Homogenization of ODE's in $\mathbb{R}^N$

Roberto Peirone¹

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Abstract

The homogenizability of ordinary equations with periodic data is investigated. Some results about the homogenizability have been previously given in Karakhanyan and Shahgholian (Ann Mat Pura Appl 196(6):2167–2183, 2017). In the present paper, the homogenizability is proved under much milder conditions than in [8].

Keywords Homogenization · Ordinary differential equations · Rotation number

Mathematics Subject Classification 74Q10 · 37E45 · 34C40

1 Introduction

In this paper, the convergence of solutions of ODE's with periodic data depending on a small parameter ε is investigated. The precise problem is the following. Let f be a function of class C^1 from $\mathbb{R}^N$ into itself, 1-periodic in all variables. This means that $f(x + v) = f(x)$ for every $x \in \mathbb{R}^N$ and every $v \in \mathbb{Z}^N$. Let us consider the following equation

$$y'(t) = f\left(\frac{y(t)}{\varepsilon}\right). \quad (E_{f;\varepsilon})$$

Let $T_{f;\varepsilon}^t(x)$ be the solution of the Cauchy problem

$$\begin{cases} y'(t) = f\left(\frac{y(t)}{\varepsilon}\right), \\ y(0) = x. \end{cases} \quad (P_{f;\varepsilon})$$

Here, we put for short

$$E_{f;1} =: E_f, \quad P_{f;1} =: P_f, \quad T_{f;1}^t =: T_f^t.$$

We say that the equation E_f is *homogenizable* if there exists the limit

$$\lim_{\varepsilon \rightarrow 0^+} T_{f;\varepsilon}^t(x) \in \mathbb{R}^N$$

✉ Roberto Peirone
peirone@axp.mat.uniroma2.it

¹ Dipartimento di Matematica, Università di Roma-Tor Vergata, Via della Ricerca Scientifica, 00133 Rome, Italy

for every $x \in \mathbb{R}^N$ and every $t \in \mathbb{R}$. As well known, the solution $T_{f;\varepsilon}^t(x)$ can be easily reduced to $T_f^t(x)$. Namely, we have

$$T_{f;\varepsilon}^t(x) = \varepsilon T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right). \quad (1.1)$$

In view of (1.1), we have a simple sufficient condition for the homogenizability. In fact, it is easy to see E_f is homogenizable if $\frac{T_f^t(x) - x}{t}$ uniformly converge to a constant $\bar{r} \in \mathbb{R}^N$ for $t \rightarrow +\infty$ (see Lemma 3.1). When $N = 1$, such a sufficient condition is always satisfied, and the constant $\bar{r} \in \mathbb{R}$ is called the *rotation number* of E_f . As a consequence, if $N = 1$ E_f is always homogenizable.

The question of when E_f is homogenizable has been asked by De Giorgi [3], p. 177. There, it was observed that if $N = 1$ E_f is homogenizable, but if $N > 1$ there are simple examples where E_f is not homogenizable.

The difficulty in the multidimensional case is related to the fact that in general there exist no strict analogs of the rotation number. In [9], some analogs of the rotation number in the multidimensional case are introduced, but with much weaker properties. In particular, the limit $\lim_{t \rightarrow +\infty} \frac{T_f^t(x) - x}{t}$ need not necessarily exist, and when exists could depend on x and is not necessarily uniform.

The problem of the convergence of $\frac{T_f^t(x) - x}{t}$ is strictly related to the convergence of the solutions of linear transport equations associated to E_f . Namely, we consider the following Cauchy problem

$$\frac{\partial u}{\partial t}(t, x) + f\left(\frac{x}{\varepsilon}\right) \cdot \frac{\partial u}{\partial x}(t, x) = 0, \quad u(0, x) = \psi(x) \quad (P_{f,\psi}^\varepsilon)$$

when $\varepsilon > 0$, and ψ is of class C^1 . Here, $\frac{\partial u}{\partial x}$ denotes the vector $(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N})$, and $\cdot$ denotes the scalar product in $\mathbb{R}^N$. Then, the solution of $P_{f,\psi}^\varepsilon$ is easily seen to be given by

$$u_{\varepsilon,f,\psi}(t, x) = \psi(T_{-f;\varepsilon}^t(x)).$$

Here, the convergence is meant to be in a weak sense. More precisely, let $\tilde{E}_f$ be the equation in $P_{f,\psi}^\varepsilon$ for $\varepsilon = 1$. We say that $\tilde{E}_f$ is homogenizable if the solutions $u_{\varepsilon,f,\psi}$ of $P_{f,\psi}^\varepsilon$ are convergent in the weak* topology of L_{loc}^∞ for every ψ of class C^1 . There is a rather extensive literature about the homogenizability of $\tilde{E}_f$ and the form of the limit of the solutions. See for example [1,4,6,10,11,14].

