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Embedding theorems for Sobolev and Hardy–Sobolev spaces and estimates of Fourier transforms

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Abstract
We prove embeddings of Sobolev and Hardy–Sobolev spaces into Besov spaces built upon certain mixed norms. This gives an improvement of the known embeddings into usual Besov spaces. Applying these results, we obtain Oberlin-type estimates of Fourier transforms for functions in Sobolev spaces $W_1^1(\mathbb{R}^n)$.

Keywords Embeddings · Moduli of continuity · Sobolev spaces · Hardy spaces · Mixed norms · Fourier transforms

Mathematics Subject Classification Primary 46E35 · 26D10; Secondary 46E30

1 Introduction

This paper is devoted to the study of some inequalities for functions in the Sobolev spaces $W_p^1(\mathbb{R}^n)$ and Hardy–Sobolev spaces $HW_1^1(\mathbb{R}^n)$.

The Sobolev space $W_p^1(\mathbb{R}^n)$ $(1 \le p < \infty)$ is defined as the class of all functions $f \in L^p(\mathbb{R}^n)$ for which every first-order weak derivative exists and belongs to $L^p(\mathbb{R}^n)$. The classical Sobolev theorem (see [26, Ch. V]) states the following.

Theorem 1.1 *Let $n \ge 2$, $1 \le p < n$, and $p^* = np/(n-p)$. Then for any $f \in W_p^1(\mathbb{R}^n)$*

$$||f||_{p^*} \le c\|\nabla f\|_p. \tag{1.1}$$

The Lebesgue norm at the left-hand side of (1.1) can be replaced by the stronger Lorentz norm. Namely, for any $f \in W_p^1(\mathbb{R}^n)$, $n \ge 2$, $1 \le p < n$,

$$||f||_{p^*,p} \le c||\nabla f||_p \tag{1.2}$$

(see [1,21,24,25]).

Let a function f be defined on $\mathbb{R}^n$ and let $k \in \{1, \dots, n\}$. Set

$$\Delta_k(h)f(x) = f(x + he_k) - f(x), \quad x \in \mathbb{R}^n,\ h \in \mathbb{R} \tag{1.3}$$

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(e_k is the kth unit coordinate vector).

The following theorem holds.

Theorem 1.2 *Let $n \in \mathbb{N}$. Assume that $1 < p < \infty$ and $n \ge 1$, or $p = 1$ and $n \ge 2$. If $p < q < \infty$ and $s = 1 - n(1/p - 1/q) > 0$, then for any $f \in W_p^1(\mathbb{R}^n)$*

$$\sum_{k=1}^n \left(\int_0^\infty h^{-sp} ||\Delta_k(h) f||_{q,p}^p \frac{dh}{h} \right)^{1/p} \le c \sum_{k=1}^n ||D_k f||_p. \tag{1.4}$$

For $p > 1$ inequality (1.4) (with the weaker norm $||\Delta_k(h) f||_q$ at the left-hand side) was obtained by Herz [10]. For $p = 1, n \ge 2$ Theorem 1.2 was proved in [11] (see also [12]). The case $p = 1$ is of special interest; we stress that Theorem 1.2 fails for $p = n = 1$. However, this theorem holds for any function f from the Hardy space $H^1(\mathbb{R})$ such that $f' \in H^1(\mathbb{R})$, if we replace the L^1-norm of f' by its H^1-norm (see [11,22]).

One of the main results of this paper is the refinement of the inequality (1.4) given in terms of mixed norms.

Let $x = (x_1, \dots, x_n)$. Denote by $\widehat{x}_k$ the $(n-1)$-dimensional vector obtained from the n-tuple x by removal of its kth coordinate. We shall write $x = (x_k, \widehat{x}_k)$.

If $X(\mathbb{R})$ and $Y(\mathbb{R}^{n-1})$ are Banach function spaces, and $k \in \{1, \dots, n\}$, we denote by $Y[X]_k$ the mixed norm space obtained by taking first the norm in X with respect to x_k, and then the norm in Y with respect to $\widehat{x}_k \in \mathbb{R}^{n-1}$.

We prove the following theorem.

Theorem 1.3 *Let $1 < p < \infty$ and $n \ge 2$, or $p = 1$ and $n \ge 3$. If $p < q < \infty$ and $\alpha = 1 - (n-1)(1/p - 1/q) > 0$, then for any $f \in W_p^1(\mathbb{R}^n)$*

$$\sum_{k=1}^n \left(\int_0^\infty h^{-\alpha p} ||\Delta_k(h) f||_{L^{q,p}[L^p]_k}^p \frac{dh}{h} \right)^{1/p} \le c \sum_{k=1}^n ||D_k f||_p. \tag{1.5}$$

We show that the left-hand side of (1.4) is majorized by the left-hand side of (1.5). Thus, for the indicated values of n and p, Theorem 1.3 provides a refinement of Theorem 1.2. We stress that inequality (1.5) holds for $n = 2$, $p > 1$. However, *the question of the validity of this inequality for $n = 2$, $p = 1$ remains open.*

As we have observed above, Theorem 1.2 fails for $p = n = 1$, but in this case there holds a weaker inequality with L^1-norm of f' replaced by its H^1-norm. Similarly, we supplement Theorem 1.3 by the following result.

As usual, for any $1 \le p \le \infty$ we denote $p' = p/(p-1)$.

Theorem 1.4 *Let $f \in W_1^1(\mathbb{R}^n)$ $(n \ge 2)$ and assume that all partial derivatives $D_j f$ $(j = 1, \dots, n)$ belong to the Hardy space $H^1(\mathbb{R}^n)$. Then for any $1 < q < (n-1)/(n-2)$*

$$\sum_{k=1}^n \int_0^\infty h^{(n-1)/q'-1} ||\Delta_k(h) f||_{L^{q,1}[L^1]_k} \frac{dh}{h} \le c \sum_{k=1}^n ||D_k f||_{H^1}. \tag{1.6}$$

That is, inequality (1.5) holds for $p = 1, n = 2$ if the L^1-norms of the derivatives are replaced by the Hardy H^1-norms. Of course, for $n \ge 3$ (1.6) follows from (1.5).

We should note that this work was partly inspired by the Oberlin estimate [20] of Fourier transforms of functions in the Hardy space $H^1(\mathbb{R}^n)$. We apply inequality (1.5) to obtain an analogue of this estimate for the derivatives of functions in $W_1^1(\mathbb{R}^n)$. In particular, we prove the following result.

Theorem 1.5 *Let $f \in W_1^1(\mathbb{R}^n)$ $(n \geq 3)$. Then*

$$\sum_{k\in\mathbb{Z}} 2^{k(2-n)} \sup_{2^k \leq r \leq 2^{k+1}} \int_{S_r} |\widehat{f}(\xi)| \mathrm{d}\sigma(\xi) \leq c||\nabla f||_1, \tag{1.7}$$

where S_r is the sphere of the radius r centered at the origin in $\mathbb{R}^n$ and $\mathrm{d}\sigma(\xi)$ is the canonical surface measure on S_r.

For $n \geq 3$ this theorem gives a refinement of the Hardy-type inequality

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)||\xi|^{1-n} \,\mathrm{d}\xi \leq c||\nabla f||_1,$$

which was proved for $f \in W_1^1(\mathbb{R}^n)$ $(n \geq 2)$ by Bourgain [4] and Pełczyński and Wojciechowski [23].

As in the case $p = 1$ in Theorem 1.3, *it is an open question whether Theorem 1.5 is true for $n = 2$.*

The paper is organized as follows: We give some definitions and auxiliary results in Sect. 2. In Sect. 3 we prove inequalities between Besov norms built upon the spaces $L^{p,\nu}(\mathbb{R}^n)$ and $L^{p,\nu}(\mathbb{R}^{n-1})[L^r(\mathbb{R})]$, $1 \leq r, \nu \leq p$. In Sect. 4 we prove Theorem 1.3. Section 5 contains the proof of Theorem 1.4. Section 6 is devoted to estimates of Fourier transforms of functions in $W_1^1(\mathbb{R}^n)$.

2 Some definitions and auxiliary results

Denote by $S_0(\mathbb{R}^n)$ the class of all measurable and almost everywhere finite functions f on $\mathbb{R}^n$ such that

$$\lambda_f(y) = |\{x \in \mathbb{R}^n : |f(x)| > y\}| < \infty \quad \text{for each } y > 0.$$

A nonincreasing rearrangement of a function $f \in S_0(\mathbb{R}^n)$ is a nonnegative and nonincreasing function f^* on $\mathbb{R}_+ = (0, +\infty)$ which is equimeasurable with $|f|$, that is, $\lambda_{f^*} = \lambda_f$. The rearrangement f^* can be defined by the equality

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} |f(x)|, \quad 0 < t < \infty \tag{2.1}$$

(see [5, p. 32]).

The following relation holds [2, p. 53]

$$\sup_{|E|=t} \int_E |f(x)|\mathrm{d}x = \int_0^t f^*(u)\mathrm{d}u. \tag{2.2}$$

In what follows we denote

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\mathrm{d}u. \tag{2.3}$$

For any $t > 0$ there is a subset $E \subset \mathbb{R}^n$ with $|E| = t$ such that

$$\frac{1}{t}\int_E |f(x)|\mathrm{d}x = f^{**}(t) \tag{2.4}$$

(see [2, p. 53]).

Let $0 < p, r < \infty$. A function $f \in S_0(\mathbb{R}^n)$ belongs to the Lorentz space $L^{p,r}(\mathbb{R}^n)$ if

$$\|f\|_{L^{p,r}} = \|f\|_{p,r} = \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^r \frac{\mathrm{d}t}{t}\right)^{1/r} < \infty.$$

We have that $||f||_{p,p} = ||f||_p$. For a fixed p, the Lorentz spaces $L^{p,r}$ strictly increase as the secondary index r increases; that is, the strict embedding $L^{p,r} \subset L^{p,s} \quad (r < s)$ holds (see [2, Ch. 4]).

