Structure of prescribed gradient domains for non-integrable vector fields

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Abstract

Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $f \in C^2(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ with n even. We describe the structure of the set of points in Ω at which the equality $Df = F$ and a certain non-integrability condition on F hold. This result generalizes the second statement of Balogh (J Reine Angew Math 564:63–83, 2003, Theorem 3.1).

Keywords Prescribed gradient domains · Characteristic set · Real analysis on submanifolds

Mathematics Subject Classification Primary 28A75 · 15A69 · 47H60

1 Introduction

Let us consider an open subset Ω of $\mathbb{R}^n$ and a vector field $F \in C^1(\Omega, \mathbb{R}^n)$. Then, whatever the choice of $f \in C^2(\Omega)$, the set

$$A_{f,F} := \{x \in \Omega \mid Df(x) = F(x)\}$$

has to be somehow “scarcely dense” at each point $x \in \Omega$ satisfying condition

$$(DF(x))^t \neq DF(x). \quad (1.1)$$

Actually, if such a condition occurs then the point x cannot be in the interior of $A_{f,F}$, by the Schwarz theorem on mixed derivatives. More interestingly, by [3, Theorem 2.1], the point x cannot be a $(n+1)$ -density point of the set $A_{f,F}$, i.e., it must be

$$\limsup_{r \rightarrow 0+} \frac{\mathcal{L}^n(B(x, r) \setminus A_{f,F})}{r^{n+1}} > 0.$$

In the special case when $n = 2m$ and

$$F_0(x_1, \dots, x_{2m}) = (2x_{m+1}, \dots, 2x_{2m}, -2x_1, \dots, -2x_m)$$

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condition (1.1) is trivially satisfied everywhere by F_0 and, according to the Balogh's result [1, Theorem 3.1], the set A_{f,F_0} is covered by countably many m -dimensional Lipschitz graphs. In particular, the Hausdorff dimension of A_{f,F_0} cannot exceed m . This theorem implies immediately that, given a C^2 hypersurface S in $\mathbb{R}^{2m+1}$, the set of points at which the tangent space of S coincides with the space spanned by the left-invariant horizontal vector fields of the Heisenberg group $\mathbb{H}^m$ over $\mathbb{R}^{2m+1}$ (namely the characteristic set of S) has Hausdorff dimension less or equal to m , compare [1, Theorem 1.2].

The main goal of this short paper is a theorem generalizing Balogh's result above and describing the structure of the set of points in $A_{f,F}$ satisfying a certain condition which is stronger than (1.1). In order to formulate such a condition and to state our result, let us consider the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and a real $n \times n$ matrix M . Then define

$$\Gamma(M) := \sum_{i=1}^n M e_i \wedge e_i$$

and (if h is a positive integer)

$$\Gamma(M)^h := \underbrace{\Gamma(M) \wedge \dots \wedge \Gamma(M)}_{h \text{ times}}.$$

Now we can finally state our structure theorem. It establishes that *if $n = 2m$, then the set $A_{f,F}^*$ of all points $x \in A_{f,F}$ such that*

$$\Gamma(DF(x))^m \neq 0 \tag{1.2}$$

is covered by a finite family of m -dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$ (Theorem 4.1 below).

In the special case considered by Balogh, i.e., when $F = F_0$, the condition (1.2) is verified at every $x \in \mathbb{R}^n$. Indeed a standard computation yields

$$\Gamma(DF_0(x))^m = C(m) e_1 \wedge \dots \wedge e_{2m}$$

for all $x \in \mathbb{R}^n$, where $C(m)$ is a positive constant depending only on m . Hence $A_{f,F_0} = A_{f,F_0}^*$, thus Balogh's result follows trivially from our theorem (Corollary 4.1 below).

The proof of our structure theorem is provided in Sect. 4 and is an elementary argument combining the classical implicit function theorem with some basic results from multilinear algebra which are developed in Sect. 3.

We have eventually to mention a few main results strictly related to this subject that have been published after [1]. In paper [4], Franchi et al. extended the Balogh's covering type argument to all stratified groups of step two. The generalization to arbitrary stratified groups has been proved by V. Magnani through a different approach based on a coarea inequality [7]. More recently, an estimate for the size of tangencies of submanifolds with respect to a non-involutive distribution, which generalizes the Balogh's theorem above, has been provided in [2].

