Uniform support function of the weighted L^2 integrations with optimal asymptoticity

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Received: 22 January 2018 / Accepted: 2 November 2018 / Published online: 16 November 2018
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Abstract

In the present article, we obtain a uniform support function of the weighted L^2 integrations on the superlevel sets of the weights with optimal asymptoticity, which is an analogue of Demailly's strong openness conjecture.

Keywords Multiplier ideal sheaf · Weighted L^2 integration · Effectiveness · Strong openness conjecture

Mathematics Subject Classification 32D15 · 32E10 · 32L10 · 32U05 · 32W05

1 Introduction

Let X be a domain in $\mathbb{C}^n$, and let φ be a negative plurisubharmonic function on X . The multiplier ideal sheaf $\mathcal{I}(\varphi)$ can be defined as the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable, which plays an important role in several complex variables, complex geometry and algebraic geometry, e.g., see [2, 3, 11–17].

It is well known that $\mathcal{I}(\varphi)$ is a coherent analytic sheaf (see [3]). Let $\mathcal{I}_+(\varphi) := \bigcup_{\varepsilon>0} \mathcal{I}((1+\varepsilon)\varphi)$. The so-called strong openness conjecture posed by Demailly (see [3]) is that $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ holds for any weight φ . If $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness conjecture

Qi'an Guan was partially supported by NSFC-11522101, NSFC-11431013 and NSFC-11825101, Jiafu Ning was partially supported by NSFC-11501058 and NSFC-11501059.

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degenerates to a very important case the so-called openness conjecture posed by Demailly and Kollár (see [2]).

For the case of dimension two, the openness conjecture and strong openness conjecture were proved by Favre and Jonsson [4] and Jonsson and Mustata [9], respectively, by using the theory of valuative tree. In [1], Berndtsson proved the openness conjecture by using his complex Brunn–Minkowski inequality. In [6] (see also [8, 10]), Guan and Zhou proved Demailly's strong openness conjecture by using L^2 extension theorem dynamically.

Note that the known approaches to the strong openness conjecture are dependent on L^2 extension theorem; then, it is natural to consider some other approaches, i.e., the asymptoticity of the superlevel sets of the weights in the present article.

Let z_0 be a point in a pseudoconvex domain $D \subset X$, and let F be a holomorphic function near z_0 . Let $D_t = \{z \in D \mid \varphi \geq -t\}$ the superlevel set of weight φ , and let $C_{F,\varphi,t}(z_0)$ be the infimum of $\int_{D_t} |F_1|^2 d\lambda_n$ for all $F_1 \in \mathcal{O}(D)$ satisfying condition $(F_1 - F, z_0) \in \mathcal{I}(\varphi)_{z_0}$, where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$.

When $C_{F,\varphi,t}(z_0) = 0$ or ∞ , set $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F,\varphi,t}(z_0)} = +\infty$. Then the strong openness conjecture is equivalent to the statements that for any pseudoconvex domain $D \subset X$, any point $z_0 \in D$ and any holomorphic F on D , if $\lim_{t \rightarrow +\infty} \frac{\int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n}{C_{F,(1+\varepsilon)\varphi,t}(z_0)} = +\infty$ holds for any $\varepsilon > 0$, then

$$\lim_{t \rightarrow +\infty} \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n}{C_{F,(1+\varepsilon)\varphi,t}(z_0)} \right) = +\infty \quad (1.1)$$

(for details, see Sect. 2.1).

In the present article, we establish the following uniform support function of $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F,\varphi,t}(z_0)}$ (independent of φ and F) with optimal asymptoticity.

Theorem 1.1 *For any pseudoconvex domain D in $\mathbb{C}^n$, any holomorphic function F and any negative plurisubharmonic function φ on D , the inequality*

$$\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F,\varphi,t}(z_0)} \geq h^{-1}(t-3) \quad (1.2)$$

holds for any $t \in (0, +\infty)$, where $h(x) = x + \log x$, and $h^{-1}(t-3)$ is the support function.

The following remark gives the optimal asymptoticity of the support function $h^{-1}(t-3)$ in inequality (1.2).

Remark 1.2 Take $D = \Delta \subset \mathbb{C}$, $z_0 = o$ the origin of $\mathbb{C}$, $F \equiv 1$ and $\varphi = 2 \log |z|$. Note that $C_{F,\varphi,t}(z_0) = \pi(1 - e^{-t})$ and $\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n = t\pi$, then $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F,\varphi,t}(z_0)} = \frac{t}{1-e^{-t}}$. One can obtain $\lim_{t \rightarrow +\infty} \frac{h^{-1}(t-3)}{\frac{t}{1-e^{-t}}} = 1$, which implies the optimal asymptoticity of the support function $h^{-1}(t-3)$ in inequality (1.2), when t goes to $+\infty$.