We will use the following Hardy's inequality (see [2, p. 124]).

Proposition 2.1 *Let φ be a nonnegative measurable function on $(0, \infty)$ and suppose $-\infty < \lambda < 1$ and $1 \le p < \infty$. Then*

$$\left(\int_0^\infty \left(t^{\lambda-1}\int_0^t \varphi(u)\mathrm{d}u\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p} \le \frac{1}{1-\lambda}\left(\int_0^\infty \left(t^\lambda \varphi(t)\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p}.$$

Applying Hardy's inequality with $p > 1$, $\lambda = 1/p$, we obtain that the operator $f \mapsto f^{**}$ is bounded in L^p for $p > 1$,

$$||f^{**}||_p \le \frac{p}{p-1}||f||_p, \quad 1 < p \le \infty. \tag{2.5}$$

We say that a measurable function ψ on $(0, \infty)$ is quasi-decreasing if there exists a constant $c > 0$ such that $\psi(t_1) \le c\psi(t_2)$, whenever $0 < t_2 < t_1 < \infty$.

It is well known that in the case $0 < p < 1$ Hardy-type inequalities are true for quasi-decreasing functions. We will use the following proposition (a short proof can be found, e.g., in [17]).

Proposition 2.2 *Let ψ be a nonnegative, quasi-decreasing function on $(0, \infty)$. Suppose also that $\alpha > 0$, $\beta > -1$ and $0 < p < 1$. Then*

$$\int_0^\infty u^{-\alpha-1}\Big(\int_0^u \psi(t)t^\beta \mathrm{d}t\Big)^p \mathrm{d}u \le c\int_0^\infty u^{-\alpha-1}\big(\psi(u)u^{\beta+1}\big)^p \mathrm{d}u.$$

Let a function $\varphi \in L^p(\mathbb{R})$. Set

$$\Delta(h)\varphi(x) = \varphi(x+h) - \varphi(x), \quad h \in \mathbb{R}, \tag{2.6}$$

and

$$\omega(\varphi; t)_p = \sup_{|h|\le t} ||\Delta(h)\varphi||_p, \quad t \ge 0.$$

Ul'yanov [28] proved the following estimate: for any $\varphi \in L^p(\mathbb{R})$, $1 \le p < \infty$

$$\varphi^{**}(t) - \varphi(t) \le 2t^{-1/p}\omega(\varphi; t)_p.$$

It easily follows that

$$\varphi^*(t) \le 2\int_t^\infty s^{-1/p}\omega(\varphi; s)_p \frac{\mathrm{d}s}{s} \tag{2.7}$$

(see also [14, p. 149], [27]). Using these estimates, Ul'yanov obtained that if $1 \le p < q < \infty$ and $\varphi \in L^p(\mathbb{R})$, then

$$||\varphi||_q \le c\left(\int_0^\infty t^{-q/p}||\Delta(t)\varphi||_p^q \,\mathrm{d}t\right)^{1/q} \tag{2.8}$$

and

$$\omega(\varphi;\delta)_q \le c\left(\int_0^\delta t^{-q/p}||\Delta(t)\varphi||_p^q\,\mathrm{d}t\right)^{1/q} \tag{2.9}$$

(some discussions and generalizations of these results can be found in [14] and [16]).

In the next section we consider functions $(x, y) \mapsto f(x, y)$, where $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, and we denote

$$\Delta_1(h) f(x, y) = f(x+h, y) - f(x, y), \quad h \in \mathbb{R}. \tag{2.10}$$

Let $V = V(\mathbb{R}^n)$ be a Banach function space over $\mathbb{R}^n$ (see [2, Ch. 1]). We shall assume that V is translation invariant, that is, whenever $f \in V$, then $\tau_h f \in V$ and $||\tau_h f||_V = ||f||_V$ for all $h \in \mathbb{R}^n$, where $\tau_h f(x) = f(x-h)$. Let $f \in V$. Set

$$\omega_1(f;\delta)_V = \sup_{|h|\le\delta} ||\Delta_1(h) f||_V, \quad \delta \ge 0.$$

In these notations, the subindex 1 indicates that the difference is taken with respect to the first variable x.

We have the following inequality

$$\omega_1(f;\delta)_V \le \frac{3}{\delta}\int_0^\delta ||\Delta_1(h) f||_V\,\mathrm{d}h. \tag{2.11}$$

Indeed, if $t, h \in [0, \delta]$, then

$$||\Delta_1(t) f||_V \le ||\Delta_1(h) f||_V + ||\Delta_1(t-h) f||_V.$$

Integrating with respect to h in $[0, \delta]$ (for a fixed $t \in [0, \delta]$), and then taking supremum over t, we obtain (2.11).

3 Different norm inequalities

Throughout this paper we use the notation (1.3).

Let $0 < \alpha < 1$, $1 \le p < \infty$, and $1 \le \theta < \infty$. The Besov space $B^\alpha_{p,\theta}(\mathbb{R}^n)$ consists of all functions $f \in L^p(\mathbb{R}^n)$ such that

$$\|f\|_{B^\alpha_{p,\theta}} = ||f||_p + \sum_{k=1}^n \left(\int_0^\infty \left(t^{-\alpha}||\Delta_k(t) f||_p\right)^\theta \frac{\mathrm{d}t}{t}\right)^{1/\theta} < \infty.$$

The classical different norm embedding theorem states that if $1 \le p < q < \infty$ and $\alpha > n(1/p - 1/q)$, then for any $1 \le \theta < \infty$

$$B^\alpha_{p,\theta}(\mathbb{R}^n) \subset B^\beta_{q,\theta}(\mathbb{R}^n), \quad \text{where} \quad \beta = \alpha - n(1/p - 1/q),$$

and for any $f \in B^\alpha_{p,\theta}(\mathbb{R}^n)$

$$||f||_{B^\beta_{q,\theta}} \le c||f||_{B^\alpha_{p,\theta}} \tag{3.1}$$

(see [19, Ch. 6]).

Roughly speaking, passing from L^p to L^q, we lose $n(1/p - 1/q)$ in the smoothness exponent.

We shall be especially interested in the one-dimensional case of this theorem. Note that for $n = 1$ (3.1) follows immediately from (2.8), (2.9) and Hardy's inequality.

In this section we obtain different norm inequalities for the Besov spaces defined in some mixed norms. First of all, we are interested in these results in connection with embeddings of Sobolev spaces (in particular, for the comparison of Theorems 1.3 and 1.2).

We keep notations introduced in Sect. 2. Namely, we use the notation $\Delta(h)\varphi$ for functions of one variable (see (2.6)). The notation $\Delta_1(h)f$ (see (2.10)) is applied to functions $(x,y)\mapsto f(x,y)$, where $x\in\mathbb{R},\ \ y\in\mathbb{R}^{n-1}\ \ (n\ge 2)$.

Let $1\le\theta<\infty,\ \ 0<\alpha<1$. Let $V=V(\mathbb{R}^n)\ \ (n\ge 2)$ be a translation invariant Banach function space. Denote by $B^\alpha_{\theta;1}(V)$ the class of all functions $f\in V$ such that

$$||f||_{B^\alpha_{\theta;1}(V)}=||f||_V+\left(\int_0^\infty[h^{-\alpha}\omega_1(f;h)_V]^\theta\frac{\mathrm{d}h}{h}\right)^{1/\theta}<\infty.$$

As above, the subindex 1 indicates that the difference is taken with respect to the first variable x. Applying (2.11) and Hardy's inequality, we obtain that

$$\int_0^\infty[h^{-\alpha}\omega_1(f;h)_V]^\theta\frac{\mathrm{d}h}{h}\le c\int_0^\infty[h^{-\alpha}||\Delta_1(h)f||_V]^\theta\frac{\mathrm{d}h}{h}. \tag{3.2}$$

As in Sect. 1, if $X(\mathbb{R})$ and $Y(\mathbb{R}^{n-1})$ are Banach function spaces, we denote by $Y[X]_1$ the mixed norm space obtained by taking first the norm in $X(\mathbb{R})$ with respect to the variable x, and then the norm in $Y(\mathbb{R}^{n-1})$ with respect to y. In this section the interior norm will be taken only in variable x. Therefore, in this section we write simply $Y[X]$ (omitting the subindex 1).

First, we have the following simple proposition.

Proposition 3.1 *Let* $1\le\theta<\infty$, $1\le r<p<\infty$, *and* $1/r-1/p<\alpha<1$. *Set* $\beta=\alpha-1/r+1/p$. *Then* $B^\alpha_{\theta;1}(L^p[L^r])\subset B^\beta_{\theta;1}(L^p(\mathbb{R}^n))$*; more exactly, for any* $f\in B^\alpha_{\theta;1}(L^p[L^r])$

$$||f||_p\le c||f||_{B^\alpha_{\theta;1}(L^p[L^r])} \tag{3.3}$$

and

$$\int_0^\infty h^{-\theta\beta}||\Delta_1(h)f||_p^\theta\frac{\mathrm{d}h}{h}\le c\int_0^\infty h^{-\theta\alpha}||\Delta_1(h)f||^\theta_{L^p[L^r]}\frac{\mathrm{d}h}{h}. \tag{3.4}$$

Proof Denote $V=L^p[L^r]$. Let $f\in B^\alpha_{\theta;1}(V)$. For a fixed $y\in\mathbb{R}^{n-1}$, set $f_y(x)=f(x,y),\ \ x\in\mathbb{R}$. By (2.8), we have

$$||f_y||_p^p\le c\int_0^\infty t^{-p/r}||\Delta(t)f_y||_r^p\mathrm{d}t.$$

Integrating with respect to y gives

$$||f||_p^p\le c\int_0^\infty t^{-p/r}||\Delta_1(t)f||_V^p\mathrm{d}t.$$

Applying standard reasonings (see, e.g., [2, Ch. 5.4]), we get

$$\left(\int_0^\infty t^{-p/r}||\Delta_1(t)f||_V^p\mathrm{d}t\right)^{1/p}$$
$$\le c\left[||f||_V+\left(\int_0^1[t^{-\theta\alpha}||\Delta_1(t)f||_V]^\theta\frac{\mathrm{d}t}{t}\right)^{1/\theta}\right].$$

These estimates imply (3.3).

