n^{-\alpha-1}$$
$$+\sum_{k=2}^{M} d_n^{k-\alpha-2}\Bigg\{p\sigma_0^{p-1}\sigma_k$$
$$+\sigma_0^p\sum_{j=2}^{k}\left[\binom{p}{j}\sigma_0^{-j}\sum_{i_1+\cdots+i_j=k}\sigma_{i_1}\cdots\sigma_{i_j}\right]\Bigg\} + R(x)d_n^{M-\alpha-1}$$
$$\text{in } \Omega_{\delta_0} \text{ and } n \geq n_0.$$

Now, it becomes clear that the choice of $\sigma_0, \ldots, \sigma_M$ in (1.15) is the unique for which

$$\big|\Delta S_n(x) - |S_n(x)|^{p-1}S_n(x)\big|d_n(x) = \big|r(x) - R(x)\big|d_n^{M-\alpha}(x)$$
$$\leq C d_n^{M-\alpha}(x) \quad \text{in } \Omega_{\delta_0} \text{ and } n \geq n_0,$$

and moreover

$$\big|\nabla(\Delta S_n - |S_n|^{p-1}S_n)d_n^2(x)\big| \leq (\alpha+1-M)\big|\nabla(r(x)-R(x))\big|d_n^{M-\alpha}(x) \leq C d_n^{M-\alpha}(x)$$
$$\text{in } \Omega_{\delta_0} \text{ and } n \geq n_0,$$

with $C = C(\sigma_0, \ldots, \sigma_M)$. The estimate in $\Omega\backslash\Omega_{\delta_0}$ is straightforward thanks to the regularity of σ_k. □

Remark 2.2 For the proof of Theorem 1.3, we take $M = [\alpha] + 1$, so that (2.1) becomes

$$|(\Delta S_n - |S_n|^{p-1}S_n)d_n| + |\nabla(\Delta S_n - |S_n|^{p-1}S_n)d_n^2| \leq C d_n^{1+[\alpha]-\alpha} \quad \text{in } \Omega.$$

Since $f \in W^{1,\infty}(\Omega)$, recalling the definition (1.23) of $\widetilde{f}_n$, it follows

$$\exists \bar{C} = \bar{C}(\sigma_0, \ldots, \sigma_M, f) \quad \text{such that} \qquad d_n|\widetilde{f}_n| + d_n^2|\nabla\widetilde{f}_n| \leq \bar{C} d_n^{1+[\alpha]-\alpha} \leq \widetilde{C}\,. \tag{2.2}$$

We anticipate that for the implementation of the Bernstein technique we only need that the quantity on the left-hand side above is bounded and not infinitesimal near $\partial\Omega$. Despite this fact, we need anyway to use $[\alpha] + 2$ terms in the definition of S_n because, unless $\alpha \in \mathbb{N}$, $[\alpha] - \alpha < 0$. Anyway the extra information that $d_n|\widetilde{f}_n| + d_n^2|\nabla\widetilde{f}_n|$ goes to zero has x approaches the boundary is used in the second part of Theorem 2.9.

Remark 2.3 For the sake of completeness, we explicitly compute the expression for σ_2:

$$\sigma_2(x) = \frac{(\alpha+2)\sigma_0^{p-2}\sigma_1^2(x)d(x) + (\alpha-1)[\sigma_1(x)\,d(x)\Delta d(x) + \nabla\sigma_1(x)\nabla d(x)]}{(2-\alpha)(1-\alpha) - (2+\alpha)(\alpha+1)}.$$

Of course, σ_0 and $\sigma_1(x)$ coincide with the ones already known in the literature.

2.2 Existence and preliminary estimates for u_n

In this section, we find suitable sub- and super-solutions for problem (1.21) in order to prove both existence and some preliminary estimates on the solutions u_n of (1.21).
We first observe that

$$\frac{\partial S_n}{\partial \nu}\bigg|_{\partial\Omega} = \alpha\sigma_0 n^{\alpha+1} + n^\alpha\sum_{k=1}^{M}\Big[(\alpha-k)\sigma_k n^{1-k} + \nabla\sigma_k\cdot\nu n^{-k}\Big]$$
$$= \alpha\sigma_0 n^{\alpha+1} + n^\alpha\psi_n \quad \text{if } \alpha \neq 1,$$

while
$$\left.\frac{\partial S_n}{\partial \nu}\right|_{\partial\Omega} = \sigma_0 n^2 + \nabla\sigma_1\cdot\nu + \nabla\sigma_2\cdot\nu\frac{1}{n} - \sigma_2 = \sigma_0 n^2 + \psi_n \quad \text{if } \alpha = 1,$$

where $\psi_n \in C(\partial\Omega)$ is uniformly bounded (with respect to n). More precisely,

$$\exists\, T = T(N,\alpha,\partial\Omega) : \|\psi_n\|_{L^\infty(\partial\Omega)} \le T \quad \forall n \in \mathbb{N}. \tag{2.3}$$

Such a bound is crucial in order to prove that problem (1.21) admits a pair of sub- and super-solutions.

Proposition 2.4 *Let $p > 1$, $f \in W^{1,\infty}(\Omega)$, S_n as (1.20) and T as in (2.3). Hence, problem (1.21) admits a pair of (classical) sub- and super-solutions.*

Proof *Case $\alpha > 1$, sub-solution.* We prove that it is possible to chose M_1 and M_2 positive constants such that $w_n := \sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha} - M_2$ is a sub-solution for (1.21). Fix at first

$$M_1 \ge \max\left\{\frac{(p-1)\alpha\sigma_0||\Delta d||_{L^\infty(\Omega)}}{(p+3)}, \frac{T}{\alpha-1}\right\}$$

and observe that this choice of M_1 implies that

$$\frac{\partial w_n}{\partial\nu} - \alpha\sigma_0 n^{\alpha+1} - \psi_n n^\alpha = [-(\alpha-1)M_1 - \psi_n]n^\alpha < 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

Moreover, using the monotonicity of the function $s \to |s|^{p-1}s$, let us fix $\delta_0 = \delta_0(M_1) < \bar{\delta}$ and $n_0 = n_0(\delta_0)$ so that

$$\begin{aligned}|w_n|^{p-1}w_n &\le |\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha}|^{p-1}\left(\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha}\right) = \left(\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha}\right)^p \\ &= \sigma_0^p d_n^{-\alpha-2}\left(1 - \frac{M_1}{\sigma_0}d_n\right)^p \\ &= \sigma_0^p d_n^{-\alpha-2}\left[1 - p\frac{M_1}{\sigma_0}d_n + O(d_n^2)\right] \quad \text{in } \Omega_{\delta_0},\ n > n_0.\end{aligned}$$

On the other hand, an easy computation shows that

$$\begin{aligned}\Delta w_n = \alpha(\alpha+1)\sigma_0 d_n^{-\alpha-2} - \alpha d_n^{-\alpha-1}\Big[\sigma_0\Delta d + (\alpha-1)M_1\Big] \\ +(\alpha-1)M_1 d_n^{-\alpha}\Delta d \quad \text{in } \Omega_{\delta_0},\ n > n_0.\end{aligned}$$

Recalling that

$$p\sigma_0^{p-1} - \alpha(\alpha-1) = p\alpha^2 + p\alpha - \alpha^2 + \alpha = 2\frac{p+3}{p-1},$$

we deduce that

$$\begin{aligned}-\Delta w_n + |w_n|^{p-1}w_n - f &\le -\Delta w_n + (\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha})^p - f \\ &\le \left(-2\frac{p+3}{p-1}M_1 + \alpha\sigma_0\Delta d\right)d_n^{-\alpha-1} + O(d_n^{-\alpha}) \\ &\le 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0,\end{aligned}$$

where the last inequality holds true thanks to the choice of M_1. Now, taking

$$M_2 \ge \sigma_0\delta_0^{-\alpha} + \left(\|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}}$$

and using once more the monotonicity of $s \to |s|^{p-1}s$, it follows that

$$\begin{aligned} -\Delta w_n + |w_n|^{p-1}w_n - f &\le -\Delta w_n - \|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} - \|f\|_{L^\infty(\Omega)} - f \\ &\le 0 \quad \text{in } \Omega\setminus\Omega_{\delta_0} \end{aligned} \tag{2.5}$$

and we conclude that w_n is a sub-solution of problem (1.21).

