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A nonlinear elliptic eigenvalue–transmission problem with Neumann boundary condition

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Abstract
Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain which is divided into two sub-domains Ω_1 and Ω_2. Consider in Ω an eigenvalue–transmission problem associated with the p-Laplacian acting in Ω_1 and the q-Laplacian acting in Ω_2, $1 < p < q$, with Dirichlet–Neumann conditions on the interface separating the two sub-domains Ω_1 and Ω_2 [see (1.1)]. The main result Theorem 2.1 states the existence of a sequence of eigenvalues for this eigenvalue problem. The proof is based on the Lusternik–Schnirelmann principle. Using the method of Lagrange multipliers for constrained minimization problems, we show (see Theorem 2.2) that if $2 \leq p < q$ then there exists an eigenfunction in any set of the form

$$\left\{u \in W^{1,p}(\Omega);\ u|_{\Omega_2} \in W^{1,q}(\Omega_2),\ \frac{1}{p}\int_{\Omega_1} |u|^p + \frac{1}{q}\int_{\Omega_2} |u|^q = \alpha\right\}, \quad \alpha > 0.$$

The case of Robin conditions on $\partial\Omega$ and the Riemannian setting are also addressed.

Keywords Eigenvalues · Transmission problem · Neumann boundary condition · Sobolev space · Lusternik–Schnirelmann principle · Lagrange multipliers · Robin boundary condition · Riemannian setting

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1 Introduction

Consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, which is divided into two Lipschitz sub-domains Ω_1 and Ω_2 by a Lipschitz closed hypersurface H. We further assume that $H \cap \partial\Omega$ is an $(N-2)$-dimensional manifold. In the differentiable category this is the case whenever H and $\partial\Omega$ intersect transversally. In other words, $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Gamma = H \cap \Omega$. The standard example we have in mind is the disc D^N divided by some coordinate hyperplane in two open components, i.e. the two open semidiscs. Deformations of this divided disc are a good enough source of further examples. The boundary of Ω is assumed smooth enough and is divided into two pieces $\partial\Omega_1$ and $\partial\Omega_2$ in such a way that $\partial\Omega_1$ is the union $\Gamma_1 \cup \Gamma$ and $\partial\Omega_2$ is the union $\Gamma_2 \cup \Gamma$. To this picture we consider the eigenvalue problem

$$\begin{cases} -\Delta_p u_1 = \lambda \mid u_1 \mid^{p-2} u_1 \quad \text{in } \Omega_1, & (1)\\ -\Delta_q u_2 = \lambda \mid u_2 \mid^{q-2} u_2 \quad \text{in } \Omega_2, & (2)\\ \dfrac{\partial u_1}{\partial \nu_p} = 0 \ \text{ on } \ \Gamma_1, \ \dfrac{\partial u_2}{\partial \nu_q} = 0 \ \text{ on } \ \Gamma_2, & (3)\\ u_1 = u_2, \ \ \dfrac{\partial u_1}{\partial \nu_p} = \dfrac{\partial u_2}{\partial \nu_q} \ \text{ on } \Gamma, & (4) \end{cases} \tag{1.1}$$

where Δ_r stands for the r-Laplace operator, namely $\Delta_r w := \operatorname{div}\big(\mid \nabla w \mid^{r-2} \nabla w\big)$ and $\dfrac{\partial}{\partial \nu_r}$ denotes the boundary operator defined by

$$\frac{\partial w}{\partial \nu_r} :=\mid \nabla w \mid^{r-2} \frac{\partial w}{\partial \nu} \quad \text{for } r = p, q, \ 1 < p < q.$$

The solution $u = (u_1, u_2)$ of the problem (1.1) is understood in a weak sense, i.e. u is an element of the space

$$W := \big\{u \in W^{1,p}(\Omega) : u|_{\Omega_2} \in W^{1,q}(\Omega_2)\big\}, \tag{1.2}$$

where $u_i = u|_{\Omega_i}$ satisfies the nonlinear problem $(1.1)_i$ on Ω_i in the sense of distributions, $i = 1, 2$, and u_1, u_2 satisfy the boundary and transmission conditions $(1.1)_{3,4}$ in the sense of traces. Recall that, for any domain $\hat{\Omega} \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\hat{\Omega}$, the trace operator

$$Tr^{\hat{\Omega}} : W^{1,p}(\hat{\Omega}) \to W^{1-1/p,p}(\partial\hat{\Omega})$$

is a linear and bounded operator, for $1 \leq p < \infty$ (see Gagliardo [8]). For linear transmission problems, involving the Laplace operator or some perturbed Stokes operators, treated by using the layer potential technique, we refer the reader to [5,9], respectively.

Definition 1.1 A scalar $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of the problem (1.1) whenever (1.1) admits a nontrivial solution $u = (u_1, u_2) \in W$. In that case $u = (u_1, u_2)$ is called an *eigenfunction/eigencouple* of the problem (1.1) (which corresponds to the eigenvalue λ) and the pair (u, λ) an *eigenpair* of the problem (1.1). Note that $W^{1,q}(\Omega)$ is a subspace of W, as $W^{1,q}(\Omega)$ is a subspace of $W^{1,p}(\Omega)$.

We endow W with the norm

$$\| u \|_W:=\| u|_{\Omega_1} \|_{W^{1,p}(\Omega_1)} + \| u|_{\Omega_2} \|_{W^{1,q}(\Omega_2)}, \quad \forall\, u \in W,$$

where $\| \cdot \|_{W^{1,p}(\Omega_1)}$ and $\| \cdot \|_{W^{1,q}(\Omega_2)}$ are the usual norms of the Sobolev spaces $W^{1,p}(\Omega_1)$ and $W^{1,q}(\Omega_2)$, respectively.

Remark 1.1 The space W defined before can be identified with the space

$$\widetilde{W} := \left\{(u_1, u_2) \in W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_2);\ \mathrm{Tr}^{\Omega_1} u_1 = \mathrm{Tr}^{\Omega_2} u_2 \text{ on } \Gamma\right\}, \tag{1.3}$$

which shows that W is a reflexive Banach space, as $\tilde{W}$ is a closed subspace of the reflexive product $W^{1,p}(\Omega_1) \times W^{1,q}(\Omega_2)$ with reflexive factors.

While the inclusion $W \subseteq \widetilde{W}$ is obvious, for the opposite one we consider $(u_1, u_2) \in \widetilde{W}$ and define

$$u(x) = \begin{cases} u_1(x), & x \in \Omega_1, \\ u_2(x), & x \in \Omega_2. \end{cases}$$

Let us show that $u \in W$. Obviously, u belongs to $L^p(\Omega)$, and its (distributional) derivatives verify the equalities:

$$\begin{aligned} \left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle &= -\left\langle u, \frac{\partial \varphi}{\partial x_i} \right\rangle = -\int_{\Omega_1 \cup \Omega_2} u \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}x \\ &= \int_{\Omega_1} \frac{\partial u}{\partial x_i} \varphi\, \mathrm{d}x - \int_{\partial \Omega_1} u \nu_{1i} \varphi\, \mathrm{d}\sigma + \int_{\Omega_2} \frac{\partial u}{\partial x_i} \varphi\, \mathrm{d}x - \int_{\partial \Omega_2} u \nu_{2i} \varphi\, \mathrm{d}\sigma, \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$, where $\nu_1 = (\nu_{11}, \ldots, \nu_{1n})$ and $\nu_2 = (\nu_{21}, \ldots, \nu_{2n})$ are the outward pointing unit normal fields to boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Clearly, the integral terms on the two boundaries cancel each other as $u_1 = u_2$ and $\nu_{1i} + \nu_{2i} = 0$, $\forall\, i = \overline{1, n}$, on Γ. Thus,

$$\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle = \int_{\Omega_1} \frac{\partial u}{\partial x_i} \varphi\, \mathrm{d}x + \int_{\Omega_2} \frac{\partial u}{\partial x_i} \varphi\, \mathrm{d}x, \quad \forall\, \varphi \in C_0^\infty(\Omega),$$

which shows that

$$\left.\frac{\partial u}{\partial x_i}\right|_{\Omega_1} = \frac{\partial u_1}{\partial x_i} \text{ and } \left.\frac{\partial u}{\partial x_i}\right|_{\Omega_2} = \frac{\partial u_2}{\partial x_i},$$

for all $i = \overline{1, n}$, and the desired claim follows now easily.