The relationship between the homogenizability of E_f and that of $\tilde{E}_f$ is the following. On the one hand, we have seen that the homogenizability of E_f can be proved as a consequence of the *uniform* convergence of $\frac{T_f^t(x) - x}{t}$ to a constant. On the other, in view of the form of the solution $u_{\varepsilon,f,\psi}$ and of (1.1), it can be proved that a sufficient condition for the homogenizability of $\tilde{E}_f$ is that $\frac{T_g^t(x) - x}{t}$ converge *almost everywhere* for both $g = f$ and for $g = -f$ (see [1,10]).

In particular, this is the case when f is divergence-free as a consequence of Birkhoff's ergodic Theorem (see [1] and also [10]). Another case where $\tilde{E}_f$ is homogenizable is when $N = 2$ and f is non-vanishing. In this case we have, more strongly, the pointwise convergence of $\frac{T_f^t(x) - x}{t}$ (Theorem 3.1 in [11], where arguments of [7] are used).

In view of the previous considerations, in [3], the problems of the homogenizability of E_f and of $\tilde{E}_f$ are considered related. In particular, in [3], it was conjectured that $\tilde{E}_f$ is always homogenizable. However, in [11], Sect. 4, an example is given of a non-homogenizable $\tilde{E}_f$ with $N = 3$. Moreover, in [12], a more complicated example is given of a non-homogenizable $\tilde{E}_f$ with $N = 2$. In the latter example, of course, f attains the value 0 somewhere.

Concerning the problem in the present paper, it seems that the only paper where the homogenizability of E_f has been investigated is [8]. In fact, in [8], some sufficient conditions for the homogenizability of E_f have been given.

In this paper, I prove that such conditions can be considerably relaxed. In particular, in Sect. 3, I prove that a sufficient condition for homogenizability is that f is a (not necessarily constant) multiple of a vector whose components are linearly independent over $\mathbb{Q}$. Note that in [8], the homogenizability of E_f was proved but under much stronger assumptions. However, the condition considered in the present paper makes the function f one-dimensional. Such a condition, although much weaker than the conditions in [8], is nevertheless relatively strong.

On the other hand, in general, it appears to be hard to obtain the homogenizability under mild conditions. However, in Sect. 4, I prove that when $N = 2$ the homogenizability takes place under much milder assumptions, imitating the arguments in [11], Sect. 3 for the homogenizability of $\tilde{E}_f$. Also, in [8], there is a specific result when $N = 2$, but under much stronger assumptions than in the present paper. In [8], some results about the speed of convergence are also discussed.

2 Notation

In this section, I describe the notation used in this paper, in some cases recalling definitions already given in Introduction. I denote by $C_N^1(\mathbb{T})$ the set of functions f of class C^1 from $\mathbb{R}^N$ to itself which are $\mathbb{Z}^N$ -periodic, that is such that $f(x + v) = f(x)$ for every $x \in \mathbb{R}^N$ and every $v \in \mathbb{Z}^N$. If $f \in C_N^1(\mathbb{T})$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, I denote by $T_f^t(x)$ the value attained at t by the solution of

$$\begin{cases} y'(t) = f(y(t)) \\ y(0) = x. \end{cases}$$

In particular, $T_f^0(x) = x$. The following three properties can be easily verified and will be used in the sequel without explicit mention. We mean that the properties hold for every $t, t_1, t_2 \in \mathbb{R}$.

$T_f^t : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a homeomorphism of the form $Id + \mathbb{Z}^N$ -periodic,

$$T_f^{t_1+t_2} = T_f^{t_1} \circ T_f^{t_2}, \quad (2.1)$$

$$\|T_f^{t_1}(x) - T_f^{t_2}(x)\| \leq |t_1 - t_2| \max \|f\|. \quad (2.2)$$

More generally, I denote by $T_{f;\varepsilon}^t(x)$ the value attained at t by the solution of

$$\begin{cases} y'(t) = f\left(\frac{y(t)}{\varepsilon}\right) \\ y(0) = x. \end{cases}$$

In this paper, we are interested in whether E_f is homogenizable. Recall that we say that the equation E_f is *homogenizable* if

$$\exists \lim_{\varepsilon \rightarrow 0^+} T_{f,\varepsilon}^t(x) \in \mathbb{R}^N \quad \forall x \in \mathbb{R}^N \quad \forall t \in \mathbb{R}.$$

Here, we denote by E_f the equation $y'(t) = f(y(t))$.