Case $\alpha > 1$, *super-solution.* Let us show now that it is possible to take $N_1 \ge M_1$ such that $v_n := \sigma_0 d_n^{-\alpha} + N_1 d_n^{1-\alpha}$ turns out to be a super-solution for (1.21). As far as the boundary condition is concerned, we have

$$\left.\frac{\partial v_n}{\partial \nu}\right|_{\partial\Omega} - \sigma_0 n^{\alpha+1} - \psi_n n^\alpha = [(\alpha-1)N_1 - \psi_n]n^\alpha > 0 \quad \text{on } \partial\Omega,$$

where the last inequality follows from the previous choice of N_1.
Since v_n is positive, thanks to the convexity of the function $(1+s)^p$ with $p > 1$, we have that

$$|v_n|^{p-1}v_n = \sigma_0^p d_n^{-\alpha-2}\left(1 + \frac{N_1}{\sigma_0}d_n\right)^p \ge \sigma_0^p d_n^{-\alpha-2}\left(1 + p\frac{N_1}{\sigma_0}d_n\right).$$

As in the previous case, it follows that

$$-\Delta v_n + v_n^p - f \ge \left(2\frac{p+3}{p-1}N_1 + \alpha\sigma_0\Delta d_n\right)d_n^{-\alpha-1} + O(d_n^{-\alpha}) \quad \text{in } \Omega_{\bar\delta},\ n > n_0,$$

where we have used that $|\nabla d|^2 = 1$ in $\Omega_{\bar\delta}$. Thanks, once again, to the choice of N_1, we can conclude that $-\Delta v_n + v_n^p \ge f$ in $\Omega_{\bar\delta}$.
Finally, we have

$$\begin{aligned} -\Delta v_n + v_n^p - f &= -\Delta v_n + (\sigma_0 d_n^{-\alpha} + N_1 d_n^{1-\alpha})^p - f \\ &\ge -\Delta v_n + N_1^p d_n^{p(1-\alpha)} - f \\ &\ge C_1 N_1^p - C_2 N_2 - C_3 \quad \text{in } \Omega\setminus\Omega_{\bar\delta}, \end{aligned}$$

where the last inequality comes from the fact that in $\Omega\setminus\Omega_{\bar\delta}$ $d_n \ge \bar\delta$ and that in $-\Delta v_n$ only linear powers of N_1 appear. So by increasing, if necessary, the value of N_1, we have $-\Delta v_n + v_n^p \ge f$ in $\Omega\setminus\Omega_{\bar\delta}$. It is then possible to conclude that v_n is a super-solution of (1.21) in Ω and that $v_n \ge w_n$.

For the range $0 < \alpha \le 1$, the proof is similar and we just stress the main differences.

Case $\alpha = 1$. Note that, with this choice of α, we have $p = 2$, $\sigma_0 = \sqrt{2}$. We claim that $w_n := \sqrt{2}d_n^{-1} + M_3 \log d_n - M_4$ and $v_n := \sqrt{2}d_n^{-1} - N_3 \log d_n + N_4$, with $M_3, M_4, N_3, N_4 > 0$, are a sub- and a super-solution for (1.21). Let us choose $M_3 \ge T$ in order to obtain

$$\left.\frac{\partial w_n}{\partial \nu}\right|_{\partial\Omega} - \sqrt{2}n^2 - \psi_n n = [-M_3 - \psi_n]n < 0.$$

Then, we fix $\delta_0 = \delta_0(M_3) < \min\{\bar\delta, 1\}$ (so that $\log(\delta_0) < 0$) and $n_0 = n_0(\delta_0)$ such that

$$\begin{aligned} w_n^3 \le 2^{\frac{3}{2}} d_n^{-3}\left(1 + \frac{M_3}{\sqrt{2}}d_n \log(d_n)\right)^3 &= 2^{\frac{3}{2}} d_n^{-3} + 6M_3 d_n^{-2}\log(d_n) \\ &\quad + o(d_n^2 \log(d_n)) \quad \text{in } \Omega_{\delta_0},\ n > n_0. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} -\Delta w_n + |w_n|^{p-1}w_n - f &\le +6M_3 d_n^{-2}\log d_n + o(d_n^2 \log(d_n)) \\ &\le 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0. \end{aligned}$$

Now, we fix

$$M_4 \geq \sqrt{2}\delta_0^{-1} + \left(\|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}}$$

that implies $-\Delta w_n + |w_n|^{p-1}w_n - f \leq 0$ in $\Omega\setminus\Omega_{\delta_0}$ and, in turn, that w_n is a sub-solution of problem (1.21).

For the super-solution v_n, we consider $N_3 \geq T$. Thus, exactly as in the previous case, we get

$$\left.\frac{\partial v_n}{\partial \nu}\right|_{\partial\Omega} - \sigma_0 n^2 - \psi_n n = [N_3 - \psi_n]n > 0.$$

Noticing that v_n is positive and using the convexity of the function $(1+s)^3$, we obtain

$$v_n^3 = 2^{\frac{3}{2}} d_n^{-3}\left(1 - \frac{N_3}{\sqrt{2}} d_n \log(d_n) + N_4\right)^3 \geq 2^{\frac{3}{2}} d_n^{-3} - 6N_3 d_n^{-2}\log(d_n).$$

Moreover, we fix $\delta_0 \leq \bar{\delta}$ and $n_0 = n_0(\delta_0)$ so that

$$-\Delta v_n + v_n^p - f \geq -6N_3 d_n^{-2}\log d_n + o(d_n^2 \log(d_n)) \geq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

At this point, choosing

$$N_4 \geq \sqrt{2}\delta_0^{-1} + \left(\|\Delta v_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}}$$

it follows that $-\Delta v_n + v_n^p - f \geq 0$ in $\Omega\setminus\Omega_{\delta_0}$ and we conclude.

Case $\alpha < 1$. Finally, we consider $w_n = \sigma_0 d_n^{-\alpha} - M_5 + M_6 d_n^{-\alpha+1}$ and $v_n = \sigma_0 d_n^{-\alpha} + N_5 - N_6 d_n^{-\alpha+1}$ with $M_5, M_6, N_5, N_6 > 0$. Let us fix $M_6 \geq \frac{T}{1-\alpha}$, in order to have

$$\left.\frac{\partial w_n}{\partial \nu}\right|_{\partial\Omega} - \alpha\sigma_0 n^{\alpha+1} - \psi_n n^\alpha < 0.$$

Moreover, taking $M_5 > 0$ it is possible to select $\delta_0 \leq \bar{\delta}$ and $n_0 = n_0(\delta_0)$ such that

$$|w_n|^{p-1}w_n = \sigma_0^p d_n^{-\alpha-2} - p\sigma_0^{p-1} M_5 d_n^2 + o(d_n^2)$$

and that

$$-\Delta w_n + |w_n|^{p-1}w_n - f \leq -p\sigma_0^{p-1} M_5 d_n^{-2} + o(d_n^{-2}) \leq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

Finally, increasing the value of M_5 so that

$$M_5 \geq \sigma_0\delta_0 + M_6\delta_0^{-\alpha+1} + \left(\|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}},$$

it follows

$$-\Delta w_n + |w_n|^{p-1}w_n - f \leq 0 \quad \text{in } \Omega\setminus\Omega_{\delta_0}.$$

As far as v_n is concerned, let us fix as before $N_6 \geq \frac{\|\psi_n\|_\infty}{1-\alpha}$ in order to get

$$\left.\frac{\partial v_n}{\partial \nu}\right|_{\partial\Omega} - \alpha\sigma_0 n^{\alpha+1} - \psi_n n^\alpha > 0.$$

Take now $N_5 > 0$ and fix $\delta_0 < \bar{\delta}$ and $n_0 = n_0(\delta_0)$ such that

$$|v_n|^{p-1} v_n = \sigma_0^p d_n^{-\alpha-2} \left(1 + \frac{N_5}{\sigma_0} d_n^{\alpha} - \frac{N_6}{\sigma_0} d_n\right)^p$$
$$\geq \sigma_0^p d_n^{-\alpha-2} + p\sigma_0^{p-1} N_5 d_n^{-2} + o(d_n^{-2}) \quad \text{in } \Omega_{\delta_0},\ n > n_0$$

and that

$$-\Delta v_n + |v_n|^{p-1} v_n - f \geq p\sigma_0^{p-1} N_5 d_n^{-2} + o(d_n^{-2}) \geq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

As in the previous cases, by increasing if necessary the value of N_5, we have $-\Delta v_n + |v_n|^{p-1} v_n - f \geq 0$ in $\Omega \setminus \Omega_{\delta_0}$ and that v_n is a super-solution of (1.21).