By equality (1.1), it follows that Theorem 1.1 implies the strong openness conjecture.

Corollary 1.3 [6] $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$.

2 Preparations

2.1 An equivalent form of Demailly's strong openness conjecture

We first show the strong openness conjecture implies (1.1).

By the definition of $C_{F, (1+\varepsilon)\varphi, t}(z_0)$, it follows that $\lim_{\varepsilon \rightarrow 0^+} C_{F, (1+\varepsilon)\varphi, t}(z_0) =$ the infimum of $\int_{D_t} |F_1|^2 d\lambda_n$ for all $F_1 \in \mathcal{O}(D)$ satisfying condition $(F_1 - F, z_0) \in \mathcal{J}_+(\varphi)_{z_0}$. We have $C_{F, \varphi, t}(z_0) \leq C_{F, (1+\varepsilon)\varphi, (1+\varepsilon)t}(z_0)$, and $C_{F, \varphi, t}(z_0)$ is increasing w.r.t. t . Note that $C_{F, \varphi, t}(z_0) = 0$ for all $t > 0$, if and only if $(F, z_0) \in \mathcal{J}_+(\varphi)_{z_0}$.

If F is holomorphic on D , and if $\lim_{t \rightarrow +\infty} \frac{\int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n}{C_{F, (1+\varepsilon)\varphi, t}(z_0)} = +\infty$ holds for any $\varepsilon > 0$.

(1) If $C_{F, \varphi, t}(z_0) = 0$ for all $t > 0$, by the strong openness conjecture, we have $C_{F, (1+\varepsilon)\varphi, t}(z_0) = 0$ for $\varepsilon > 0$ small enough, and for all $t > 0$. (1.1) holds.

(2) If $C_{F, \varphi, t}(z_0) \geq a_0$ for all $t > t_0$ for some fixed $a_0 > 0$, then $C_{F, (1+\varepsilon)\varphi, t}(z_0) \geq a_0$ for all $t > (1+\varepsilon)t_0$. $C_{F, \varphi, t}(z_0)$ and $C_{F, (1+\varepsilon)\varphi, t}(z_0) > 0$ are bounded above by $\int_D |F|^2 < \infty$. Therefore, $\lim_{t \rightarrow +\infty} (\lim_{\varepsilon \rightarrow 0^+} \frac{\int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n}{C_{F, (1+\varepsilon)\varphi, t}(z_0)}) = +\infty$ if and only if $\lim_{t \rightarrow +\infty} (\lim_{\varepsilon \rightarrow 0^+} \int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n) = +\infty$. Assume that

$$\lim_{t \rightarrow +\infty} \left(\lim_{\varepsilon \rightarrow 0^+} \int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n \right) < +\infty.$$

By the dominated convergence theorem, it follows that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n = \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n. \quad (2.1)$$

So, $\lim_{t \rightarrow +\infty} (\lim_{\varepsilon \rightarrow 0^+} \int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n) = \lim_{t \rightarrow +\infty} \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n < +\infty$. we get $(F, z_0) \in \mathcal{J}_+(\varphi)_{z_0}$, and $C_{F, \varphi, t}(z_0) = 0$ for all $t > 0$, contradiction.

By combining (1) and (2), one sees that the strong openness conjecture implies (1.1).

Then it remains to consider that equality (1.1) implies the strong openness conjecture.

If not, i.e., there exists $z_0 \in X$ satisfying $\mathcal{J}_+(\varphi)_{z_0} \subsetneq \mathcal{J}_+(\varphi)_{z_0}$, then there exists a holomorphic function F near z_0 such that $(F, z_0) \in \mathcal{J}_+(\varphi)_{z_0}$ and $(F, z_0) \notin \mathcal{J}_+(\varphi)_{z_0}$. Choosing D small enough, it follows from $(F, z_0) \in \mathcal{J}_+(\varphi)_{z_0}$ that $\int_D |F|^2 e^{-\varphi} d\lambda_n < +\infty$, which implies

$$\lim_{t \rightarrow +\infty} \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n = \int_D |F|^2 e^{-\varphi} d\lambda_n < +\infty. \quad (2.2)$$