Next, I recall the notion of *rotation number* of a map $\phi : \mathbb{R} \rightarrow \mathbb{R}$ continuous, strictly increasing and such that $\phi - Id$ is 1-periodic. Denote the set of such maps by $D(\mathbb{T})$. For such type of maps, there exists

$$\lim_{n \rightarrow +\infty} \frac{\phi^n(x) - x}{n} =: \rho(\phi).$$

Such a limit $\rho(\phi)$ exists and does not depend on $x \in \mathbb{R}$ and is called *rotation number* of ϕ . Here, ϕ^n denotes the n -th iterate of ϕ . Note that in Introduction we have defined the rotation number of the equation E_f . As expected, there is a relationship between the two notions, that is, the rotation number of E_f amounts to the rotation number of the Poincaré map of f . We will use the following well-known properties of the rotation number of a map.

Lemma 2.1 *If $\phi \in D(\mathbb{T})$ then*

- (i) $\rho(\phi) = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q > 0$ if and only if ϕ has a q -periodic point, more precisely there exists $x \in \mathbb{R}$ such that $\phi^q(x) = x + p$.
- (ii) For every positive integer n and every integer m , we have

$$\phi^n(x + m) = \phi^n(x) + m.$$

- (iii) For every positive integer n and every $x \in \mathbb{R}$ we have

$$|\phi^n(x) - x - n\rho(\phi)| < 1.$$

Proof (i) See for example, Prop. 11.1.4 in [7]. (ii) Since $\phi^n \in D(\mathbb{T})$ as well, it suffices to prove the statement for $n = 1$. Since the fact that $\phi - Id$ is 1-periodic immediately implies that $\phi(x + 1) = \phi(x) + 1$, we easily obtain $\phi(x + m) = \phi(x) + m$ for every integer m . (iii) By contradiction, if for example $\phi^n(x) \geq x + n\rho(\phi) + 1$ for some $x \in \mathbb{R}$, let $m \in \mathbb{Z}$ be such that

$$n\rho(\phi) < m \leq n\rho(\phi) + 1. \quad (2.3)$$

Then, $\phi^n(x) \geq x + m$, thus, using (ii) and the assumption that ϕ is increasing we obtain

$$\phi^{2n}(x) = \phi^n(\phi^n(x)) \geq \phi^n(x + m) = \phi^n(x) + m \geq x + 2m.$$

By an analogous inductive argument, we obtain that $\phi^{sn}(x) \geq x + sm$ for every positive integer s , and we easily deduce that $\rho(\phi) \geq \frac{m}{n}$, but this contradicts (2.3). $\square$

3 Homogenizability for general N

In order to investigate the homogenizability of E_f , recall that, as seen in Introduction,

$$T_{f,\varepsilon}^t(x) = \varepsilon T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right). \quad (3.1)$$

The following is a simple criterion for homogenizability.

Lemma 3.1 Suppose

$$\exists \bar{r} \in \mathbb{R}^N : \frac{T_f^t(x) - x}{t} \xrightarrow[t \rightarrow +\infty]{} \bar{r} \text{ uniformly in } x \in \mathbb{R}^N. \quad (3.2)$$

Then E_f is homogenizable. More precisely, we have

$$T_{f,\varepsilon}^t(x) \xrightarrow[\varepsilon \rightarrow 0^+]{} x + \bar{r}t \quad \forall x \in \mathbb{R}^N \quad \forall t \in \mathbb{R}. \quad (3.3)$$

Proof By the hypothesis, for any given $\eta > 0$, we have

$$\exists \bar{t} : \left| \frac{T_f^t(x) - x}{t} - \bar{r} \right| < \eta \quad \forall t \geq \bar{t} \quad \forall x \in \mathbb{R}^N.$$

In particular, for every $t \in \mathbb{R}$, every $\varepsilon > 0$ and every $x \in \mathbb{R}^N$ such that $\frac{t}{\varepsilon} \geq \bar{t}$, we have

$$\left| \frac{T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right) - \frac{x}{\varepsilon}}{\frac{t}{\varepsilon}} - \bar{r} \right| < \eta,$$

thus,

$$\left| \varepsilon T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right) - (x + \bar{r}t) \right| < \eta |t|.$$

In view of (3.1), (3.3) easily follows for $t > 0$. If $t < 0$, the argument is analogous, noting that

$$\frac{T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right) - \frac{x}{\varepsilon}}{\frac{t}{\varepsilon}} = \frac{T_f^{\frac{-t}{\varepsilon}}\left(T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right)\right) - T_f^{\frac{t}{\varepsilon}}\left(\frac{x}{\varepsilon}\right)}{\frac{-t}{\varepsilon}}$$

and for $t = 0$ (3.3) is trivial. $\square$

Often, it is simpler to verify the uniform convergence as in (3.2) not for $t \rightarrow +\infty$, but along an increasing sequence tending to $+\infty$. So, we are lead to give the following definition. We say that a sequence of non-negative functions $\beta_n : \mathbb{R}^N \rightarrow \mathbb{R}$ is *regularly increasing* if the following conditions (i) and (ii) hold.

