



A global bifurcation theorem for critical values in Banach spaces

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Abstract

We present a global bifurcation result for critical values of C^1 maps in Banach spaces. The approach is topological based on homotopy equivalence of pairs of topological spaces. For C^2 maps, we prove a particular global bifurcation result, based on the notion of spectral flow.

Keywords Global bifurcation · Critical values · Spectral flow

Mathematics Subject Classification 58E05 · 58E07 · 58J30

1 Introduction

In this paper we present three global bifurcation results for critical values of C^1 maps in Banach spaces and of C^2 maps in Hilbert spaces. We proceed in the general spirit of the family of works that uses topological methods, whose origin can be found in the textbook of Krasnoselskij [19] in 1964 and the paper of Rabinowitz [25] in 1971, even though, we must emphasize, their results concern bifurcation of *solutions* of particular equations, while ours are related to bifurcation of *critical values*, that is, *target values* of a particular function.

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Krasnoselskij obtains the following local bifurcation theorem, which we recall in a simplified version. Let X be a real Banach space. Consider a map $f : \mathbb{R} \times X \rightarrow X$ of the form

$$f(\lambda, x) = x - \lambda C(x),$$

where C is nonlinear, compact, Fréchet differentiable at $x = 0$ and such that $C(0) = 0$. We use the term “compact” for a continuous map sending bounded subsets of the domain to relatively compact subsets of the target space. The solutions of the equation

$$f(\lambda, x) = 0 \tag{1.1}$$

of the form $(\lambda, 0)$ are called *trivial*, and a real number λ_0 is called a *bifurcation point* of (1.1) if every neighborhood of $(\lambda_0, 0)$ in $\mathbb{R} \times X$ contains nontrivial solutions. It is immediate to notice that a necessary condition for λ_0 to be a bifurcation point is that the linear operator $Id - \lambda_0 C'(0)$ is not invertible, that is, λ_0 is a characteristic value of the Fréchet derivative $C'(0)$ of C at zero (which is a compact linear operator).

Krasnoselskij proves that λ_0 is a bifurcation point of (1.1) if it is a characteristic value of $C'(0)$ of odd algebraic multiplicity. Rabinowitz extends this result, proving a so-called global bifurcation theorem, i.e., showing that there exists a connected set R of nontrivial solutions whose closure contains $(\lambda_0, 0)$ and such that at least one of the two alternatives is verified:

- (i) R is unbounded,
- (ii) the closure of R meets a point of the form $(\lambda_1, 0)$ with $\lambda_0 \neq \lambda_1$.

It is obvious why Krasnoselskij’s result is usually called *local*, while Rabinowitz’s one *global*. The approaches of the two authors are based on the application of the Leray–Schauder degree. It is not possible to explain here such a method in details. We limit ourselves to recall the following idea: take $\lambda \in \mathbb{R}$. If $Id - \lambda C'(0)$ is an automorphism of X , we simply denote by the symbol $\text{deg}_{\text{LS}}(Id - \lambda C'(0))$ the Leray–Schauder degree of the triple $(Id - \lambda C'(0), U, 0)$, where U is any open bounded subset of X containing the origin. Such a value could be 1 or -1 , while the Leray–Schauder degree of any triple $(Id - \hat{\lambda} C'(0), U, 0)$ is not defined when $\hat{\lambda}$ is a characteristic value of $C'(0)$. The degree is also locally constant, when defined, with respect to λ . It can be proven that, when λ crosses a characteristic value $\hat{\lambda}$, the number $\text{deg}_{\text{LS}}(Id - \lambda C'(0))$ changes sign if and only if $\hat{\lambda}$ has odd algebraic multiplicity. This sign jump guarantees the bifurcation at $\hat{\lambda}$. If, otherwise, the algebraic multiplicity of $\hat{\lambda}$ is even, this point could be (or not) a bifurcation point, but the degree does not help to give an answer.

Now, two interesting facts happen:

- (a) If, in Eq. (1.1), X is a real separable Hilbert space and $C'(0)$ is a symmetric (i.e., self-adjoint) operator, then every characteristic value of $C'(0)$ is a bifurcation point;
- (b) In some cases, the bifurcation points that are characteristic values of $C'(0)$ of even algebraic multiplicity do not produce a “global bifurcation branch” in the sense of Rabinowitz’s theorem.

Some questions have been quite naturally stimulated in the last decades and in recent years by the above facts: if one tackles a more general problem than (1.1), is it possible to find a more general degree theory to detect local or global bifurcation? Can we use more sophisticated topological methods than degree theories? How can we explain that, in some cases, there is local and not global bifurcation?

More general topological degree theories have been introduced, extending the Leray–Schauder degree to compact and noncompact perturbations—also multivalued perturbations—of nonlinear Fredholm maps between Banach spaces (see, e.g., [5–7,11,12,14,15,21,24,

30]). Consequently, local and global bifurcation results have been obtained for more general problems than (1.1). Actually there is an enormous literature on the subject.

Consider for example a Banach space X and a C^1 map $f : \mathbb{R} \times X \rightarrow X$. Assume $f(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$. Suppose that, for any (λ, x) , the Fréchet derivative $\partial_2 f(\lambda, x)$ of f with respect to the second variable at (λ, x) is a Fredholm operator of index zero. With a particular notion of orientation for Fredholm maps in (possibly infinite dimensional) Banach spaces, it is possible to define a topological degree for any partial map $f(\lambda, \cdot)$ (see [6,13,15]). Given $\lambda \in \mathbb{R}$, denote by L_λ the operator $\partial_2 f(\lambda, 0)$. Suppose λ_0 is such that L_λ is an isomorphism for $|\lambda - \lambda_0|$ small and nonzero. If the degree of L_λ has a sign jump when λ crosses λ_0 , then λ_0 turns out to be a bifurcation point of $f(\lambda, x) = 0$ with a global bifurcation behavior. Analogously to the case of compact perturbations of the identity studied by Krasnoselskij and Rabinowitz, also in this case the lack of sign jump of the degree does not say anything about bifurcation.

In the self-adjoint case, the Morse index is a useful tool to detect local bifurcation (but not global, see the above remark b) in some cases for which the degree does not help. Consider a separable real Hilbert space H . It is known that, given a self-adjoint Fredholm operator $T : H \rightarrow H$, there exists a unique orthogonal splitting of H ,

$$H = V^-(T) \oplus V^+(T) \oplus \ker T,$$

such that $V^-(T)$ and $V^+(T)$ are T -invariant, the quadratic form $x \mapsto \langle Tx, x \rangle$ is negative definite on $V^-(T)$ and positive definite on $V^+(T)$.

With a slight abuse of notation, we will refer to $V^-(T)$ and $V^+(T)$ as the *negative* and the *positive eigenspaces* of T , respectively. The *Morse index* of T , denoted by $\mu(T)$, is defined as the dimension of $V^-(T)$ when it is finite. The following local bifurcation result can be found in the textbook [22] by Mawhin and Willem (see also [16,26]). They consider a compact interval $[a, b]$ and an open neighborhood U of $[a, b] \times \{0\}$ in $\mathbb{R} \times H$. Given a C^2 map $\psi : U \rightarrow \mathbb{R}$, denote by L_λ the Hessian of $\psi_\lambda := \psi(\lambda, \cdot)$ at zero, that is, the second derivative of ψ with respect to second variable at the point $(\lambda, 0)$.

Theorem A *In the above notation, assume that $0 \in H$ is a critical point of the functional ψ_λ for every $\lambda \in [a, b]$. In addition, assume that $L_\lambda : H \rightarrow H$ is a Fredholm operator and suppose that the negative eigenspace $V^-(L_\lambda)$ is finite dimensional for every $\lambda \in [a, b]$. If*

$$\mu(L_a) \neq \mu(L_b),$$

then the interval $[a, b]$ contains a bifurcation point.

The above important result does not apply in the important case when the operators L_λ are so-called strongly indefinite, that is, when their positive and negative eigenspaces have infinite dimension. In order to extend Theorem A to the strongly indefinite case, in a series of papers by Fitzpatrick et al. [16,17,23] a bifurcation problem for a Hamiltonian system is investigated by the application of the spectral flow. The spectral flow has been introduced by Athiyah et al. [3], and it is a topologically invariant integer number associated with a continuous path of self-adjoint Fredholm operators, $L_\lambda, \lambda \in [a, b]$, in a separable real Hilbert space H . The spectral flow can be defined by different equivalent methods. In the next section we will summarize its construction, following the approach of Fitzpatrick et al. [16]. Here, we limit ourselves to observe that

$$\text{sf}(L, [a, b]) = \mu(L_a) - \mu(L_b)$$

when both sides of the above equality are meaningful. In [16] it is proven the following extension of Theorem A.