The ordered sub- and super-solutions obtained in the previous proposition allow us to prove existence of a solution for problem (1.21) and, on the other hand, give us a control on the behaviour of u_n (and in turn of z_n) near $\partial\Omega$, which is crucial in order to prove the results of the next section.

Theorem 2.5 *Let $p > 1$, $f \in W^{1,\infty}(\Omega)$, S_n as in (1.20). Then, problem (1.21) has a unique classical solution u_n for every $n \in \mathbb{N}$. Moreover,*

$$\exists C = C(\alpha, N, \partial\Omega, f) : \left|\frac{z_n(x)}{S_n(x)}\right| = \left|\frac{u_n(x)}{S_n(x)} - 1\right| \leq C\varepsilon(d_n(x)) \tag{2.6}$$

where

$$\varepsilon(s) = \begin{cases} s & \text{if } \alpha > 1 \\ s(1 + |\log s|) & \text{if } \alpha = 1 \\ s^{\alpha} & \text{if } \alpha < 1. \end{cases} \tag{2.7}$$

Proof The proof of the existence and uniqueness is standard, and we sketch it here for the convenience of the reader. In Proposition 2.4, for every $\alpha > 0$ we have constructed a pair of ordered sub- and super-solutions for problem (1.21)

$$w_n \leq v_n \quad \text{in } \Omega.$$

Let us set $v_n^0 = v_n$, $C := \max\{||v_n||_{L^\infty(\Omega)}, ||w_n||_{L^\infty(\Omega)}\}$, $m > pC^{p-1}$, and let us define v_n^i for $i = 1, 2, \ldots$ as the solutions of

$$\begin{cases} -\Delta v_n^i + m v_n^i = m v_n^{i-1} - |v_n^{i-1}|^{p-1} v_n^{i-1} + f, & \text{in } \Omega, \\ \frac{\partial v_n^i}{\partial \nu} = \frac{\partial S_n}{\partial \nu}, & \text{on } \partial\Omega. \end{cases}$$

The choice of m allows us to say that the function $s \to ms - |s|^{p-1}s$ is increasing in $[-C, C]$ so that we can apply the standard procedure of the sub- and super-solution method for existence of solutions (see for instance [17]). We claim that $v_n^{i-1} \geq v_n^i$ for every $i = 1, 2, \ldots$. Indeed, for $i = 1$ we have that the function $w := v_n^1 - v_n^0$ satisfies

$$\begin{cases} -\Delta w + mw \leq 0, & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Hopf's lemma and the strong maximum principle assure us that $w \leq 0$, which implies $v_0 \geq v_1$, and we can conclude the proof of the claim by induction. Similarly, we can prove that $w_n \leq v_n^i$ for every $i = 1, 2, \ldots$ Then, we have that $v_n^i \searrow u_n$ a.e in Ω as $i \to \infty$ and that

$$w_n \leq u_n \leq v_n;$$

moreover, by compactness and regularity arguments (see, respectively, [1,2]) it is possible to prove that $u_n \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves (1.21). The uniqueness follows by Theorem 3.6 of [16].
In order to prove (2.7), we first consider the case $\alpha > 1$. We have that

$$\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha} - M_2 \leq u_n \leq \sigma_0 d_n^{-\alpha} + N_1 d_n^{1-\alpha},$$

where M_1, M_2 and N_1 are the constants given by Proposition 2.4. Subtracting S_n, we get

$$-(M_1 - \sigma_1)\, d_n^{1-\alpha} + O\left(d_n^{2-\alpha}\right) \leq u_n - S_n \leq (N_1 - \sigma_1)\, d_n^{1-\alpha} + o\left(d_n^{1-\alpha}\right) + O\left(d_n^{2-\alpha}\right)$$

with b and B bounded functions. Thanks to the choice of M_1 and N_1, it follows that there exists a positive constant $C = C(\alpha, N, \partial\Omega, f)$ such that

$$\left|\frac{u_n(x)}{S_n(x)} - 1\right| \leq C d_n(x) \quad \text{in } \Omega.$$

The case $\alpha \leq 1$ follows similarly using the respective sub- and super-solutions, and for brevity we omit the proof.

We close this section proving the following proposition.

Proposition 2.6 *The sequence u_n of solutions of problem* (1.21) *converges in $C^2_{\mathrm{loc}}(\Omega)$ to the solution of problem* (1.10).

Proof Let us define $\psi_n := u_n - u_{n+1}$, which satisfies

$$\begin{cases} -\Delta\psi_n + |u_n|^{p-1}u_n - |u_{n+1}|^{p-1}u_{n+1} = 0, & \text{in } \Omega, \\ \frac{\partial \psi_n}{\partial \nu} < 0, & \text{on } \partial\Omega. \end{cases}$$

The Neumann boundary condition assures us that the maximum of ψ_n cannot be reached on $\partial\Omega$. So let it be $x_0 \in \Omega$ the maximum point for ψ_n. This implies that $-\Delta\psi_n(x_0) \geq 0$, and then we obtain from the equation the following information:

$$|u_n(x_0)|^{p-1}u_n(x_0) - |u_{n+1}(x_0)|^{p-1}u_{n+1}(x_0) \leq 0.$$

But, since $s \to |s|^{p-1}s$ is increasing, it has to be $\psi_n(x_0) = u_n(x_0) - u_{n+1}(x_0) \leq 0$. Being x_0 the maximum point, it follows that $u_n \leq u_{n+1}$ in Ω. So we know that the sequence u_n is increasing and converges pointwise to some function u. Moreover, we know, thanks to Theorem 2.5, that each u_n is between the relative sub- and super-solutions $w_n \leq u_n \leq v_n$. Thus, we have that for any ω compactly contained in Ω there exists $c = c(\omega, \alpha, N)$ such that

$$u_n(x) \leq u_{n+1}(x) \leq \cdots \leq u(x) \leq v(x) \leq c \quad \forall\, x \in \omega,$$

where v is the limit as n diverges of the super-solutions v_n. On the other hand, we also have that

$$w(x) \leq u(x) \quad \forall\, x \in \Omega,$$

where w is the limit of the sub-solutions. Thus, using standard compactness and interior elliptic regularity arguments, we have that for every $\omega \subset\subset \Omega$ $u_n \to u$ in $C^2(\omega)$. Thus, we can pass to the limit with respect to n in (1.21), and moreover we also obtain

$$\lim_{x \to \partial\Omega} u(x) = \infty.$$

□

2.3 Estimates of z_n and $|\nabla z_n|$ in $L^\infty(\Omega)$

Now, we are ready to prove the uniform estimate in $W^{1,\infty}(\Omega)$ for $z_n := u_n - S_n$, where u_n are the solutions of problem (1.21) and S_n has been defined in (1.20). Note that thanks to Proposition 2.6 we already know that for every ω compactly contained in Ω we have that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega : \|z_n\|_{W^{1,\infty}(\omega)} \le \|u_n\|_{W^{1,\infty}(\omega)} + \|S_n\|_{W^{1,\infty}(\omega)} \le c_\omega.$$

Thus, the main concern here is to obtain a Lipschitz control in Ω_{δ_0} for some $\delta_0 > 0$ small enough.

Let us start with the bound in $L^\infty(\Omega_{\delta_0})$.

Theorem 2.7 *Let z_n be as above. Then, there exists $\delta_0 = \delta_0(\sigma_0, \dots, \sigma_{[\alpha]+1})$ and a constant $C = C(p, N, \partial\Omega, f, \delta_0)$ such that*

$$||z_n||_{L^\infty(\Omega_{\delta_0})} \le C. \tag{2.8}$$

Proof We build barriers in a neighbourhood of $\partial\Omega$ through sub- and super-solutions method.