(i) β_0 is bounded, that is there exists c_0 such that

$$0 \leq \beta_0(x) \leq c_0 \quad \forall x \in \mathbb{R}^N. \quad (3.4)$$

(ii) There exist c_1, c_2 with $0 < c_1 < c_2$ such that

$$c_1 \leq \beta_{n+1}(x) - \beta_n(x) \leq c_2. \quad (3.5)$$

Note that it follows from (3.4) and (3.5) that

$$\beta_n \xrightarrow[n \rightarrow +\infty]{} +\infty \text{ uniformly on } \mathbb{R}^N. \quad (3.6)$$

Note that we do not require the continuity of β_n . We have

Lemma 3.2 Suppose β_n is regularly increasing and $\bar{r} \in \mathbb{R}^N$. Then the following are equivalent.

$$(a) \quad \frac{T_f^t(x) - x}{t} \xrightarrow[t \rightarrow +\infty]{} \bar{r} \text{ uniformly in } x \in \mathbb{R}^N,$$

- (b) $\frac{T_f^{\beta_n(x)}(x) - x}{\beta_n(x)} \xrightarrow{n \rightarrow +\infty} \bar{r}$ uniformly in $x \in \mathbb{R}^N$,
- (c) $\frac{T_f^{\beta_n(x)}(x) - T_f^{\beta_0(x)}(x)}{\beta_n(x) - \beta_0(x)} \xrightarrow{n \rightarrow +\infty} \bar{r}$ uniformly in $x \in \mathbb{R}^N$.

Proof In view of (3.6) (a) $\Rightarrow$ (b) is trivial. Next, by (3.4), (3.6) again and (2.2) we have

$$\frac{T_f^{\beta_0(x)}(x) - x}{\beta_n(x)} \xrightarrow{n \rightarrow +\infty} 0 \text{ uniformly in } x \in \mathbb{R}^N.$$

Thus, by (3.4) and (3.6), we easily deduce (b) $\Longleftrightarrow$ (c). Thus, it suffices to prove that (b) implies (a). Suppose (b) holds, thus for every given $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\left| \frac{T_f^{\beta_n(x)}(x) - x}{\beta_n(x)} - \bar{r} \right| < \varepsilon \quad \forall n \geq \bar{n} \quad \forall x \in \mathbb{R}^N. \quad (3.7)$$

Let $t > c_0$. Thus, for every $x \in \mathbb{R}^N$ we have $t > \beta_0(x)$, hence there exists $n = n(t, x)$ such that

$$\beta_n(x) \leq t < \beta_{n+1}(x). \quad (3.8)$$

Since, by (3.4) and (3.5) we have $\beta_n(x) \leq nc_2 + c_0$, we deduce

$$n(t, x) \xrightarrow{t \rightarrow +\infty} +\infty \text{ uniformly in } x \in \mathbb{R}^N. \quad (3.9)$$

Moreover, by the definition of a regularly increasing sequence, we have $0 \leq t - \beta_n(x) \leq \beta_{n+1}(x) - \beta_n(x) \leq c_2$, thus by (2.2) we have

$$\left\| T_f^t(x) - T_f^{\beta_{n(t,x)}(x)}(x) \right\| \leq c_2 \max \|f\|. \quad (3.10)$$

Also, by (3.5) and (3.8) we have $\frac{t}{\beta_{n(t,x)}} \xrightarrow{t \rightarrow +\infty} 1$ uniformly in $x \in \mathbb{R}^N$. Thus, (a) easily follows from (3.7), (3.9) and (3.10). $\square$

We will now prove the main theorem in this section. In order to prove the homogenizability of E_f for arbitrary N , we need a rather strong condition, in particular, f has to be a (non-constant) multiple of some vector. Namely, we say that $f \in C^1(\mathbb{T})$ is *ergodically multiple* if the following holds.

There exists $v \in \mathbb{R}^N$ such that the numbers $v_1, v_2, \dots, v_N$, are rationally independent, and moreover for every $x \in \mathbb{R}^N$ there exists a positive real number $w(x) > 0$ such that $f(x) = w(x)v$.