Note that $C_{F, (1+\varepsilon)\varphi, t}(z_0)$ is upper bounded with respect to t for D small enough, it follows that

$$\lim_{t \rightarrow +\infty} \frac{\int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n}{C_{F, (1+\varepsilon)\varphi, t}(z_0)} = +\infty$$

for any $\varepsilon > 0$. As $(F, z_0) \notin \mathcal{J}_+(\varphi)_{z_0}$, it follows that the infimum of $\int_{D_t} |F_1|^2 d\lambda_n$ for all $F_1 \in \mathcal{O}(D)$ satisfying condition $(F_1 - F, z_0) \in \mathcal{J}_+(\varphi)_{z_0}$ is bigger than a positive constant C_0 for t large enough. Combining with inequality (2.2), one can obtain that and

$$\lim_{t \rightarrow +\infty} \left(\lim_{\varepsilon \rightarrow 0^+} \frac{\int_{D_t} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda_n}{C_{F, (1+\varepsilon)\varphi, t}(z_0)} \right) \leq \lim_{t \rightarrow +\infty} \frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_0} = \frac{\int_D |F|^2 e^{-\varphi} d\lambda_n}{C_0} < +\infty,$$

which contradicts equality (1.1). Then equality (1.1) implies the strong openness conjecture.

2.2 $\bar{\partial}$ -equation with L^2 estimates

We prove Proposition 2.3 by the following Lemma, whose various forms have already appeared in [5,7].

Lemma 2.1 *Let $B_0 \in (0, 1]$ be arbitrarily given. Let U_v be a strongly pseudoconvex domain relatively compact in pseudoconvex domain $D \subseteq \mathbb{C}^n$ containing o . Let F be a holomorphic function on D . Let φ be a negative plurisubharmonic function on D , such that $u(o) = -\infty$. Then there exists a holomorphic function F_{v,t_0} on U_v , such that*

$$(F_{v,t_0} - F, o) \in \mathcal{I}(\varphi)_o$$

and

$$\begin{aligned} & \int_{U_v} |F_{v,t_0} - (1 - b_{t_0}(\varphi))F|^2 d\lambda_n \\ & \leq \left(1 - e^{-(t_0+B_0)}\right) \int_{U_v} \frac{1}{B_0} (\mathbb{1}_{\{-t_0-B_0 < t < -t_0\}} \circ \varphi) |F|^2 e^{-\varphi} d\lambda_n, \end{aligned} \quad (2.3)$$

where $b_{t_0}(t) = \int_{-\infty}^t \frac{1}{B_0} \mathbb{1}_{\{-t_0-B_0 < s < -t_0\}} ds$, and t_0 is a positive number.

Remark 2.2 Replacing the strong pseudoconvexity of U_v by pseudoconvexity, and o by $z_0 \in U_v$ for any v , Lemma 2.1 still holds. We may take $U_v \subset \subset U_{v+1}$ and $\cup U_v = D$. In the following, by approximation, we may take D instead of U_v .

2.3 A useful proposition

Proposition 2.3 *Let C_1, C_2 be two positive constants and $t \geq \log \frac{C_1}{C_2}$. Let $p > 0$. We consider the set of (F, φ, p) satisfying*

$$(1) \int_{D_t} |F|^2 e^{-\varphi} \leq C_1; \quad (2) C_{F,p\varphi,t}(z_0) \geq C_2;$$

where $C_{F,p\varphi,t}(z_0)$ is the infimum of $\int_{D_t} |F_1|^2 d\lambda_n$ for all $F_1 \in \mathcal{O}(D)$ satisfying condition $(F_1 - F, z_0) \in \mathcal{I}(p\varphi)_{z_0}$, and $D_t = \{z \in D \mid \varphi \geq -t\}$.

Then, for $p \neq 1$ and $1/2$, we have

$$\begin{aligned} & \left(1 - \frac{1}{p}\right)^{-1} \left(\frac{C_2}{C_1}\right) \left(\left(\frac{C_2}{C_1}\right)^{p-1} - e^{-(p-1)t} \right) - 2 \left(1 - \frac{1}{2p}\right)^{-1} \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} \\ & \left(\left(\frac{C_2}{C_1}\right)^{p-\frac{1}{2}} - e^{-(p-\frac{1}{2})t} \right) \\ & + \left(\left(\frac{C_2}{C_1}\right)^p - e^{-pt} \right) \leq 1. \end{aligned} \quad (2.4)$$

For $p = 1$, we have

$$\left(\frac{C_2}{C_1}\right) \left(t + \log \frac{C_2}{C_1}\right) - 4 \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} \left(\left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \right) + \left(\left(\frac{C_2}{C_1}\right) - e^{-t} \right) \leq 1. \quad (2.5)$$

For $p = 1/2$, we have

$$\frac{C_2}{C_1} \left(e^{\frac{t}{2}} - \left(\frac{C_2}{C_1}\right)^{-\frac{1}{2}} \right) - \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} \left(t + \log \frac{C_2}{C_1} \right) + \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \leq 1.$$

Proof As $C_{F,p\varphi,t}(z_0) \geq C_2 > 0$, then we obtain $\varphi(z_0) = -\infty$.