Proposition 1.1 *The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of the problem (1.1) if and only if there exists $u = u_\lambda \in W \setminus \{0\}$ such that*

$$\begin{aligned} &\int_{\Omega_1} \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w\, dx + \int_{\Omega_2} \mid \nabla u \mid^{q-2} \nabla u \cdot \nabla w\, dx \\ &= \lambda \Big(\int_{\Omega_1} \mid u \mid^{p-2} uw\, dx + \int_{\Omega_2} \mid u \mid^{q-2} uw\, dx \Big), \quad \forall\, w \in W. \end{aligned} \tag{1.4}$$

Proof Indeed, if $u \in W$ is a solution of the problem (1.1), then we have for all $w \in W$

$$\begin{aligned} &\int_{\Omega_1} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) w\, \mathrm{d}x + \int_{\Omega_2} \operatorname{div}\left(|\nabla u|^{q-2} \nabla u\right) w\, \mathrm{d}x \\ &= -\lambda \int_{\Omega_1} |u|^{p-2} uw\, \mathrm{d}x - \lambda \int_{\Omega_2} |u|^{q-2} uw\, \mathrm{d}x \end{aligned}$$

or, equivalently,

$$\begin{aligned} &-\int_{\Omega_1} |\nabla u|^{p-2} \nabla u \cdot \nabla w\, \mathrm{d}x + \int_\Gamma w |\nabla u|^{p-2} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma - \int_{\Omega_2} |\nabla u|^{q-2} \nabla u \cdot \nabla w\, \mathrm{d}x \\ &- \int_\Gamma w |\nabla u|^{q-2} \frac{\partial u}{\partial \nu}\, \mathrm{d}\sigma = -\lambda \int_{\Omega_1} |u|^{p-2} uw\, \mathrm{d}x - \lambda \int_{\Omega_2} |u|^{q-2} uw\, \mathrm{d}x, \end{aligned}$$

which is equivalent to (1.4).

Conversely, assume that $u \in W$ satisfies (1.4) and consider $w \in W$ such that $w|_{\Omega_1} = \varphi$ for some arbitrary $\varphi \in C_0^\infty(\Omega_1)$ and $w|_{\Omega_2} = 0$. We obtain

$$\int_{\Omega_1} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \lambda \int_{\Omega_1} |u|^{p-2} u\varphi \, \mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega_1).$$

By using the formula of integration by parts, we obtain

$$-\int_{\Omega_1} \mathrm{div}\big(|\nabla u|^{p-2} \nabla u\big)\varphi \, \mathrm{d}x = \lambda \int_{\Omega_1} |u|^{p-2} u\varphi \, \mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega_1),$$

which shows that $-\Delta_p u = \lambda \mid u \mid^{p-2} u$ in Ω_1. Similarly, $-\Delta_q u = \lambda \mid u \mid^{q-2} u$ in Ω_2.

We next assume that $w \in C^1(\overline{\Omega})$ and $w|_{\Omega_2} = 0$. With such a choice of w, using the integration by parts formula, the fact that $w|_\Gamma = 0$ and the equation $-\Delta_p u = \lambda \mid u \mid^{p-2} u$ in Ω_1 obtained above, the relation (1.4) implies

$$0 = \int_{\partial\Omega_1} w \mid \nabla u \mid^{p-2} \frac{\partial u}{\partial \nu} \, \mathrm{d}\sigma = \int_{\Gamma_1} w \mid \nabla u \mid^{p-2} \frac{\partial u}{\partial \nu} \, \mathrm{d}\sigma$$

for all $w \in C^1(\overline{\Omega}_1),\ w|_\Gamma = 0$, therefore $\dfrac{\partial u}{\partial \nu_p} = \mid \nabla u \mid^{p-2} \dfrac{\partial u}{\partial \nu} = 0$ on Γ_1. One can similarly show that $\dfrac{\partial u}{\partial \nu_q} = \mid \nabla u \mid^{q-2} \dfrac{\partial u}{\partial \nu} = 0$ on Γ_2.

It remains to obtain the transmission conditions on Γ. First of all, it is obvious that $\mathrm{Tr}^{\Omega_1}(u|_{\Omega_1}) = \mathrm{Tr}^{\Omega_2}(u|_{\Omega_2})$ on Γ. Finally, we take in (1.4) $w = \varphi$, where φ is an arbitrary function in $C_0^\infty(\Omega)$. Using again the integration by parts formula (in particular, on Γ we have $\nu_1 + \nu_2 = 0$, the normal vector ν_k being chosen to point towards the exterior of Ω_k, $k = 1, 2$) and the equations and equalities proved before, we derive

$$\int_\Gamma \varphi \frac{\partial u}{\partial \nu_p} \, \mathrm{d}\sigma + \int_\Gamma \varphi \frac{\partial u}{\partial \nu_q} \, \mathrm{d}\sigma = 0, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Thus, the transmission relation

$$\frac{\partial u}{\partial \nu_p} = \frac{\partial u}{\partial \nu_q} \quad \text{on } \Gamma$$

is satisfied. This completes the proof. □

If we choose $w = u$ in (1.4), we see that there exist no negative eigenvalues of problem (1.1). It is also obvious that $\lambda_0 = 0$ is an eigenvalue of this problem and the corresponding eigenfunctions are the nonzero constant functions. So any other eigenvalue belongs to $(0, \infty)$.

Obviously, u corresponding to any eigenvalue $\lambda > 0$ cannot be a constant function (see (1.4) with $w = u$).

If we assume that $\lambda > 0$ is an eigenvalue of problem (1.1) and choose $w \equiv 1$ in (1.4), we deduce that every eigenfunction u corresponding to λ satisfies the equation

$$\int_{\Omega_1} |u|^{p-2} u \, \mathrm{d}x + \int_{\Omega_2} |u|^{q-2} u \, \mathrm{d}x = 0.$$

So all eigenfunctions corresponding to positive eigenvalues necessarily belong to the set

$$\mathscr{C} := \Big\{ u \in W;\ \int_{\Omega_1} |u|^{p-2} u \, \mathrm{d}x + \int_{\Omega_2} |u|^{q-2} u \, \mathrm{d}x = 0 \Big\}. \tag{1.5}$$

Using the Sobolev's embedding theorem and [11, Lemma A_1]), we can see that $\mathscr{C}$ is a weakly closed subset of W. This set has nonzero elements. To show this, we choose $x_1, x_2 \in \Omega_1,\ x_1 \neq x_2, r > 0$, such that $B_r(x_1) \cap B_r(x_2) = \emptyset,\ B_r(x_k) \subset \Omega_1$, and consider the test functions $u_k : \Omega \to \mathbb{R},\ k = 1, 2,$

$$u_k(x) = \begin{cases} e^{-\dfrac{1}{r^2 - \mid x - x_k \mid^2}}, & \text{if } x \in B_r(x_k), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $u_k \in W$, $k = 1, 2$. Denote

$$\theta_k = \int_\Omega u_k^{p-1}\, \mathrm{d}x.$$

Obviously, $\theta_k > 0$, $k = 1, 2$. Define $\sigma_k = \theta_k^{\frac{-1}{p-1}}$, $\ k = 1, 2$. It is then easily seen that the function $w = \sigma_1 u_1 - \sigma_2 u_2$ belongs to $\mathscr{C} \backslash \{0\}$.