Here, rationally independent means linearly independent over $\mathbb{Q}$. Note that in particular, $v_i \neq 0$ for every $i = 1, \dots, N$. Moreover, we can and do assume that also $v_1, v_2, \dots, v_N, 1$ are rationally independent. In fact, it is easy to see that if $v_1, v_2, \dots, v_N$ are rationally independent and $\bar{s} > 0$ is such that $\frac{1}{\bar{s}}$ does not belong to the linear space over $\mathbb{Q}$ generated by $\{v_1, v_2, \dots, v_N\}$, then $\bar{s}v_1, \bar{s}v_2, \dots, \bar{s}v_N, 1$ are rationally independent. Of course the function w described above is C^1 and $\mathbb{Z}^N$ -periodic. The notation *ergodically multiple* is suggested by the fact that a translation by a vector v with $v_1, v_2, \dots, v_N, 1$ rationally independent is ergodic (see, e.g., Proposition 4.2.2 in [7]). We will now prove that then E_f is P -homogenizable.

Lemma 3.3 *Let $v \in \mathbb{R}^N$ be such that the numbers $v_1, v_2, \dots, v_N, 1$ are rationally independent, and let θ be a continuous and $\mathbb{Z}^N$ -periodic function from $\mathbb{R}^N$ into $\mathbb{R}$. Then the time average*

$$\frac{1}{n} \sum_{k=0}^{n-1} \theta(x + kv)$$

tends, for $n \rightarrow +\infty$, to a constant vector in $\mathbb{R}^N$ uniformly in $x \in \mathbb{R}^N$.

Proof Since we can interpret θ as a continuous function from the N -dimensional torus into $\mathbb{R}$, this immediately follows from Theorem 2 in Section 1.8 and Theorem 2 in Section 3.1 of [2]. $\square$

Theorem 3.4 *If f is ergodically multiple, then E_f is homogenizable.*

Proof Note that the function $w : \mathbb{R}^N \rightarrow \mathbb{R}$, mentioned in the definition of an ergodically multiple function, is continuous and $\mathbb{Z}^N$ -periodic, thus it has a maximum and a minimum. Thus, for every $x \in \mathbb{R}^N$ there exists a unique $\theta(x) > 0$ such that

$$T_f^{\theta(x)}(x) = x + v.$$

and, of course, θ is continuous and $\mathbb{Z}$ -periodic. Let

$$\begin{aligned} \tilde{S}_n(x) &:= \sum_{k=0}^{n-1} \theta(x + kv), \\ \tilde{A}_n(x) &:= \frac{1}{n} \tilde{S}_n(x). \end{aligned}$$

By Lemma 3.3 we have

$$\tilde{A}_n \xrightarrow{n \rightarrow +\infty} c \text{ uniformly on } \mathbb{R}^N$$

where $c \in \mathbb{R}$ is a suitable constant. On the other hand, we have

$$T_f^{\tilde{S}_n(x)}(x) = x + nv,$$

for every positive integer n , thus,

$$\frac{T_f^{\tilde{S}_n(x)}(x) - x}{\tilde{S}_n(x)} \xrightarrow{n \rightarrow +\infty} \frac{v}{c}$$

and the limit is uniform in $x \in \mathbb{R}^N$. On the other hand, since θ is positive and continuous, it has positive minimum and maximum, thus by definition, $\tilde{S}_n$ is regularly increasing. We now conclude using Lemmas 3.1 and 3.2. $\square$

4 Homogenizability in the two-dimensional case

In this section, we will prove that when $N = 2$ we have the homogenizability under much milder conditions than for generic N . We say that a function $v : \mathbb{R} \rightarrow \mathbb{R}^N$ is $\mathbb{T}$ -periodic if its lift on the N -torus is periodic in other words if there exists $T > 0$ such that $v(t + T) - v(t) \in \mathbb{Z}^N$ for every $t \in \mathbb{R}$. We suppose $N = 2$, $f(x) \neq 0$ for every $x \in \mathbb{R}^2$, and E_f has no $\mathbb{T}$ -periodic solutions.

The basic tool for the proof is a curve transverse to f introduced by Siegel in [13]. That is, there exists a C^1 function $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ (Siegel's curve) which is one-to-one, *transverse to f* , in the sense that $\gamma'(t)$ is not a multiple of $f(\gamma(t))$ for all $t \in \mathbb{R}$, and $\mathbb{T}$ -periodic. In other words

$$\gamma(t+1) = \gamma(t) + \bar{v} \quad (4.1)$$

for all $t \in \mathbb{R}$ and for some $\bar{v} \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. We can also assume that conversely $\gamma(t_1) - \gamma(t_2) \in \mathbb{Z}^2$ if and only if $t_1 - t_2 \in \mathbb{Z}$. Note that, in this situation, then $\gamma(t) \xrightarrow[t \rightarrow \pm\infty]{} \infty$, so that $\gamma(\mathbb{R}) \cup \{\infty\}$ can be seen as a Jordan curve in S^2 (identified to $\mathbb{R}^2 \cup \{\infty\}$).

Therefore, we have two standard results (cf. for example [5], Theorem 4.6 and its proof and Lemma 4.4). The first is that $\gamma(\mathbb{R})$ separates $\mathbb{R}^2$; this means that $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$ has precisely two components.