Let $\psi := p\varphi$.

Using Lemma 2.1 and Remark 2.2, we obtain that

$$\begin{aligned} & \left(\left(\int_{D_t} |F_{v,t_0}|^2 d\lambda_n \right)^{1/2} - \left(\int_{D_t} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n \right)^{1/2} \right)^2 \\ & \leq \int_{D_t} |F_{v,t_0} - (1 - b_{t_0}(\psi))F|^2 d\lambda_n \\ & \leq \int_D |F_{v,t_0} - (1 - b_{t_0}(\psi))F|^2 d\lambda_n \\ & \leq \left(1 - e^{-(t_0+B_0)} \right) \int_D \frac{1}{B_0} (\mathbb{1}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 e^{-\psi} d\lambda_n. \end{aligned} \quad (2.6)$$

Note that

$$\begin{aligned} & \left(1 - e^{-(t_0+B_0)} \right) \int_D \frac{1}{B_0} (\mathbb{1}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 e^{-\psi} d\lambda_n \\ & \leq \left(e^{t_0+B_0} - 1 \right) \int_D \frac{1}{B_0} (\mathbb{1}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n. \end{aligned} \quad (2.7)$$

Then we have

$$\begin{aligned} & \left(\left(\int_{D_t} |F_{v,t_0}|^2 d\lambda_n \right)^{1/2} - \left(\int_{D_t} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n \right)^{1/2} \right)^2 \\ & \leq \left(e^{t_0+B_0} - 1 \right) \int_D \frac{1}{B_0} (\mathbb{1}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n. \end{aligned} \quad (2.8)$$

Note that

$$e^{\frac{1}{p}t_0} \int_{D_t} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n \leq e^{\frac{1}{p}t_0} \int_{D_t} |\mathbb{1}_{\{\frac{1}{p}\psi \leq -\frac{1}{p}t_0\}} F|^2 d\lambda_n \leq \int_{D_t} |F|^2 e^{-\frac{1}{p}\psi} d\lambda_n.$$

As

$$C_1 \geq \int_{D_t} |F|^2 e^{-\frac{1}{p}\psi} d\lambda_n = \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n$$

and $\inf\{\int_{D_t} |F_1|^2 d\lambda_n \mid F_1 \in \mathcal{O}(D), (F_1 - F, z_0) \in \mathcal{J}(\psi)_{z_0}\} \geq C_2$, when

$$e^{\frac{1}{p}t_0} \geq \frac{C_1}{C_2},$$

then we have

$$\begin{aligned} & \left(C_2^{1/2} - \left(C_1 e^{-\frac{1}{p}t_0} \right)^{1/2} \right)^2 \\ & \leq \left(\left(\int_{D_t} |F_{v,t_0}|^2 d\lambda_n \right)^{1/2} - \left(\int_{D_t} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n \right)^{1/2} \right)^2. \end{aligned} \quad (2.9)$$

It follows that

$$\begin{aligned}
 & \left(C_2^{1/2} - \left(C_1 e^{-\frac{1}{p} t_0} \right)^{1/2} \right)^2 \\
 & \leq \left(e^{t_0 + B_0} - 1 \right) \int_D \frac{1}{B_0} (\mathbb{1}_{\{-t_0 - B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n \\
 & \leq e^{t_0 + B_0} \int_D \frac{1}{B_0} (\mathbb{1}_{\{-t_0 - B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n.
 \end{aligned} \tag{2.10}$$

Replacing t_0 by $k B_0$, and assuming that $e^{\frac{1}{p} k B_0} \geq \frac{C_1}{C_2}$, we obtain that

$$\begin{aligned}
 & \left(C_2^{1/2} - \left(C_1 e^{-\frac{1}{p} k B_0} \right)^{1/2} \right)^2 \\
 & \leq e^{(k+1) B_0} \int_D \frac{1}{B_0} (\mathbb{1}_{\{-(k+1) B_0 < t < -k B_0\}} \circ \psi) |F|^2 d\lambda_n.
 \end{aligned} \tag{2.11}$$

It follows that

$$\begin{aligned}
 & B_0 e^{-(k+1) B_0} e^{\frac{1}{p} k B_0} \left(C_2^{1/2} - \left(C_1 e^{-\frac{1}{p} k B_0} \right)^{1/2} \right)^2 \\
 & \leq e^{\frac{1}{p} k B_0} \int_D (\mathbb{1}_{\{-(k+1) B_0 < t < -k B_0\}} \circ \psi) |F|^2 d\lambda_n \\
 & \leq \int_D (\mathbb{1}_{\{-(k+1) B_0/p < t < -k B_0/p\}} \circ \varphi) |F|^2 e^{-\varphi} d\lambda_n,
 \end{aligned} \tag{2.12}$$

the last inequality holds because $\psi = p\varphi$.