Our next goal is to prove, via the Lusternik–Schnirelmann principle, that there exists a sequence of positive eigenvalues of problem (1.1). Note, however, that this sequence might not cover the whole eigenvalue set.

2 Results

In what follows we make use of a version of Lusternik–Schnirelmann principle (see [2], [19, Section 44.5, Remark 44.23] and [11]) in order to establish the existence of a sequence of eigenvalues for problem (1.1).

Define the functionals $F, G : W \longrightarrow \mathbb{R}$ by

$$F(u) := \frac{1}{p} \int_{\Omega_1} \mid u \mid^p \mathrm{d}x + \frac{1}{q} \int_{\Omega_2} \mid u \mid^q \mathrm{d}x, \tag{2.1}$$

$$\begin{aligned} G(u) &:= \frac{1}{p} \int_{\Omega_1} \left(\mid \nabla u \mid^p + \mid u \mid^p \right) \mathrm{d}x + \frac{1}{q} \int_{\Omega_2} \left(\mid \nabla u \mid^q + \mid u \mid^q \right) \mathrm{d}x \\ &= F(u) + \frac{1}{p} \int_{\Omega_1} \mid \nabla u \mid^p \ \mathrm{d}x + \frac{1}{q} \int_{\Omega_2} \mid \nabla u \mid^q \ \mathrm{d}x. \end{aligned} \tag{2.2}$$

It is easily seen that functionals F and G are of class C^1 on W (see Remark 2.1) and obviously F, G are even with $F(0) = G(0) = 0$. We also have

$$\langle F'(u), w \rangle = \int_{\Omega_1} \mid u \mid^{p-2} uw \, \mathrm{d}x + \int_{\Omega_2} \mid u \mid^{q-2} uw \, \mathrm{d}x, \tag{2.3}$$

$$\langle G'(u), w \rangle = \langle F'(u), w \rangle + \int_{\Omega_1} \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w \, \mathrm{d}x + \int_{\Omega_2} \mid \nabla u \mid^{q-2} \nabla u \cdot \nabla w \, \mathrm{d}x, \tag{2.4}$$

for all $w \in W$. We denote by $S_G(1)$ the level set of G, $S_G(1) := \{u \in W;\ G(u) = 1\}$.

We have the following auxiliary result:

Lemma 2.1 *The functionals F and G satisfy the following properties:*

(h_1) *F' is strongly continuous, i.e. $u_n \rightharpoonup u$ (meaning $u_n \to u$ weakly) in $W \Rightarrow F'(u_n) \to F'(u)$ and*

$$\langle F'(u), u \rangle = 0 \Rightarrow u = 0;$$

(h_2) *G' is bounded and satisfies condition (S_0), i. e.,*

$$u_n \rightharpoonup u,\ G'(u_n) \rightharpoonup w,\ \langle G'(u_n), u_n\rangle \to \langle w, u\rangle \ \Rightarrow\ u_n \to u;$$

(h_3) *$S_G(1)$ is bounded and if $u \neq 0$ then*

$$\langle G'(u), u\rangle > 0, \quad \lim_{t\to\infty} G(tu) = \infty, \quad \inf_{u\in S_G(1)} \langle G'(u), u\rangle > 0.$$

Proof (h_1) Assume that $u_n \rightharpoonup u$ in W. Hölder's inequality yields

$$\begin{aligned}
|\ \langle F'(u_n) - F'(u), w\rangle\ | &\le \|\,|\ u_n\ |^{p-2}\ u_n - |\ u\ |^{p-2}\ u\ \|_{L^{\frac{p}{p-1}}(\Omega_1)} \|\ w\ \|_{L^p(\Omega_1)} \\
&\quad + \|\,|\ u_n\ |^{q-2}\ u_n - |\ u\ |^{q-2}\ u\ \|_{L^{\frac{q}{q-1}}(\Omega_2)} \|\ w\ \|_{L^q(\Omega_2)} \\
&\le \Big(\|\,|\ u_n\ |^{p-2}\ u_n - |\ u\ |^{p-2}\ u\ \|_{L^{\frac{p}{p-1}}(\Omega_1)} \\
&\quad + \|\,|\ u_n\ |^{q-2}\ u_n - |\ u\ |^{q-2}\ u\ \|_{L^{\frac{q}{q-1}}(\Omega_2)} \Big) \|\ w\ \|_W, \qquad (2.5)
\end{aligned}$$

for all $w \in W$. This shows that the linear functionals $F'(u_n) - F'(u)$ are all bounded and

$$\begin{aligned}
\|F'(u_n) - F'(u)\| &\le \|\,|\ u_n\ |^{p-2}\ u_n - |\ u\ |^{p-2}\ u\ \|_{L^{\frac{p}{p-1}}(\Omega_1)} \\
&\quad + \|\,|\ u_n\ |^{q-2}\ u_n - |\ u\ |^{q-2}\ u\ \|_{L^{\frac{q}{q-1}}(\Omega_2)}, \qquad (2.6)
\end{aligned}$$

for all $n \ge 1$. Since $u_n \rightharpoonup u$ in W, it follows that $\{u_n\}$ as well as the sequences of restrictions $\{u_n\big|_{\Omega_1}\}$ and $\{u_n\big|_{\Omega_2}\}$ are bounded (see [1, Proposition 3.5, p. 58]). Consequently, $u_n \to u$ in $L^p(\Omega)$, $u_n|_{\Omega_1} \to u|_{\Omega_1}$ in $L^p(\Omega_1)$ and $u_n|_{\Omega_2} \to u|_{\Omega_2}$ in $L^q(\Omega_2)$, as the canonical injections $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, $W^{1,p}(\Omega_1) \hookrightarrow L^p(\Omega_1)$ and $W^{1,q}(\Omega_2) \hookrightarrow L^q(\Omega_2)$ are all compact (see [20, Proposition 21.29, p. 262]). The convergence $\|u_n\|_{L^p(\Omega_1)} \longrightarrow \|u\|_{L^p(\Omega_1)}$ is equivalent with

$$\int_{\Omega_1} |\ |\ u_n\ |^{p-2}\ u_n|^{\frac{p}{p-1}}\ \mathrm{d}x \to \int_{\Omega_1} |\ |\ u\ |^{p-2}\ u|^{\frac{p}{p-1}}\ \mathrm{d}x. \qquad (2.7)$$

As the set of weak cluster points of the sequence $\{|\ u_n\ |^{p-2}\ u_n\}$ in $L^{p/(p-1)}(\Omega_1)$ is the singleton $\{|\ u\ |^{p-2}\ u\}$, it follows that in fact this sequence is strongly convergent in $L^{p/(p-1)}(\Omega_1)$ to $|\ u\ |^{p-2}\ u$ (see, for example, [1, Prop. 3.32, p. 78]).