The second result is that given a curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ of class C^1 , that for simplicity we assume to be one-to-one and that touches γ transversely at $\bar{t}$, then α cuts γ at $\bar{t}$.

Here, we say that α touches γ transversely at $\bar{t}$ if there exists $\bar{\tau} \in \mathbb{R}$ such that $\alpha(\bar{t}) = \gamma(\bar{\tau})$ and $\alpha'(\bar{t})$ is not a multiple of $\gamma'(\bar{\tau})$. We say that α cuts γ at $\bar{t}$ if there exists $\varepsilon > 0$ such that $\alpha(t)$ lies in one component C of $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$ for $t \in]\bar{t}, \bar{t} + \varepsilon[$, and in the other for $t \in]\bar{t} - \varepsilon, \bar{t}[$. Here, we will say that α enters in C at $\gamma(\bar{\tau})$.

Theorem 4.1 *If $N = 2$, $f(x) \neq 0$ for every $x \in \mathbb{R}^2$ and E_f has no $\mathbb{T}$ -periodic solution, then E_f is homogenizable.*

Proof We will prove that Lemma 3.1 holds imitating the proof of Second Case of Theorem 3.1 in [11], which in turns was based on arguments in [7]. In [11], the proof was merely sketched. Here, since on the one hand the statement is different and on the other, the hypothesis is a bit different from that in [7], for clarity I will give more details than in [11]. I will divide the proof into several steps.

Step 1 Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ be a Siegel curve. Let us denote by C_1 and C_2 the two components of $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$. We will say that E_f enters in C_1 (resp. in C_2) at $\gamma(\tau)$ if the curve $t \mapsto T_f^t(\gamma(\tau))$ enters in C_1 (resp. in C_2) at $\gamma(\tau)$. By connectedness, there exists $i = 1, 2$ such that E_f enters in C_i at $\gamma(\tau)$ for every $\tau \in \mathbb{R}$ and we can suppose $i = 1$. Thus, for every $x \in \mathbb{R}^2$ there exists at most one $t \in \mathbb{R}$ such that $T_f^t(x) \in \gamma(\mathbb{R})$, and the same holds if we replace γ with $\gamma + v$ with $v \in \mathbb{Z}^2$. It easily follows that, if $\tau \in \mathbb{R}$, we have then $T_f^t(\gamma(\tau)) \in C_1$ for every $t > 0$.

Step 2 Since $\bar{v} \neq (0, 0)$, then either $\bar{v}_1 \neq 0$ or $\bar{v}_2 \neq 0$. We can and do assume $\bar{v}_1 > 0$. Let $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as

$$v(x) = x_2 - \frac{\bar{v}_2}{\bar{v}_1} x_1.$$

Note that

$$v(x + (0, a)) = v(x) + a \quad \forall x \in \mathbb{R}^2, \forall a \in \mathbb{R}. \quad (4.2)$$

In view of (4.1), the function $v \circ \gamma$ is 1-periodic, and clearly, is also continuous. Thus, it is bounded. It follows that there exists $\bar{a} > 0$ such that if $|a| > \bar{a}$, then the curves $\gamma_a := \gamma + (0, a)$ do not intersect γ . Thus, for every $a > \bar{a}$ the curves γ_a lie in $C_{i(+)}$, where $i(+)$ amounts to 1 or to 2. Recall that C_1 and C_2 are the components of $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$. Of course the component where γ_a lies does not depend on $a > \bar{a}$. Similarly, for every $a < -\bar{a}$ the curves γ_a lie in $C_{i(-)}$, where $i(-)$ amounts to 1 or to 2.

Step 3 Of course, $i(+)$ and $i(-)$ do not necessarily coincide. We will now prove that in fact, $i(+)$ $\neq$ $i(-)$. Note that the C^1 map $\gamma_1 : \mathbb{R} \rightarrow \mathbb{R}$ contains some (in fact infinitely

many) non-critical values by the Sard Lemma. Let $\tilde{x}_1$ be one of these non-critical values. Thus, if for some $t \in \mathbb{R}$ we have $\gamma_1(t) = \tilde{x}_1$, then $\gamma'_1(t) \neq 0$, therefore the line r defined by $x = \tilde{x}_1$ is transverse to γ . Since by (4.1) and the fact that $\bar{v}_1 > 0$ we have $\gamma_1(t) \xrightarrow[t \rightarrow +\infty]{} +\infty$, $\gamma_1(t) \xrightarrow[t \rightarrow -\infty]{} -\infty$ then γ intersects r an odd number of times. Moreover, by the transversality of r to γ , every time that r crosses γ , it changes component passing from C_1 to C_2 or conversely. In sum, if t_1 is such that $\gamma_1(t_1) = \tilde{x}_1$, the points of the form $\gamma(t_1) + (0, a)$ and $\gamma(t_1) - (0, a)$, for large a lie in different components. This proves the assertion.