2 Notation

If m, n are positive integers with $m \leq n$ then $I(n, m)$ is the set of integer multi-indices $(\alpha_1, \dots, \alpha_m)$ such that $1 \leq \alpha_1 < \dots < \alpha_m \leq n$. If $\alpha \in I(n, m)$ then $\bar{\alpha}$ is the complement in $I(n, n - m)$ of α . Moreover $\sigma(\alpha, \bar{\alpha})$ is the sign of the permutation of $(1, \dots, n)$ into $(\alpha, \bar{\alpha})$. The space of m -vectors in $\mathbb{R}^n$ is denoted by $\Lambda^m \mathbb{R}^n$. If $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$ and

$\alpha = (\alpha_1, \dots, \alpha_m) \in I(n, m)$, then we put $e_\alpha := e_{\alpha_1} \wedge \dots \wedge e_{\alpha_m}$. Recall that $\{e_\alpha\}_{\alpha \in I(n, m)}$ is a basis of $\Lambda^m \mathbb{R}^n$. Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ matrices with real entries and consider $M \in M_n(\mathbb{R})$. Then M^t is the transpose of M . Moreover, if $1 \leq m \leq n$, define the linear operator $\Lambda^m M : \Lambda^m \mathbb{R}^n \rightarrow \Lambda^m \mathbb{R}^n$ as the one such that $\Lambda^m M(e_\alpha) = M e_{\alpha_1} \wedge \dots \wedge M e_{\alpha_m}$, for all $\alpha = (\alpha_1, \dots, \alpha_m) \in I(n, m)$. Observe that, for all $\beta \in I(n, m)$, one has

$$\Lambda^m M(e_\beta) = \sum_{\alpha \in I(n, m)} (\det M_{\alpha, \beta}) e_\alpha$$

where $M_{\alpha, \beta}$ is the $m \times m$ submatrix of M with rows $\alpha_1, \dots, \alpha_m$ and columns $\beta_1, \dots, \beta_m$. The symmetric group of degree m is denoted by S_m . Finally, if Ω is an open subset of $\mathbb{R}^n$, $x \in \Omega$ and $\psi \in C^1(\Omega, \mathbb{R}^m)$, then $D\psi(x)$ denotes the matrix of the differential of ψ at x (with respect to the canonical basis).

3 Some basic preliminaries from multilinear algebra

Proposition 3.1 *Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. Consider the operator $\Gamma : M_n(\mathbb{R}) \rightarrow \Lambda^2 \mathbb{R}^n$ defined as*

$$\Gamma(M) := \sum_{i=1}^n M e_i \wedge e_i, \quad M \in M_n(\mathbb{R}).$$

Then Γ has the following properties:

(1) *It is linear, namely if $r, s \in \mathbb{R}$ and $M, N \in M_n(\mathbb{R})$ then*

$$\Gamma(rM + sN) = r\Gamma(M) + s\Gamma(N).$$

(2) *For all $M \in M_n(\mathbb{R})$ one has*

$$\Gamma(M) = \sum_{\substack{i, j=1 \\ i < j}}^n (M_{ij} - M_{ji}) e_i \wedge e_j$$

where $M_{ij} := (M e_j) \cdot e_i$. Hence:

- (i) $\Gamma(M^t) = -\Gamma(M)$;
- (ii) $\Gamma(M) = 0$ if and only if M is symmetric.

(3) *If $\{u_1, \dots, u_n\}$ is any arbitrary orthonormal basis in $\mathbb{R}^n$ and $M \in M_n(\mathbb{R})$, then*

$$\sum_{i=1}^n M u_i \wedge u_i = \Gamma(M).$$

Moreover, in the special case when n is even, i.e., $n = 2m$, the following identities hold for all $M \in M_{2m}(\mathbb{R})$:

$$\begin{aligned} \Gamma(M)^m &:= \underbrace{\Gamma(M) \wedge \dots \wedge \Gamma(M)}_{m \text{ times}} = C(m) \sum_{\alpha \in I(2m, m)} [(\Lambda^m M) e_\alpha] \wedge e_\alpha \\ &= C(m) \left(\sum_{\alpha \in I(2m, m)} \sigma(\alpha, \bar{\alpha}) \det M_{\alpha, \bar{\alpha}} \right) e_1 \wedge \dots \wedge e_{2m} \end{aligned}$$

with $C(m) := m! (-1)^{\frac{m(m-1)}{2}}$.

Proof Assertion (1) follows immediately from the definition of Γ , while (2) is a standard computation:

$$\begin{aligned}\Gamma(M) &= \sum_{i,j=1}^n M_{ji} e_j \wedge e_i \\ &= \sum_{\substack{i,j=1 \\ i < j}}^n M_{ji} e_j \wedge e_i + \sum_{\substack{i,j=1 \\ i > j}}^n M_{ji} e_j \wedge e_i \\ &= \sum_{\substack{i,j=1 \\ i < j}}^n M_{ji} e_j \wedge e_i + \sum_{\substack{i,j=1 \\ i < j}}^n M_{ij} e_i \wedge e_j \\ &= \sum_{\substack{i,j=1 \\ i < j}}^n (M_{ij} - M_{ji}) e_i \wedge e_j.\end{aligned}$$