Further, inequality (2.9) gives that

$$||\Delta(h) f_y||_p^p \le c \int_0^h ||\Delta(t) f_y||_r^p t^{-p/r} \mathrm{d}t.$$

Integrating with respect to y, we get

$$\int_{\mathbb{R}^n} |\Delta_1(h) f(x, y)|^p(x, y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{n-1}} ||\Delta(h) f_y||_p^p \mathrm{d}y$$
$$\le c \int_{\mathbb{R}^{n-1}} \int_0^h ||\Delta(t) f_y||_r^p t^{-p/r} \mathrm{d}t\mathrm{d}y = c \int_0^h ||\Delta_1(t) f||_V^p t^{-p/r} \mathrm{d}t.$$

This implies that

$$\int_0^\infty h^{-\theta\beta} ||\Delta_1(h) f||_p^\theta \frac{\mathrm{d}h}{h}$$
$$\le c \int_0^\infty h^{-\theta\beta} \left(\int_0^h ||\Delta_1(t) f||_V^p t^{-p/r} \mathrm{d}t \right)^{\theta/p} \frac{\mathrm{d}h}{h}.$$

If $\theta \ge p$, then we apply Proposition 2.1 and we obtain (3.4). Let $\theta < p$. Observe that the function $\psi(t) = \omega_1(f; t)_V / t$ is quasi-decreasing. Hence, applying Proposition 2.2 and inequality (3.2), we get

$$\int_0^\infty h^{-\theta\beta} ||\Delta_1(h) f||_p^\theta \frac{\mathrm{d}h}{h}$$
$$\le c \int_0^\infty h^{-\theta\beta} \left(\int_0^h \omega_1(f; t)_V^p t^{-p/r} \mathrm{d}t \right)^{\theta/p} \frac{\mathrm{d}h}{h}$$
$$\le c' \int_0^\infty h^{-\theta\alpha} \omega_1(f; h)_V^\theta \frac{\mathrm{d}h}{h} \le c'' \int_0^\infty h^{-\theta\alpha} ||\Delta_1(h) f||_V^\theta \frac{\mathrm{d}h}{h}.$$

This implies (3.4). □

Note that, in contrast to (3.1), the loss in the smoothness exponent given by (3.4) is only $1/r - 1/p$. It is natural because the integrability exponent changes in only one variable.

Now, we replace the L^p-norm in (3.3) and (3.4) by the $L^{p,\nu}$-Lorentz norm. In this case simple arguments similar to those given above cannot be applied. Indeed, it was shown by Cwikel [6] that if $p \ne \nu$, then neither of the spaces $L^{p,\nu}(\mathbb{R}^2)$ and $L^{p,\nu}(\mathbb{R})[L^{p,\nu}(\mathbb{R})]$ is contained in the other. Therefore, we apply different methods; namely, we shall use iterated rearrangements.

Let $g \in S_0(\mathbb{R}^n)$, $n \ge 2$. For a fixed $y \in \mathbb{R}^{n-1}$, denote by $\mathcal{R}_1 g(s, y)$ the nonincreasing rearrangement of the function $g_y(x) = g(x, y)$, $x \in \mathbb{R}$. Further, for a fixed $s > 0$, let $\mathcal{R}_{1,2} g(s, t)$ be the nonincreasing rearrangement of the function $y \mapsto \mathcal{R}_1 g(s, y)$, $y \in \mathbb{R}^{n-1}$.

The iterated rearrangement $\mathcal{R}_{1,2} g$ is defined on $\mathbb{R}^2_+$. It is nonnegative, nonincreasing in each variable, and equimeasurable with $|g|$ function (see [3,15,16]).

Let $0 < p, \nu < \infty$, and $n \ge 2$. For a function $g \in S_0(\mathbb{R}^n)$, denote

$$\|g\|_{\mathcal{L}^{p,\nu}} = \left(\int_{\mathbb{R}^2_+} (st)^{\nu/p-1} \mathcal{R}_{1,2} g(s, t)^\nu \, \mathrm{d}s\mathrm{d}t \right)^{1/\nu}$$

(see [3]). The following inequalities hold [29]:

$$\|g\|_{p,\nu} \le c \|g\|_{\mathcal{L}^{p,\nu}} \quad \text{if } 0 < \nu \le p < \infty \tag{3.5}$$

and

$$\|g\|_{\mathcal{L}^{p,\nu}} \le c'\|g\|_{p,\nu} \quad \text{if } 0 < p \le \nu < \infty. \tag{3.6}$$

Proposition 3.2 *Let* $1 \le \theta < \infty$, $1 \le \nu \le p < \infty$, $1 \le r < p$, *and* $1/r - 1/p < \alpha < 1$. *Set* $\beta = \alpha - 1/r + 1/p$. *Then* $B^{\alpha}_{\theta;1}(L^{p,\nu}[L^r]) \subset B^{\beta}_{\theta;1}(L^{p,\nu})$; *more exactly, for any* $f \in B^{\alpha}_{\theta;1}(L^{p,\nu}[L^r])$

$$||f||_{L^{p,\nu}} \le c||f||_{B^{\alpha}_{\theta;1}(L^{p,\nu}[L^r])} \tag{3.7}$$

and

$$\int_0^\infty h^{-\theta\beta}||\Delta_1(h)f||^{\theta}_{L^{p,\nu}}\frac{\mathrm{d}h}{h} \le c\int_0^\infty h^{-\theta\alpha}||\Delta_1(h)f||^{\theta}_{L^{p,\nu}[L^r]}\frac{\mathrm{d}h}{h}. \tag{3.8}$$

Proof Let $f \in B^{\alpha}_{\theta,1}(L^{p,\nu}[L^r])$. Set $\varphi_h(x, y) = |\Delta_1(h)f(x, y)|$. Let s and h be fixed positive numbers. We consider the function $y \mapsto \mathcal{R}_1\varphi_h(s, y),\ \ y \in \mathbb{R}^{n-1}$. As in Sect. 2 above (see (2.4)), we can state that for any $t > 0$ there exists a set $E = E_{s,t,h} \subset \mathbb{R}^{n-1}$ with $\mathrm{mes}_{n-1}\, E = t$ such that

$$\mathcal{R}_{1,2}\varphi_h(s, t) \le \frac{1}{t}\int_E \mathcal{R}_1\varphi_h(s, y)\,\mathrm{d}y. \tag{3.9}$$

By (2.7), for any $s > 0$

$$\mathcal{R}_1\varphi_h(s, y) \le 2\int_s^\infty \omega(\varphi_h(\cdot, y); u)_r\frac{\mathrm{d}u}{u^{1+1/r}}. \tag{3.10}$$

Set $g_{u,h}(y) = \omega(\varphi_h(\cdot, y); u)_r$. By (2.2), we have

$$\frac{1}{t}\int_E g_{u,h}(y)\,\mathrm{d}y \le g^{**}_{u,h}(t). \tag{3.11}$$

Applying inequalities (3.9), (3.10), and (3.11), we obtain

$$\begin{aligned}\mathcal{R}_{1,2}\varphi_h(s, t) &\le \frac{2}{t}\int_s^\infty\int_E g_{u,h}(y)\,\mathrm{d}y\frac{\mathrm{d}u}{u^{1+1/r}}\\ &\le 2\int_s^\infty g^{**}_{u,h}(t)\frac{\mathrm{d}u}{u^{1+1/r}}.\end{aligned}$$

Further, we shall estimate

$$||\Delta_1(h)f||^{\nu}_{\mathcal{L}^{p,\nu}} = \int_0^\infty\int_0^\infty (st)^{\nu/p-1}\mathcal{R}_{1,2}\varphi_h(s, t)^{\nu}\mathrm{d}s\mathrm{d}t.$$

Fix $t > 0$. Applying Hardy's inequality, we have

$$\begin{aligned}\int_0^\infty s^{\nu/p-1}\mathcal{R}_{1,2}\varphi_h(s, t)^{\nu}\mathrm{d}s &\le 2^{\nu}\int_0^\infty s^{\nu/p-1}\left(\int_s^\infty g^{**}_{u,h}(t)\frac{\mathrm{d}u}{u^{1+1/r}}\right)^{\nu}\mathrm{d}s\\ &\le c\int_0^\infty s^{\nu/p-\nu/r-1}g^{**}_{s,h}(t)^{\nu}\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}||\Delta_1(h)f||^{\nu}_{\mathcal{L}^{p,\nu}} &= \int_{\mathbb{R}^2_+}(st)^{\nu/p-1}\mathcal{R}_{1,2}\varphi_h(s, t)^{\nu}\,\mathrm{d}s\mathrm{d}t\\ &\le c\int_0^\infty s^{\nu/p-\nu/r-1}\int_0^\infty t^{\nu/p-1}g^{**}_{s,h}(t)^{\nu}\mathrm{d}t\mathrm{d}s\\ &\le c'\int_0^\infty s^{\nu/p-\nu/r-1}||g_{s,h}||^{\nu}_{L^{p,\nu}}\mathrm{d}s.\end{aligned}$$

By (2.11), we have

$$g_{s,h}(y) = \omega(\varphi_h(\cdot, y); s)_r \le \frac{c}{s}\int_0^s ||\Delta(u)\varphi_h(\cdot, y)||_r \mathrm{d}u.$$