Taking k_0 such that

$$e^{\frac{1}{p} k_0 B_0} \geq \frac{C_1}{C_2} \geq e^{\frac{1}{p} (k_0 - 1) B_0},$$

and taking the sum over k of our previous inequality, we obtain

$$\begin{aligned}
 & \sum_{k=k_0}^{[pt/B_0]-1} B_0 e^{-(k+1) B_0} e^{\frac{1}{p} k B_0} \left(C_2^{1/2} - \left(C_1 e^{-\frac{1}{p} k B_0} \right)^{1/2} \right)^2 \\
 & \leq \sum_{k=k_0}^{[pt/B_0]-1} \int_D (\mathbb{1}_{\{-(k+1) B_0/p < t < -k B_0/p\}} \circ \varphi) |F|^2 e^{-\varphi} d\lambda_n \\
 & \leq \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n \leq C_1.
 \end{aligned} \tag{2.13}$$

Note that when $p > 0$, $p \neq 1$ and $p \neq 1/2$, we have

$$\begin{aligned}
 & \sum_{k=k_0}^{[pt/B_0]-1} B_0 e^{-(k+1)B_0} e^{\frac{1}{p}k B_0} \left(C_2^{1/2} - \left(C_1 e^{-\frac{1}{p}k B_0} \right)^{1/2} \right)^2 \\
 &= \sum_{k=k_0}^{[pt/B_0]-1} B_0 e^{-B_0} \left(e^{\left(\frac{1}{p}-1\right)k B_0} C_2 - 2e^{\left(\frac{1}{2p}-1\right)k B_0} C_2^{1/2} C_1^{1/2} + e^{-k B_0} C_1 \right) \\
 &= B_0 e^{-B_0} \left(C_2 \frac{e^{-(1-\frac{1}{p})k_0 B_0} - e^{-(1-\frac{1}{p})\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-(1-\frac{1}{p})B_0}} \right. \\
 &\quad \left. - 2C_2^{1/2} C_1^{1/2} \frac{e^{-k_0\left(1-\frac{1}{2p}\right)B_0} - e^{-(1-\frac{1}{2p})\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-(1-\frac{1}{2p})B_0}} + C_1 \frac{e^{-k_0 B_0} - e^{\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-B_0}} \right) \quad (2.14) \\
 &\geq B_0 e^{-B_0} \left(C_2 \frac{e^{-(1-\frac{1}{p})B_0} \left(\frac{C_2}{C_1}\right)^{p-1} - e^{-(1-\frac{1}{p})\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-(1-\frac{1}{p})B_0}} \right. \\
 &\quad \left. - 2C_2^{1/2} C_1^{1/2} \frac{\left(\frac{C_2}{C_1}\right)^{p\left(1-\frac{1}{2p}\right)} - e^{-(1-\frac{1}{2p})\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-(1-\frac{1}{2p})B_0}} + C_1 \frac{e^{-B_0} \left(\frac{C_2}{C_1}\right)^p - e^{\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-B_0}} \right).
 \end{aligned}$$

Let us take the limit

$$\begin{aligned}
 & \lim_{B_0 \rightarrow 0} B_0 e^{-B_0} \left(C_2 \frac{e^{-(1-\frac{1}{p})B_0} \left(\frac{C_2}{C_1}\right)^{p-1} - e^{-(1-\frac{1}{p})\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-(1-\frac{1}{p})B_0}} \right. \\
 &\quad \left. - 2C_2^{1/2} C_1^{1/2} \frac{\left(\frac{C_2}{C_1}\right)^{p\left(1-\frac{1}{2p}\right)} - e^{-(1-\frac{1}{2p})\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-(1-\frac{1}{2p})B_0}} + C_1 \frac{e^{-B_0} \left(\frac{C_2}{C_1}\right)^p - e^{\left[\frac{pt}{B_0}\right] B_0}}{1 - e^{-B_0}} \right) \\
 &= C_2 \frac{\left(\frac{C_2}{C_1}\right)^{p-1} - e^{-(1-\frac{1}{p})pt}}{1 - \frac{1}{p}} \quad (2.15) \\
 &\quad - 2C_2^{1/2} C_1^{1/2} \frac{\left(\frac{C_2}{C_1}\right)^{p\left(1-\frac{1}{2p}\right)} - e^{-(1-\frac{1}{2p})pt}}{1 - \frac{1}{2p}} + C_1 \left(\left(\frac{C_2}{C_1}\right)^p - e^{-pt} \right) \\
 &= C_2 \frac{\left(\frac{C_2}{C_1}\right)^{p-1} - e^{-(p-1)t}}{1 - \frac{1}{p}} \\
 &\quad - 2C_2^{1/2} C_1^{1/2} \frac{\left(\frac{C_2}{C_1}\right)^{p-\frac{1}{2}} - e^{-(p-\frac{1}{2})t}}{1 - \frac{1}{2p}} + C_1 \left(\left(\frac{C_2}{C_1}\right)^p - e^{-pt} \right).
 \end{aligned}$$