Theorem B (Fitzpatrick, Pejsachowicz, Recht) *Let H be a separable real Hilbert space and let $\psi : \mathbb{R} \times H \rightarrow \mathbb{R}$ be a C^2 function such that, for each $\lambda \in \mathbb{R}$, $x = 0$ is a critical point of the functional $\psi_\lambda := \psi(\lambda, \cdot)$. Assume that the Hessian L_λ of ψ_λ at 0 is Fredholm and that L_a and L_b are nonsingular for suitable a, b . If $\text{sf}(L, [a, b]) \neq 0$, then every neighborhood of $[a, b] \times \{0\}$ contains points (λ, x) such that x is a nonzero critical point of ψ_λ .*

Formula (1.2) gives an interesting algebraic relation between the spectral flow and the Leray–Schauder degree and helps to understand why Theorems B (and also Theorem A) improves Krasnoselskij’s local bifurcation result. If $L_\lambda, \lambda \in [a, b]$, is a continuous path of self-adjoint Fredholm operators in a separable real Hilbert space H , and L_λ is of the form

$$L_\lambda = Id - K_\lambda,$$

where Id is, as previously, the identity and any K_λ is symmetric and compact, then one can prove (see [16]) that

$$(-1)^{\text{sf}(L, [a, b])} = \text{deg}_{\text{LS}}(L_a) \cdot \text{deg}_{\text{LS}}(L_b). \tag{1.2}$$

The above equality explains why the spectral flow is a finer invariant than the degree to detect bifurcation, even if it can be applied in a more restricted context. The spectral flow could be nonzero with a lack of sign jump of the degree. In other words, it detects bifurcation in some cases when the degree does not.

On the other hand, the spectral flow helps to prove *local* bifurcation results, as in Theorem B, and it seems unable to provide *global* bifurcation results. The reason is probably due to the fact that the spectral flow is defined for *linear operators*, while the degree works in nonlinear maps (see Example 3.5). A possible nonlinear version of the spectral flow could help to obtain global bifurcation results, but to the best of our knowledge it does not exist, and its construction (if possible) is an interesting and challenging open problem.

Motivated by these difficulties and, at the same time, by the purpose of studying global bifurcation, in this paper we face a different problem focusing our attention on bifurcation of target values of a suitable function. One of the results we present in this paper, Theorem 3.3, shows the existence of a global bifurcation branch of critical values of a C^1 map $f : \mathbb{R} \times X \rightarrow \mathbb{R}$, where X is a real Banach space and some topological conditions are verified. It includes, as a second result, the particular case in which X is a separable Hilbert space, f is C^2 , the Hessians of f with respect to the second variable at the points $(\lambda, 0)$,

$$L_\lambda := \frac{\partial^2 f}{\partial x^2}(\lambda, 0) : H \rightarrow H,$$

are Fredholm, and a sufficient condition to obtain bifurcation is given in terms of Morse index (Theorem 3.4). We also obtain a third global bifurcation result, Theorem 6.1, also for the C^2 case, when the Hessians L_λ are strongly indefinite and the Morse index is not defined. Adding a particular compactness assumption, which at this moment we are not able to remove, we prove a global bifurcation result if the spectral flow of L_λ in a suitable interval is nonzero. Theorem 6.1 clearly draws inspiration from the seminal papers [16,17] by Fitzpatrick, Pejsachowicz and Rabier and uses the spectral flow in a context of a global bifurcation problem.

In our first theorem, we obtain the bifurcation result assuming that two suitable topological pairs of inverse images of the map f_λ are not homotopy equivalent, for two different values of the parameter λ . This condition is sufficient to give bifurcation when combined with other assumptions (see below), such as a special Palais–Smale-type condition.

The paper is organized as follows. In Sect. 2 we recall some basic notions of homotopy equivalence of topological pairs and we summarize the construction of the spectral flow.

In Sect. 3 we present the bifurcation problem and we state our main results, Theorems 3.3 and 3.4. In Sect. 4 we show some technical results concerning deformation of level sets, which are used in the proofs of our main results. Such deformation results are original and have in our opinion some independent interest. Section 5 is devoted to the proof of Theorems 3.3 and 3.4. In Sect. 6 we provide a bifurcation theorem for strongly indefinite functionals where the corresponding invariant for the Hessian is the spectral flow. Finally, in Sect. 7 we illustrate the applicability of Theorem 3.4 with a second-order ODE.

2 Preliminaries

First of all, let us summarize the construction of the spectral flow in the approach followed by Fitzpatrick et al. [16]. Let H be an infinite-dimensional separable real Hilbert space. Consider an orthogonal decomposition

$$H = H_+ \oplus H_-, \tag{2.1}$$

with H_+ and H_- of infinite dimension. We call *symmetry* the linear operator $\mathcal{J} : H \rightarrow H$ which can be represented, in splitting (2.1), by the block matrix of operators

$$\begin{pmatrix} Id_{H_+} & 0 \\ 0 & -Id_{H_-} \end{pmatrix}.$$

Observe that we have infinitely many symmetries of H , depending on splittings like (2.1), and that $\mathcal{J}^2 = Id$ for any symmetry \mathcal{J} . Let $\{e_n^\pm, n \in \mathbb{N}\}$ be two Hilbert bases of H_+ and H_- , respectively, and call H_n the $2n$ -dimensional subspace of H generated by $\{e_k^\pm, k \leq n\}$. Denote by $P_n : H \rightarrow H_n$ the orthogonal projection. Given a self-adjoint compact linear operator $K : H \rightarrow H$, consider $L = \mathcal{J} + K$. Let $L_n : H_n \rightarrow H_n$ be the operator given by

$$L_n x = P_n(Lx)$$

and call *signature* of L_n the integer number

$$\text{sign } L_n = \mu(-L_n) - \mu(L_n),$$

where, as already said, $\mu(\cdot)$ is the Morse index of the considered operator. In [16, Lemma 1.1] the following result is proven.

Lemma (Fitzpatrick, Pejsachowicz, Recht) *Suppose that $L = \mathcal{J} + K$ is an automorphism of H . Then, there is a positive integer N such that $\text{sign } L_n$ is constant if $n \geq N$.*

The above eventually constant integer is called *generalized signature* of L with respect to \mathcal{J} and is denoted by $\text{sign}_{\mathcal{J}}(L)$. It is possible to prove that this integer actually depends on the symmetry \mathcal{J} (as the notation suggests), but not on the chosen Hilbert bases $\{e_n^\pm\}$ of the subspaces of H related to \mathcal{J} . In other words, even if the definition of any L_n clearly depends on the chosen bases $\{e_n^\pm\}$, one can prove that $\text{sign}_{\mathcal{J}}(L)$ does not.

Consider now a continuous path $K_\lambda, \lambda \in [a, b]$, of self-adjoint compact operators of H . Given a symmetry \mathcal{J} of H , take the path $L_\lambda = \mathcal{J} + K_\lambda$ and suppose that L_a and L_b are automorphisms. The *spectral flow* of the path L is defined as

$$\text{sf}(L, [a, b]) = \frac{\text{sign}_{\mathcal{J}}(L_b) - \text{sign}_{\mathcal{J}}(L_a)}{2}. \tag{2.2}$$

One can prove that the above formula does not depend on \mathcal{J} even though $\text{sign}_{\mathcal{J}}(L_a)$ and $\text{sign}_{\mathcal{J}}(L_b)$ do. The definition of spectral flow can be extended to any continuous path of

self-adjoint Fredholm operators $L_\lambda, \lambda \in [a, b]$, such that L_a and L_b are invertible. In the particular case when $L_\lambda = T + K_\lambda$, where T is Fredholm and self-adjoint, and any K_λ is compact and self-adjoint, the spectral flow is defined as

$$\text{sf}(L, [a, b]) = \dim(V^-(L_a) \cap V^+(L_b)) - \dim(V^-(L_b) \cap V^+(L_a)), \tag{2.3}$$

which is finite. In the case of a general path of self-adjoint Fredholm operators L_λ , it can be proven the existence of a path M_λ of automorphisms of H (called *cogredient parametrix*) such that

$$M_\lambda^* L_\lambda M_\lambda = T + K_\lambda,$$

where M_λ^* is the adjoint of M_λ , T is Fredholm and self-adjoint and any K_λ is compact and self-adjoint. Hence, the definition $\text{sf}(L, [a, b])$ is given by (2.3) applied to $T + K_\lambda$ and this does not depend on the choice of M_λ . The reader can see [16] for an accurate study of the properties of the spectral flow.

We now recall a basic definition of homotopy theory. Consider a pair of topological spaces, that is, a pair (X, A) such that X is any topological space and $A \subseteq X$. Given a continuous map $F : X \rightarrow Y$ between topological spaces, to simplify the notation and avoid the introduction of new symbols, we will denote by $F : (X, A) \rightarrow (Y, B)$ the corresponding continuous maps between pairs (if, of course, it is well defined, that is, if $F(A) \subseteq B$).