Thus, by the Minkowski inequality,

$$||g_{s,h}||_{L^{p,\nu}} \le \frac{c}{s}\int_0^s ||\Delta_1(u)\varphi_h||_V \mathrm{d}u, \quad \text{where} \quad V = L^{p,\nu}[L^r].$$

Using this estimate and applying Hardy's inequality, we obtain

$$\begin{aligned} ||\Delta_1(h)f||^{\nu}_{\mathcal{L}^{p,\nu}} &\le c\int_0^\infty s^{\nu/p-\nu/r-1}\left(\frac{1}{s}\int_0^s ||\Delta_1(u)\varphi_h||_V \mathrm{d}u\right)^{\nu} \mathrm{d}s \\ &\le c'\int_0^\infty s^{\nu/p-\nu/r-1}||\Delta_1(s)\varphi_h||^{\nu}_V \mathrm{d}s. \end{aligned}$$

Obviously,

$$||\Delta_1(s)\varphi_h||_V \le 2||\Delta_1(\min(s, h))f||_V.$$

Thus,

$$\begin{aligned} &\int_0^\infty h^{-\theta\beta}||\Delta_1(h)f||^{\theta}_{\mathcal{L}^{p,\nu}}\frac{\mathrm{d}h}{h} \\ &\le c\int_0^\infty h^{-\theta\beta}\left(\int_0^\infty s^{\nu/p-\nu/r-1}||\Delta_1(\min(s,h))f||^{\nu}_V \mathrm{d}s\right)^{\theta/\nu}\frac{\mathrm{d}h}{h} \\ &\le c'\left[\int_0^\infty h^{-\theta\beta}\left(\int_0^h s^{\nu/p-\nu/r-1}||\Delta_1(s)f||^{\nu}_V \mathrm{d}s\right)^{\theta/\nu}\frac{\mathrm{d}h}{h}\right. \\ &\quad \left. + \int_0^\infty h^{-\theta\beta}||\Delta_1(h)f||^{\theta}_V\left(\int_h^\infty s^{\nu/p-\nu/r-1}\mathrm{d}s\right)^{\theta/\nu}\frac{\mathrm{d}h}{h}\right] \equiv c(I_1 + I_2). \end{aligned}$$

First,

$$I_2 = c\int_0^\infty h^{-\theta\alpha}||\Delta_1(h)f||^{\theta}_V\frac{\mathrm{d}h}{h}. \tag{3.12}$$

Further, if $\theta > \nu$, then by Proposition 2.1 we obtain

$$I_1 \le c\int_0^\infty h^{-\theta\alpha}||\Delta_1(h)f||^{\theta}_V\frac{\mathrm{d}h}{h}. \tag{3.13}$$

If $\theta \le \nu$, we obtain estimate (3.13) exactly as in Proposition 3.1. Namely, using the fact that the function $\psi(t) = \omega_1(f; t)_V/t$ is quasi-decreasing, we apply Proposition 2.2 and inequality (3.2). Estimates (3.12) and (3.13) give that

$$\int_0^\infty h^{-\theta\beta}||\Delta_1(h)f||^{\theta}_{\mathcal{L}^{p,\nu}}\frac{\mathrm{d}h}{h} \le c\int_0^\infty h^{-\theta\alpha}||\Delta_1(h)f||^{\theta}_V\frac{\mathrm{d}h}{h}.$$

Since $\nu \le p$, the latter inequality implies (3.8) (see (3.5)).

Inequality (3.7) follows by similar arguments; we omit the details. □

Remark 3.3 In this work we apply Proposition 3.2 only for $\nu = r < p$. It would be interesting to consider other cases and further generalizations in this direction.

4 Embeddings of Sobolev spaces $W_p^1(\mathbb{R}^n)$

In this section we prove a refinement of Theorem 1.2. For $1 \le p, q < \infty$ and $k = 1, \dots, n$, denote by $V_{q,p,k}(\mathbb{R}^n)$ the mixed norm space $L^{q,p}(\mathbb{R}^{n-1})[L^p(\mathbb{R})]_k$ obtained by taking first the norm in $L^p(\mathbb{R})$ with respect to the variable x_k, and then the norm in $L^{q,p}(\mathbb{R}^{n-1})$ with respect to $\widehat{x}_k$.

We shall use the following simple fact.

Proposition 4.1 *Let a function φ be defined on $\mathbb{R}$ and assume that φ is locally absolutely continuous (that is, φ is absolutely continuous in each bounded interval $[a, b] \subset \mathbb{R}$). Let $\psi = |\varphi|$. Then, ψ also is locally absolutely continuous and*

$$|\psi'(x)| \le |\varphi'(x)| \quad \textit{for almost } x \in \mathbb{R}.$$

Indeed, this statement follows immediately from the inequality

$$|\psi(x+h) - \psi(x)| \le |\varphi(x+h) - \varphi(x)|.$$

Theorem 4.2 *Let $1 < p < \infty$ and $n \ge 2$, or $p = 1$ and $n \ge 3$. If $p < q < \infty$ and $\alpha = 1 - (n-1)(1/p - 1/q) > 0$, then for any $f \in W_p^1(\mathbb{R}^n)$*

$$\sum_{k=1}^n \left(\int_0^\infty h^{-\alpha p} ||\Delta_k(h) f||_{V_{q,p,k}}^p \frac{\mathrm{d}h}{h} \right)^{1/p} \le c||\nabla f||_p. \tag{4.1}$$

Proof We estimate the last term of the sum in (4.1). Set

$$\varphi_h(\widehat{x}_n) = \left(\int_{\mathbb{R}} |\Delta_n(h) f(x)|^p \mathrm{d}x_n \right)^{1/p}$$

and

$$\psi_j(\widehat{x}_n) = \left(\int_{\mathbb{R}} |D_j f(x)|^p \mathrm{d}x_n \right)^{1/p}, \quad j = 1, \dots, n.$$

We consider the integral

$$J = \int_0^\infty h^{-\alpha p} K(h) \frac{\mathrm{d}h}{h}, \tag{4.2}$$

where

$$K(h) = ||\Delta_n(h) f||_{V_{q,p,n}}^p = \int_0^\infty t^{p/q-1} \varphi_h^*(t)^p \mathrm{d}t.$$

Set

$$K_1(h) = \int_{h^{n-1}}^\infty t^{p/q-1} \varphi_h^*(t)^p \mathrm{d}t, \quad K_2(h) = \int_0^{h^{n-1}} t^{p/q-1} \varphi_h^*(t)^p \mathrm{d}t. \tag{4.3}$$

For any $h > 0$

$$|\Delta_n(h) f(x)| \le \int_0^h |D_n f(x + ue_n)| \mathrm{d}u.$$

Raising to the power p, integrating over x_n in $\mathbb{R}$, and applying Hölder's inequality, we obtain

$$\varphi_h(\widehat{x}_n)^p \le \int_{\mathbb{R}} \left(\int_0^h |D_n f(x + ue_n)| \mathrm{d}u \right)^p dx_n \le h^p \psi_n(\widehat{x}_n)^p.$$

Thus,

$$\varphi_h^*(t) \le h\psi_n^*(t). \tag{4.4}$$

From here (see (4.3))

$$K_1(h) \le h^p \int_{h^{n-1}}^{\infty} t^{p/q-1}\psi_n^*(t)^p \mathrm{d}t$$

and therefore

$$\begin{aligned} J_1 &= \int_0^\infty h^{-\alpha p} K_1(h)\frac{\mathrm{d}h}{h} \le \int_0^\infty h^{(1-\alpha)p}\int_{h^{n-1}}^\infty t^{p/q-1}\psi_n^*(t)^p \mathrm{d}t\frac{\mathrm{d}h}{h} \\ &= \int_0^\infty t^{p/q-1}\psi_n^*(t)^p \int_0^{t^{1/(n-1)}} h^{(1-\alpha)p}\frac{\mathrm{d}h}{h}\mathrm{d}t \\ &= ((1-\alpha)p)^{-1}\int_0^\infty \psi_n^*(t)^p \mathrm{d}t = c||D_n f||_p^p. \end{aligned}$$

This estimate holds for all $p \ge 1$ and $n \ge 2$.

Estimating $K_2(h)$, we first assume that $p = 1$ and $n \ge 3$. Set

$$g(\widehat{x}_n) = \int_{\mathbb{R}} |f(x)|\, \mathrm{d}x_n.$$

Then $||g||_{L^1(\mathbb{R}^{n-1})} = ||f||_{L^1(\mathbb{R}^n)}$. Moreover, $g \in W_1^1(\mathbb{R}^{n-1})$ and

$$||D_j g||_{L^1(\mathbb{R}^{n-1})} \le ||D_j f||_{L^1(\mathbb{R}^n)}, \quad j = 1, \dots, n-1. \tag{4.5}$$

Indeed, since $f \in W_p^1(\mathbb{R}^n)$, then for any $j = 1, \dots, n$ and almost all $\widehat{x}_j \in \mathbb{R}^{n-1}$ the function f is locally absolutely continuous with respect to x_j (see, e.g., [30, 2.1.4]). Thus, we can apply Proposition 4.1.