Bound from above. Let us fix $\delta_0 = \delta_0(\sigma_0, \dots, \sigma_{[\alpha]+1})$ and $n_0 = n_0(\delta_0)$ such that $S_n \ge 0$ in Ω_{δ_0} and $n > n_0$. By definition of S_n there exists $K = K(\delta_0)$ such that $pS_n^{p-1}d_n^2 \ge K$ in Ω_{δ_0}. Choose now $B > \frac{\delta_0 \widetilde{C}}{k}$, where $\widetilde{C}$ is the constant given by (2.2). Taking advantage of the convexity of $s \to (1+s)^p$, we have

$$\begin{aligned} &-\Delta B + |S_n|^{p-1} S_n \left[\left|1 + \frac{B}{S_n}\right|^{p-1} \left(1 + \frac{B}{S_n}\right) - 1 \right] - \widetilde{f}_n \\ &= S_n^p \left[\left(1 + \frac{B}{S_n}\right)^p - 1 \right] - \widetilde{f}_n \ge pBS_n^{p-1} - |\widetilde{f}_n| \\ &\ge \frac{BK}{d_n^2} - |\widetilde{f}_n| \ge \frac{BK - \widetilde{C} d_n}{d_n^2} \ge 0 \quad \text{in } \Omega_{\delta_0} \text{ and } n > n_0, \end{aligned}$$

where we have used (2.2) and the choice of B. Then, if we define $\bar{w} = B + \max_{\{x\in\Omega : d(x)=\delta_0\}} |z_n|$ it easily follows that $z_n \le \bar{w}$ in Ω.

Bound from below. For any positive fixed constant $B > 0$, thanks to (2.7) there exist $\delta_0 = \delta(\sigma_0, \dots, \sigma_{[\alpha]+1}, B)$, $n_0 = n_0(\delta_0)$ and a constant $C = C(\delta_0)$ such that

$$\begin{aligned} |S_n|^{p-1} S_n \left[\left|1 - \frac{B}{S_n}\right|^{p-1} \left(1 - \frac{B}{S_n}\right) - 1 \right] &= S_n^p(x) \left[\left(1 - \frac{B}{S_n(x)}\right)^p - 1 \right] \\ &= S_n^p(x) \left[-\frac{pB}{S_n(x)} + O\left(d_n^{2\alpha} B^2\right) \right] \\ &\le -\frac{K_1 B}{d_n^2(x)} + O\left(d_n^{\alpha-2}\right) B^2 \\ &\le \frac{-K_1 B + O\left(d_n^{\alpha}\right) B^2}{d_n^2(x)} \quad \text{in } \Omega_{\delta_0} \text{ and } n > n_0. \end{aligned}$$

Thus, it follows that

$$-\Delta(-B) + |S_n|^{p-1} S_n \left[\left|1 - \frac{B}{S_n}\right|^{p-1} \left(1 - \frac{B}{S_n}\right) - 1 \right]$$
$$-\widetilde{f}_n \leq \frac{-K_1 B + K_2 B^2 d_n^\alpha + \widetilde{C} d_n}{d_n^2} \leq 0$$
$$\text{in } \Omega_{\delta_0} \text{ and } n > n_0.$$

At this point, reasoning exactly as in the first part, it follows that $\underline{w} = -B - \max_{\{x\in\Omega:d(x)=\delta_0\}} |z_n|$ controls z_n from below in Ω and the proof is concluded. □

We can now state and prove our main result.

Theorem 2.8 *Let z_n be the functions defined in (1.22). Then, there exists $\delta_0 = \delta_0(\sigma_0, \ldots, \sigma_{[\alpha]+1})$ and $C = C(\alpha, N, \partial\Omega, f, \delta_0)$ such that*

$$\|\nabla z_n\|_{L^\infty(\Omega_{\delta_0})} \leq C.$$

Proof We divide the proof in two steps.

Step 1. Inequality satisfied by $|\nabla z_n|^2$.

Step 2. Application of maximum principle to $w_n := |\nabla z_n|^2 e^{\lambda d_n}$ in Ω_{δ_0}.

Step 1. Thanks to (2.6), there exist $\delta_0 < \bar{\delta}$ and $n_0 = n_0(\delta_0)$ such that

$$\left.\begin{array}{l} 0 < C_1 \leq \left(1 + \dfrac{z_n}{S_n}\right)^{p-1} \leq C_2 \\ \left|\left(1 + \dfrac{z_n(x)}{S_n(x)}\right)^p - 1\right| \leq C_3 \dfrac{|z_n(x)|}{S_n(x)} \end{array}\right. \quad \forall n > n_0 \quad \forall x \in \Omega_{\delta_0}, \tag{2.9}$$

where the positive constants C_1, C_2 and C_3 depend only on $\alpha, N, \partial\Omega$. Moreover, from the definition of S_n and thanks to the boundary condition on z_n it follows that

$$|\nabla S_n \nabla z_n| \leq C \left[\frac{|\nabla d_n||\nabla z_n|}{d_n^{\alpha+1}} + \frac{|\nabla z_n|}{d_n^\alpha} \right] \leq C \left[\frac{\varepsilon(d_n)}{d_n^{\alpha+1}} + \frac{|\nabla z_n|}{d_n^\alpha} \right], \tag{2.10}$$

where $\varepsilon(s)$ is defined in (2.7). All the computations performed from now on are meant on Ω_{δ_0} and with $n > n_0$. At first, let us recover the equation satisfied by $|\nabla z_n|^2$ (see [24] and reference therein). In order to do it, it is useful to recall that

$$\nabla(|\nabla z_n|^2) = 2D^2 z_n \nabla z_n \quad \text{and that} \quad \Delta(|\nabla z_n|^2) = 2\nabla(\Delta z_n)\nabla z_n + 2|D^2 z_n|^2.$$

Hence, through Schwarz inequality, we get

$$\Delta(|\nabla z_n|^2) \geq 2\nabla\left[(z_n + S_n)^p - S_n^p\right]\nabla z_n - 2\nabla \widetilde{f}_n \nabla z_n + \frac{2}{N}(\Delta z_n)^2.$$

Now, we consider separately each one of the terms on the right-hand side above.

First term. We rewrite it as

$$\begin{aligned} \nabla\left[(z_n + S_n)^p - S_n^p\right]\nabla z_n &= \nabla\left[S_n^p \left[\left(1 + \frac{z_n}{S_n}\right)^p - 1\right]\right]\nabla z_n \\ &= pS_n^{p-1}\left(1 + \frac{z_n}{S_n}\right)^{p-1} |\nabla z_n|^2 + p\left[S_n\left[\left(1 + \frac{z_n}{S_n}\right)^p - 1\right]\right. \\ &\quad \left. -z_n\left(1 + \frac{z_n}{S_n}\right)^{p-1}\right] S_n^{p-2}\nabla S_n \nabla z_n. \end{aligned} \tag{2.11}$$

Note that in the right-hand side above the first term is the coercive one, while the other has to be absorbed. Thanks to (2.9), the coercive term of (2.11) becomes

$$\exists\gamma > 0 : pS_n^{p-1}\left(1+\frac{z_n}{S_n}\right)^{p-1}|\nabla z_n|^2 \geq 3\gamma\frac{|\nabla z_n|^2}{d_n^2}.$$

Recalling (2.7), Theorem 2.7 and (2.9), the last term of (2.11) can be controlled as follows:

$$\begin{aligned}
&p\left|S_n\left[\left(1+\frac{z_n}{S_n}\right)^p-1\right]-z_n\left(1+\frac{z_n}{S_n}\right)^{p-1}\right|S_n^{p-2}|\nabla S_n\nabla z_n|\\
&\leq C|z_n|S_n^{p-2}\left[\frac{\varepsilon(d_n)}{d_n^{\alpha+1}}+\frac{|\nabla z_n|}{d_n^\alpha}\right]\\
&\leq C|z_n|\left[\frac{\varepsilon(d_n)}{d_n^3}+\frac{|\nabla z_n|}{d_n^2}\right]\\
&\leq C\frac{|z_n|}{d_n^3}+\gamma\frac{|\nabla z_n|^2}{d_n^2}+C_\gamma\frac{z_n^2}{d_n^2},
\end{aligned}$$

where we have used both (2.9) and (2.10). Then, we get

$$\nabla\left[(z_n+S_n)^p-S_n^p\right]\nabla z_n \geq 3\gamma\frac{|\nabla z_n|^2}{d_n^2}-C\frac{|z_n|}{d_n^3}-C\frac{z_n^2}{d_n^2}.$$