Two functions $F, G : (X, A) \rightarrow (Y, B)$ are *homotopic* if there exists a continuous map $H : [0, 1] \times X \rightarrow Y$ such that $H(0, x) = F(x)$, $H(1, x) = G(x)$ and $H(t, x) \in B$ for all $x \in A$ and all $t \in [0, 1]$. Two pairs (X, A) and (Y, B) are *homotopy equivalent* if there exist two maps $F : (X, A) \rightarrow (Y, B)$ and $G : (Y, B) \rightarrow (X, A)$ such that $G \circ F$ and $F \circ G$ are homotopic to the identity (in the sense described above). If this is the case, we have isomorphisms in the relative homology groups $H_i(X, A) \cong H_i(Y, B)$ for every i , see for example [18, pag. 118]. We consider here singular homology groups with coefficients in \mathbb{R} .

Let now X be a real Banach space. Consider a C^1 function $\phi : X \rightarrow \mathbb{R}$, an isolated critical point p of ϕ and any $k \in \mathbb{N}$. Denote $c = \phi(p)$ and let $\phi^{(c)}$ be the set $\{x \in X : \phi(x) \leq c\}$. Following Chang [9] (see also [22]), we define the k th-local critical group $C_k(\phi, p)$ as

$$C_k(\phi, p) := H_k\left(\phi^{(c)} \cap U, (\phi^{(c)} \setminus \{p\}) \cap U\right) \tag{2.4}$$

where, as previously, we consider here singular homology with coefficients in \mathbb{R} . The set U is an open neighborhood of p not containing any other critical point of ϕ . Let us point out that, by general properties in singular homology theory, the above definition does not depend on U .

As we will see, the key ingredient to prove local bifurcation of critical points is the invariance of the critical groups under small perturbations. The next theorem is proven in [22, Section 8.9].

Theorem 2.1 *Let U be an open neighborhood of a given point v in a Hilbert space H and consider a map $\phi \in C^2(U, \mathbb{R})$ having v as the only critical point and satisfying the Palais–Smale condition over a closed ball $B(v, r) \subseteq U$. Then, there exists $\eta > 0$, depending only upon ϕ , such that, for any ψ satisfying the same assumptions, the condition*

$$\sup_{u \in U} (|\psi(u) - \phi(u)| + (|\nabla \psi(u) - \nabla \phi(u)|)) \leq \eta$$

implies

$$\dim C_k(\psi, v) = \dim C_k(\phi, v), \quad k \in \mathbb{N}.$$

As a consequence, if a function f as in Theorem A has nonvanishing spectral flow in the interval $[a, b]$, then we have

$$\dim C_k(f_a, 0) \neq \dim C_k(f_b, 0)$$

and thus $0 \in X$ cannot be isolated as a critical point for every λ .

We will consider a global version of the local critical groups explained by Theorem 2.2 that can be found in [9, Theorem 5.1.27].

Theorem 2.2 *Let X be a real Banach space. Assume that $\varphi : X \rightarrow \mathbb{R}$ is C^1 and satisfies the Palais–Smale condition. Suppose that c is an isolated critical value of φ , where the critical points of φ in $\varphi^{-1}(c)$ are z_1, \dots, z_m . Then for sufficiently small $\varepsilon > 0$ we have*

$$H_k \left(\varphi^{(c+\varepsilon)}, \varphi^{(c-\varepsilon)} \right) = \bigoplus_{j=1}^m C_k(\varphi, z_j)$$

for every $k = 0, 1, 2, \dots$

Theorem 2.2 plays a central role in the proof of our main results.

3 Main results

Let us start by presenting the following conditions which will be considered throughout the paper.

Definition 3.1 Let X be a real Banach space. We say that a C^1 map $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ satisfies the *basic condition* if

$$f(\lambda, 0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(\lambda, 0) = f'_\lambda(0) = 0, \quad \forall \lambda \in \mathbb{R}. \tag{3.1}$$

The map $x \in X \mapsto f(\lambda, x)$, defined for a given real λ , will be also denoted by f_λ and its Fréchet derivative at a point x by $f'_\lambda(x)$. The following subset of \mathbb{R}^2

$$S_f = \{(\lambda, y) \in \mathbb{R}^2 : f_\lambda(x) = y, f'_\lambda(x) = 0 \text{ for some } x \in X\}$$

will be called the set of *critical pairs*. In other words, S_f is the set of pairs (λ, y) such that y is a critical value of f_λ . The line $Z = \mathbb{R} \times \{0\} \subseteq S_f$ is regarded as the set of the *trivial critical pairs*. We say that a trivial critical pair $(\lambda_0, 0) \in Z$ is a *bifurcation point* if every neighborhood of $(\lambda_0, 0)$ contains nontrivial critical pairs. Then, we see that

$$E_f := \overline{S_f} \setminus Z \tag{3.2}$$

is the union of the bifurcation points and the nontrivial critical pairs. We will also consider a subset of trivial critical pairs:

$$Z_{\text{in}} = [-1, 1] \times \{0\} \subset \mathbb{R}^2. \tag{3.3}$$

Definition 3.2 (*(PS)-type conditions*). Let $I \subset \mathbb{R}$ be a compact interval and $c \in \mathbb{R}$ be given. We say that f satisfies the $(\text{PS})_{I,c}$ condition if, for every sequence (λ_n, x_n) in $I \times X$ such that $f_{\lambda_n}(x_n) \rightarrow c$ and such that $f'_{\lambda_n}(x_n) \rightarrow 0 \in X^*$, there exists a subsequence (x_{n_k}) converging to a point $x_0 \in X$.

Let J be a subset of \mathbb{R} . We say that f satisfies the $(\text{PS})_{I,J}$ condition if it satisfies the $(\text{PS})_{I,c}$ condition for every $c \in J$. We say that f satisfies the $(\text{PS})_{\mathbb{R},\mathbb{R}}$ condition if it satisfies the $(\text{PS})_{[-N,N],\mathbb{R}}$ condition for every $N \in \mathbb{N}$.

We are now in a position to state the following two theorems which, associated with Theorem 6.1, are our main results. Actually, Theorem 3.4 is a consequence of Theorem 3.3, particularly important in applications. The proof of both results is given in Sect. 5.

Theorem 3.3 *Let $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ be a C^1 function verifying basic condition (3.1). Denote by $\partial_1 f : \mathbb{R} \times X \rightarrow \mathbb{R}$ the map*

$$\partial_1 f(\lambda, x) = \frac{\partial f}{\partial \lambda}(\lambda, x).$$

Assume that the following conditions hold:

- (i) *f satisfies the $(PS)_{\mathbb{R}, \mathbb{R}}$ condition.*
- (ii) *$\partial_1 f$ is bounded in the sets of the form*

$$f_{[-N, N]}^{-1}([-N, N]) := \{(\lambda, x) \in [-N, N] \times X : f(\lambda, x) \in [-N, N]\}, \quad \forall N \in \mathbb{N}.$$

- (iii) *The trivial critical pairs $(-1, 0)$, $(1, 0)$ are not bifurcation points of f .*
- (iv) *There exists $\varepsilon_* > 0$ such that for every $0 < \varepsilon < \varepsilon_*$ the pairs of spaces $(f_{-1}^{(\varepsilon)}, f_{-1}^{(-\varepsilon)})$ and $(f_1^{(\varepsilon)}, f_1^{(-\varepsilon)})$ are not homotopy equivalent.*

Then, E_f contains a connected subset intersecting Z_{in} which either

1. *is unbounded in \mathbb{R}^2 , or else*
2. *intersects $\{-1, 1\} \times \mathbb{R}$.*

Theorem 3.4 *Let H be a separable real Hilbert space and consider a C^2 function $f : \mathbb{R} \times H \rightarrow \mathbb{R}$ satisfying basic condition (3.1).*

Denote by I the interval $[-1, 1]$ and assume that the following assumptions hold:

- (i) *f satisfies the $(PS)_{\mathbb{R}, \mathbb{R}}$ condition.*
- (ii) *$\partial_1 f$ is bounded in the sets of the form $f_{[-N, N]}^{-1}([-N, N])$ for every $N \in \mathbb{N}$.*
- (iii) *For $i = -1, 1$, the point $0 \in H$ is a nondegenerate critical point and the only critical point of f_i with value 0.*
- (iv) *Assume that, for every $\lambda \in I$, the Hessian of f at zero*

$$L_\lambda := \frac{\partial^2 f}{\partial x^2}(\lambda, 0) : H \rightarrow H$$

is Fredholm and suppose

$$\mu(L_{-1}) \neq \mu(L_1).$$

where μ denotes the Morse index. Here we assume that $\mu(L_{-1})$ and $\mu(L_1)$ are finite.

Then the conclusion of Theorem 3.3 holds.

One may ask whether the above theorems work with critical points instead of critical values, as in the bifurcation theorems of Rabinowitz [25]. The answer is negative as proven by R. Böhme [8] (see also [1,28]). We directly construct, and show below, a counterexample regarding this lack of global behavior in the conditions of Theorem 3.4.

Example 3.5 A classic problem in analysis consists in proving the existence of a *cone eversion*. This is a smooth function $c : I \times C \rightarrow \mathbb{R}$, where

$$I = [-1, 1], \quad C = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\},$$

such that

$$c(-1, x) = \|x\|, \quad c(1, x) = -\|x\|$$

and such that c_λ has no critical points in C for $\lambda \in I$. The existence of such a function may seem counterintuitive but actually is guaranteed by the parametric holonomic approximation theorem, see Example 4.1.1 in [10]. An explicit formula for c was computed in [27]. Taking \mathbb{R}^2 as the Hilbert space H in Theorem 3.4, we construct $f : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as follows.