We have

$$\varphi_h(\widehat{x}_n) \le \int_{\mathbb{R}} |f(x)|\mathrm{d}x_n + \int_{\mathbb{R}} |f(x+he_n)|\mathrm{d}x_n = 2g(\widehat{x}_n).$$

Thus (see (4.3)),

$$K_2(h) \le 2\int_0^{h^{n-1}} t^{1/q-1} g^*(t)\mathrm{d}t$$

and

$$\begin{aligned} J_2 &= \int_0^\infty h^{-\alpha} K_2(h)\frac{\mathrm{d}h}{h} \le 2\int_0^\infty h^{-\alpha}\int_0^{h^{n-1}} t^{1/q-1} g^*(t)\mathrm{d}t\frac{\mathrm{d}h}{h} \\ &= 2\int_0^\infty t^{1/q-1} g^*(t)\int_{t^{1/(n-1)}}^\infty h^{(1-1/q)(n-1)-1}\frac{\mathrm{d}h}{h} \\ &= c\int_0^\infty t^{-1/(n-1)} g^*(t)\mathrm{d}t = c||g||_{(n-1)',1}. \end{aligned}$$

Taking into account (4.5) and applying inequality (1.2), we get

$$J_2 \le c||g||_{(n-1)',1} \le c'\sum_{j=1}^{n-1} ||D_j f||_1.$$

Together with the estimate $J_1 \le c||D_n f||_1$ obtained above, this gives (4.1) for $p = 1$, $n \ge 3$.

Let now $p > 1$, $n \ge 2$. In what follows we write $x = (u, x_n)$, $u = \widehat{x}_n \in \mathbb{R}^{n-1}$.

For a fixed $u \in \mathbb{R}^{n-1}$ and $t > 0$, denote by $Q_u(t)$ the cube in $\mathbb{R}^{n-1}$ centered at u with the side length $(4t)^{1/(n-1)}$. Let

$$A_{u,t,h} = \{v \in Q_u(t) : \varphi_h(v) \le \varphi_h^*(2t)\}.$$

Then $\mathrm{mes}_{n-1} A_{u,t,h} \ge 2t$. Thus, we have

$$\begin{aligned}\varphi_h(u) - \varphi_h^*(2t) &\le \varphi_h(u) - \frac{1}{\mathrm{mes}_{n-1} A_{u,t,h}} \int_{A_{u,t,h}} \varphi_h(v) \mathrm{d}v \\ &\le \frac{1}{2t} \int_{Q_u(t)} |\varphi_h(u) - \varphi_h(v)| \mathrm{d}v. \end{aligned} \tag{4.6}$$

Further,

$$\begin{aligned}|\varphi_h(u) - \varphi_h(v)| &= \left| \left(\int_{\mathbb{R}} |f(u, x_n + h) - f(u, x_n)|^p \mathrm{d}x_n \right)^{1/p} \right. \\ &\quad \left. - \left(\int_{\mathbb{R}} |f(v, x_n + h) - f(v, x_n)|^p \mathrm{d}x_n \right)^{1/p} \right| \\ &\le 2 \left(\int_{\mathbb{R}} |f(u, x_n) - f(v, x_n)|^p \mathrm{d}x_n \right)^{1/p}. \end{aligned}$$

We have (see [18, p. 143])

$$|f(u, x_n) - f(v, x_n)| \le |u - v| \sum_{j=1}^{n-1} \int_0^1 |D_j f(u + \tau(v - u), x_n)| \mathrm{d}\tau.$$

If $v \in Q_u(t)$, then $|u - v| \le \sqrt{n-1}(2t)^{1/(n-1)}$. Thus, by the Minkowski inequality, for any $v \in Q_u(t)$

$$\begin{aligned}&|\varphi_h(u) - \varphi_h(v)| \\ &\le ct^{1/(n-1)} \sum_{j=1}^{n-1} \left(\int_{\mathbb{R}} \left(\int_0^1 |D_j f(u + \tau(v - u), x_n)| \mathrm{d}\tau \right)^p \mathrm{d}x_n \right)^{1/p} \\ &\le ct^{1/(n-1)} \sum_{j=1}^{n-1} \int_0^1 \left(\int_{\mathbb{R}} |D_j f(u + \tau(v - u), x_n)|^p \mathrm{d}x_n \right)^{1/p} \mathrm{d}\tau \\ &= ct^{1/(n-1)} \sum_{j=1}^{n-1} \int_0^1 \psi_j(u + \tau(v - u)) \mathrm{d}\tau. \end{aligned}$$

From here and (4.6),

$$\varphi_h(u) - \varphi_h^*(2t) \le ct^{1/(n-1)-1} \sum_{j=1}^{n-1} \int_{Q_0(t)} \int_0^1 \psi_j(u + \tau z) \mathrm{d}\tau \mathrm{d}z. \tag{4.7}$$

Taking into account that

$$\varphi_h^*(t) \le \sup_{\mathrm{mes}_{n-1} E = t} \frac{1}{t} \int_E \varphi_h(u) \mathrm{d}u,$$

and applying (4.7), we get

$$\varphi_h^*(t) - \varphi_h^*(2t) \le \sup_{\mathrm{mes}_{n-1} E = t} \frac{1}{t}\int_E [\varphi_h(u) - \varphi_h^*(2t)]\mathrm{d}u$$
$$\le ct^{1/(n-1)-1}\sum_{j=1}^{n-1} \sup_{\mathrm{mes}_{n-1} E = t}\int_{Q_0(t)}\int_0^1 \frac{1}{t}\int_E \psi_j(u+\tau z)\mathrm{d}u\mathrm{d}\tau\mathrm{d}z.$$

Let $E \subset \mathbb{R}^{n-1}$, $\mathrm{mes}_{n-1} E = t$. Then for all $\tau \in [0, 1]$ and $z \in Q_0(t)$

$$\frac{1}{t}\int_E \psi_j(u+\tau z)\mathrm{d}u \le \psi_j^{**}(t).$$

Thus, we have that

$$\varphi_h^*(t) - \varphi_h^*(2t) \le ct^{1/(n-1)}\sum_{j=1}^{n-1}\psi_j^{**}(t). \tag{4.8}$$

Now, for any $\varepsilon > 0$, we have

$$J_2(\varepsilon)^{1/p} = \left(\int_\varepsilon^{1/\varepsilon} h^{-\alpha p}\int_{\varepsilon^{n-1}}^{h^{n-1}} t^{p/q-1}\varphi_h^*(t)^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p}$$
$$\le \left(\int_0^\infty h^{-\alpha p}\int_0^{h^{n-1}} t^{p/q-1}[\varphi_h^*(t) - \varphi_h^*(2t)]^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p}$$
$$+\left(\int_\varepsilon^{1/\varepsilon} h^{-\alpha p}\int_{\varepsilon^{n-1}}^{h^{n-1}} t^{p/q-1}\varphi_h^*(2t)^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p} \equiv I' + I''(\varepsilon).$$

By (4.8) and (2.5),

$$I' \le c\sum_{j=1}^{n-1}\left(\int_0^\infty h^{-\alpha p}\int_0^{h^{n-1}} t^{p/q+p/(n-1)-1}\psi_j^{**}(t)^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p}$$
$$= c\sum_{j=1}^{n-1}\left(\int_0^\infty t^{p/q+p/(n-1)-1}\psi_j^{**}(t)^p\int_{t^{1/(n-1)}}^\infty h^{-\alpha p}\frac{\mathrm{d}h}{h}\mathrm{d}t\right)^{1/p}$$
$$= c'\sum_{j=1}^{n-1}\left(\int_0^\infty \psi_j^{**}(t)^p\mathrm{d}t\right)^{1/p} \le c''\sum_{j=1}^{n-1}||\psi_j||_p = c''\sum_{j=1}^{n-1}||D_jf||_p.$$

Further,

$$I''(\varepsilon) = \left(2^{-p/q}\int_\varepsilon^{1/\varepsilon} h^{-\alpha p}\int_{2\varepsilon^{n-1}}^{2h^{n-1}} t^{p/q-1}\varphi_h^*(t)^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p}$$
$$\le 2^{-1/q}\left(\int_\varepsilon^{1/\varepsilon} h^{-\alpha p}\int_{\varepsilon^{n-1}}^{h^{n-1}} t^{p/q-1}\varphi_h^*(t)^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p}$$
$$+2^{-1/q}\left(\int_0^\infty h^{-\alpha p}\int_{h^{n-1}}^\infty t^{p/q-1}\varphi_h^*(t)^p\mathrm{d}t\frac{\mathrm{d}h}{h}\right)^{1/p}$$
$$= 2^{-1/q}\left(J_2(\varepsilon)^{1/p} + J_1^{1/p}\right).$$

As we have proved above, $J_1^{1/p} \le c||D_n f||_p$. Thus,

$$J_2(\varepsilon)^{1/p} \le I' + I''(\varepsilon) \le 2^{-1/q} J_2(\varepsilon)^{1/p} + c\sum_{j=1}^n ||D_j f||_p$$

and

$$J_2(\varepsilon)^{1/p} \le c' \sum_{j=1}^n ||D_j f||_p.$$

This implies that

$$J_2^{1/p} = \left(\int_0^\infty h^{-\alpha p} \int_0^{h^{n-1}} t^{p/q-1} \varphi_h^*(t)^p \mathrm{d}t \frac{\mathrm{d}h}{h}\right)^{1/p} \le c' \sum_{j=1}^n ||D_j f||_p.$$

Thus, we have (see notations (4.2) and (4.3))

$$J^{1/p} \le J_1^{1/p} + J_2^{1/p} \le c'' \sum_{j=1}^n ||D_j f||_p.$$

In turn, this yields (4.1) for $p > 1,\ n \ge 2$. □

Remark 4.3 By Proposition 3.2, inequality (4.1) gives a refinement of (1.4).

We stress that (4.1) is true for $p > 1, n = 2$. As it was already observed, *we do not know whether this inequality is true for* $p = 1,\ n = 2$. However, we shall show that similar inequality holds for $p = 1,\ n = 2$ if we replace the L^1-norms of derivatives by the Hardy H^1-norms.