Also (3) is very easy:

$$\begin{aligned}\sum_{i=1}^n M e_i \wedge e_i &= \sum_{i,h,k=1}^n (e_i \cdot u_h)(e_i \cdot u_k) M u_h \wedge u_k \\ &= \sum_{h,k=1}^n \left[\sum_{i=1}^n (e_i \cdot u_h)(e_i \cdot u_k) \right] M u_h \wedge u_k \\ &= \sum_{h,k=1}^n (u_h \cdot u_k) M u_h \wedge u_k \\ &= \sum_{h=1}^n M u_h \wedge u_h.\end{aligned}$$

Finally, one has

$$\begin{aligned}\Gamma(M)^m &= \left(\sum_{i_1=1}^{2m} M e_{i_1} \wedge e_{i_1} \right) \wedge \cdots \wedge \left(\sum_{i_m=1}^{2m} M e_{i_m} \wedge e_{i_m} \right) \\ &= (-1)^{1+2+\cdots+m-1} \sum_{i_1, \dots, i_m=1}^{2m} M e_{i_1} \wedge \cdots \wedge M e_{i_m} \wedge e_{i_1} \wedge \cdots \wedge e_{i_m} \\ &= (-1)^{\frac{m(m-1)}{2}} \sum_{\alpha \in I(2m, m)} \sum_{\lambda \in S_m} M e_{\alpha_{\lambda(1)}} \wedge \cdots \wedge M e_{\alpha_{\lambda(m)}} \wedge e_{\alpha_{\lambda(1)}} \wedge \cdots \wedge e_{\alpha_{\lambda(m)}} \\ &= (-1)^{\frac{m(m-1)}{2}} \sum_{\alpha \in I(2m, m)} \#(S_m) [(\wedge^m M) e_\alpha] \wedge e_\alpha \\ &= (-1)^{\frac{m(m-1)}{2}} m! \sum_{\alpha \in I(2m, m)} [(\wedge^m M) e_\alpha] \wedge e_\alpha\end{aligned}$$

and

$$\begin{aligned} [(\Lambda^m M) e_\alpha] \wedge e_\alpha &= \sum_{\beta \in I(2m, m)} (\det M_{\beta, \alpha}) e_\beta \wedge e_\alpha = (\det M_{\tilde{\alpha}, \alpha}) e_{\tilde{\alpha}} \wedge e_\alpha \\ &= \sigma(\tilde{\alpha}, \alpha) (\det M_{\tilde{\alpha}, \alpha}) e_1 \wedge \cdots \wedge e_{2m}. \end{aligned}$$

□

Remark 3.1 From (1) and (2) of Proposition 3.1, it follows at once the following identity which will be useful below:

$$\Gamma(M - M^t) = 2\Gamma(M) = -2\Gamma(M^t)$$

for all $M \in M_n(\mathbb{R})$.

Remark 3.2 Let $h \geq 2$ and $n \geq 2h$. Then the identity $\Gamma(M)^h = 0$ holds whenever M is symmetric (by (2) of Proposition 3.1), but it can occur even when M is nonsymmetric. For example, for

$$M_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

one has $\Gamma(M) = e_1 \wedge e_2$, hence $\Gamma(M)^h = 0$.

4 The structure theorem: statement and proof

Definition 4.1 Given $f \in C^2(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^{2m})$, where Ω is an open subset of $\mathbb{R}^{2m}$, define the sets

$$A_{f,F} := \{x \in \Omega \mid Df(x) = F(x)\}$$

and

$$A_{f,F}^* := \{x \in A_{f,F} \mid \Gamma(DF(x))^m \neq 0\}.$$

Remark 4.1 As we have already pointed out in Introduction, under the assumptions of Definition 4.1, a point $x_0 \in \Omega$ cannot be a $(2m+1)$ -density point of the set $A_{f,F}$ if $\Gamma(DF(x_0)) \neq 0$ (by [3, Theorem 2.1]). In particular one has the following property: There is no point $x_0 \in A_{f,F}^*$ which is a $(2m+1)$ -density point of the set $A_{f,F}^*$. In our main result, namely Theorem 4.1, we will prove that something very much stronger than this property actually holds for $A_{f,F}^*$. In particular, the set $A_{f,F}^*$ must be so thin that its Hausdorff dimension cannot exceed m .

We need the following lemma which will be proved by a simple argument combining Proposition 3.1 and the implicit function theorem.

Lemma 4.1 Let $\Phi \in C^1(\Omega, \mathbb{R}^{2m})$, where Ω is an open subset of $\mathbb{R}^{2m}$. Then

$$\{x \in \Omega \mid \Phi(x) = 0, \Gamma(D\Phi(x))^m \neq 0\} \quad (4.1)$$

is a relatively closed subset of Ω which is covered by a finite family $\{\Sigma_\alpha \mid \alpha \in I(2m, m)\}$ of m -dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$.