The formula for c in polar coordinates as given in [27] is

$$c(\lambda, (\alpha, r)) = 2t + g(\lambda, \alpha) + (r - 2)h(\lambda, \alpha)$$

for two functions g, h satisfying

$$\left(\frac{\partial g}{\partial \alpha}(\lambda, \alpha), h(\lambda, \alpha) \right) \neq (0, 0)$$

for all (λ, α) . Given $a, b \in \mathbb{R}$, consider $p_{a,b}(r) = (3a - b + 1)r^2 + (-2a + b - 2)r^3 + r^4$, which is the polynomial function satisfying the properties

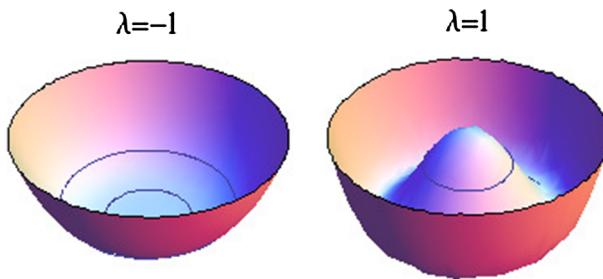
$$p_{a,b}(0) = p'_{a,b}(0) = 0, \quad p_{a,b}(1) = a, \quad p'_{a,b}(1) = b,$$

and define

$$f(\lambda, (\alpha, r)) = p_{g(\lambda,\alpha),3h(\lambda,\alpha)}(r).$$

Putting, as usual, $f_\lambda := f(\lambda, \cdot)$, we easily verify

- (i) $f(\lambda, x), \nabla f_\lambda(x) \rightarrow +\infty$ uniformly in λ , as $\|x\| \rightarrow \infty$.
- (ii) $f(\lambda, 0) = 0, \nabla f_\lambda(0) = 0$ for all $\lambda \in I$.
- (iii) $f(-1, (\alpha, r)) = 4r^2 - 3r^3 + r^4, f(1, (\alpha, r)) = -2r^2 - r^3 + r^4$.
- (iv) $\nabla f_\lambda(x) \neq 0$ for $\|x\| = 1, \lambda \in I$.



Property (iii) implies

$$\mu \left(\frac{\partial^2 f}{\partial x^2}(-1, 0) \right) = 0 \quad \text{and} \quad \mu \left(\frac{\partial^2 f}{\partial x^2}(1, 0) \right) = 2.$$

In addition, the path $\lambda \mapsto \nabla f_\lambda(0)$ does not satisfy the necessary hypothesis for the “classical” global bifurcation theorem (in the sense of Rabinowitz). Also, one may check that f is in the conditions of Theorem 3.3. Let

$$K_f = \{(\lambda, x) \in I \times X : \nabla f_\lambda(x) = 0\}$$

and notice that condition (iv) forces $K_f \cap (I \times \partial B(0, 1)) = \emptyset$. Then, the connected component of $\overline{K_f} \setminus (I \times \{0\})$ containing bifurcation points $(\lambda, 0)$ belongs to $I \times B(0, 1)$ and does not intersect the subspaces $\lambda = -1, 1$. That is, there is *local* but not *global* bifurcation.

The function f_1 has a set of nontrivial critical points in X of the form $\partial B(0, R)$ with $R \in (1, 2)$. Actually the connected component of K_f containing $\{1\} \times \partial B(0, R)$ lies outside $I \times B(0, 1)$, but its image by f crosses the value 0.

This example shows that it is not possible to prove *global* bifurcation of *critical points* relying only on the fact that the Morse index is nonconstant of Morse index (and, in the infinite-dimensional case, nonvanishing spectral flow).

The proofs of Theorems 3.3 and 3.4 require several lemmas. We start with some results concerning the level sets and the $(PS)_{I,c}$ condition.

4 Deformation theorems

We present here some technical lemmas which will play a fundamental role in the proof of Theorems 3.3 and 3.4. Next lemma is a sort of characterization of the $(PS)_{I,c}$ property. In this section, I denotes a real compact interval and the function f is not required to verify basic condition (3.1).

Lemma 4.1 *Let $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ be a C^1 function satisfying the $(PS)_{I,a}$ condition for a given $a \in \mathbb{R}$ which is a regular value for every f_λ , with $\lambda \in I$. Then, there exist $\varepsilon, \delta > 0$ such that $\|f'_\lambda(x)\| \geq \varepsilon$ for every $(\lambda, x) \in f_I^{-1}((a - \delta, a + \delta))$.*

Proof Assume by contradiction that there exists a sequence $(\lambda_n, x_n) \in I \times X$ such that $\|f'_{\lambda_n}(x_n)\| \rightarrow 0$ and $f(\lambda_n, x_n) \rightarrow a$. Taking a convergent subsequence (λ_{n_j}) and a convergent subsequence (x_{n_j}) given by the $(PS)_{I,a}$ condition, we deduce that a is a singular value. \square

Lemma 4.2 *Consider a closed set $J \subset \mathbb{R}$ and let $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ be a C^1 function satisfying the $(PS)_{I,J}$ condition. Then the set*

$$K = \{(\lambda, c) \in I \times J : \exists x \in X, f_\lambda(x) = c, f'_\lambda(x) = 0\}$$

is closed.

Proof Take a convergent sequence $(\lambda_n, c_n) \rightarrow (\lambda, c) \in I \times J$, $(\lambda_n, c_n) \in K$, and consider $x_n \in X$, for any n , such that $f(\lambda_n, x_n) = c_n, f'_\lambda(x_n) = 0$. By the $(PS)_{I,c}$ condition there exists a convergent subsequence of (x_n) . Thus, the continuity of f'_λ shows that $(\lambda, c) \in K$. \square

Lemma 4.3 *Let $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ be a C^1 function satisfying $(PS)_{I,\{a,b\}}$ where $a < b$ are regular values of f_λ for $\lambda \in I$. Denote $U_t = (a - t, a + t) \cup (b - t, b + t)$. Assume that $\partial_1 f$ is bounded in $f_I^{-1}(U_\delta)$ for some $\delta > 0$. Then, there exists a function $v : \mathbb{R} \times X \rightarrow X$ such that*

- (i) v is bounded and locally Lipschitz.
- (ii) $\partial_1 f(\lambda, x) + f'_\lambda(x)[v(\lambda, x)] < 0$ if $\lambda \in I$ and $f_\lambda(x) = a$ or b .

Remark 4.4 The reader can understand the notation $f_I^{-1}(U_\delta)$ by the analogous one in the statement of Theorem 3.3.

Proof First, by Lemma 4.1 we may assume, taking a smaller $\delta > 0$ if necessary, that, for some $\varepsilon > 0$, $\|f'_\lambda(x)\| \geq \varepsilon$ for all $(\lambda, x) \in f_I^{-1}(U_\delta)$. Let $v : \mathbb{R} \rightarrow [0, 1]$ be a continuous function equal to 0 in $U_{\delta/2}$ and equal to 1 in $\mathbb{R} \setminus U_\delta$. We shall construct $v(\lambda, x)$ satisfying the inequality

$$\partial_1 f(\lambda, x) + f'_\lambda(x)[v(\lambda, x)] < v(f_\lambda(x)) (|\partial_1 f(\lambda, x)| + 1). \tag{4.1}$$

Thus, condition (ii) will follow. To this purpose, fix $(\lambda, x) \in I \times X$ and assume $f_\lambda(x) \in U_\delta$, so that $\|f'_\lambda(x)\| \geq \varepsilon$. There exists $w_{\lambda,x} \in X$ such that

$$\|w_{\lambda,x}\| \leq \frac{2}{\varepsilon} (|\partial_1 f(\lambda, x)| + 1)$$

and

$$f'_\lambda(x)[w_{\lambda,x}] \leq -(|\partial_1 f(\lambda, x)| + 1).$$

Hence, we have $\partial_1 f(\lambda, x) + f'_\lambda(x)[w_{\lambda,x}] < 0$. On the other hand, if $f_\lambda(x) \notin U_\delta$, we define $w_{\lambda,x} = 0$. In both cases, we obtain

$$\partial_1 f(\lambda, x) + f'_\lambda(x)[w_{\lambda,x}] < v(f_\lambda(x)) (|\partial_1 f(\lambda, x)| + 1).$$