Second term. We apply Young's inequality and use (2.2) to obtain

$$-\nabla\widetilde{f}_n\nabla z_n \geq -\gamma\frac{|\nabla z_n|^2}{d_n^2}-C_\gamma|\nabla\widetilde{f}_n|^2d_n^2 \geq -\gamma\frac{|\nabla z_n|^2}{d_n^2}-\frac{C_\gamma}{d_n^2}.$$

Third term. Using the easy inequality $(a-b)^2 \geq \frac{a^2}{4}-b^2$, we obtain

$$\begin{aligned}
\frac{2}{N}(\Delta z_n)^2 &= \frac{2}{N}\left[(z_n+S_n)^p-S_n^p-\widetilde{f}_n\right]^2\\
&\geq \frac{S_n^{2p}}{2N}\left[\left(1+\frac{z_n}{S_n}\right)^p-1\right]^2-\frac{2}{N}\widetilde{f}_n^2.
\end{aligned}$$

Moreover, using the fact that the function $(1+s)^p-1$ is convex for $p>1$ and has strictly positive derivative in zero and recalling (2.2) we have

$$\frac{2}{N}(\Delta z_n)^2 \geq \frac{C}{N}\frac{z_n^2}{d_n^4}-\frac{C_N}{d_n^2}.$$

Hence, gathering together the inequalities above, we have that

$$\Delta(|\nabla z_n|^2) \geq \gamma\frac{|\nabla z_n|^2}{d_n^2}+\frac{C_3}{N}\frac{z_n^2}{d_n^4}-C_2\frac{|z_n|}{d_n^3}-C\frac{z_n^2}{d_n^2}-\frac{C}{d_n^2} \quad \forall n>n_0 \quad \forall x\in\Omega_{\delta_0}.$$

Using Young's inequality we get, for $\epsilon>0$, that

$$\frac{|z_n|}{d_n^3} \leq \epsilon\frac{z_n^2}{d_n^4}+\frac{C_\epsilon}{d_n^2}.$$

So up to a decrease in δ_0 and an increase in n_0, we finally obtain

$$\Delta(|\nabla z_n|^2) \geq \gamma\frac{|\nabla z_n|^2}{d_n^2}-\frac{C_1}{d_n^2} \quad \text{in } \Omega_{\delta_0}, \quad \forall n>n_0. \tag{2.12}$$

Step 2. As in [22] let us consider now $w_n := |\nabla z_n|^2 e^{\lambda d_n}$ with $\lambda > 2\|\Delta d\|_{L^\infty(\Omega)}$. Its boundary behaviour is described in Lemma 2.4 of [22] (using in turn an idea of [25]). For the convenience of the reader, we report here the computations. Notice at first that the boundary condition $\frac{\partial z_n}{\partial \nu} = \nabla z_n \cdot \nu = 0$ implies that there exists a function $\mu \in L^\infty(\partial\Omega)$ such that

$$\nabla(\nabla z_n \nabla d_n)|_{\partial\Omega} = \mu\nu.$$

To get convinced of this fact, just observe that the regular function $\nabla z_n \cdot d_n$ takes the value 0 on Ω (recall that $\nu = -\nabla d_n$). Then, its gradient evaluated on the boundary cannot have any tangential component, otherwise the condition for $\nabla v_n \cdot d_n$ would be violated. Hence, we have

$$\mu\nu \cdot \nabla z_n = \nabla(\nabla z_n \nabla d_n)\nabla z_n = D^2 z_n \nabla z_n \nabla d_n + D^2 d_n \nabla z_n \nabla z_n \quad \text{on } \partial\Omega.$$

But the left-hand side above is zero, so that

$$\frac{\partial |\nabla z_n|^2}{\partial \nu} = 2D^2 d_n \nabla z_n \cdot \nu \le 2||D^2 d||_\infty |\nabla z_n|^2$$

and as a consequence

$$\begin{aligned}\frac{\partial w_n}{\partial \nu} &= \nabla(|\nabla z_n|^2 e^{\lambda d_n}) \cdot \nu = \lambda w_n \nabla d_n \cdot \nu + e^{\lambda d_n} \nabla(|\nabla z_n|^2) \cdot \nu \\ &\le \left[-\lambda + 2\|\Delta d\|_{L^\infty(\Omega)}\right] w_n \quad \text{on } \partial\Omega.\end{aligned} \tag{2.13}$$

Hence, we can take λ large enough to have

$$\frac{\partial w_n}{\partial \nu} \le 0 \quad \text{on } \partial\Omega. \tag{2.14}$$

Taking into account (2.12), it follows that w_n satisfies

$$\Delta w_n \ge \left(\lambda^2 + \lambda \Delta d_n\right) w_n + 2\lambda \nabla w_n \nabla d_n - 2\lambda^2 w_n + \gamma \frac{w_n}{d_n^2} - \frac{C_1}{d_n^2},$$

that is,

$$-\Delta w_n + \left[\gamma - \left(\lambda^2 + \lambda ||\Delta d_n||_{L^\infty(\Omega)}\right) d_n^2\right] \frac{w_n}{d_n^2} + 2\lambda \nabla w_n \nabla d_n \le \frac{C_1}{d_n^2}.$$

Hence, up to a decrease in δ_0 and an increase in n_0, we get

$$-\Delta w_n + 2\lambda \nabla w_n \nabla d_n + \frac{\gamma}{2} \frac{w_n}{d_n^2} \le \frac{C_2}{d_n^2} \quad \text{in} \quad \Omega_0 \quad \text{and} \quad n > n_0. \tag{2.15}$$

Coupling equation (2.15) together with the boundary condition (2.14), we take advantage of the maximum principle to conclude that

$$\sup_{\Omega_{\delta_0}} w_n \le C + \max_{\partial\Omega_0 \setminus \partial\Omega} w_n \le C + C_1 \max_{\partial\Omega_0 \setminus \partial\Omega} |\nabla z_n|^2.$$

Being the last term above uniformly bounded thanks to Proposition 2.6, the theorem is proved. □

2.4 Boundary behaviour of *z*

Thanks to the results of the previous sections, we deduce that z solves

$$\begin{cases} -\Delta z + |z+S|^{p-1}(z+S) - |S|^{p-1}S = \widetilde{f}, & \text{in } \Omega, \\ z \in W^{1,\infty}(\Omega), \end{cases} \tag{2.16}$$

and that moreover

$$\left|\frac{z}{S}\right| \leq 0(1) \quad \text{as} \quad d(x) \to 0 \quad \text{and} \quad |\widetilde{f}|d + |\nabla \widetilde{f}|d^2 \leq \bar{C} d^{1+[\alpha]-\alpha} \leq \widetilde{C}, \tag{2.17}$$

where $S = \lim_{n\to\infty} S_n$ and $\widetilde{f} = \lim_{n\to\infty} \widetilde{f}_n$. It is worth to stress that so far we do not know if z satisfies some sort of boundary conditions: indeed, the only information we have is that z solves an equation and that it is Lipschitz up to the boundary. Anyway, this is enough to define the function z on $\partial\Omega$ and to try to deduce its boundary value by means of a scaling argument.

Indeed, we are going to prove that the fact that z solves (2.16) with (2.17) implies that z satisfies both Dirichlet and Neumann boundary conditions. Here, we follow the approach of [4] (see also [30]) in order to deduce the following result.