One can similarly show that $|\ u_n\ |^{q-2}\ u_n \to |\ u\ |^{q-2}\ u$ in $L^{q/(q-1)}(\Omega_2)$. Thus, the convergence $F'(u_n) \to F'(u)$ in W^* follows by using (2.6).

If $\langle F'(u), u\rangle = 0$ then obviously $u = 0$.

Note that the strong continuity of G can be similarly derived.

(h_2) Let us first prove that for all $u, w \in W$ the following relations hold:

$$\begin{aligned}
&\langle G'(u) - G'(w), u - w\rangle \\
&\ge \big(\ \|\ u_1\ \|^{p-1}_{W^{1,p}(\Omega_1)} - \|\ w_1\ \|^{p-1}_{W^{1,p}(\Omega_1)}\ \big)\big(\ \|\ u_1\ \|_{W^{1,p}(\Omega_1)} - \|\ w_1\ \|_{W^{1,p}(\Omega_1)}\ \big) \\
&\quad + \big(\ \|\ u_2\ \|^{q-1}_{W^{1,q}(\Omega_2)} - \|\ w_2\ \|^{q-1}_{W^{1,q}(\Omega_2)}\ \big)\big(\ \|\ u_2\ \|_{W^{1,q}(\Omega_2)} - \|\ w_2\ \|_{W^{1,q}(\Omega_2)}\ \big) \ge 0, \qquad (2.8)
\end{aligned}$$

where u_1, w_1, u_2, w_2 stand for $u\big|_{\Omega_1}, w\big|_{\Omega_1}, u\big|_{\Omega_2}, w\big|_{\Omega_2}$, respectively. Moreover,

$$\langle G'(u) - G'(w), u - w\rangle = 0 \ \Leftrightarrow\ u = w \text{ a. e. in } \Omega. \qquad (2.9)$$

It is obvious that

$$\begin{aligned}\langle G'(u) - G'(w), u - w\rangle = &\parallel u_1 \parallel^p_{W^{1,p}(\Omega_1)} + \parallel w_1 \parallel^p_{W^{1,p}(\Omega_1)} \\ &+ \parallel u_2 \parallel^q_{W^{1,q}(\Omega_2)} + \parallel w_2 \parallel^q_{W^{1,q}(\Omega_2)} \\ &- (T_1 + T_2) - (T_3 + T_4),\end{aligned} \tag{2.10}$$

where we have denoted

$$T_1 := \int_{\Omega_1} \big(\mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w + \mid u \mid^{p-2} uw\big)\, \mathrm{d}x,$$

$$T_2 := \int_{\Omega_1} \big(\mid \nabla w \mid^{p-2} \nabla w \cdot \nabla u + \mid w \mid^{p-2} wu\big)\, \mathrm{d}x,$$

and T_3, T_4 are similarly defined, by replacing p and Ω_1 with q and Ω_2. Using the Hölder inequality, we obtain that

$$\begin{aligned}T_1 &\le \Big(\int_{\Omega_1} \mid \nabla u \mid^p \mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\Omega_1} \mid \nabla w \mid^p \mathrm{d}x\Big)^{\frac{1}{p}} + \Big(\int_{\Omega_1} \mid u \mid^p \mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\Omega_1} \mid w \mid^p \mathrm{d}x\Big)^{\frac{1}{p}} \\ &\le \Big(\int_{\Omega_1} (\mid \nabla u \mid^p + \mid u \mid^p)\mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\Omega_1} (\mid \nabla w \mid^p + \mid w \mid^p)\mathrm{d}x\Big)^{\frac{1}{p}} \\ &= \parallel u_1 \parallel^{p-1}_{W^{1,p}(\Omega_1)} \parallel w_1 \parallel_{W^{1,p}(\Omega_1)},\end{aligned} \tag{2.11}$$

where we have also used the inequality

$$\alpha^s \gamma^{1-s} + \beta^s \delta^{1-s} \le (\alpha + \beta)^s (\gamma + \delta)^{1-s}, \quad \forall\, \alpha, \beta, \gamma, \delta > 0,\ s \in (0, 1).$$

Similar inequalities can be obtained for the other terms, T_2, T_3, T_4, and using (2.10) we derive (2.8).

Now by (2.8) we see that $\langle G'(u) - G'(w), u - w\rangle = 0$ implies

$$\parallel u_1 \parallel_{W^{1,p}(\Omega_1)} = \parallel w_1 \parallel_{W^{1,p}(\Omega_1)},\ \parallel u_2 \parallel_{W^{1,q}(\Omega_2)} = \parallel w_2 \parallel_{W^{1,q}(\Omega_2)}, \tag{2.12}$$

and also we have equalities in Hölder inequalities; therefore, there exist positive constants, k_1, k_2 such that $\mid u_i \mid = k_i \mid w_i \mid,\ i = 1, 2$. On the other hand, we have equality in (2.11); thus,

$$T_1 = k_1^{p-1} \parallel w_1 \parallel^p_{W^{1,p}(\Omega_1)} \Rightarrow u_1 = k_1 w_1 \ \text{ a. e. in } \Omega_1.$$

Similarly, we can derive that $u_2 = k_2 w_2$ a. e. in Ω_2 and taking into account (2.12) we derive (2.9).

In order to prove that G' is bounded, we can use again the Hölder inequality and straightforward computations lead us to

$$\mid \langle G'(u), w\rangle \mid \le \big(\parallel u_1 \parallel^{p-1}_{W^{1,p}(\Omega_1)} + \parallel u_2 \parallel^{q-1}_{W^{1,q}(\Omega_2)} \big) \parallel w \parallel_W, \quad \forall\, u, w \in W.$$

Moreover, a similar argument to the one we used to prove (h_1) would imply the continuity of G'.

Finally, let us prove that G' verifies condition (S_0), i.e.,

$$u_n \rightharpoonup u,\ G'(u_n) \rightharpoonup w,\ \langle G'(u_n), u_n\rangle \to \langle w, u\rangle \text{ implies } u_n \to u,$$

for some $u \in W,\ w \in W^*$. Indeed, as $u_n \rightharpoonup u$ in W, we have $u_n|_{\Omega_1} \to u|_{\Omega_1}$ in $L^p(\Omega_1)$ and $u_n|_{\Omega_2} \to u|_{\Omega_2}$ in $L^q(\Omega_2)$. Since W is a reflexive Banach space, using the Lindenstrauss–Asplund–Troyanski theorem (see [18]), it is enough to prove that $\parallel u_n \parallel_W \to \parallel u \parallel_W$ in order to obtain the strong convergence $u_n \to u$. This convergence is a simple consequence of the equality

$$\lim_{n\to\infty} \langle G'(u_n) - G'(u), u_n - u\rangle = \lim_{n\to\infty} \big(\langle G'(u_n), u_n\rangle - \langle G'(u_n), u\rangle - \langle G'(u), u_n - u\rangle\big) = 0$$

and the inequality (2.8).