By continuity, every $(\lambda, x) \in I \times X$ has a neighborhood $V^{\lambda,x}$ in $\mathbb{R} \times X$ such that

$$\partial_1 f(\alpha, y) + f'_\alpha(y)[w_{\lambda,x}] < v(f_\alpha(y)) (|\partial_1 f(\alpha, y)| + 1) \tag{4.2}$$

for every $(\alpha, y) \in V^{\lambda,x}$. Since $I \times X$ is paracompact and $\{V^{\lambda_i,x_i}\}_{\lambda_i,x_i}$ is an open covering of $I \times X$, we obtain a countable, locally finite refinement $V_i \subseteq V^{\lambda_i,x_i}$ covering $I \times X$, and a locally Lipschitz partition of unity, that is, a collection of (locally Lipschitz) nonnegative functions $\eta_i : \mathbb{R} \times X \rightarrow \mathbb{R}$ with support in V_i and such that

$$\sum_{i \in \mathbb{N}} \eta_i(\lambda, x) = 1 \quad \forall (\lambda, x) \in I \times X,$$

where the above sum is locally finite. Recalling (4.2), we have, for any i and any $(\alpha, y) \in I \times X$,

$$\eta_i(\alpha, y) \partial_1 f(\alpha, y) + f'_\alpha(y)[\eta_i(\alpha, y)w_{\lambda,x}] \leq \eta_i(\alpha, y) v(f_\alpha(y)) (|\partial_1 f(\alpha, y)| + 1),$$

with the strict inequality if $\eta_i(\alpha, y) > 0$. Then, we obtain

$$\sum_{i \in \mathbb{N}} \eta_i(\alpha, y) (\partial_1 f(\alpha, y) + f'_\alpha(y)[w_{\lambda,x}]) < \sum_{i \in \mathbb{N}} \eta_i(\alpha, y) v(f_\alpha(y)) (|\partial_1 f(\alpha, y)| + 1).$$

Now, define

$$v(\lambda, x) = \sum_{i \in \mathbb{N}} \eta_i(\lambda, x) w_{\lambda_i,x_i}.$$

We have

$$\partial_1 f(\alpha, y) + f'_\alpha(y)[v(\lambda, x)] < v(f_\alpha(y)) (|\partial_1 f(\alpha, y)| + 1).$$

Since $\|w_{\lambda,x}\| \leq \frac{2}{\varepsilon} (|\partial_1 f(\lambda, x)| + 1)$ for every $(\lambda, x) \in I \times X$, we have, for any $(\alpha, y) \in \mathbb{R} \times X$,

$$\|v(\alpha, y)\| \leq \sum_{i \in \mathbb{N}} \eta_i(\alpha, y) \|w_{\lambda_i,x_i}\| \leq \frac{2}{\varepsilon} \left(\sup_{(\lambda,x) \in f_I^{-1}(U_\delta)} |\partial_1 f(\lambda, x)| + 1 \right)$$

and v is bounded. Finally, it is immediate to observe that v is locally Lipschitz and this concludes the proof. \square

The next two theorems are key to the study of deformation of one-parameter families of functions. Theorems 4.5 and 4.6 generalize analogous results in the textbook [9], where they appear under much stronger conditions.

Theorem 4.5 (Deformation Theorem). *Let $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ be C^1 and $a < b$ regular values of f_λ for $\lambda \in I$. Assume that f satisfies (PS) $_{I, \{a,b\}}$ and that $\partial_1 f$ is bounded in $f_I^{-1}(U_\delta)$ for some $\delta > 0$. Then the pairs $(f_{-1}^{(b)}, f_{-1}^{(a)})$ and $(f_1^{(b)}, f_1^{(a)})$ are homotopy equivalent.*

Proof The vector field v constructed in Lemma 4.3 is locally Lipschitz and bounded, so it generates a globally defined flow

$$\phi : \mathbb{R} \times \mathbb{R} \times X \rightarrow X,$$

which satisfies

$$\frac{\partial}{\partial t} \phi(t, \lambda, x) = v(t + \lambda, \phi(t, \lambda, x)).$$

For $(\lambda_0, x_0) \in I \times X$, the curve $x(t) = \phi(t, \lambda_0, x_0)$ is the solution of the initial value problem

$$\begin{cases} x'(t) = v(\lambda_0 + t, x(t)) \\ x(\lambda_0) = x_0. \end{cases} \tag{4.3}$$

For any $(\lambda, x) \in \mathbb{R} \times X$ define $\varphi(t) = f_{t+\lambda}(\phi(t, \lambda, x))$, which verifies

$$\varphi'(t) = \partial_1 f(t + \lambda, \phi(t, \lambda, x)) + f'_{t+\lambda}(\phi(t, \lambda, x))[v(t + \lambda, \phi(t, \lambda, x))].$$

Notice that, by condition (ii) of Lemma 4.3, $\varphi(t) = a$ implies $\varphi'(t) < 0$. So, if $\varphi(0) \leq a$, then we have $\varphi(t) < a$ for every $t > 0$. If $x \in f_\lambda^{(a)}$, then, by the definition of φ and the previous consideration, $\phi(t, \lambda, x) \in f_{t+\lambda}^{(a)}$. We can write it as

$$\phi(t, \lambda, f_\lambda^{(a)}) \subseteq f_{t+\lambda}^{(a)}.$$

Similarly, for b we obtain

$$\phi(t, \lambda, f_\lambda^{(b)}) \subseteq f_{t+\lambda}^{(b)}.$$

Applying the same reasoning to the function $\bar{f}(\lambda, x) := f(-\lambda, x)$, we obtain a (globally defined) flow $\bar{\phi} : \mathbb{R} \times \mathbb{R} \times X \rightarrow X$ such that

$$\bar{\phi}(t, \lambda, \bar{f}_\lambda^{(a)}) \subseteq \bar{f}_{t+\lambda}^{(a)}, \quad \bar{\phi}(t, \lambda, \bar{f}_\lambda^{(b)}) \subseteq \bar{f}_{t+\lambda}^{(b)}$$

for every $t > 0$.

The functions $F(x) = \phi(2, -1, x)$, $G(x) = \bar{\phi}(2, -1, x)$ can be seen, with an abuse of notation (see Preliminaries), as continuous functions of pairs

$$\begin{aligned} F &: (f_{-1}^{(b)}, f_{-1}^{(a)}) \rightarrow (f_1^{(b)}, f_1^{(a)}) \\ G &: (f_1^{(b)}, f_1^{(a)}) \rightarrow (f_{-1}^{(b)}, f_{-1}^{(a)}). \end{aligned}$$

Define

$$H_t(x) = \bar{\phi}(t, 1 - t, \phi(t, -1, x)), \quad \bar{H}_t(x) = \phi(t, 1 - t, \bar{\phi}(t, -1, x)).$$

Then, we verify that H_t, \bar{H}_t are functions of pairs

$$H_t : (f_{-1}^{(b)}, f_{-1}^{(a)}) \rightarrow (f_{-1}^{(b)}, f_{-1}^{(a)}), \quad \bar{H}_t : (f_1^{(b)}, f_1^{(a)}) \rightarrow (f_1^{(b)}, f_1^{(a)})$$

for $t > 0$. In addition, $H_0(x) = x, H_2(x) = G(F(x)), \bar{H}_0(x) = x, \bar{H}_2(x) = F(G(x))$. Thus, F and G are homotopy equivalences. □

Theorem 4.6 *Let (r, s) be an open bounded interval and let $a, b : I \rightarrow (r, s)$ be two continuous functions such that $a(\lambda) < b(\lambda)$ for any λ , and $f : \mathbb{R} \times X \rightarrow \mathbb{R}$ a C^1 function satisfying the $(PS)_{I, [r, s]}$ -condition.*

Assume that $\partial_1 f : I \times X \rightarrow \mathbb{R}$ is bounded in the set $f_I^{-1}(r, s)$. Assume also that $a(\lambda), b(\lambda)$ are regular values of f_λ for every $\lambda \in I$. Then the pairs $(f_{-1}^{(b(-1))}, f_{-1}^{(a(-1))})$ and $(f_1^{(b(1))}, f_1^{(a(1))})$ are homotopy equivalent.

Proof By hypothesis, the graphs of a, b in $I \times \mathbb{R}$ do not intersect the set

$$K = \{(\lambda, c) \in I \times [r, s] : \exists x \in X, f_\lambda(x) = c, f'_\lambda(x) = 0\}.$$

Since K is closed by Lemma 4.2, the graphs of the functions a, b can be approximated by C^1 functions with the same endpoints $a(\pm 1)$ and $b(\pm 1)$. Thus we may assume a, b are C^1 . We consider the function

$$g(\lambda, x) = \frac{f(\lambda, x) - a(\lambda)}{b(\lambda) - a(\lambda)}.$$

Thus, g satisfies

$$f_\lambda^{(a(\lambda))} = g_\lambda^{(0)}, \quad f_\lambda^{(b(\lambda))} = g_\lambda^{(1)},$$

and it is easy to check that g verifies the conditions of Theorem 4.5 with $a = 0, b = 1$. □

5 Proofs of Theorems 3.3 and 3.4

First, we prove two technical lemmas.