5 Embeddings of Hardy–Sobolev spaces

For a function $f \in L^1(\mathbb{R}^n)$ the Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-i2\pi x\cdot\xi}\, \mathrm{d}x, \quad \xi \in \mathbb{R}^n.$$

Let $f \in L^1(\mathbb{R}^n)$. The Riesz transforms $R_j f$ $(j = 1, \dots, n)$ of f are defined by the equality

$$R_j f(x) = \lim_{\varepsilon\to 0+} c_n \int_{|y|\ge\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\mathrm{d}y, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.$$

The space $H^1(\mathbb{R}^n)$ is the class of all functions $f \in L^1(\mathbb{R}^n)$ such that $R_j f \in L^1(\mathbb{R}^n)$ $(j = 1, \dots, n)$. The H^1-norm is defined by

$$||f||_{H^1} = ||f||_1 + \sum_{j=1}^n ||R_j f||_1$$

(see [7, p. 144], [8, Ch. III.4]).

If $f \in H^1(\mathbb{R}^n)$, then we have (see [8, p. 197])

$$(R_j f)^\wedge(\xi) = -\frac{i\xi_j}{|\xi|} \widehat{f}(\xi).$$

Let P_t be the Poisson kernel in $\mathbb{R}^n$. We consider $n+1$ harmonic functions in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$

$$u_0(x,t) = (P_t * f)(x), \quad u_j(x,t) = (P_t * R_j f)(x) \quad (j = 1, \dots, n). \tag{5.1}$$

These functions satisfy the equations of conjugacy

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \quad (0 \le j, k \le n), \quad \sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0 \quad (x_0 = t) \tag{5.2}$$

(see [8, Ch. III.4]).

For any $x \in \mathbb{R}^n$, denote by $\Gamma(x)$ the cone

$$\Gamma(x) = \left\{ (y,t) \in \mathbb{R}^{n+1}_+ : |x - y| \le t \right\}.$$

Let $f \in L^1(\mathbb{R}^n)$. The nontangential maximal function Nf is defined by

$$Nf(x) = \sup_{(y,t) \in \Gamma(x)} |(P_t * f)(y)|.$$

A function $f \in L^1(\mathbb{R}^n)$ belongs to $H^1(\mathbb{R}^n)$ if and only if $Nf \in L^1(\mathbb{R}^n)$. In this case

$$c||f||_{H^1} \le ||Nf||_1 \le c'||f||_{H^1} \quad (c > 0) \tag{5.3}$$

(see [8, Ch. III.4], [9, Th. 6.7.4]).

The nontangential maximal function Nf is controlled by the vertical maximal function

$$N_v f(x) = \sup_{t>0} |(P_t * f)(x)|.$$

Namely, $Nf \in L^1(\mathbb{R}^n)$ if and only if $N_v f \in L^1(\mathbb{R}^n)$, and in this case

$$||N_v f||_1 \le ||Nf||_1 \le c||N_v f||_1 \tag{5.4}$$

(see [7, p.170], [9, Th. 6.4.4]).

Furthermore, if $f \in H^1(\mathbb{R}^n)$, then

$$\sum_{j=0}^{n} ||N_v f_j||_1 \le c||f||_{H^1}, \tag{5.5}$$

where $f_0 = f$, $f_j = R_j f \ \ (j = 1, \dots, n)$ (see [26, Ch. VII.3.2])).

Inequalities (5.3)–(5.5) imply that for any $f \in H^1(\mathbb{R}^n)$ its Riesz transforms $R_j f$ $(j = 1, \dots, n)$ belong to $H^1(\mathbb{R}^n)$ and

$$||R_j f||_{H^1} \le c||f||_{H^1} \quad (j = 1, \dots, n) \tag{5.6}$$

(see also [8, pp. 288, 322]).

Denote by $HW^1_1(\mathbb{R}^n)$ the space of all functions $f \in H^1(\mathbb{R}^n)$ for which all weak partial derivatives $D_j f$ exist and belong to $H^1(\mathbb{R}^n)$.

Lemma 5.1 *Let $f \in HW^1_1(\mathbb{R}^n)$ and let $u(x,t) = (P_t * f)(x)$, $t > 0$. Set*

$$\widetilde{N} f(x) = \sup_{(y,t) \in \Gamma(x)} \left| \frac{\partial u}{\partial t}(y,t) \right|. \tag{5.7}$$

Then

$$\widetilde{N}f(x) \le \sum_{j=1}^{n} N(R_j(D_j f))(x) \tag{5.8}$$

and

$$\|\widetilde{N}f\|_1 \le c\sum_{j=1}^{n} \|D_j f\|_{H^1}. \tag{5.9}$$

Proof Let $u_j(x,t) = P_t * (R_j f)(x)$ $(j = 1, \dots, n)$. By the Fourier inversion,

$$u_j(x,t) = -\int_{\mathbb{R}^n} \widehat{f}(\xi)\frac{i\xi_j}{|\xi|}\mathrm{e}^{2\pi i\xi\cdot x}\mathrm{e}^{-2\pi|\xi|t}\mathrm{d}\xi.$$

Further,

$$\frac{\partial u_j}{\partial x_j}(x,t) = -\int_{\mathbb{R}^n} 2\pi i\xi_j\widehat{f}(\xi)\frac{i\xi_j}{|\xi|}\mathrm{e}^{2\pi i\xi\cdot x}\mathrm{e}^{-2\pi|\xi|t}\mathrm{d}\xi.$$

Indeed, differentiation under the integral sign is justified by the convergence of the integral

$$\int_{\mathbb{R}^n} |\xi||\widehat{f}(\xi)|\mathrm{e}^{-2\pi|\xi|t}\mathrm{d}\xi, \quad t > 0.$$

Thus,

$$\frac{\partial u_j}{\partial x_j}(x,t) = (P_t * (R_j(D_j f)))(x) \quad (j = 1, \dots, n). \tag{5.10}$$

By (5.2),

$$\left|\frac{\partial u}{\partial t}(x,t)\right| \le \sum_{j=1}^{n}\left|\frac{\partial u_j}{\partial x_j}(x,t)\right|. \tag{5.11}$$

Applying (5.11) and (5.10), we get (5.8). By (5.3) and (5.6), this implies (5.9). □

As it was mentioned above, the following theorem holds.

Theorem 5.2 *Assume that* $f \in HW_1^1(\mathbb{R}^n)$ $(n \in \mathbb{N})$ *and* $1 < q < n'$. *Then*

$$\sum_{k=1}^{n}\int_0^\infty h^{n/q'-1}\|\Delta_k(h)f\|_q\frac{\mathrm{d}h}{h} \le c\sum_{k=1}^{n}\|D_k f\|_{H^1}.$$

For $n \ge 2$ this result follows from Theorem 4.2; for $n = 1$ it was proved in [22] (see also [11]).

In this section we obtain a refinement of Theorem 5.2 for $n \ge 2$. For $1 < q < \infty$ and $k = 1, \dots, n$, denote by $V_{q,k}$ the mixed norm space $L^{q,1}(\mathbb{R}^{n-1})[L^1(\mathbb{R})]_k$ obtained by taking first the norm in $L^1(\mathbb{R})$ with respect to the variable x_k, and then the norm in $L^{q,1}(\mathbb{R}^{n-1})$ with respect to $\widehat{x}_k$.

Theorem 5.3 *Assume that* $f \in HW_1^1(\mathbb{R}^n)$ $(n \ge 2)$. *Let* $1 < q < (n-1)'$ *and* $\alpha = 1 - (n-1)/q'$. *Then*

$$\sum_{k=1}^{n}\int_0^\infty h^{-\alpha}\|\Delta_k(h)f\|_{V_{q,k}}\frac{\mathrm{d}h}{h} \le c\sum_{k=1}^{n}\|D_k f\|_{H^1}. \tag{5.12}$$

Proof For $n \geq 3$ (5.12) follows from the stronger inequality (4.1). We assume that $n = 2$. Set

$$\varphi_h(x) = \int_{\mathbb{R}} |f(x, y+h) - f(x, y)| \mathrm{d}y, \quad h > 0.$$

We consider the integral

$$J = \int_0^\infty h^{-1/q-1} \int_0^\infty s^{1/q-1} \varphi_h^*(s) \mathrm{d}s \mathrm{d}h. \tag{5.13}$$

We have

$$\begin{aligned} J &= \int_0^\infty h^{-1/q-1} \int_h^\infty s^{1/q-1} \varphi_h^*(s) \mathrm{d}s \mathrm{d}h \\ &\quad + \int_0^\infty h^{-1/q-1} \int_0^h s^{1/q-1} \varphi_h^*(s) \mathrm{d}s \mathrm{d}h \equiv J_1 + J_2. \end{aligned}$$

As in Theorem 4.2, we have the estimate

$$\varphi_h^*(s) \leq h g^*(s), \quad \text{where} \quad g(x) = \int_{\mathbb{R}} |D_2 f(x, y)| \mathrm{d}y. \tag{5.14}$$

Applying this estimate, we immediately get that

$$J_1 \leq c||D_2 f||_1. \tag{5.15}$$

Further, for the simplicity, we may assume that $J_2 < \infty$ (otherwise we can apply the same arguments as ones given at the final part of the proof of Theorem 4.2 for estimation of J_2). We first consider the difference $\varphi_h^*(s) - \varphi_h^*(2s)$. Denote

$$\begin{aligned} \psi(x) &= \int_{\mathbb{R}} N(D_1 f)(x, y) \mathrm{d}y, \\ \psi_1(x) &= \int_{\mathbb{R}} N(R_1(D_1 f))(x, y) \mathrm{d}y, \quad \psi_2(x) = \int_{\mathbb{R}} N(R_2(D_2 f))(x, y) \mathrm{d}y, \end{aligned}$$

and $\Psi = \psi + \psi_1 + \psi_2$.