The properties (h_3) follow immediately from the definition of the functional G. Thus, the proof is complete. □

Remark 2.1 For the convenience of the reader, we recall that:

1. the C^1-smooth regularity of the functionals F and G follows by computing the Gâteaux derivatives
$$\langle F'(u), w\rangle = \frac{d}{dt}\Big|_{t=0} F(u+tw) \text{ and } \langle G'(u), w\rangle = \frac{d}{dt}\Big|_{t=0} G(u+tw)$$
of F and G at $u \in W$ in the direction $w \in W$ and showing that they have the forms (2.3) and (2.4), respectively. The existence of the Gâteaux derivatives of F and G at every point of W and all directions of W combined with the strong continuity of F' and G' shows the Fréchet differentiability of F and G and therefore the C^1-smooth regularity of F and G.
2. The weak closedness of the set $\mathscr{C}$ defined by (1.5) follows also from the strong continuity of F' and the representation of $\mathscr{C}$ as $\{u \in W \mid \langle F'(u), 1\rangle = 0\}$.

Due to the properties $(h_1) - (h_3)$, verified by the functionals F and G, combined with their properties to be even and to vanish at zero, it follows, according to the Lusternik–Schnirelmann principle, that the eigenvalue problem

$$F'(u) = \mu G'(u), \quad u \in S_G(1) \tag{2.13}$$

admits a sequence of eigenpairs $\{(u_n, \mu_n)\}$ such that $u_n \rightharpoonup 0$ and $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\mu_n \neq 0$, for all n. In fact, $\{\mu_n\}$ is a decreasing sequence of non-negative reals (which converges to zero) and

$$\mu_n = \sup_{H\in\mathbb{A}_n} \inf_{u\in H} F(u), \quad \forall\, n \in \mathbb{N}, \tag{2.14}$$

where $\mathbb{A}_n$ is the class of all compact, symmetric subsets K of $S_G(1)$ such that $F(u) > 0$ on K and $\gamma(K) \geq n$, where $\gamma(K)$ denotes the genus of K, i.e.,

$$\gamma(K) := \inf\{k \in \mathbb{N};\ \exists\, h : K \to \mathbb{R}^k \setminus \{0\} \text{ such that h is continuous and odd}\}.$$

The problem (2.13) consists in finding $u \in S_G(1)$ such that

$$\begin{aligned}&\int_{\Omega_1} |\, u\,|^{p-2}\, uw \,\mathrm{d}x + \int_{\Omega_2} |\, u\,|^{q-2}\, uw \,\mathrm{d}x \\ &= \mu\Big(\langle F'(u), w\rangle + \int_{\Omega_1} |\, \nabla u\,|^{p-2}\, \nabla u \cdot \nabla w \,\mathrm{d}x + \int_{\Omega_2} |\, \nabla u\,|^{q-2}\, \nabla u \cdot \nabla w \,\mathrm{d}x\Big),\end{aligned}$$

for all $w \in W$, or equivalently, in finding $u \in S_G(1)$, such that

$$\begin{aligned}&\int_{\Omega_1} |\, \nabla u\,|^{p-2}\, \nabla u \cdot \nabla w \,\mathrm{d}x + \int_{\Omega_2} |\, \nabla u\,|^{q-2}\, \nabla u \cdot \nabla w \,\mathrm{d}x \\ &= (1/\mu - 1)\Big(\int_{\Omega_1} |\, u\,|^{p-2}\, uw \,\mathrm{d}x + \int_{\Omega_2} |\, u\,|^{q-2}\, uw \,\mathrm{d}x\Big), \quad \forall\, w \in W.\end{aligned} \tag{2.15}$$

Observe that (2.15) is the variational formulation of problem (1.1). We therefore get the following consequence of the Lusternik–Schnirelmann principle associated with the transmission problem (1.1):

Theorem 2.1 *The sequence* $\{\mu_n\}$ *of eigenvalues of the problem* (2.13) *produces a nondecreasing sequence* $\lambda_n = \dfrac{1}{\mu_n} - 1$ *of eigenvalues of the problem* (1.1) *and obviously* $\lambda_n \to \infty$ *as* $n \to \infty$.

In what follows we shall use the Lagrange multipliers rule to show that every positive level set of the functional F defined by (2.1) contains an eigenfunction of the problem (1.1) and we shall find its corresponding eigenvalue in terms of the pointed out eigenfunction. Such an eigenfunction will appear as a solution of the minimum problem

$$\min_{u \in \mathscr{C} \cap S_F(\alpha)} H(u), \tag{2.16}$$

where H is defined by

$$H : W \to [0, \infty),\ H(u) := \frac{1}{p} \int_{\Omega_1} |\ \nabla u\ |^p \,\mathrm{d}x + \frac{1}{q} \int_{\Omega_2} |\ \nabla u\ |^q \,\mathrm{d}x, \quad \forall\, u \in W, \tag{2.17}$$

$\mathscr{C}$ is defined by (1.5) and $S_F(\alpha)$ is the set at the level $\alpha > 0$ of F, i.e.

$$S_F(\alpha) := \{u \in W;\ F(u) = \alpha\}, \quad \forall\, \alpha > 0.$$

In this respect we first recall the Lagrange multipliers principle (see, for example, [14, Thm. 2.2.18, p. 78]):

Lemma 2.2 *Let* X, Y *be real Banach spaces and let* $f : D \to \mathbb{R}$ *be Fréchet differentiable,* $g \in C^1(D, Y)$*, where* $D \subseteq X$ *is a nonempty open set. If* v_0 *is a local minimizer of the constraint problem*

$$\min f(w), \quad g(w) = 0,$$

and $\mathscr{R}(g'(v_0))$ *(the range of* $g'(v_0)$*) is closed, then there exist* $\lambda^* \in \mathbb{R}$ *and* $y^* \in Y^*$*, at least one of which is non zero, such that*

$$\lambda^* f'(v_0) + y^* \circ g'(v_0) = 0,$$

where Y^* *stands for the dual of* Y.

Note that $\lambda^* \neq 0$ whenever $g'(v_0)$ is onto and can be therefore chosen to be 1 in this particular case.

The eigenvalue problem corresponding to the minimum problem (2.16), via the Lagrange multipliers, is:

$$H'(u_\alpha) = \lambda_\alpha F'(u_\alpha),\ \lambda_\alpha > 0, u_\alpha \neq 0. \tag{2.18}$$

Its variational version is (1.4).

Theorem 2.2 *Let* F *and* H *be the functionals defined by* (2.1) *and* (2.17)*. For every* $2 \leq p < q$*,* $\alpha > 0$*, the minimization problem* (2.16) *has a solution* u_α *which is an eigenfunction of the eigenvalue problem* (2.18) *and therefore a solution of the variational version* (1.4) *of the initial eigenvalue problem* (1.1).

Proof Let us first show that the set $\mathscr{C} \cap S_F(\alpha)$ is nonempty for every $\alpha > 0$. Indeed, if we choose $w \in \mathscr{C} \cap C_0^\infty(\Omega_1)$, nonzero, then $\alpha w/F(w) \in \mathscr{C} \cap S_F(\alpha)$.