Lemma 5.1 *Let $D \subset \mathbb{R}^2$ be a closed rectangle, A_0, A_1 be two opposite sides of D and B_0, B_1 be the other two. Let $S \subset D$ be a compact set. If S does not contain a connected component intersecting B_0 and B_1 , then there exists a continuous curve $\gamma : [0, 1] \rightarrow D$ with $\gamma(0) \in A_0$ and $\gamma(1) \in A_1$ and not intersecting S, B_0 and B_1 .*

Proof Without loss of generality, we can assume that A_0, A_1 are vertical sides and B_0, B_1 horizontal. Let $S_i = S \cap B_i, i = 0, 1$ and assume there is no connected component of S which intersects S_0 and S_1 at the same time. We may assume that $S_i \neq \emptyset$ because, otherwise, a horizontal line close to B_i would be the desired curve. By Whyburn’s lemma there is a separation $S = C_0 \cup C_1$ where $S_i \subseteq C_i$ and C_0, C_1 are two nonempty disjoint compact sets. The sets $C_i \cup B_i$ are also nonempty disjoint and compact. By the smooth Urysohn lemma [29, corollary of Theorem 1.11] there exists a C^∞ smooth function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $g(x) = i$ for all $x \in C_i \cup B_i, i = 0, 1$.

Take $\alpha \in (0, 1)$ a regular value of the three functions $g, g|_{A_0}, g|_{A_1}$ and consider

$$L = g^{-1}(\alpha) \cap (D \setminus (B_0 \cup B_1))$$

which is a one-dimensional differentiable manifold with boundary, with $\partial L = L \cap (A_0 \cup A_1)$ and $L \cap S = \emptyset$. Since $g(x) = i$ for all $x \in B_i, i = 0, 1$, and α is a regular value of $g|_{A_i}, i = 0, 1$, then the cardinality of $L \cap A_i$ is odd, for $i = 0, 1$. Therefore, there is a curve of L having one endpoint in each of the A_i 's. \square

Lemma 5.2 *Let $a, b : [-1, 1] \rightarrow [-1, 1]$ be two continuous functions satisfying*

$$a(-1) = b(-1) = -1, \quad a(1) = b(1) = 1.$$

Then for every $\varepsilon > 0$ there exist continuous functions $\tilde{a}, \tilde{b}, c, d : [-1, 1] \rightarrow [-1, 1]$ such that

1. $\tilde{a}(-1) = \tilde{b}(-1) = -1, \quad \tilde{a}(1) = \tilde{b}(1) = 1,$
2. $c(-1) = d(-1) = -1, \quad c(1) = d(1) = 1,$
3. $\|\tilde{a} - a\|_\infty < \varepsilon, \quad \|\tilde{b} - b\|_\infty < \varepsilon,$
4. $\tilde{a}(c(t)) = \tilde{b}(d(t)).$

Proof We approximate a, b by smooth functions $\tilde{a}, \tilde{b} : [-1, 1] \rightarrow [-1, 1]$ satisfying conditions 1, 3 above and

- (i) $\tilde{a}'(-1), \tilde{b}'(-1), \tilde{a}'(1), \tilde{b}'(1) > 0,$
- (ii) the critical values of \tilde{a} and \tilde{b} are disjoint,
- (iii) $\tilde{a}(x), \tilde{b}(x) \in (-1, 1)$ for all $x \in (-1, 1)$.

Let $D = [-1, 1] \times [-1, 1]$ and $\phi : D \rightarrow \mathbb{R}$, defined as $\phi(x, y) = \tilde{a}(x) - \tilde{b}(y)$. Condition (ii) guarantees that 0 is a regular value of $\phi|_{D^0}$, where D^0 stands for the interior of D . Let $L = \phi^{-1}(0)$. Since condition (iii) implies

$$\begin{aligned} \phi(1, t) &> 0, & \phi(t, -1) &> 0, \\ \phi(-1, t) &< 0, & \phi(t, 1) &< 0, \end{aligned}$$

for all $t \in (-1, 1)$, we have $L \cap \partial D = \{(-1, -1), (1, 1)\}$. Even if D is not a manifold with boundary (because of the vertices) by condition (i), L can be parameterized near $(-1, -1)$ and $(1, 1)$ with curves entering D^0 . Thus, L is a differentiable manifold of dimension 1 with boundary $\{(-1, -1), (1, 1)\}$ which must connect these two points. Parameterize the curve inside L connecting $(-1, -1)$ and $(1, 1)$ by $(c(t), d(t))$. Then c, d satisfy

$$0 = \phi(c(t), d(t)) = \tilde{a}(c(t)) - \tilde{b}(d(t)).$$

\square

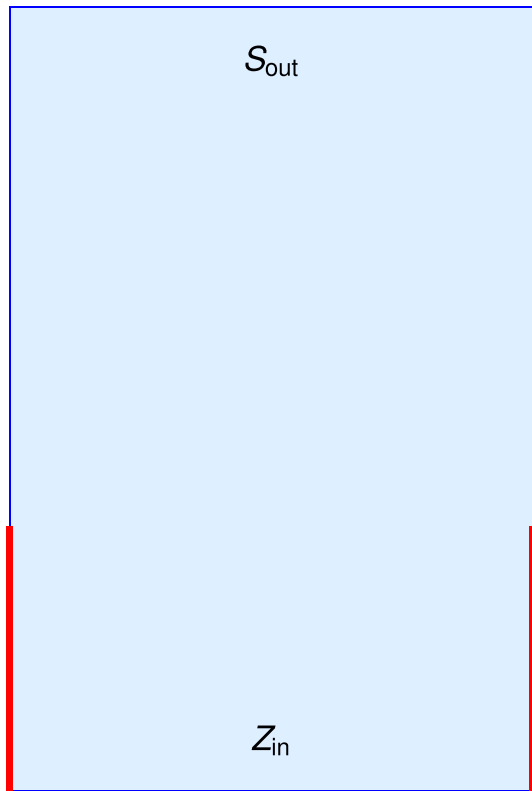
Proof of Theorem 3.3 Let $\varepsilon > 0$ be small enough so that the two disks $B_{-1} = B((-1, 0), \varepsilon)$ and $B_1 = B((1, 0), \varepsilon)$ contain only trivial critical pairs.

Let G_f be the connected component of $E_f \cup Z_{in}$ containing Z_{in} [recall formulas (3.2) and (3.3) for the definitions of E_f and Z_{in}]. Assume by contradiction that none of alternatives 1 and 2 in the statement of the theorem, are satisfied for G_f . Then, there is $R > 0$ such that $G_f \subseteq [-1, 1] \times [-R, R]$.

The set $D_+ = [-1, 1] \times [0, R]$ is homeomorphic to a closed rectangle, where the vertical sides correspond to $\{-1, 1\} \times [0, \varepsilon]$ and the horizontal lines to the rest of ∂D_+ , this is Z_{in} and the three segments

$$S_{out} := (\{-1, 1\} \times [\varepsilon, R]) \cup ([-1, 1] \times \{R\}).$$

Fig. 1 The red sides correspond to the vertical sides of the square and the blue sides, to the horizontal ones (colour figure online)



The reader can see the above Fig. 1. If claim 2 of the theses of the theorem fails, then there is no connected set inside E_f that intersects at the same time both horizontal sides of the rectangle. Therefore, applying Lemma 5.1 which is invariant by homeomorphisms, as it is easy to see, we conclude that there exists a continuous curve

$$u_+ : [-1, 1] \rightarrow D_+ \setminus E_f$$

such that

$$u_+(-1) = (-1, \varepsilon/2), \quad u_+(1) = (1, \varepsilon/2).$$

Similarly, define D_- in the lower semiplane and a map

$$u_- : [-1, 1] \rightarrow D_- \setminus E_f$$

such that

$$u_-(-1) = (-1, -\varepsilon/2), \quad u_-(1) = (1, -\varepsilon/2),$$

and having analogous properties to u_+ . Denote

$$u_+(t) = (\lambda_+(t), y_+(t)), \quad u_-(t) = (\lambda_-(t), y_-(t)).$$

By Lemma 5.2 with $\varepsilon < \text{dist}(S_f, \text{Im}(u_{\pm}))$ and a, b replaced by λ_-, λ_+ , we obtain functions $\tilde{\lambda}_-, \tilde{\lambda}_+, c, d : [-1, 1] \rightarrow [-1, 1]$ such that

1. $\tilde{u}_{\pm}(t) := (\tilde{\lambda}_{\pm}(t), y_{\pm}(t)) \in \mathbb{R}^2 \setminus S_f$ for all $t \in [-1, 1]$,
2. $\tilde{\lambda}_-(c(t)) = \tilde{\lambda}_+(d(t))$.