By inequalities (2.13), (2.14) and (2.15), it follows that

$$\begin{aligned} & \left(1 - \frac{1}{p}\right)^{-1} \left(\frac{C_2}{C_1}\right) \left(\left(\frac{C_2}{C_1}\right)^{p-1} - e^{-(p-1)t} \right) - 2 \left(1 - \frac{1}{2p}\right)^{-1} \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} \\ & \left(\left(\frac{C_2}{C_1}\right)^{p-\frac{1}{2}} - e^{-(p-\frac{1}{2})t} \right) \\ & + \left(\left(\frac{C_2}{C_1}\right)^p - e^{-pt} \right) \leq 1. \end{aligned} \quad (2.16)$$

When $p = 1$, we can take $p = 1$ to (2.13), by the same way, with some changes, and we can get (2.5). We write down details below.

Taking k_0 , such that

$$e^{k_0 B_0} \geq \frac{C_1}{C_2} \geq e^{(k_0-1)B_0}.$$

Let $p = 1$ in (2.12), and take sum, then we have

$$\begin{aligned} & \sum_{k=k_0}^{[t/B_0]-1} B_0 e^{-(k+1)B_0} e^{kB_0} (C_2^{1/2} - (C_1 e^{-kB_0})^{1/2})^2 \\ & \leq \sum_{k=k_0}^{[t/B_0]-1} \int_D (\mathbb{1}_{\{-(k+1)B_0 < t < -kB_0\}} \circ \varphi) |F|^2 e^{-\varphi} d\lambda_n \\ & \leq \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n \leq C_1. \end{aligned} \quad (2.17)$$

It follows that

$$\begin{aligned} & \sum_{k=k_0}^{[t/B_0]-1} B_0 e^{-(k+1)B_0} e^{kB_0} (C_2^{1/2} - (C_1 e^{-kB_0})^{1/2})^2 \\ & = \sum_{k=k_0}^{[t/B_0]-1} B_0 e^{-B_0} (C_2 - 2e^{-\frac{1}{2}kB_0} C_2^{1/2} C_1^{1/2} + e^{-kB_0} C_1) \\ & = B_0 e^{-B_0} (C_2 \left(\left[\frac{t}{B_0} \right] - k_0 \right) \\ & \quad - 2C_2^{1/2} C_1^{1/2} \frac{e^{-\frac{1}{2}k_0 B_0} - e^{-\frac{1}{2} \left[\frac{t}{B_0} \right] B_0}}{1 - e^{-\frac{1}{2}B_0}} + C_1 \frac{e^{-k_0 B_0} - e^{-\left[\frac{t}{B_0} \right] B_0}}{1 - e^{-B_0}}) \\ & \geq B_0 e^{-B_0} \left(C_2 \left(\left[\frac{t}{B_0} \right] + \frac{1}{B_0} \log \frac{C_2}{C_1} - 1 \right) \right. \\ & \quad \left. - 2C_2^{1/2} C_1^{1/2} \frac{\left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{1}{2} \left[\frac{t}{B_0} \right] B_0}}{1 - e^{-\frac{1}{2}B_0}} + C_1 \frac{e^{-B_0} \frac{C_2}{C_1} - e^{-\left[\frac{t}{B_0} \right] B_0}}{1 - e^{-B_0}} \right). \end{aligned} \quad (2.18)$$

Take limitation

$$\begin{aligned}
 & \lim_{B_0 \rightarrow 0} B_0 e^{-B_0} \left(C_2 \left(\left\lfloor \frac{t}{B_0} \right\rfloor + \frac{1}{B_0} \log \frac{C_2}{C_1} - 1 \right) \right. \\
 & \quad \left. - 2C_2^{1/2} C_1^{1/2} \frac{\left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{1}{2} \left\lfloor \frac{t}{B_0} \right\rfloor B_0}}{1 - e^{-\frac{1}{2} B_0}} + C_1 \frac{e^{-B_0 \frac{C_2}{C_1}} - e^{-\left\lfloor \frac{t}{B_0} \right\rfloor B_0}}{1 - e^{-B_0}} \right) \\
 & = C_2 \left(C_2 t + \log \frac{C_2}{C_1} \right) \\
 & \quad - 4C_2^{1/2} C_1^{1/2} \left(\left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{1}{2} p t} \right) + C_1 \left(\frac{C_2}{C_1} - e^{-t} \right).
 \end{aligned} \tag{2.19}$$