Step 4 Suppose for example $i(+) = 1$, the case $i(+) = 2$ being completely analogous. By definition we have $\gamma(t_1) + (0, a) \in C_2$ for $a < -\bar{a}$. Moreover, $v(\gamma(t_1) + (0, a)) \xrightarrow[a \rightarrow -\infty]{} -\infty$ by (4.2) and $v \circ \gamma$ is bounded. As a consequence, there exists $a' \in \mathbb{R}$ such that $x \in C_2$ if $v(x) < a'$. Similarly, there exists $a'' \in \mathbb{R}$ such that $x \in C_1$ if $v(x) > a''$.

Step 5 Let $x \in \mathbb{R}^2$. Since $v(x + (0, m)) = v(x) + m > a''$ for suitable $m \in \mathbb{Z}$, for such m we have $x + (0, m) \in C_1$ by step 4. Hence, $T_f^t(x) + (0, m) = T_f^t(x + (0, m)) \notin C_2$ for every $t > 0$ by step 1. Therefore, $v(T_f^t(x)) + m = v(T_f^t(x) + (0, m)) \geq a'$. In other words, the set of values $\tilde{G}_x = \{v(T_f^t(x)), t > 0\}$ with $t > 0$ is bounded from below.

On the other hand, $\tilde{G}_x$ cannot be bounded, since every solution $\bar{\gamma}$ of E_f cannot lie in a strip of the form $v^{-1}([a, b])$. In fact, if this would occur, by an argument similar to that used in [11] Section 3, using the Poincaré–Bendixon Theorem, we would deduce that E_f has a $\mathbb{T}$ -periodic solution, contrary to our assumption.

We have thus proved that $\tilde{G}_x$ is unbounded from above. Next, let $\mu_2 \in \mathbb{Z}$ be such that $\mu_2 > v(x) - a'$, and let $\mu = (0, \mu_2)$. We have $v(x - \mu) = v(x) - \mu_2 < a'$, thus $T_f^0(x - \mu) = x - \mu \in C_2$.

On the other hand, since $\tilde{G}_x$ is unbounded from above, we have

$$v(T_f^t(x - \mu)) = v(T_f^t(x)) - \mu_2 > a'',$$

hence $T_f^t(x - \mu) \in C_1$, for some $t > 0$. Hence, there exists $t > 0$ such that $T_f^t(x - \mu) \in \gamma(\mathbb{R})$, in other words, $T_f^t(x)$ has the form $\gamma(\tilde{\tau}) + \mu$, $\tilde{\tau} \in \mathbb{R}$.

In particular, if $x = \gamma(\tau)$, we can choose μ independent of τ as $v \circ \gamma$ is bounded. Thus, for every $\tau \in \mathbb{R}$ there exist (unique) $\psi(\tau) > 0$ and $\phi(\tau) \in \mathbb{R}$ such that

$$T_f^{\psi(\tau)}(\gamma(\tau)) = \gamma(\phi(\tau)) + \mu. \quad (4.3)$$

Step 6 We have

$$\begin{aligned} T_f^{\psi(\tau)}(\gamma(\tau + 1)) &= T_f^{\psi(\tau)}(\gamma(\tau) + \bar{v}) \\ &= T_f^{\psi(\tau)}(\gamma(\tau)) + \bar{v} \\ &= \gamma(\phi(\tau)) + \mu + \bar{v} \\ &= \gamma(\phi(\tau) + 1) + \mu. \end{aligned}$$

Thus, we have

$$\psi(\tau + 1) = \psi(\tau), \quad (4.4)$$

$$\phi(\tau + 1) = \phi(\tau) + 1. \quad (4.5)$$

Of course, ϕ and ψ are continuous by the transversality of γ to f . Moreover, ϕ is one-to-one. In fact, if $\tau_1 \neq \tau_2$ and $\phi(\tau_1) = \phi(\tau_2) := \bar{\phi}$, then the solution of E_f passing through $\bar{\phi}$ would

touch γ at two different points, namely $\gamma(\tau_1)$ and $\gamma(\tau_2)$, but this is not possible (see step 1). In view of (4.5), ϕ is continuous, strictly increasing and of the form $\text{Id} + \mathbb{Z}$ -periodic. On the other hand, by (4.4) $\psi : \mathbb{R} \rightarrow]0, +\infty[$ is continuous and $\mathbb{Z}$ -periodic.