Now define

$$\begin{aligned} \lambda(t) &= \tilde{\lambda}_-(c(t)) = \tilde{\lambda}_+(d(t)), \\ a(t) &= y_-(c(t)), \\ b(t) &= y_+(d(t)). \end{aligned}$$

By the properties of the curves \tilde{u}_{\pm} we know that $a(t)$ and $b(t)$ are regular values of $f_{\lambda(t)} : X \rightarrow \mathbb{R}$, for any $t \in [0, 1]$. Applying Theorem 4.6 to the family of maps $f_{\lambda(t)}$, we get the homotopy equivalence of pairs

$$\left(f_{-1}^{(b(-1))}, f_{-1}^{(a(-1))} \right) \cong \left(f_1^{(b(1))}, f_1^{(a(1))} \right).$$

In view of assumption (iii), we have

$$\left(f_{-1}^{(\delta)}, f_{-1}^{(-\delta)} \right) \cong \left(f_1^{(\delta)}, f_1^{(-\delta)} \right)$$

for any $\delta \in (0, \varepsilon/2)$, which contradicts hypothesis (iv). □

Proof of Theorem 3.4 We want to prove that conditions 3.4(iii) and 3.4(iv) imply conditions 3.4(iii) and 3.4(iv). [Actually conditions 3.4(i) and 3.4(ii) coincide with 3.4(i) and 3.4(ii).] Here, the reader could notice that we have used a simplified notation to refer to the assumptions of Theorems 3.3 and 3.4.

The facts that f is C^2 and that $0 \in H$ is a nondegenerate critical point [condition 3.4(iii)] imply 3.4(iii).

It remains to show that conditions 3.4(iii) and 3.4(iv) imply condition 3.4(iv). Assume the pairs of spaces $\left(f_{-1}^{(\delta)}, f_{-1}^{(-\delta)} \right)$ and $\left(f_1^{(\delta)}, f_1^{(-\delta)} \right)$ are homotopy equivalent for $\delta > 0$ arbitrarily small. Then, by Theorem 2.2 and the fact that $0 \in H$ is the only critical point with value 0 [condition 3.4(iii)], we can compute the critical groups for every $k \in \mathbb{N}$ as

$$C_k(f_i, 0) \cong H_k \left(f_i^{(\varepsilon/2)}, f_i^{(-\varepsilon/2)} \right)$$

for $i = -1, 1$.

The remainder of the proof is standard. Condition 3.4(iv) permits to reduce the computation of the critical groups to finite-dimensional spaces, and the condition

$$\mu(L_{-1}) \neq \mu(L_1)$$

implies that $C_k(f_{-1}, 0)$ is not isomorphic to $C_k(f_1, 0)$ for some k , which is a contradiction. □

6 Strongly indefinite functions

In this section we will prove a global bifurcation result for a class of nonlinear functionals verifying analogous conditions of Theorem 3.4, except for the fact that the Hessian operators of the functionals have here infinite-dimensional negative eigenspaces and thus the Morse index cannot be defined. Condition (iv) of Theorem 3.4 will be replaced in Theorem 6.1 by a more general one involving the spectral flow of the Hessian operators. On the other hand, Theorem 6.1 is not strictly considered as an extension of Theorem 3.4 because it requires a

special compactness assumption, as we will see below. With a slight abuse of notation, in the sequel $\nabla f(\lambda, x)$ will denote the Fréchet derivative of f with respect to the second variable.

Theorem 6.1 *Let $f : \mathbb{R} \times H \rightarrow \mathbb{R}$ be a C^2 function such that*

$$f(\lambda, 0) = 0 \quad \text{and} \quad \nabla f(\lambda, 0) = 0, \quad \forall \lambda \in \mathbb{R}.$$

Suppose that, for every $(\lambda, x) \in \mathbb{R} \times H$, one has $\nabla f(\lambda, x) = \mathcal{J}(x) - K(\lambda, x)$ where \mathcal{J} a symmetry of H and the range of $K : \mathbb{R} \times H \rightarrow H$ is contained in a compact set. Denote by I the interval $[-1, 1]$ and assume that the following assumptions hold:

- (i) *f satisfies the $(PS)_{\mathbb{R}, \mathbb{R}}$ condition.*
- (ii) *$\partial_1 f$ is bounded in the sets of the form $f_{[-N, N]}^{-1}([-N, N])$ for every $N \in \mathbb{N}$.*
- (iii) *For $i = -1, 1, 0 \in H$ is a nondegenerate critical point and the only critical point of f_i with value 0.*
- (iv) *Assume that, for every $\lambda \in I$, the Hessian of f with respect to the second variable at $x = 0$,*

$$L_\lambda := \partial_2 \nabla f(\lambda, 0),$$

is Fredholm for every $\lambda \in [-1, 1]$ and suppose

$$\text{sf}(L, I) \neq 0.$$

Then the conclusion of Theorem 3.3 holds.

Recalling Sect. 2, let $H = H_+ \oplus H_-$ be the orthogonal splitting produced by \mathcal{J} and let $\{e_n^\pm, n \in \mathbb{N}\}$ be two Hilbert bases of H_+ and H_- , respectively. Call H_n the $2n$ -dimensional subspace of H generated by $\{e_k^\pm, k \leq n\}$. As in Sect. 2, denote by $P_n : H \rightarrow H_n$ the orthogonal projection and define f_n as the restriction of $P_n f$ to $\mathbb{R} \times H_n$. Clearly we have

$$\begin{aligned} \nabla f(\lambda, x) &= \mathcal{J}(x) - K(\lambda, x), \\ \nabla f_n(\lambda, x) &= \mathcal{J}(x) - K_n(\lambda, x), \\ \partial_2 \nabla f_n(\lambda, x) &= \mathcal{J} - \partial_2 K_n(\lambda, x), \end{aligned}$$

where $K_n(\lambda, x) = P_n K(\lambda, x)$. We need three technical lemmas.

Lemma 6.2 *There are $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$, the balls $B((\pm 1, 0), \varepsilon) \subseteq \mathbb{R}^2$ contain only trivial critical pairs of f_m .*

Proof Assume the contrary. Then, we have a sequence $(\lambda_n, x_n) \in \mathbb{R} \times H$ such that

$$\begin{aligned} x_n &\in H_n \setminus H_{n-1} \\ \lambda_n &\rightarrow \lambda^* = \pm 1 \\ f(\lambda_n, x_n) &\rightarrow 0 \\ \nabla f(\lambda_n, x_n) &= 0. \end{aligned}$$

Since $\mathcal{J}(H_n) = H_n$ and $P_n \rightarrow Id$ uniformly in compact sets, and since K has compact image, we have a subsequence, we still call (x_n) , such that

$$x_n = \mathcal{J} P_n K(\lambda_n, x_n) \rightarrow x^*.$$

(In the above equality, we also use the fact that the inverse of \mathcal{J} is \mathcal{J} .) By the continuity of f and ∇f ,

$$f(\lambda^*, x^*) = 0, \quad \nabla f(\lambda^*, x^*) = 0.$$

By the assumption (iii), we have $x^* = 0$. Now, since f is C^2 , we have

$$\begin{aligned} 0 &= \nabla f_n(\lambda_n, x_n) = \partial_2 \nabla f_n(\lambda^*, 0) x_n + o(\|x_n\|) \\ 0 &= \partial_2 \nabla f(\lambda^*, 0) x_n + (Id - P_n) \partial_2 K(\lambda^*, 0) x_n + o(\|x_n\|). \end{aligned}$$

Since the Frechet derivative of a compact function is a compact operator [20, Theorem 17.1], and since $P_n \rightarrow Id$ uniformly in compact sets, we have (modulo a subsequence)

$$\|(Id - P_n) \partial_2 K(\lambda^*, 0) x_n\| = o(\|x_n\|).$$

Hence, we conclude that $\partial_2 \nabla f(\lambda^*, 0) x_n = o(\|x_n\|)$ which contradicts the invertibility of $\partial_2 \nabla f(\lambda^*, 0)$. The lemma is thus proven. □

The following lemma can be easily verified.

Lemma 6.3 *For any m , the function f_m satisfies $(PS)_{\mathbb{R}, \mathbb{R}}$.*

Proof Assume there is a sequence $(\lambda_n, x_n) \in \mathbb{R} \times H_m$ such that

$$\begin{aligned} f_m(\lambda_n, x_n) &\rightarrow c \\ \nabla f_m(\lambda_n, x_n) &= \mathcal{J}(x_n) - K_m(\lambda_n, x_n) \rightarrow 0. \end{aligned}$$

Then, by the compactness of K_m , there is a subsequence, again called (x_n) , such that $\mathcal{J}K_m(\lambda_n, x_n)$ converges, and thus

$$x_n = \mathcal{J}(\nabla f_m(\lambda_n, x_n)) + \mathcal{J}K_m(\lambda_n, x_n)$$

also converges. □

Also we prove the following result.

Lemma 6.4 *If $(z_n) \subset \mathbb{R}^2$ is a convergent sequence such that $z_n \in S_{f_n}$, then $z = \lim z_n \in S_f$.*

Proof Take $(\lambda_n, x_n) \in \mathbb{R} \times H_n$ such that $(\lambda_n, f(t_n, x_n)) = z_n$ and $\nabla f_n(\lambda_n, x_n) = 0$. We have

$$x_n = \mathcal{J}K_n(\lambda_n, x_n).$$

Thus, taking a subsequence of $\mathcal{J}K_n(\lambda_n, x_n)$, we have

$$x = \lim_{n \rightarrow \infty} x_n = \mathcal{J}K(\lambda, x)$$

and $(\lambda, f(\lambda, x)) = z$. □

For sufficiently large m , the spectral flow of the Hessian operators of f , $\text{sf}(L, I)$, which is nonzero according to the assumptions of Theorem 6.1, is computed as $\mu(L_{m,-1}) - \mu(L_{m,1})$, where $\mu(L_{m,\lambda})$ is the Morse index of the Hessian of $f_m(\lambda, \cdot)$. Therefore, considering Lemma 6.3, the function f_m is in the hypotheses of Theorem 3.4. We obtain for every $m \geq m_0$ a family of closed connected sets $C_m \subseteq S_{f_m}$ satisfying the conclusion of Theorem 3.3. Now we can prove the main result of this section.