By inequalities (2.17) (2.18) and (2.19), we have

$$\left(\frac{C_2}{C_1} \right) \left(t + \log \frac{C_2}{C_1} \right) - 4 \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} \left(\left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \right) + \left(\left(\frac{C_2}{C_1} \right) - e^{-t} \right) \leq 1.$$

When $p = 1/2$, take k_0 , such that

$$e^{2k_0 B_0} \geq \frac{C_1}{C_2} \geq e^{2(k_0-1)B_0}.$$

Let $p = 1/2$ in (2.12), and take the sum, then we have

$$\begin{aligned}
 & \sum_{k=k_0}^{\left\lfloor \frac{t}{2B_0} \right\rfloor - 1} B_0 e^{-(k+1)B_0} e^{2kB_0} (C_2^{1/2} - (C_1 e^{-2kB_0})^{1/2})^2 \\
 & \leq \sum_{k=k_0}^{\left\lfloor \frac{t}{2B_0} \right\rfloor - 1} \int_D (\mathbb{1}_{\{-2(k+1)B_0 < t < -2kB_0\}} \circ \varphi) |F|^2 e^{-\varphi} d\lambda_n \\
 & \leq \int_{D_t} |F|^2 e^{-\varphi} d\lambda_n \leq C_1.
 \end{aligned} \tag{2.20}$$

As

$$\begin{aligned}
 & \sum_{k=k_0}^{\left\lfloor \frac{t}{2B_0} \right\rfloor - 1} B_0 e^{-(k+1)B_0} e^{2kB_0} \left(C_2^{1/2} - (C_1 e^{-2kB_0})^{1/2} \right)^2 \\
 & = \sum_{k=k_0}^{\left\lfloor \frac{t}{2B_0} \right\rfloor - 1} B_0 e^{-B_0} \left(e^{kB_0} C_2 - C_2^{\frac{1}{2}} C_1^{\frac{1}{2}} + C_1 e^{-kB_0} \right) \\
 & = B_0 e^{-B_0} \left(C_2 \frac{e^{B_0 \left(\frac{C_2}{C_1} \right)^{-\frac{1}{2}}} - e^{\left\lfloor \frac{t}{2B_0} \right\rfloor B_0}}{1 - e^{B_0}} \right)
 \end{aligned}$$

$$\begin{aligned}
& -2C_2^{\frac{1}{2}}C_1^{\frac{1}{2}}\left(\left[\frac{t}{2B_0}\right]-k_0\right)+C_1\frac{e^{-k_0B_0}-e^{-\left[\frac{t}{2B_0}\right]B_0}}{1-e^{-B_0}}\right) \\
& \geq B_0e^{-B_0}\left(C_2\frac{e^{B_0}\left(\frac{C_2}{C_1}\right)^{\frac{1}{2}}-e^{\left[\frac{t}{2B_0}\right]B_0}}{1-e^{B_0}}\right. \\
& \quad \left.-2C_2^{\frac{1}{2}}C_1^{\frac{1}{2}}\left(\left[\frac{t}{2B_0}\right]+\frac{1}{2}\log\frac{C_2}{C_1}\right)+C_1\frac{e^{-B_0}\left(\frac{C_2}{C_1}\right)^{\frac{1}{2}}-e^{-\left[\frac{t}{2B_0}\right]B_0}}{1-e^{-B_0}}\right).
\end{aligned} \tag{2.21}$$

We take again the limit

$$\begin{aligned}
& \lim_{B_0 \rightarrow 0} B_0 e^{-B_0} \left(C_2 \frac{e^{B_0} \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{\left[\frac{t}{2B_0} \right] B_0}}{1 - e^{B_0}} \right. \\
& \quad \left. - 2C_2^{\frac{1}{2}}C_1^{\frac{1}{2}} \left(\left[\frac{t}{2B_0} \right] + \frac{1}{2} \log \frac{C_2}{C_1} \right) + C_1 \frac{e^{-B_0} \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\left[\frac{t}{2B_0} \right] B_0}}{1 - e^{-B_0}} \right) \\
& = C_2 \left(e^{\frac{t}{2}} - \left(\frac{C_2}{C_1} \right)^{-\frac{1}{2}} \right) - 4C_2^{\frac{1}{2}}C_1^{\frac{1}{2}} \left(t + \log \frac{C_2}{C_1} \right) + C_1 \left(\left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \right).
\end{aligned} \tag{2.22}$$