Theorem 2.9 *Under the assumptions of Theorem 1.3, it follows that*

$$z(\bar{x}) = 0 \quad \textit{and} \quad \lim_{\delta\to 0} \frac{z(\bar{x} - \delta\nu(\bar{x}))}{\delta} = 0 \quad \forall\, \bar{x} \in \partial\Omega.$$

Proof *Behaviour of* z *on* $\partial\Omega$. Let us consider a point $x_0 \in \partial\Omega$ and let us identify it as the origin $\mathcal{O}_\eta$ of a new system of coordinates $(\eta_1, \ldots, \eta_N) = (\eta_1, \eta')$ such that $e_{\eta_1} = \nabla d(x_0) = -\nu(x_0)$, where e_{η_1} is the versor of the η_1-axis. The equation for z remains unchanged by such a transformation being the Laplacian invariant under rotation and translation. In order to perform a blow-up near the origin $\mathcal{O}_\eta$, let us consider, for $\delta_0, \delta > 0$ and $0 < \sigma < \frac{1}{2}$, the set

$$U_\delta = B\left(\delta_0 e_{\eta_1}, \delta_0\right) \cap B(0, \delta^{1-\sigma})$$

that, through the change of variable $\xi = \frac{\eta}{\delta}$, becomes

$$W_\delta = \left\{(\xi_1, \xi') : \left(\xi_1 - \frac{\delta_0}{\delta}\right)^2 + |\xi'|^2 < \left(\frac{\delta_0}{\delta}\right)^2, |\xi| \leq \delta^{-\sigma}\right\}.$$

We take δ_0 small enough in order to have $U_\delta \subset \Omega$; this is always possible due to the smoothness of Ω. Moreover, notice that U_δ collapses to x_0 as δ goes to zero; meanwhile, thanks to the choice of σ, W_δ converges to the half space $\mathbb{R}^N_+$. Now, we can define

$$v_\delta(\xi) := z(\delta\xi) \quad \text{in } W_\delta$$

that inherits the following properties from z

$$\|v_\delta\|_{L^\infty(W_\delta)} = \|z\|_{L^\infty(U_\delta)} \quad \text{and} \quad \|\nabla v_\delta\|_{L^\infty(W_\delta)} = \delta \|\nabla z\|_{L^\infty(U_\delta)}.$$

From this information and from the fact that $W_\delta \nearrow \mathbb{R}^N_+$, we can infer that there exists a sequence $\delta_n \to 0$ such that $\{v_{\delta_n}\} \to v$ in $C^{0,\iota}_{\text{loc}}(\mathbb{R}^N_+)$ with $\iota \in (0,1)$ as $n \to \infty$. Moreover, thanks to the second estimate above we have that $v \equiv \text{const}$ and hence the theorem is proved if we show that $v \equiv 0$.

Choosing δ_0 small enough, the equation satisfied by v_{δ_n} becomes

$$-\Delta v_{\delta_n} + \left[(v_{\delta_n} + S)^p - S^p\right]\delta_n^2 = \widetilde{f}\delta_n^2 \quad \text{in } W_{\delta_n}.$$

It is important to stress that in W_{δ_n} the lower-order term above and the datum are singular only at the origin $\mathcal{O}_\eta$. In order to pass to the limit in the equation above, note that

$$d(\delta_n \xi) = \delta_n \xi_1 + O\left(\delta_n^2 |\xi|^2\right) \quad \text{uniformly in } W_{\delta_n} \quad \text{as} \quad n \to \infty.$$

Hence,

$$\begin{aligned}\left[(v_{\delta_n} + S)^p - S^p\right]\delta_n^2 &= S^p\left[\left(\frac{v_{\delta_n}}{S} + 1\right)^p - 1\right]\delta_n^2 = S^p\left[p\frac{v_{\delta_n}}{S} + O(d^{2\alpha})\right]\delta_n^2 \\ &= \left[pS^{p-1}v_{\delta_n} + O(d^{\alpha-2})\right]\delta_n^2 = \left[p\sigma_0\frac{v_{\delta_n}}{d^2} + o(d^{-2})\right]\delta_n^2 \\ &\qquad \text{uniformly in } W_{\delta_n} \quad \text{as} \quad n \to \infty,\end{aligned}$$

and for the datum

$$\widetilde{f}\delta_n^2 = \widetilde{f}d\frac{\delta_n^2}{d} \le C\frac{\delta_n}{\xi_1 + O(\delta|\xi|^2)} \quad \text{uniformly in } W_{\delta_n} \text{ as } n \to \infty.$$

Passing to the limit w.r.t n, we deduce that v satisfies

$$-\Delta v + p\sigma_0 \frac{v}{\xi_1^2} = 0 \quad \text{in } \mathbb{R}^N_+ \tag{2.18}$$

that admits $v \equiv 0$ as the unique constant solution. Being the previous argument independent of the considered sequence, we deduce that, if $\delta_n \to 0$ as n diverges and for any sequence $\{\eta_n\} \subset \Omega$ such that $\eta_n \in U_{\delta_n}$, it follows that

$$\lim_{n\to\infty} z(\eta_n) = \lim_{n\to\infty} v_{\delta_n}(\xi_n) = 0.$$

Behaviour of $\frac{\partial z}{\partial \nu}$. In order to prove that z satisfies also the Neumann boundary condition, we prove that there exist $1 < \beta_1, \beta_2 < 2$ and $A_1, A_2 > 0$ such that

$$-A_1 d^{\beta_1}(x) \le z(x) \le A_2 d^{\beta_2}(x) \quad \forall\, x \in \Omega. \tag{2.19}$$

Indeed, thanks to the previous step, (2.19) implies that

$$\lim_{\delta\to 0}\frac{z(\bar{x} - \delta\nu(\bar{x})) - z(\bar{x})}{\delta} = \lim_{\delta\to 0}\frac{z(\bar{x} - \delta\nu(\bar{x}))}{\delta} = 0 \quad \forall\, \bar{x} \in \partial\Omega.$$

Using (2.17) and by the definition of S, there exists $\delta_0 \le \bar{\delta}$ such that

$$|z + S|^{p-1}(z + S) - |S|^{p-1}S = S^p\left[\left(1 + \frac{z}{S}\right)^p - 1\right] \quad \text{in } \Omega_{\delta_0}.$$

Moreover, for any small $\epsilon > 0$ fixed, decreasing the value of δ_0 if needed, it holds true that

$$(1-\epsilon)p\sigma_0^{p-1}\frac{z}{d^2} \le S^p\left[\left(1 + \frac{z}{S}\right)^p - 1\right] \le (1+\epsilon)p\sigma_0^{p-1}\frac{z}{d^2} \quad \text{in } \Omega_{\delta_0}. \tag{2.20}$$

This implies that, for every $\delta \le \delta_0$, we have a comparison principle for sub- and super-solutions in $C^2(\Omega_{\tilde{\delta}}) \cap C(\bar{\Omega}_{\tilde{\delta}})$ for the problem

$$\begin{cases} -\Delta z + S^p\left[\left(1 + \frac{z}{S}\right)^p - 1\right] = \widetilde{f}, & \text{in } \Omega_{\tilde{\delta}}, \\ z = g, & \text{on } \partial\Omega_{\tilde{\delta}}, \end{cases}$$

for any $g \in C(\partial\Omega_{\tilde\delta})$. Let us prove that there exist β_1, A_1 positive constant such that $w := -A_1 d^{\beta_1}$ is a sub-solution of the problem above. Noticing that $-1 < [\alpha] - \alpha \le 0$, it is possible to choose β_1, β_2 in such a way that $1 < \beta_1, \beta_2 < 2 + [\alpha] - \alpha \le 2$ and that

$$\gamma_1 := -\beta_1^2 + \beta_1 + (1+\epsilon)p\sigma_0^{p-1} > 0 \quad \text{and} \quad \gamma_2 := -\beta_2^2 + \beta_2 + (1-\epsilon)p\sigma_0^{p-1} > 0.$$

Let us take moreover $\tilde\delta \le \delta_0$ such that

$$1 - \frac{\beta_i}{\gamma_i}\|\Delta d\|_{L^\infty(\Omega)} d - \bar C d^{2+\alpha-\alpha-\beta} > 0 \quad \text{for } i = 1, 2 \quad \text{in } \Omega_{\tilde\delta},$$

where $\bar C$ is given by (2.17), and finally fix $A_i := \max\{\tilde\delta^{-\beta_i}\|z\|_{L^\infty(\Omega)}, 1\}$ for $i = 1, 2$. With this choice, let us consider $w = -A_1 d^{\beta_1}$. Simple computations show that

$$\begin{aligned} -\Delta w + (1+\epsilon)p\sigma_0^{p-1}\frac{w}{d^2} - \widetilde f &\le -\gamma A_1 d^{\beta_1-2} + \beta_1 A_1 |\Delta d| d^{\beta_1-1} + A_1|\widetilde f| \\ &\le -A_1 d^{\beta_1-2}(\gamma_1 - \beta_1|\Delta d| d - |\widetilde f| d^{2-\beta_1}) \le 0 \quad \text{in } \Omega_{\tilde\delta}. \end{aligned}$$

Thanks to the inequality above, the choice of A_1 and (2.20) we infer that $w \le z$ in $\Omega_{\tilde\delta}$ (actually in all Ω thanks to the choice of A_1). In the very same way we prove that $z \le v := A_2 d^{\beta_2}$ and thus (2.19) is proved. □

Let us now give the proof of Theorem 1.3.