Proof of Theorem 6.1 As in the proof of Theorem 3.3, assume none of the alternatives 1 and 2 are satisfied. Take $G_f \subseteq \mathbb{R}^2$, $R > 0$, D_+ , D_- and S_{out} as in the proof of Theorem 3.3, and $\varepsilon > 0$ as in Lemma 6.2. The set G_f is contained in D_+ , contains Z_{in} and does not intersect S_{out} . Again by Whyburn’s lemma, there exists an open set A containing G_f and such that

$$\begin{aligned} \bar{A} &\in D_+, \\ \partial A \cap (E_f \cup S_{\text{out}}) &= \emptyset. \end{aligned}$$

For every $m > m_0$ we have nonempty intersections $z_m \in C_m \cap \partial A$. By Lemma 6.2, we have $z_m \notin B_{\pm 1}$. Taking a convergent subsequence given by the compactness of ∂A , we may assume $z_m \rightarrow z$ with $z \in \partial A$. Finally, by Lemma 6.4 we obtain $z \in S_f$. Since $z \notin Z_{\text{in}} \cup S_{\text{out}}$ we deduce $z \in E_f$, which is a contradiction. \square

7 Example of application in a second-order ODE

In order to show the applicability of our method, we consider the following parameterized second-order differential equation with periodic boundary conditions:

$$\begin{cases} x''(t) = -g'(x(t))p_\lambda(t) \\ x(0) = x(2\pi) \\ x'(0) = x'(2\pi). \end{cases} \tag{7.1}$$

where we assume that

- (i) the map $g : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 with $g(0) = g'(0) = 0$;
- (ii) the limits $\lim_{s \rightarrow -\infty} g'(s) < 0 < \lim_{s \rightarrow +\infty} g'(s)$ exist and are finite;
- (iii) the real map $p(\lambda, t) = p_\lambda(t)$ is continuous and bounded on $[-1, 1] \times \mathbb{R}$;
- (iv) any p_λ is 2π -periodic;
- (v) p_{-1} and p_1 are constant;
- (vi) $\int_0^{2\pi} p_\lambda(t) dt \neq 0$ for every $\lambda \in [-1, 1]$.

Call $H = H^1_{per}([0, 2\pi], \mathbb{R})$ the subspace of the Hilbert space $H^1([0, 2\pi], \mathbb{R})$ consisting of 2π -periodic absolutely continuous functions with derivative in L^2 . Then define the action functional

$$\begin{aligned} f : [-1, 1] \times H &\rightarrow \mathbb{R} \\ f_\lambda(x) &= \frac{1}{2} \int_0^{2\pi} x'(t)^2 dt - \int_0^{2\pi} g(x(t))p_\lambda(t) dt. \end{aligned}$$

Compute the derivatives

$$\begin{aligned} f'_\lambda(x)[v] &= \int_0^{2\pi} x'(t)v'(t) dt - \int_0^{2\pi} g'(x(t))p_\lambda(t)v(t) dt \\ f''_\lambda(0)[v, w] &= \int_0^{2\pi} v'(t)w'(t) dt - c \int_0^{2\pi} p_\lambda(t)v(t)w(t) dt \end{aligned}$$

so that $x \in H$ is a (classic) solution of (7.1) if and only if $f'(x) = 0$. Let us show that f satisfies the hypotheses of Theorem 3.4 for some values of $a_{\pm 1}$ and c , where we denote $p_{-1}(t) = a_{-1}$, $p_1(t) = a_1$ and $g''(0) = c$. For convenience denote $\kappa_i = \sqrt{a_i c}$.

First of all, the fact that f is C^2 and satisfies the (PS) $_{\mathbb{R}, \mathbb{R}}$ condition can be verified thanks to the smoothness and growth condition of g . The proof is standard but rather technical, and we only describe the main lines. We follow the proof of [2, Lemma 3.1].

Assume that

$$|f_{\lambda_n}(y_n)| \rightarrow c, \quad \|f'_{\lambda_n}(y_n)\|_{H^*} := \varepsilon_n \rightarrow 0, \quad \lambda_n \rightarrow \lambda \tag{7.2}$$

where H^* denotes the dual space of H . We claim that y_n is bounded: indeed, otherwise we may suppose that $\|y_n\|_H \rightarrow \infty$. Set $v_n = \frac{y_n}{\|y_n\|_H}$, then taking a subsequence we may assume that $v_n \rightarrow v$ weakly and $v_n \rightarrow v$ uniformly. Moreover, from the inequality

$$\frac{1}{2} \int_0^{2\pi} y_n'(t)^2 dt - \int_0^{2\pi} g(y_n(t)) p_{\lambda_n}(t) dt \leq C$$

and $|g(u)| \leq C|u|$ we obtain

$$\int_0^{2\pi} \frac{y_n'(t)^2}{\|y_n\|_H^2} dt \rightarrow 0.$$

We also have that

$$\|y_n - y_n(0)\|_\infty \leq C \|y_n'\|_{L^2}$$

for some constant C . Thus, if we write

$$v_n = \frac{y_n - y_n(0)}{\|y_n\|_H} + \frac{y_n(0)}{\|y_n\|_H},$$

the first term goes to 0. Then, taking a subsequence we may assume that $v_n \rightarrow c_0$ uniformly for some constant $c_0 \neq 0$. Furthermore, $v_n \rightarrow 0$ in L^2 and we conclude that $v_n \rightarrow c_0$ in H . This means that y_n diverges either to $+\infty$ or $-\infty$. Then the limit condition of g' can be exploited as in [2, Lemma 3.1] to conclude that $f_{\lambda_n}'(y_n)[v_n]$ cannot converge to 0 and this is a contradiction.

Now that y_n is bounded in H , taking a subsequence we may assume that $y_n \rightarrow y$ weakly in H^1 and uniformly for some y . As $\|y_n\|_H$ is bounded,

$$|f_{\lambda_n}'(y_n)(y_n - y)| \leq \varepsilon_n \|y_n - y\| \rightarrow 0.$$

Moreover, as g' is bounded, the uniform convergence of $y_n \rightarrow y$ implies that the second term of $f_{\lambda_n}'(y_n)(y_n - y)$ tends to 0, and hence

$$\int_0^{2\pi} y_n'(y_n' - y') dt \rightarrow 0.$$

Then, as $y_n \rightarrow y$ weakly in H , we conclude that

$$\int_0^{2\pi} (y_n' - y')^2 dt \rightarrow 0,$$

and thus $y_n \rightarrow y$ in H . This proves that f_λ satisfies (PS) $_{\mathbb{R}, \mathbb{R}}$.

For $i = -1, 1$ we compute

$$f_i''(0)[v, w] = \int_0^{2\pi} w'(t)v'(t)dt - ca_i \int_0^{2\pi} w(v)v(t)dt.$$

Writing the elements of H as Fourier series $x(t) = \sum_{n \in \mathbb{Z}} x_n e^{int}$ we have an isomorphism

$$H \cong \left\{ (x_n) \in \ell^2(\mathbb{C}) \mid \sum_{n \in \mathbb{Z}} (nx_n)^2 < \infty, x_n = \overline{x_{-n}} \right\}$$

and in these coordinates

$$f_i''(0)[v, w] = \sum (n^2 - ca_i)v_n \overline{w_n}$$

then we have that $\mu(f_i''(0))$ is the number of negative coefficients in the sum, this is

$$\mu(f_i''(0)) = \begin{cases} 1 + 2 \lfloor \kappa_i \rfloor & \text{if } ca_i \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

provided $\kappa_i \notin \mathbb{N}_0$, which corresponds to the nondegeneracy of $f_i''(0)$.

Condition 3.4(iii) of Theorem 3.4 can be verified in different ways in terms of g since equation (7.1) is conservative for $\lambda = \pm 1$. For example, assume for simplicity that the area $A(E) = |\{(x, p) \in \mathbb{R}^2 / \frac{1}{2}p^2 + g(x) \leq E\}|$ is a convex function. Then one can check that the minimal period

$$T(E) = \sqrt{2} \int_{g(x) \leq E} \frac{dx}{\sqrt{E - g(x)}}$$

of the unique periodic solution x_E of energy E is an increasing function of E (see [4, Chapter 2]). We also have that $T(E) \rightarrow \frac{2\pi}{\kappa_i}$ when $E \rightarrow 0$, so there are at most $\lfloor \kappa_i \rfloor$ -many periodic solutions of period 2π , and it suffices to verify that $f_i(x_E) \neq 0$ if $T(E)$ divides 2π .

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