Step 7 Now, putting $\Sigma_n(\tau) = \sum_{i=0}^{n-1} \psi(\phi^i(\tau))$ for $n = 0, 1, 2, \dots$, where of course we mean that $\Sigma_0(\tau) = 0$, from (4.3) we easily get

$$T_f^{\Sigma_n(\tau)}(\gamma(\tau)) = \gamma(\phi^n(\tau)) + n\mu. \quad (4.6)$$

Note that the rotation number $\rho(\phi)$ of ϕ is irrational. In fact, in the opposite case, by Lemma 2.1 i) there would exist $\tau \in \mathbb{R}$ and integers n and p with $n > 0$ such that $\phi^n(\tau) = \tau + p$, thus, by (4.6) and (4.1) there would exist a $\mathbb{T}$ -periodic solution of E_f , contrary to our assumption.

Step 8 Hence, using Theorem 11.2.9 and Prop. 4.1.13 of [7] (or Theorem 2 in Section 1.8 of [2]), there exists $c \in \mathbb{R}$ such that the sequence $\frac{1}{n}\Sigma_n$ is uniformly convergent on $[0, 1]$, thus by periodicity on $\mathbb{R}$, for $n \rightarrow +\infty$, to a constant $c \in \mathbb{R}$. Note that since ψ is continuous, positive and periodic, thus greater than a positive constant, then we have $c > 0$.

Step 9 Moreover, there exists $A > 0$ such that, given $x \in \mathbb{R}^2$ there exist $t_x > 0$, $\tau_x \in \mathbb{R}$ and $\mu_x \in \mathbb{Z}^2$ such that

$$T_f^{t_x}(x) = \gamma(\tau_x) + \mu_x, \quad (4.7)$$

$$0 \leq t_x \leq A. \quad (4.8)$$

Note that (4.7) follows from step 5.

Next, we prove that we can choose $t_x \in \mathbb{R}$, $\tau_x \in \mathbb{R}$ and $\mu_x \in \mathbb{Z}^2$ so that for some $A > 0$ (4.8) holds. The argument in step 5 shows that we can take μ_x constant on a fixed compact subset K of $\mathbb{R}^2$, and we choose $K := [0, 1] \times [0, 1]$. Moreover, τ_x and t_x continuously depend on $x \in K$ by the transversality of f to γ . As a consequence, t_x is bounded on K . By the definition of K , we can express every $x \in \mathbb{R}^2$ as $x = y + v$, with $y \in K$, $v \in \mathbb{Z}^2$. Thus,

$$T_f^{t_y}(x) = T_f^{t_y}(y) + v = \gamma(\tau_y) + (v + \mu_y)$$

so that (4.7) holds with $t_x = t_y$, $\tau_x = \tau_y$, $\mu_x = v + \mu_y$. In this way the functions t_x and τ_x could be discontinuous in x , but this is not a problem as we have not required the continuity of such functions. In any case, t_x is bounded and (4.8) holds.

Step 10 For every $n = 0, 1, 2, \dots$, let now

$$\beta_n(x) = t_x + \Sigma_n(\tau_x).$$

By (4.8) and since ψ is continuous and periodic, thus bounded, the sequence β_n is regularly increasing. In view of Lemma 3.2, in order to prove (3.2) and thus the Theorem, it suffices to prove that (c) in Lemma 3.1 holds for some $\alpha \in \mathbb{R}^N$. Since by (4.7) we have

$$T_f^{\beta_n(x)}(x) = T_f^{\Sigma_n(\tau_x)}(\gamma(\tau_x)) + \mu_x$$

it suffices to prove that for some $\bar{r} \in \mathbb{R}^N$, we have

$$\frac{T_f^{\Sigma_n(\tau)}(\gamma(\tau)) - \gamma(\tau)}{\Sigma_n(\tau)} \longrightarrow \bar{r} \text{ uniformly in } \tau \in \mathbb{R}. \quad (4.9)$$

By Lemma 2.1 (iii) for every $\tau \in \mathbb{R}$, we have

$$\tau + [n\rho(\phi)] - 1 \leq \tau + n\rho(\phi) - 1 \leq \phi^n(\tau) \leq \tau + n\rho(\phi) + 1 \leq \tau + [n\rho(\phi)] + 2.$$

Thus, since γ is Lipschitz, for suitable $c_3 > 0$ we have

$$|\gamma(\phi^n(\tau)) - \gamma(\tau + [n\rho(\phi)])| \leq c_3 |\phi^n(\tau) - (\tau + n\rho(\phi))| \leq 2c_3.$$

Now, by (4.1) we have $\gamma(\tau + [n\rho(\phi)]) = \gamma(\tau) + [n\rho(\phi)]\bar{\nu}$, thus by (4.6) and Step 8, we easily obtain (4.9) with $\bar{r} = \frac{1}{c}(\rho(\phi)\bar{\nu} + \mu)$ and the Theorem is proved. $\square$

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