By combining inequalities (2.20) (2.21) and (2.22), we have

$$\frac{C_2}{C_1} \left(e^{\frac{t}{2}} - \left(\frac{C_2}{C_1} \right)^{-\frac{1}{2}} \right) - \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} \left(t + \log \frac{C_2}{C_1} \right) + \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \leq 1.$$

□

Remark 2.4 For $p = 1$, let $p_j > 1$, $j = 1, 2, \dots$ and $\lim_{j \rightarrow \infty} p_j = 1$, then from $C_{F, p_j \varphi, t} \geq C_{F, \varphi, t} \geq C_2$. Therefore, inequality (2.4) holds with $p = p_j$. Taking $j \rightarrow \infty$, we can get

$$\left(\frac{C_2}{C_1} \right) \left(t + \log \frac{C_2}{C_1} \right) - 4 \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} \left(\left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \right) + \left(\left(\frac{C_2}{C_1} \right) - e^{-t} \right) \leq 1.$$

For $p = 1/2$, let $p_j > 1/2$, $j = 1, 2, \dots$ and $\lim_{j \rightarrow \infty} p_j = 1/2$, then from $C_{F, p_j \varphi, t} \geq C_{F, \frac{1}{2} \varphi, t} \geq C_2$. Therefore, inequality (2.4) holds with $p = p_j$. Taking $j \rightarrow \infty$, we can get

$$\frac{C_2}{C_1} \left(e^{\frac{t}{2}} - \left(\frac{C_2}{C_1} \right)^{-\frac{1}{2}} \right) - \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} (t + \log t) + \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}} - e^{-\frac{t}{2}} \leq 1.$$

3 Proof of Theorem 1.1

It suffices to prove the following proposition

Proposition 3.1 Let C_1, C_2 be two positive constants and $t > \log \frac{C_1}{C_2}$. If the set of (F, φ) satisfies

$$(1) \int_{D_t} |F|^2 e^{-\varphi} = C_1; \quad (2) C_{F, \varphi, t}(z_0) = C_2.$$

Then $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F, \varphi, t}(z_0)} \geq h^{-1}(t-3)$, where $h(x) = x + \log x$.

Note that

- (1) When $t > \log \frac{C_1}{C_2}$ and $C_{F, \varphi, t}(z_0) > 0$, Proposition 3.1 implies $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F, \varphi, t}(z_0)} \geq h^{-1}(t-3)$;
- (2) When $t \leq \log \frac{C_1}{C_2}$ and $C_{F, \varphi, t}(z_0) > 0$, it follows that $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F, \varphi, t}(z_0)} = \frac{C_1}{C_2} \geq e^t \geq h^{-1}(t-3)$,
- (3) When $C_{F, \varphi, t}(z_0) = 0$, set $\frac{\int_{D_t} |F|^2 e^{-\varphi} d\lambda_n}{C_{F, \varphi, t}(z_0)} = +\infty$,

then Proposition 3.1 implies Theorem 1.1.

Proof of Proposition 3.1 By Proposition 2.3 (case $p = 1$), we have

$$\left(\frac{C_2}{C_1}\right) \left(t + \log \frac{C_2}{C_1}\right) - 4 \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} \left(\left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} - e^{-\frac{t}{2}}\right) + \left(\left(\frac{C_2}{C_1}\right) - e^{-t}\right) \leq 1. \quad (3.1)$$

Since

$$\begin{aligned} & \left(\frac{C_2}{C_1}\right) \left(t + \log \frac{C_2}{C_1}\right) - 4 \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} \left(\left(\frac{C_2}{C_1}\right)^{\frac{1}{2}} - e^{-\frac{t}{2}}\right) + \left(\left(\frac{C_2}{C_1}\right) - e^{-t}\right) \\ & \geq \frac{C_2}{C_1} \left(t + \log \frac{C_2}{C_1}\right) - 3 \frac{C_2}{C_1}, \end{aligned} \quad (3.2)$$

combining (3.1) and (3.2), we get

$$\frac{C_2}{C_1} \left(t + \log \frac{C_2}{C_1}\right) - 3 \frac{C_2}{C_1} \leq 1, \quad (3.3)$$

hence

$$\frac{C_1}{C_2} + \log \frac{C_1}{C_2} \geq t - 3.$$

Set $h(x) = x + \log x$ for $x > 0$, it is easy that $h(x)$ is increasing. By (3.3), we get $\frac{C_1}{C_2} \geq h^{-1}(t-3)$. We may take $C_1 = \int_{D_t} |F|^2 e^{-\varphi}$, then the proposition follows. $\square$

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