Let $x \in \mathbb{R}$ and $s > 0$. There exists $\tau = \tau(x, s) \in (0, 2s)$ such that

$$\varphi_h(x + 2\tau) \leq \varphi_h^*(2s) \quad \text{and} \quad \Psi(x + 2\tau) \leq \Psi^*(s). \tag{5.16}$$

Indeed, let A be the set of all $u \in (0, 4s)$ such that at least one of the inequalities

$$\varphi_h(x + u) > \varphi_h^*(2s) \quad \text{or} \quad \Psi(x + u) > \Psi^*(s) \tag{5.17}$$

holds. Then $\mathrm{mes}_1 A \leq 3s$ and therefore there exists $u \in (0, 4s)$ for which both the inequalities (5.17) fail.

Further, we have

$$\begin{aligned} \varphi_h(x) - \varphi_h^*(2s) &\leq \varphi_h(x) - \varphi_h(x + 2\tau) \\ &\leq 2 \int_{\mathbb{R}} |f(x + 2\tau, y) - f(x, y)| \mathrm{d}y. \end{aligned} \tag{5.18}$$

For fixed x, y, and s, consider the cones

$$\Gamma_1 = \Gamma(x, y) \quad \text{and} \quad \Gamma_2 = \Gamma(x + 2\tau, y).$$

The point $(x+\tau, y, \tau)$ belongs to both of them. Let $u = P_t * f$. Then

$$\begin{aligned}
&|f(x+2\tau, y) - f(x,y)| \\
&\quad \le |f(x,y) - u(x+\tau, y, \tau)| + |f(x+2\tau, y) - u(x+\tau, y, \tau)| \\
&\quad \le \int_0^\tau \left(\left| \frac{\partial u}{\partial x}(x+t, y, t) \right| + \left| \frac{\partial u}{\partial t}(x+t, y, t) \right| \right) \mathrm{d}t \\
&\quad\quad + \int_0^\tau \left(\left| \frac{\partial u}{\partial x}(x+\tau+s, y, \tau - s) \right| + \left| \frac{\partial u}{\partial t}(x+\tau+s, y, \tau - s) \right| \right) \mathrm{d}s \\
&\quad \le \tau \sup_{(x',y',t)\in\Gamma_1} \left(\left| \frac{\partial u}{\partial x}(x', y', t) \right| + \left| \frac{\partial u}{\partial t}(x', y', t) \right| \right) \\
&\quad\quad + \tau \sup_{(x',y',t)\in\Gamma_2} \left(\left| \frac{\partial u}{\partial x}(x', y', t) \right| + \left| \frac{\partial u}{\partial t}(x', y', t) \right| \right) \\
&\quad \le \tau \left[N(D_1 f)(x,y) + N(D_1 f)(x+2\tau, y) + \widetilde{N} f(x,y) + \widetilde{N} f(x+2\tau, y) \right]
\end{aligned}$$

(we have used the notation (5.7)). Applying (5.8), we have

$$\begin{aligned}
|f(x+2\tau, y) - f(x,y)| &\le \tau\, (N(D_1 f)(x,y) + N(D_1 f)(x+2\tau, y) \\
&+ N(R_1(D_1 f))(x,y) + N(R_1(D_1 f))(x+2\tau, y) \\
&+ N(R_2(D_2 f))(x,y) + N(R_2(D_2 f))(x+2\tau, y))\,.
\end{aligned}$$

By (5.18), this implies that

$$\varphi_h(x) - \varphi_h^*(2s) \le 2\tau(\Psi(x) + \Psi(x+2\tau)),$$

where $\Psi = \psi + \psi_1 + \psi_2$. Taking into account (5.16), we obtain

$$\varphi_h^*(s) - \varphi_h^*(2s) \le 8s\Psi^*(s).$$

From here

$$\begin{aligned}
J_2' &= \int_0^\infty h^{-1/q-1} \int_0^h s^{1/q-1} [\varphi_h^*(s) - \varphi_h^*(2s)] \mathrm{d}s \mathrm{d}h \\
&\le 8 \int_0^\infty h^{-1/q-1} \int_0^h s^{1/q} \Psi^*(s) \mathrm{d}s \mathrm{d}h = 8q \int_0^\infty \Psi^*(s) \mathrm{d}s = 8q||\Psi||_1.
\end{aligned}$$

Applying (5.3) and (5.6), we get

$$\begin{aligned}
||\Psi||_1 &= ||N(D_1 f)||_1 + ||N(R_1(D_1 f))||_1 + ||N(R_2(D_2 f))||_1 \\
&\le c(||D_1 f||_{H^1} + ||D_2 f||_{H^1}).
\end{aligned}$$

Thus,

$$J_2' \le c'(||D_1 f||_{H^1} + ||D_2 f||_{H^1}). \tag{5.19}$$

Further, we consider

$$J_2'' = \int_0^\infty h^{-1/q-1} \int_0^h s^{1/q-1} \varphi_h^*(2s) \mathrm{d}s \mathrm{d}h.$$

We have (see (5.13))

$$J_2'' = 2^{-1/q} \int_0^\infty h^{-1/q-1} \int_0^{2h} s^{1/q-1} \varphi_h^*(s) \mathrm{d}s \mathrm{d}h \le 2^{-1/q} J. \tag{5.20}$$

Using estimates (5.15), (5.19), and (5.20), we obtain

$$J \le 2^{-1/q} J + c(||D_1 f||_{H^1} + ||D_2 f||_{H^1}). \tag{5.21}$$

We assumed that $J_2 < \infty$ and hence $J = J_1 + J_2 < \infty$. Thus, (5.21) implies (5.12) for $n = 2$. □

6 Estimates of Fourier transforms

By Hardy's inequality, for any $f \in H^1(\mathbb{R}^n)$ $(n \in \mathbb{N})$

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|}{|\xi|^n} \, d\xi \le c||f||_{H^1}. \tag{6.1}$$

It was first discovered by Bourgain [4] that for $n \ge 2$ the Fourier transforms of the derivatives of functions in the Sobolev space $W_1^1(\mathbb{R}^n)$ satisfy Hardy's inequality. More exactly, Bourgain considered the periodic case. His studies were continued by Pełczyński and Wojciechowski [23]. The following theorem holds (Bourgain; Pełczyński and Wojciechowski).

Theorem 6.1 *Let $f \in W_1^1(\mathbb{R}^n)$ $(n \ge 2)$. Then*

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)||\xi|^{1-n} \, d\xi \le c||\nabla f||_1. \tag{6.2}$$

Equivalently,

$$\sum_{k=1}^{n} \int_{\mathbb{R}^n} \frac{|(D_k f)^\wedge(\xi)|}{|\xi|^n} d\xi \le c \sum_{k=1}^{n} \|D_k f\|_1. \tag{6.3}$$

This is Hardy-type inequality. These results were extended in [13,15].

In contrast to (6.1), inequalities (6.2) and (6.3) fail to hold for $n = 1$.

Oberlin [20] proved the following refinement of Hardy's inequality (6.1) valid for $n \ge 2$.

Theorem 6.2 *Let $f \in H^1(\mathbb{R}^n)$ $(n \ge 2)$. Then*

$$\sum_{k\in\mathbb{Z}} 2^{k(1-n)} \sup_{2^k \le r \le 2^{k+1}} \int_{S_r} |\widehat{f}(\xi)| \, d\sigma(\xi) \le c||f||_{H^1}, \tag{6.4}$$

where S_r is the sphere of the radius r centered at the origin in $\mathbb{R}^n$ and $d\sigma(\xi)$ is the canonical surface measure on S_r.

Inequality (6.4) was used in [20] to obtain the description of radial Fourier multipliers for $H^1(\mathbb{R}^n)$ $(n \ge 2)$. Observe that these results fail for $n = 1$.

In this section we prove some estimates of Fourier transforms of functions in $W_1^1(\mathbb{R}^n)$ $(n \ge 3)$. In particular, these estimates provide Oberlin-type inequalities for the Fourier transforms of the derivatives of functions in $W_1^1(\mathbb{R}^n)$.

We shall use the notation (2.3).

Theorem 6.3 *Let $f \in W_1^1(\mathbb{R}^n)$ $(n \ge 3)$. Then*

$$\sum_{j=1}^{n} \int_0^\infty F_{t,j}^{**}(t^{n-1}) \, dt \le c||\nabla f||_1, \tag{6.5}$$

where

$$F_{t,j}(\widehat{\xi_j}) = \sup_{|\xi_j|\ge t} |\widehat{f}(\xi)| \quad (t>0). \tag{6.6}$$

Proof We estimate the first term of the sum in (6.5). Set $\varphi_h(x) = \Delta_1(h)f(x)$. Then

$$\widehat{\varphi_h}(\xi) = \widehat{f}(\xi)(\mathrm{e}^{2\pi i h\xi_1} - 1).$$

Let $t>0$ and $\tau = 1/t$. Assume that $|\xi_1| \ge t$. Then

$$\begin{aligned}\frac{1}{\tau}\int_0^\tau |\mathrm{e}^{2\pi i h\xi_1} - 1|\mathrm{d}h &\ge \frac{1}{\tau}\int_0^\tau (1-\cos(2\pi\xi_1 h))\mathrm{d}h\\ &= 1 - \frac{\sin(2\pi\xi_1\tau)}{2\pi\xi_1\tau} \ge 1 - \frac{1}{2\pi|\xi_1|\tau} \ge 1 - \frac{1}{2\pi}.\end{aligned}$$

It follows that

$$\frac{2}{\tau}\int_0^\tau |\widehat{\varphi_h}(\xi)|\mathrm{d}h \ge |\widehat{f}(\xi)| \quad \text{if} \quad |\xi_1| \ge t$$

and

$$\frac{2}{\tau}\sup_{|\xi_1|\ge t}\int_0^\tau |\widehat{\varphi_h}(\xi)|\mathrm{d}h \ge F_{t,1}(\widehat{\xi_1}).$$