Proof of Theorem 1.3 Thanks to Theorems 2.7 and 2.8, we have a uniform Lipschitz bound for the sequence $z_n = u_n - S_n$ in Ω_{δ_0}, while Proposition 2.6 assures the interior regularity. Thus, we can deduce that there exists a constant $C = C(\alpha, N, \partial\Omega, f)$ such that

$$\|z_n\|_{W^{1,\infty}(\Omega)} \le \|z_n\|_{W^{1,\infty}(\Omega_{\delta_0})} + \|z_n\|_{W^{1,\infty}(\Omega\setminus\Omega_{\delta_0})} \le C$$

and passing to the limit with respect to n

$$\|u - S\|_{W^{1,\infty}(\Omega)} = \|z\|_{W^{1,\infty}(\Omega)} \le C.$$

Moreover, (1.17) is deduced from Theorem 2.9. □

2.5 Generalizations

In this last section, we give for brevity the sketch of the proof of Theorem 1.5, just stressing the main differences with respect to Theorem 1.3.

Sketch of the proof of Theorem 1.5 Let us give at first the compete expression of $\sigma_{h,0},\ldots,\sigma_{h,[\alpha]+1}$

$$
\begin{aligned}
\sigma_{h,0} &:= [\alpha(\alpha+1)]^{\frac{1}{p-1}},\\
\sigma_{h,1}(x) &:= \alpha\sigma_0 h^{-1}\frac{\alpha h^{-\frac{1}{2}}\nabla h\nabla d_n + h^{\frac{1}{2}}(N-1)H(x)}{2(1+2\alpha)},\\
\sigma_{h,k}(x) &:= \frac{L_k(\sigma_{h,k-1},\sigma_{h,k-2})+P_k(\sigma_{h,k-1},\sigma_{h,k-2})+Q_k(\sigma_{h,k-2})}{(k-\alpha)(k-\alpha-1)-(2+\alpha)(\alpha+1)}\\
&\quad+\frac{\sigma_{h,0}^{p}}{(k-\alpha)(k-\alpha-1)-(2+\alpha)(\alpha+1)}\\
&\quad\sum_{j=2}^{k}\left[\binom{p}{j}\sigma_{h,0}^{-j}\sum_{i_1+\cdots+i_j=k}\sigma_{h,i_1}(x)\cdots\sigma_{h,i_j}(x)\right]\\
&\quad\text{for} k=2\cdots[\alpha]+1 \text{ and } i_1,\ldots,i_j \text{ positive integers,}
\end{aligned}
\tag{2.21}
$$

where

$$
\begin{aligned}
L_k(\sigma_{h,k-1},\sigma_{h,k-2}) &:= (\alpha+1-k)\Big[\sigma_{k-1}\left(h^{-\frac{1}{2}}\nabla h\nabla d_n + h^{\frac{1}{2}}\Delta d_n+\Delta\sigma_{k-2}\right)\\
&\qquad +2h^{\frac{1}{2}}\nabla\sigma_{k-1}\nabla d_n\Big]h^{-1}\\
P_k(\sigma_{h,k-1},\sigma_{h,k-2}) &:= (\alpha+2-k)\Big[(k-\alpha-1)\sigma_{k-1}\nabla h\nabla d_n\\
&\qquad +\sigma_{k-2}\left(-\frac{1}{4}h^{\frac{3}{2}}|\nabla h|^2+\frac{1}{2}h^{-\frac{1}{2}}\Delta h\right)\Big]h^{-\frac{3}{2}}\\
Q_k(\sigma_{h,k-2}) &:= (\alpha+2-k)\left[\nabla\sigma_{k-2}\nabla h+\frac{(k-\alpha-3)}{4}\sigma_{k-2}h^{-\frac{9}{4}}|\nabla h|^2\right].
\end{aligned}
$$

A tedious computation shows that with such a choice, there exists a positive constant $\widetilde{C}_h=\widetilde{C}_h(\alpha,N,\partial\Omega,h,r)$ such that

$$
\begin{aligned}
&|(\Delta S_{h,n}-|S_{h,n}|^{p-1}S_{h,n})d_n|+|\nabla(\Delta S_{h,n}-|S_{h,n}|^{p-1}S_{h,n})d_n^2|\\
&\le \bar{C}_h d_n^{1+[\alpha]-\alpha}\le \widetilde{C}_h \quad \text{in } \Omega,
\end{aligned}
$$

where $S_{h,n}(x)=\sum_{k=0}^{[\alpha]+1}\sigma_{h,k}(x)\,\left(\sqrt{h(x)}d_n(x)\right)^{k-\alpha}$. Hence, we can define the approximated problems

$$
\begin{cases}
-\Delta u_{h,n}+h(x)|u_{h,n}|^{p-1}u_{h,n}=r(x,u_{h,n}), & \text{in } \Omega\\
\frac{\partial u_{h,n}}{\partial\nu}=\frac{\partial S_{h,n}}{\partial\nu} & \text{on } \partial\Omega.
\end{cases}
\tag{2.22}
$$

For the sake of clarity, we give some details of the construction of the sub-solution in the case $\alpha>1$. Let us consider the function

$$
w_{h,n}:=\sigma_0 h^{-\frac{\alpha}{2}}(x)d_n^{-\alpha}(x)-M_{h,1}h^{\frac{1-\alpha}{2}}(x)d_n^{1-\alpha}(x)-M_{h,2},
$$

with

$$
M_{h,1}\ge\alpha\sigma_0(p-1)\frac{\alpha A^{-\frac{3}{2}}\|\nabla h\|_{L^\infty(\Omega)}+A^{-\frac{1}{2}}\|\Delta d_n\|}{p+3}.
$$

Notice that thanks to (1.29)

$$|r(x,w_{h,n})d_n^{\alpha+1}| = |r(x,d_n^{-\alpha}+o(d_n^{-\alpha}))|d_n d_n^{\alpha}$$
$$\leq \sup_{0<s<1}\left\{|r(x,s^{-\alpha})|s\right\} d_n^{-\alpha} = o(1) \quad \text{as } d\to 0,\ n\to\infty.$$

Then, there exist $\delta_0=\delta_0(M_{h,1},r)$ and $n_0=n_0(\delta_0)$ such that

$$\begin{aligned}&-\Delta w_{h,n}+|w_{h,n}|^{p-1}w_{h,n}-r(x,w_{h,n})\\&\leq\left(-2\frac{p+3}{p-1}h^{\frac{-\alpha+1}{2}}M_{h,1}-\alpha^2\sigma_0 h^{-\frac{\alpha}{2}-1}\nabla h\nabla d_n+\alpha\sigma_0 h^{-\frac{\alpha}{2}}\Delta d_n\right.\\&\left.-r(x,w_{h,n})d_n^{\alpha+1}\right)d_n^{-\alpha-1}\\&\quad+O(d_n^{-\alpha})\leq 0\quad\forall n>n_0\quad\forall x\in\Omega_{\delta_0}.\end{aligned}$$

Up to an increase in the value of $M_{h,1}$ and taking the value of $M_{h,2}$ large enough, we deduce [following the same arguments that have led to (2.4) and (2.5)] that $w_{h,n}$ is a sub-solution of (2.22).
Once that sub- and super-solutions are obtained, we proceed as in Proposition 2.4, Theorem 2.5 and Proposition 2.6 in order to deduce that the solution $u_{h,n}$ of (2.22) converges (as $n\to\infty$) in $C^2_{\text{loc}}(\Omega)$ to u_h, unique solution of (1.25). Moreover, the following estimate is satisfied

$$\exists C=C(\alpha,N,\partial\Omega,h,r):\left|\frac{u_{h,n}(x)}{S_{h,n}(x)}-1\right|\leq C\varepsilon(d_n(x)), \tag{2.23}$$

where

$$\varepsilon(s)=\begin{cases}s & \text{if } \alpha>1\\ s(1+|\log s|) & \text{if } \alpha=1\\ s^{\alpha} & \text{if } \alpha<1.\end{cases}$$