Proof For $\alpha \in I(2m, m)$, define

$$\Phi_\alpha := (\Phi_{\alpha_1}, \dots, \Phi_{\alpha_m}), \quad \Sigma_\alpha := \{x \in \Omega \mid \Phi_\alpha(x) = 0, \text{rank } D\Phi_\alpha(x) = m\}.$$

Observe that each Σ_α has to be a m -dimensional regularly imbedded C^1 submanifold of $\mathbb{R}^{2m}$ by a standard application of the implicit function theorem, e.g., compare [6, Theorem 4.3.1] or [5, Ch. 1, Theorem 3.2]. The conclusion follows from the last identity in Proposition 3.1, with $M = D\Phi(x)$. $\square$

Now we are ready to prove our main result.

Theorem 4.1 (*Structure theorem*) Let $f \in C^2(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^{2m})$, where Ω is an open subset of $\mathbb{R}^{2m}$. Then $A_{f,F}^*$ is a relatively closed subset of Ω which is covered by a finite family $\{\Sigma_\alpha \mid \alpha \in I(2m, m)\}$ of m -dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$. In particular, the Hausdorff dimension of $A_{f,F}^*$ has to be less or equal to m .

Proof Consider

$$\Phi := Df - F \in C^1(\Omega, \mathbb{R}^{2m}).$$

Since

$$D\Phi = D^2f - DF$$

one has

$$\Gamma(D\Phi(x) - (D\Phi(x))^t) = \Gamma((DF(x))^t - DF(x))$$

for all $x \in \Omega$. By recalling Remark 3.1, we get

$$\Gamma(D\Phi(x)) = -\Gamma(DF(x))$$

for all $x \in \Omega$, hence

$$A_{f,F}^* = \{x \in \Omega \mid \Phi(x) = 0, \Gamma(D\Phi(x))^m \neq 0\}.$$

The conclusion follows from Lemma 4.1.

From Theorem 4.1, we obtain the following corollary.

Corollary 4.1 Consider $F_0 : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ defined as

$$F_0(x_1, \dots, x_{2m}) := (2x_{m+1}, \dots, 2x_{2m}, -2x_1, \dots, -2x_m)$$

and let Ω be an open subset of $\mathbb{R}^{2m}$. Then, for all $f \in C^2(\Omega)$, the set A_{f,F_0} is a relatively closed subset of Ω which is covered by a finite family $\{\Sigma_\alpha \mid \alpha \in I(2m, m)\}$ of m -dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$. In particular, the Hausdorff dimension of A_{f,F_0} has to be less or equal to m .

Proof Observe that

$$DF_0 = \begin{pmatrix} 0 & 2I_m \\ -2I_m & 0 \end{pmatrix}$$

where I_m denotes the identity $m \times m$ matrix. Thus, for all $x \in \mathbb{R}^{2m}$, we obtain

$$\begin{aligned}\Gamma(DF_0(x)) &= \sum_{i=1}^{2m} DF_0(x)e_i \wedge e_i = \sum_{i=1}^m (-2e_{m+i}) \wedge e_i + \sum_{i=m+1}^{2m} 2e_{i-m} \wedge e_i \\ &= 4 \sum_{i=1}^m e_i \wedge e_{m+i}\end{aligned}$$

hence

$$\begin{aligned}\Gamma(DF_0(x))^m &= 4^m \sum_{i_1, \dots, i_m=1}^m (e_{i_1} \wedge e_{m+i_1}) \wedge \dots \wedge (e_{i_m} \wedge e_{m+i_m}) \\ &= 4^m \sum_{\lambda \in S_m} (e_{\lambda(1)} \wedge e_{m+\lambda(1)}) \wedge \dots \wedge (e_{\lambda(m)} \wedge e_{m+\lambda(m)}) \\ &= 4^m \sum_{\lambda \in S_m} (e_1 \wedge e_{m+1}) \wedge \dots \wedge (e_m \wedge e_{2m}) \\ &= 4^m m! (-1)^{\frac{m(m-1)}{2}} e_1 \wedge \dots \wedge e_{2m}.\end{aligned}$$

In particular one has $\Gamma(DF_0(x))^m \neq 0$ for all $x \in \mathbb{R}^{2m}$, so that $A_{f, F_0} = A_{f, F_0}^*$. The conclusion follows from Theorem 4.1. $\square$

Remark 4.2 Corollary 4.1 improves slightly the second statement in [1, Theorem 3.1], which yields immediately the following interesting fact: the characteristic set of a codimension 1 submanifold of class C^2 in the Heisenberg group $\mathbb{H}^m$ has Hausdorff dimension less or equal to m , compare [1, Theorem 1.2].

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