By (2.2), we have

$$\begin{aligned}F_{t,1}^{**}(t^{n-1}) &\le \frac{2t^{1-n}}{\tau}\sup_{\mathrm{mes}_{n-1}E = t^{n-1}}\sup_{|\xi_1|\ge t}\int_E\int_0^\tau |\widehat{\varphi_h}(\xi)|\mathrm{d}h\mathrm{d}\widehat{\xi_1}\\ &\le \frac{2t^{1-n}}{\tau}\sup_{|\xi_1|\ge t}\int_0^\tau \sup_{\mathrm{mes}_{n-1}E = t^{n-1}}\int_E |\widehat{\varphi_h}(\xi)|\mathrm{d}\widehat{\xi_1}\mathrm{d}h.\end{aligned}$$

Let $1 < q < (n-1)'$; then $q < 2$. By Hölder's inequality, for any set $E \subset \mathbb{R}^{n-1}$ with $\mathrm{mes}_{n-1} E = t^{n-1}$ and any fixed $\xi_1 \in \mathbb{R}$

$$t^{1-n}\int_E |\widehat{\varphi_h}(\xi)|\mathrm{d}\widehat{\xi_1} \le t^{-(n-1)/q'}\left(\int_{\mathbb{R}^{n-1}} |\widehat{\varphi_h}(\xi)|^{q'}\mathrm{d}\widehat{\xi_1}\right)^{1/q'}.$$

Observe that for fixed $h>0$ and $\xi_1 \in \mathbb{R}$, $\widehat{\varphi_h}(\xi) = (\widehat{\varphi_h})_{\xi_1}(\widehat{\xi_1})$ is the Fourier transform of the function

$$\widehat{x_1} \mapsto \int_{\mathbb{R}} \Delta_1(h)f(x)\mathrm{e}^{-2\pi i\xi_1 x_1}\,\mathrm{d}x_1.$$

Applying the Hausdorff–Young inequality, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^{n-1}} |(\widehat{\varphi_h})_{\xi_1}(\widehat{\xi_1})|^{q'}\mathrm{d}\widehat{\xi_1}\right)^{1/q'} &\le \left(\int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}}\Delta_1(h)f(x)\mathrm{e}^{-2\pi i\xi_1 x_1}\,\mathrm{d}x_1\right|^q \mathrm{d}\widehat{x_1}\right)^{1/q}\\ &\le \left(\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}|\Delta_1(h)f(x)|\,\mathrm{d}x_1\right)^q \mathrm{d}\widehat{x_1}\right)^{1/q}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\frac{2t^{1-n}}{\tau}\sup_{\mathrm{mes}_{n-1}E = t^{n-1}}\sup_{|\xi_1|\ge t}\int_E\int_0^\tau |\widehat{\varphi_h}(\xi)|\mathrm{d}h\mathrm{d}\widehat{\xi_1}\\ &\quad\le 2t^{1-(n-1)/q'}\int_0^{1/t} ||\Delta_1(h)f||_{L^q[L^1]}\mathrm{d}h.\end{aligned}$$

It follows that

$$\int_0^\infty F^{**}_{t,1}(t^{n-1})\,\mathrm{d}t \le 2\int_0^\infty t^{1-(n-1)/q'}\int_0^{1/t} ||\Delta_1(h)f||_{L^q[L^1]}\mathrm{d}t\mathrm{d}t$$
$$= c\int_0^\infty h^{(n-1)/q'-1}||\Delta_1(h)f||_{L^q[L^1]}\frac{\mathrm{d}h}{h}.$$

Applying Theorem 4.2, we obtain that

$$\int_0^\infty F^{**}_{t,1}(t^{n-1})\,\mathrm{d}t \le c||\nabla f||_1.$$

□

Similarly, we have the following theorem.

Theorem 6.4 *Let* $f \in HW_1^1(\mathbb{R}^2)$. *Then*

$$\int_0^\infty [F^{**}_{t,1}(t) + F^{**}_{t,2}(t)]\mathrm{d}t \le c(||D_1 f||_{H^1} + ||D_2 f||_{H^1}),$$

where

$$F_{t,1}(\xi) = \sup_{|\eta|\ge t}|\widehat{f}(\xi,\eta)|, \quad F_{t,2}(\eta) = \sup_{|\xi|\ge t}|\widehat{f}(\xi,\eta)|.$$

Applying Theorem 6.3, we obtain the following Oberlin-type estimate.

Theorem 6.5 *Let* $f \in W_1^1(\mathbb{R}^n)$ $(n \ge 3)$. *Then*

$$\sum_{k\in\mathbb{Z}} 2^{k(2-n)} \sup_{2^k\le r\le 2^{k+1}} \int_{S_r} |\widehat{f}(\xi)|\mathrm{d}\sigma(\xi) \le c||\nabla f||_1, \tag{6.7}$$

where S_r *is the sphere of the radius* r *centered at the origin in* $\mathbb{R}^n$ *and* $\mathrm{d}\sigma(\xi)$ *is the canonical surface measure on* S_r.

Proof Let B'_r be the ball in $\mathbb{R}^{n-1}$ of the radius $r/\sqrt{n'}$ centered at the origin. Set

$$S^+_{r,j} = \left\{\xi \in S_r : \xi_j \ge \frac{r}{\sqrt{n}}\right\} \quad \text{and} \quad S^-_{r,j} = \left\{\xi \in S_r : \xi_j \le -\frac{r}{\sqrt{n}}\right\}.$$

Clearly,

$$S^+_{r,j} \cup S^-_{r,j} = \{\xi \in S_r : \widehat{\xi}_j \in B'_r\} \quad \text{and} \quad S_r = \bigcup_{j=1}^n (S^+_{r,j} \cup S^-_{r,j}). \tag{6.8}$$

The surface $S^+_{r,j}$ is given by the equation

$$\xi_j = \sqrt{r^2 - |\widehat{\xi}_j|^2}, \quad \widehat{\xi}_j \in B'_r.$$

Using notation (6.6), we have

$$\int_{S^+_{r,j}} |\widehat{f}(\xi)|\mathrm{d}\sigma(\xi) = \int_{B'_r}\left|\widehat{f}\left(\sqrt{r^2-|\widehat{\xi}_j|^2}, \widehat{\xi}_j\right)\right| \frac{r}{\sqrt{r^2-|\widehat{\xi}_j|^2}}\mathrm{d}\widehat{\xi}_j$$
$$\le \sqrt{n}\int_{B'_r} F_{r/\sqrt{n},j}(\widehat{\xi}_j)\mathrm{d}\widehat{\xi}_j.$$

Further, $\text{mes}_{n-1} B'_r = c_n r^{n-1}$. If $2^k \le r \le 2^{k+1}$, then $\text{mes}_{n-1} B'_r \asymp 2^{k(n-1)}$. It easily follows that

$$2^{k(1-n)} \sup_{2^k \le r \le 2^{k+1}} \int_{S^+_{r,j}} |\widehat{f}(\xi)| \mathrm{d}\sigma(\xi) \le c 2^{k(1-n)} \int_0^{2^{k(n-1)}} F^*_{t_k,j}(u)\mathrm{d}u \le c' F^{**}_{t_k,j}(t_k^{n-1}),$$

where $t_k = 2^k/\sqrt{n}$. Similar estimates hold for integrals over $S^-_{r,j}$. Taking into account (6.8), we obtain

$$\sum_{k\in\mathbb{Z}} 2^{k(2-n)} \sup_{2^k \le r \le 2^{k+1}} \int_{S_r} |\widehat{f}(\xi)| \mathrm{d}\sigma(\xi) \le c \sum_{j=1}^n \sum_{k\in\mathbb{Z}} 2^k F^{**}_{t_k,j}(t_k^{n-1}) \le c' \sum_{j=1}^n \int_0^\infty F^{**}_{t,j}(t^{n-1})\mathrm{d}t.$$

By Theorem 6.3, this implies (6.7). □

We observe that (6.7) is equivalent to the inequality

$$\sum_{j=1}^n \sum_{k\in\mathbb{Z}} 2^{k(1-n)} \sup_{2^k \le r \le 2^{k+1}} \int_{S_r} |(D_j f)^\wedge(\xi)| \mathrm{d}\sigma(\xi) \le c \sum_{j=1}^n ||D_j f||_1$$

which is a direct analogue of the Oberlin inequality (6.4).

Clearly, Theorem 6.3 can be used to derive other Oberlin-type estimates. For example, one can replace spheres by the surfaces of cubes. For $k \in \mathbb{Z}$ and $1 \le j \le n$, denote

$$Q_k^{(j)} = \{\widehat{\xi}_j : |\xi_m| \le 2^k, \ 1 \le m \le n, \ m \ne j\}.$$

Applying Theorem 6.3, we obtain the following

Corollary 6.6 *Let* $f \in W_1^1(\mathbb{R}^n)$ $(n \ge 3)$. *Then*

$$\sum_{j=1}^n \sum_{k\in\mathbb{Z}} 2^{k(2-n)} \sup_{2^k \le |\xi_j| \le 2^{k+1}} \int_{Q_k^{(j)}} |\widehat{f}(\xi)| \mathrm{d}\widehat{\xi}_j \le c||\nabla f||_1. \tag{6.9}$$

Let $Q_k = [-2^k, 2^k]^n$ and $P_k = Q_k \setminus Q_{k-1}$ $(k \in \mathbb{Z})$. We have

$$\sum_{j=1}^n \sup_{2^{k-1} \le |\xi_j| \le 2^k} \int_{Q_k^{(j)}} |\widehat{f}(\xi)| \mathrm{d}\widehat{\xi}_j \ge 2^{1-k} \int_{P_k} |\widehat{f}(\xi)| \mathrm{d}\xi.$$

Thus, (6.9) gives the strengthening of the inequality (6.2) (for $n \ge 3$).

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