Let us now define $z_{h,n}:=u_{h,n}-S_{h,n}$, that solves

$$\begin{cases}-\Delta z_{h,n}+|z_h+S_h|^{p-1}(z_{h,n}+S_{h,n})-|S_{h,n}|^{p-1}S_{h,n}=r(x,z_{h,n}-S_{h,n})+\widetilde{f}_{h,n}, & \text{in } \Omega,\\ \frac{\partial z_{h,n}}{\partial\nu}=0, & \text{on } \partial\Omega,\end{cases}$$

where $\widetilde{f}_n:=\Delta S_{h,n}-|S_{h,n}|^{p-1}S_{h,n}$. Concerning the $L^\infty(\Omega)$ estimate for $z_{h,n}$, we adapt the proof of Theorem 2.7 as follows. Let us fix a positive constant $B>0$, and let $\delta_0=\delta_0(S_{h,n})$ and $n_0=n_0(\delta_0)$ be such that

$$\begin{aligned}&-\Delta B+S_{h,n}^p\left[\left(1+\frac{B}{S_n}\right)^p-1\right]-r(x,\sigma_0 d_n^{-\alpha}+o(d_n^{-\alpha}))-\widetilde{f}_{h,n}\\&\geq\frac{BK}{d_n^2}-\frac{\sup_{0<s<1}\{|r(x,s^{-\alpha})|s\}}{d_n}-\widetilde{f}_{h,n}\geq\frac{BK-Cd_n-\bar{C}_h d_n}{d_n^2}\\&\geq 0\quad\text{in } \Omega_{\delta_0}\ \text{and}\ n>n_0,\end{aligned}$$

where we have used the first condition of (1.29). Thus, we can continue as in the proof of Theorem 2.7 to conclude that

$$\exists C=C(\alpha,N,\partial\Omega,h,r):\|z_{h,n}\|_{L^\infty(\Omega_\delta)}\leq C.$$

Let us now have a closer glance to the perturbed version of Theorem 2.8, for which the growth conditions (1.29) are especially designed. Exactly as in the previous section, we obtain that there exist δ_0 and n_0 such that

$$\Delta(|\nabla z_{h,n}|^2) \geq 2\nabla\left[h(z_{h,n}+S_{h,n})^p - hS_{h,n}^p\right]\nabla z_{h,n} + \frac{2}{N}(\Delta z_{h,n})^2$$
$$+2\nabla r(x, z_{h,n}+S_{h,n})\nabla z_{h,n} - 2\nabla(\widetilde{f}_{h,n})\nabla z_{h,n} \quad \text{in } \Omega_{\delta_0}, \quad \forall\, n > n_0.$$

The main concern of course is the third term on the right-hand side above; we have

$$2\nabla r(x, z_{h,n}+S_{h,n})\nabla z_{h,n} \leq 2\nabla_x r\nabla z_{h,n} + 2\frac{\partial r}{\partial s}|\nabla z_{h,n}|^2 - C\frac{\partial r}{\partial s}d_n^{-\alpha-1}\nabla d_n \nabla z_{h,n}$$
$$\leq \gamma\frac{|\nabla z_{h,n}|^2}{d_n^2} + C_\gamma |r_x|^2 d_n^2 + 2\,|r_s|\,|\nabla z_{h,n}|^2 + C\,|r_s|\,d_n^{-\alpha-1}.$$

Let us focus on the last three terms on the right-hand side above. Using assumption (1.29) and estimate (2.23), we get that for $d(x) \to 0$ and $n \to \infty$

$$\begin{aligned}|r_x(x, z_{h,n}+S_{h,n})| &= \left| r_x\left(x, S_{h,n}\left(1+\frac{z_{h,n}}{S_{h,n}}\right)\right)\right|\frac{d_n^2}{d_n^2}\\ &= |\nabla_x r(x, \sigma_0 d_n^{-\alpha} + o(d_n^{-\alpha}))|\frac{d_n^2}{d_n^2}\\ &\leq \frac{\sup_{0<s<1}\left\{|\nabla_x r(x, s^{-\alpha})|s^2\right\}}{d_n^2} \leq \frac{C}{d_n^2},\\ |r_s(x, z_{h,n}+S_{h,n})||\nabla z_{h,n}|^2 &\leq \sup_{0<s<1}\{|r_s(x, s^{-\alpha})|s^2\}\frac{|\nabla z_{h,n}|^2}{d_n^2} = o(1)\frac{|\nabla z_{h,n}|^2}{d_n^2},\\ |r_s(x, z_{h,n}+S_{h,n})|d_n^{-\alpha-1} &\leq \frac{\sup_{0<s<1}\{|r_s(x, s^{-\alpha})|s^{-\alpha+1}\}}{d_n^2} \leq \frac{C}{d_n^2}.\end{aligned}$$

Thus, up to a decrease in the value of δ_0 and an increase in n_0, we obtain

$$2\nabla r(x, z_{h,n}+S_{h,n})\nabla z_n \leq (\gamma + o(1))\frac{|\nabla z_n|^2}{d_n^2} + \frac{C}{d_n^2} \quad \text{in } \Omega_{\delta_0}.$$

At this point, it is easy to deduce the counter of (2.12), i.e. there exist some δ_0 and $n_0 = n_0(\delta_0)$ such that

$$\Delta(|\nabla z_{h,n}|^2) \geq \gamma\frac{|\nabla z_{h,n}|^2}{d_n^2} - \frac{C_1}{d_n^2} \quad \forall n > n_0 \quad \forall x \in \Omega_0.$$

From now on, the proof follows closely Theorem 2.8.

Hence, we infer that there exists $z_h \in C^2(\Omega)$, such that $z_{h,n} \to z_h$ in $C^2_{\text{loc}}(\Omega)$, that solves

$$\begin{cases} -\Delta z_h + |z_h + S_h|^{p-1}(z_h + S_h) - |S_h|^{p-1}S_h = r(x, z_h - S_h) + \widetilde{f}_h, & \text{in } \Omega,\\ z_h \in W^{1,\infty}(\Omega), \end{cases}$$

and that moreover

$$\left|\frac{z_h}{S_h}\right| \leq o(1) \quad \text{as} \quad d(x) \to 0 \quad \text{and} \quad |\widetilde{f}_h|d + |\nabla\widetilde{f}_h|d^2 \leq C.$$

As far as the boundary conditions of z_h are concerned, fixing $\bar{x} \in \partial\Omega$ and following the same notation of Theorem 2.9, let us set $v_{h,\delta}(\xi) := z_h(\delta\xi)$ in W_δ that inherits the following properties from z_h

$$\|v_{h,\delta}\|_{L^\infty(W_\delta)} = \|z_h\|_{L^\infty(U_\delta)} \quad \text{and} \quad \|\nabla v_{h,\delta}\|_{L^\infty(W_\delta)} = \delta\|\nabla z_h\|_{L^\infty(U_\delta)}.$$

Thus, the limit function v_h has to be a constant. Moreover, using the first assumption in (1.29) we have that

$$
\begin{aligned}
r(x, z_h + S_h)\delta^2 &= r\left(x, S_h\left(1+\frac{z_h}{S_h}\right)\right)\frac{d(x)}{d(x)}\delta^2 \\
&\le C\frac{\delta^2}{\delta\xi + O(\delta^2\xi^2)} \quad \text{uniformly in } W_\delta \text{ as } \delta \to 0,
\end{aligned}
$$

and thus v_h solves

$$
-\Delta v_h + p\sigma_0 h(\bar{x})\frac{v_h}{\xi_1^2} = 0 \quad \text{in } \mathbb{R}^+,
$$

whose unique constant solution is zero. From this, we infer that $z(\bar{x}) := u(\bar{x}) - S_h(\bar{x}) = 0$ for every $\bar{x} \in \partial\Omega$.

In order to recover the Neumann boundary condition, using the same notation of the second part of Theorem 2.9, we can infer that there exist $1 < \beta_{h,1} < 2$, $\gamma_{h,1} > 0$, $\tilde{\delta}$ and $A_{h,1} > 0$ such that

$$
\begin{aligned}
&-\Delta w_h + (1+\epsilon)\|h\|_{L^\infty(\Omega)} p\sigma_0^{p-1}\frac{w_h}{d^2} - r(x, w_h + S) - \widetilde{f}_h \\
&= -A_{h,1}d^{\beta_{h,1}-2}(\gamma_{h,1} - \beta_{h,1}|\Delta d|d - \sup_{0<s<1}|r(x, s^{-\alpha})s|d^{1-\beta_{h,1}} \\
&\quad -\widetilde{f}d^{2-\beta_{h,1}}) \le 0 \quad \text{in } \Omega_{\tilde{\delta}},
\end{aligned}
$$

where the last inequality is implied by assumption (1.29). The rest of the proof closely follows Theorem 2.9. □

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