Now, the functional H is coercive on the weakly closed subset $\mathscr{C} \cap S_F(\alpha)$ of the reflexive Banach space W, i.e.,

$$\lim_{\substack{\|u\|_W \to \infty \\ u \in \mathscr{C} \cap S_F(\alpha)}} H(u) = \infty.$$

This fact is a simple consequence of the equality

$$\lim_{\substack{\|u\|_W \to \infty \\ u \in S_F(\alpha)}} (\| \nabla u \|_{L^p(\Omega_1)} + \| \nabla u \|_{L^q(\Omega_2)}) = \infty.$$

On the other hand, the weakly lower semicontinuity of the norms in $L^p(\Omega_1)$ and $L^q(\Omega_2)$ implies the weakly lower semicontinuity of the functional H on $\mathscr{C} \cap S_F(\alpha)$. Then, we can apply [16, Theorem 1.2] in order to obtain the existence of a global minimum point of H over $\mathscr{C} \cap S_F(\alpha)$, say u_α, i.e., $H(u_\alpha) = \min_{u \in \mathscr{C} \cap S_F(\alpha)} H(u)$. Obviously, $u_\alpha \in \mathscr{C} \cap S_F(\alpha)$ implies that u_α is a nonconstant function. In fact, u_α is a solution of the minimization problem

$$\min_{w \in W} H(w),$$

under the restrictions

$$\begin{aligned} g(w) &:= \frac{1}{p} \int_{\Omega_1} | w |^p \, \mathrm{d}x + \frac{1}{q} \int_{\Omega_2} | w |^q \, \mathrm{d}x - \alpha = 0, \\ h(w) &:= \int_{\Omega_1} | w |^{p-2} w \, \mathrm{d}x + \int_{\Omega_2} | w |^{q-2} w \, \mathrm{d}x = 0, \quad \forall\, w \in W. \end{aligned}$$

We can apply Lemma 2.2 with $X = W,\ D = W \backslash \{0\},\ Y = \mathbb{R},\ f = H,\ g,\ h : W \to \mathbb{R}$ being the functions just defined above, and $v_0 = u_\alpha$, on the condition that $\mathscr{R}(g'(u_\alpha)), \mathscr{R}(h'(u_\alpha))$ be closed sets. In fact, we can show that $g'(u_\alpha),\ h'(u_\alpha)$ are surjective, i.e. $\forall\ c_1, c_2 \in \mathbb{R}$ there exist $w_1, w_2 \in W$ such that

$$\langle g'(u_\alpha), w_1 \rangle = c_1, \langle h'(u_\alpha), w_2 \rangle = c_2.$$

We seek w_1, w_2 of the form $w_1 = \beta u_\alpha,\ w_2 = \gamma$, with $\beta, \gamma \in \mathbb{R}$. Thus, we obtain from the above equations

$$\begin{aligned} \beta\Big(\int_{\Omega_1} | u_\alpha |^p \, \mathrm{d}x + \int_{\Omega_2} | u_\alpha |^q \, \mathrm{d}x \Big) &= c_1, \\ \gamma\Big((p-1) \int_{\Omega_1} | u_\alpha |^{p-2} \, \mathrm{d}x + (q-1) \int_{\Omega_2} | u_\alpha |^{q-2} \, \mathrm{d}x \Big) &= c_2 \end{aligned}$$

which have unique solutions β, γ since $u_\alpha \in S_F(\alpha)$ implies that

$$r_1 \int_{\Omega_1} | u_\alpha |^{p_1} \, \mathrm{d}x + r_2 \int_{\Omega_2} | u_\alpha |^{q_1} \, \mathrm{d}x > 0, \quad \forall\ p_1, q_1, r_i > 0,\ i = 1, 2.$$

Thus, by Lemma 2.2, there exist λ and $\mu \in \mathbb{R}$ such that, $\lambda^2 + \mu^2 > 0$ and for all $w \in W$,

$$\begin{aligned}&\int_{\Omega_1} \mid \nabla u_\alpha \mid^{p-2} \nabla u_\alpha \cdot \nabla w \, \mathrm{d}x + \int_{\Omega_2} \mid \nabla u_\alpha \mid^{q-2} \nabla u_\alpha \cdot \nabla w \, \mathrm{d}x \\ &- \lambda\Big(\int_{\Omega_1} \mid u_\alpha \mid^{p-2} u_\alpha w \, \mathrm{d}x + \int_{\Omega_2} \mid u_\alpha \mid^{q-2} u_\alpha w \, \mathrm{d}x \Big) \\ &- \mu\Big((p-1) \int_{\Omega_1} \mid u_\alpha \mid^{p-2} w \, \mathrm{d}x + (q-1) \int_{\Omega_2} \mid u_\alpha \mid^{q-2} w \, \mathrm{d}x \Big) = 0. \end{aligned} \tag{2.19}$$

Testing with $w = 1$ in (2.19) and observing that u_α belongs to $\mathscr{C}$, we deduce that $\mu = 0$ and therefore $\lambda \neq 0$. By choosing $w = u_\alpha$ in (2.19), we find $K_{1\alpha} - \lambda K_{2\alpha} = 0$, where $K_{1\alpha}$ and $K_{2\alpha}$ denote the constants

$$\int_{\Omega_1} \mid \nabla u_\alpha \mid^p \, \mathrm{d}x + \int_{\Omega_2} \mid \nabla u_\alpha \mid^q \mathrm{d}x \text{ and } \int_{\Omega_1} \mid u_\alpha \mid^p \, \mathrm{d}x + \int_{\Omega_2} \mid u_\alpha \mid^q \mathrm{d}x,$$

respectively, which are positive as $u_\alpha \in \mathscr{C} \cap S_F(\alpha)$. In other words (2.19) becomes

$$\begin{aligned}&\int_{\Omega_1} \mid \nabla u_\alpha \mid^{p-2} \nabla u_\alpha \cdot \nabla w \, \mathrm{d}x + \int_{\Omega_2} \mid \nabla u_\alpha \mid^{q-2} \nabla u_\alpha \cdot \nabla w \, \mathrm{d}x \\ &= \lambda_\alpha \Big(\int_{\Omega_1} \mid u_\alpha \mid^{p-2} u_\alpha w \, \mathrm{d}x + \int_{\Omega_2} \mid u_\alpha \mid^{q-2} u_\alpha w \, \mathrm{d}x \Big), \end{aligned} \tag{2.20}$$

where

$$\lambda_\alpha = \frac{K_{1\alpha}}{K_{2\alpha}} = \frac{\int_{\Omega_1} \mid \nabla u_\alpha \mid^p \, \mathrm{d}x + \int_{\Omega_2} \mid \nabla u_\alpha \mid^q \, \mathrm{d}x}{\int_{\Omega_1} \mid u_\alpha \mid^p \, \mathrm{d}x + \int_{\Omega_2} \mid u_\alpha \mid^q \, \mathrm{d}x}.$$

Thus, $(\lambda_\alpha, u_\alpha)$ is an eigenpair of problem (1.4). □

Remark 2.2 The results we have proved so far are also valid for the eigenvalue problem obtained out of (1.1) by replacing Eq. $(1.1)_2$ with the equation

$$-\Delta_q u_2 = \lambda \mid u_2 \mid^{p-2} u_2 \quad \text{in } \Omega_2, \tag{2.21}$$

for $1 < p < q$. In this case we shall consider the same space W but endowed with the norm

$$|\| u \||:=\| u \|_{W^{1,p}(\Omega_1)} + \| \nabla u \|_{L^q(\Omega_2)} + \| u \|_{L^p(\Omega_2)}, \quad \forall\, u \in W. \tag{2.22}$$

If $p \leq q$, then $|\| \cdot \||$ is a norm in W equivalent with the usual norm $\| \cdot \|_W$ of this space. This fact follows from [4, Proposition 3.9.55].

In this case, the variational version of the new eigenvalue problem is:

Find $\lambda \in \mathbb{R}$ for which there exists $u \in W \backslash \{0\}$ such that

$$\begin{aligned}&\int_{\Omega_1} \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w \, \mathrm{d}x + \int_{\Omega_2} \mid \nabla u \mid^{q-2} \nabla u \cdot \nabla w \, \mathrm{d}x \\ &= \lambda \int_{\Omega} \mid u \mid^{p-2} uw \, \mathrm{d}x, \quad \forall\, w \in W. \end{aligned} \tag{2.23}$$

In order to obtain the counterpart of Theorem 2.1 for this new eigenvalue transmission problem, we need to verify the conditions $(h_1) - (h_3)$ of Lemma 2.1. We shall define for this new context the corresponding functionals $F_p, G_p : W \to [0, \infty)$

$$F_p(u) := \frac{1}{p} \int_{\Omega} \mid u \mid^p \, \mathrm{d}x, \; G_p(u) := F_p(u) + \frac{1}{p} \int_{\Omega_1} \mid \nabla u \mid^p \, \mathrm{d}x + \frac{1}{q} \int_{\Omega_2} \mid \nabla u \mid^q \, \mathrm{d}x. \tag{2.24}$$

All calculations are similar to those we did to prove $(h_1)-(h_3)$ in the case of the eigenvalue transmission problem (1.1), except the one which verifies the property (S_0) on G'_p of (h_2). In order to prove (S_0), we define the functional $J : W \to W^*$

$$\langle J(u), w\rangle := \int_{\Omega_2} | u |^{p-2} uw \, \mathrm{d}x - \int_{\Omega_2} | u |^{q-2} uw \, \mathrm{d}x, \quad \forall \, u, w \in W.$$

One can show, by using the same type of arguments as we did to prove (h_1) and Lemma 2.1, that $J(u)$ is strongly continuous. Let us consider

$$u_n \rightharpoonup u, \; G'_p(u_n) \rightharpoonup w_p, \; \langle G'_p(u_n), u_n\rangle \to \langle w_p, u\rangle \text{ as } n \to \infty$$

for some $u \in W, \; w_p \in W^*$ and we shall show that $u_n \to u$. In this respect (see also the argument within the proof of the statement (h_2)) it is sufficient to show that $\| u_n \|_W \to \| u \|_W$, as $\| \cdot \|_W$ and $||| \cdot |||$ are equivalent norms on W. In this regard we observe that

$$G'(u_n) = G'_p(u_n) - J(u_n) \rightharpoonup w_p - J(u), \langle G'(u_n), u_n\rangle \to \langle w_p - J(u), u\rangle,$$

which combined with the (S_0) property of G' implies the desired statement.

The counterpart of Theorem 2.2 can be obtained with no difficulty, by using arguments similar to those we have used in the case of the eigenvalue transmission problem (1.1).

3 Extensions

In this section we discuss some extensions of the previous results.

An eigenvalue–transmission problem with Robin boundary conditions

Following the same type of arguments, one can actually prove the counterparts of Theorem 2.1 and Theorem 2.2 for the following more general eigenvalue–transmission problem, involving Robin conditions on Γ_1 and Γ_2, namely

$$\begin{cases} -\Delta_p u_1 = \lambda \mid u_1 \mid^{p-2} u_1 & \text{in } \Omega_1, \\ -\Delta_q u_2 = \lambda \mid u_2 \mid^{q-2} u_2 & \text{in } \Omega_2, \\ \dfrac{\partial u_1}{\partial \nu_p} + \beta_1 |u_1|^{p-2} = 0 \;\text{ on } \Gamma_1, \\ \dfrac{\partial u_2}{\partial \nu_q} + \beta_2 |u_2|^{q-2} = 0 \;\text{ on } \Gamma_2, \\ u_1 = u_2, \;\; \dfrac{\partial u_1}{\partial \nu_p} = \dfrac{\partial u_2}{\partial \nu_q} \;\; \text{on } \Gamma, \end{cases} \tag{3.1}$$

where $\beta_1, \beta_2 \geq 0$. The variational version of problem (3.1) is:

Proposition 3.1 *The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of the problem (3.1) if and only if there exists $u \in W \backslash \{0\}$ such that*

$$\begin{aligned} &\int_{\Omega_1} | \nabla u |^{p-2} \nabla u \cdot \nabla w \, dx + \int_{\Omega_2} | \nabla u |^{q-2} \nabla u \cdot \nabla w \, dx \\ &\quad + \beta_1 \int_{\partial\Omega_1} | u |^{p-2} uw \, d\sigma + \beta_2 \int_{\partial\Omega_2} | u |^{q-2} uw \, d\sigma \\ &\quad = \lambda\Big(\int_{\Omega_1} | u |^{p-2} uw \, dx + \int_{\Omega_2} | u |^{q-2} uw \, dx\Big), \quad \forall w \in W. \end{aligned} \tag{3.2}$$

While the functional playing the role of F in this setting remains unchanged, the functional playing the role of $G : W \longrightarrow \mathbb{R}$ is given by

$$G(u) := \frac{1}{p}\int_{\Omega_1} \big(\mid \nabla u \mid^p + \mid u \mid^p \big)\, \mathrm{d}x + \frac{1}{q}\int_{\Omega_2} \big(\mid \nabla u \mid^q + \mid u \mid^q \big)\, \mathrm{d}x \\ + \frac{1}{p}\int_{\partial\Omega_1} \beta_1 \mid u \mid^p \mathrm{d}\sigma + \frac{1}{q}\int_{\partial\Omega_2} \beta_2 \mid u \mid^q \mathrm{d}\sigma. \tag{3.3}$$

The counterpart of problem (1.1) in the Riemannian setting

Let (M, g) be a compact boundaryless Riemannian manifold and $\Omega \subseteq M$ be a connected open set such that $\Omega_- := M\backslash\overline{\Omega}$ is also connected. We denote Ω by Ω_+ and the common boundary of Ω_+ and Ω_- by $\partial\Omega$, which is assumed to be a hypersurface of M. We consider the following coupled problem

$$\begin{cases} -\Delta_p u_+ = \lambda \mid u_+ \mid^{p-2} u_+ & \text{in } \Omega_+, \\ -\Delta_q u_- = \lambda \mid u_- \mid^{q-2} u_- & \text{in } \Omega_-, \\ u_+ = u_-, \quad \dfrac{\partial u_+}{\partial \nu_p} = \dfrac{\partial u_-}{\partial \nu_q} & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\Delta_r w$ stands for the r-Laplace operator $\operatorname{div}\big(\mid \nabla w \mid^{r-2} \nabla w\big)$.

Proposition 3.2 *The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of the problem (3.4) if and only if there exists $u \in W_\Omega\backslash\{0\}$ such that*

$$\int_{\Omega_+} \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w\, dx + \int_{\Omega_-} \mid \nabla u \mid^{q-2} \nabla u \cdot \nabla w\, dx \\ = \lambda\Big(\int_{\Omega_+} \mid u \mid^{p-2} uw\, dx + \int_{\Omega_-} \mid u \mid^{q-2} uw\, dx \Big), \quad \forall\, w \in W_\Omega, \tag{3.5}$$

where

$$W_\Omega := \big\{ u \in W^{1,p}(M) : u|_{\Omega_-} \in W^{1,q}(\Omega_-) \big\}. \tag{3.6}$$

The proof of Proposition 3.2 works along the same lines with the proof of Proposition 1.1 and partly relies on the integration by parts formula [12, p. 383]

$$\int_X (f \operatorname{div} X) d V_g = -\int_X g(X, \operatorname{grad} f) \mathrm{d}V_g + \int_{\partial X} g(X, \nu) \mathrm{d}V_{\tilde{g}}$$

where (X, g) is a compact oriented Riemannian manifold, ν is the outward unit normal vector field on ∂X, and $\tilde{g}$ is the Riemannian metric on ∂X induced by g.

We endow W_Ω with the norm

$$\| u \|_{W_\Omega} := \| u|_{\Omega_+} \|_{W^{1,p}(\Omega_+)} + \| u|_{\Omega_-} \|_{W^{1,q}(\Omega_-)}, \quad \forall\, u \in W_\Omega$$

where $\| \cdot \|_{W^{1,p}(\Omega_+)}$ and $\| \cdot \|_{W^{1,q}(\Omega_-)}$ are the usual norms of the Sobolev spaces $W^{1,p}(\Omega_+)$ and $W^{1,q}(\Omega_-)$, respectively.

Remark 3.1 The space W_Ω defined before can be identified with the space

$$\widetilde{W}_\Omega := \{(u_+, u_-) \in W^{1,p}(\Omega_+) \times W^{1,q}(\Omega_-);\ \mathrm{Tr}^{\Omega_+} u_+ = \mathrm{Tr}^{\Omega_-} u_- \text{ on } \partial\Omega\}.$$

Note that W_Ω is a reflexive Banach space, as it is a closed subspace of the reflexive product $W^{1,p}(\Omega_+) \times W^{1,q}(\Omega_-)$ with reflexive factors (see [1, p. 70], [6, p. 11] or [7, p. 20]). Define the functionals F and G on W_Ω:

$$F(u) := \frac{1}{p}\int_{\Omega_+} |\, u\,|^p \,\mathrm{d}x + \frac{1}{q}\int_{\Omega_-} |\, u\,|^q \,\mathrm{d}x, \tag{3.7}$$

$$G(u) := \frac{1}{p}\int_{\Omega_+} \left(|\, \nabla u\,|^p + |\, u\,|^p \right) \mathrm{d}x + \frac{1}{q}\int_{\Omega_-} \left(|\, \nabla u\,|^q + |\, u\,|^q \right) \mathrm{d}x, \tag{3.8}$$

for all $u \in W_\Omega$. It is easily seen that functionals F and G are of class C^1 on W_Ω and obviously F, G are even with $F(0) = G(0) = 0$. We also have

$$\langle F'(u), w\rangle = \int_{\Omega_+} |\, u\,|^{p-2}\, uw \,\mathrm{d}x + \int_{\Omega_-} |\, u\,|^{q-2}\, uw \,\mathrm{d}x,$$

$$\langle G'(u), w\rangle = \langle F'(u), w\rangle + \int_{\Omega_+} |\, \nabla u\,|^{p-2}\, \nabla u \cdot \nabla w \,\mathrm{d}x + \int_{\Omega_-} |\, \nabla u\,|^{q-2}\, \nabla u \cdot \nabla w \,\mathrm{d}x,$$

for all $w \in W_\Omega$. We denote by $S_G(1)$ the level set $\{u \in W_\Omega;\ G(u) = 1\}$ of G.

The following auxiliary result can be proved in a similar way with Lemma 2.1.

Lemma 3.1 *The functionals F and G satisfy the following properties:*

($\mathfrak{h}_1$) *F' is strongly continuous, i.e. $u_n \rightharpoonup u$ in $W_\Omega \Rightarrow F'(u_n) \to F'(u)$ and*

$$\langle F'(u), u\rangle = 0 \Rightarrow u = 0;$$

($\mathfrak{h}_2$) *G' is bounded and satisfies condition (S_0), i.e.,*

$$u_n \rightharpoonup u,\ \ G'(u_n) \rightharpoonup w,\ \ \langle G'(u_n), u_n\rangle \to \langle w, u\rangle \ \Rightarrow\ u_n \to u;$$

($\mathfrak{h}_3$) *$S_G(1)$ is bounded and if $u \neq 0$ then*

$$\langle G'(u), u\rangle > 0, \quad \lim_{t\to\infty} G(tu) = \infty, \quad \inf_{u \in S_G(1)} \langle G'(u), u\rangle > 0.$$

According to the properties $(\mathfrak{h}_1) - (\mathfrak{h}_3)$, verified by the functionals F and G, combined with their properties to be even and to vanish at zero, it follows, via the Lusternik–Schnirelmann principle, that the eigenvalue problem

$$F'(u) = \mu G'(u),\ \ u \in S_G(1) \tag{3.9}$$

admits a sequence of eigenpairs $\{(u_n, \mu_n)\}$ such that $u_n \rightharpoonup 0$, $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\mu_n \neq 0$, for all n.

Theorem 3.1 *The sequence $\{\mu_n\}$ of eigenvalues of the problem (3.9) produces a nondecreasing sequence $\lambda_n = \dfrac{1}{\mu_n} - 1$ of eigenvalues of the problem (3.4) and obviously $\lambda_n \to \infty$ as $n \to \infty$.*

Consider the minimization problem

$$\min_{u \in \mathscr{C}_\Omega \cap S_F(\alpha)} H(u), \tag{3.10}$$

where

$$H : W_\Omega \to [0, \infty),\ H(u) := \frac{1}{p} \int_{\Omega_+} |\nabla u|^p \, \mathrm{d}x + \frac{1}{q} \int_{\Omega_-} |\nabla u|^q \, \mathrm{d}x, \quad \forall\, u \in W_\Omega, \tag{3.11}$$

$$\mathscr{C}_\Omega := \left\{ u \in W_\Omega;\ \int_{\Omega_+} |u|^{p-2} u \, \mathrm{d}x + \int_{\Omega_-} |u|^{q-2} u \, \mathrm{d}x = 0 \right\}. \tag{3.12}$$

and $S_F(\alpha)$ is the set at the level $\alpha > 0$ of F (i.e. $S_F(\alpha) := \{u \in W;\ F(u) = \alpha\}$).

The eigenvalue problem corresponding to the minimization problem (3.10), via the Lagrange multipliers, is:

$$H'(u_\alpha) = \lambda_\alpha F'(u_\alpha),\ \lambda_\alpha > 0, u_\alpha \neq 0, \tag{3.13}$$

Its variational version is (3.5).

Theorem 3.2 *Let F and H be the functionals defined by* (3.7) *and* (3.11). *For every* $2 \leq p < q$, $\alpha > 0$*, the problem* (3.10) *has a solution u_α which is an eigenfunction of the eigenvalue problem* (3.13) *and therefore a solution of the variational version* (3.5) *of the eigenvalue problem* (3.4).

Remark 3.2 Note that in (3.4) there is no boundary condition, because the ambient manifold M is boundary-free. One can think of the eigenvalue–transmission counterpart of the problem (1.1) in a more general Riemannian setting, where the ambient manifold M has nonempty boundary and the interface hypersurface is suitably chosen. Indeed, for a compact Riemannian manifold (M, g) with nonempty boundary, we consider a connected open set $\Omega_1 \subseteq M$ such that $\Omega_2 := M \setminus \overline{\Omega}_1$ is also connected and $\overline{\Omega}_1$, $\overline{\Omega}_2$ are manifolds with boundaries $\partial\Omega_1, \partial\Omega_2$. We further assume that their common boundary part $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ is a hypersurface of M (which is closed), such that $\Gamma_1 := \partial\Omega_1 \setminus \Gamma$ and $\Gamma_2 := \partial\Omega_2 \setminus \Gamma$ are also connected. With such choices the eigenvalue–transmission counterpart of the problem (1.1) in this more general Riemannian setting looks like (1.1).

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