Biharmonic Riemannian submersions

Mehmet Akif Akyol¹ · Ye-Lin Ou²

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Abstract

In this paper, we study biharmonic Riemannian submersions. We first derive bitension field of a general Riemannian submersion, and we then use it to obtain biharmonic equations for Riemannian submersions with one-dimensional fibers and Riemannian submersions with basic mean curvature vector fields of fibers. These are used to construct examples of proper biharmonic Riemannian submersions with one-dimensional fibers and to characterize warped products whose projections onto the first factor are biharmonic Riemannian submersions.

Keywords Biharmonic maps · Riemannian submersions · Biharmonic Riemannian submersions · Warped product · Twisted product

Mathematics Subject Classification 58E20 · 53C43

1 Introduction

A map $\phi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is a *biharmonic map* if $\phi|\Omega$ is a critical point of the bienergy functional

$$E_2(\phi, \Omega) = \frac{1}{2} \int_{\Omega} |\tau(\phi)|^2 dv_g$$

for every compact subset Ω of M , where $\tau(\phi) = \text{Trace}_g \nabla d\phi$ is the tension field of ϕ . Using the first variational formula (see [6]), one obtains the biharmonic map equation

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✉ Ye-Lin Ou
yelin.ou@tamuc.edu

Mehmet Akif Akyol
mehmetakifakyol@bingol.edu.tr

¹ Department of Mathematics, Faculty of Science and Arts, Bingöl University, Bingöl 12000, Turkey

² Department of Mathematics, Texas A&M University-Commerce, Commerce, TX 75429, USA

$$\tau^2(\phi) := -\Delta^\phi(\tau(\phi)) - \text{Trace}_g R^N(d\phi, \tau(\phi))d\phi = 0, \quad (1)$$

where

$$\Delta^\phi = -\text{Trace}_g(\nabla^\phi)^2 = -\text{Trace}_g(\nabla^\phi \nabla^\phi - \nabla_{\nabla^\phi}^\phi)$$

is the Laplacian on sections of the pull-back bundle $\phi^{-1}TN$ and R^N is the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla_X^N, \nabla_Y^N]Z - \nabla_{[X, Y]}^N Z.$$

Biharmonic Riemannian submersions were first studied by Oniciuc in [9] where he proved that if a Riemannian submersion $\phi : (M^m, g) \rightarrow (N^n, h)$ has basic tension field, i.e., $\tau(\phi)(p) = \tau(\phi)(q)$ provided $\phi(p) = \phi(q)$ (or the tension field is constant along the fibers), then

$$\tau_2(\phi) = \text{Tr}_h(\nabla^N)^2 \tau(\phi) + \nabla_{\tau(\phi)}^N \tau(\phi) + \text{Ric}^N(\tau(\phi)).$$

In particular, it was proved in [9] that if the tension field $\tau(\phi)$ of a Riemannian submersion $\phi : (M^m, g) \rightarrow (N^n, h)$ is a unitary Killing vector field on N , then it is a biharmonic map.

The biharmonic map equations for a more general class of maps, i.e., conformal submersions (or even more generally, horizontally weakly conformal maps), were obtained independently in [1] and [7]. As an application of these equations, a family of biharmonic Riemannian submersions were constructed in [7]. Later in [13], biharmonic Riemannian submersions from three-dimensional spaces with one-dimensional fibers were studied using the so-called integrability data. It was proved in [13] that a Riemannian submersion from a three-dimensional space form onto a surface is biharmonic if and only if it is harmonic, i.e., the fibers have to be totally geodesic.

In a recent paper [5], the authors found many examples of biharmonic Riemannian submersions in their study of generalized harmonic morphisms which are maps between Riemannian manifolds that pull back local harmonic functions to local biharmonic functions.

Example 1 (see [5] for details) The map $\phi : \mathbb{R}^4 \setminus \{(0, 0, 0, x) : x \in \mathbb{R}\} = M^4 \rightarrow \mathbb{R}^2$ with $\phi(x_1, \dots, x_4) = (\sqrt{x_1^2 + x_2^2 + x_3^2}, x_4)$ is a proper biharmonic Riemannian submersion with the mean curvature vector field of fibers being basic vector field. In fact, using cylindrical coordinates $(r, \theta, \varphi, x_4)$ the Riemannian submersion can be expressed as $\phi(r, \theta, \varphi, x_4) = (r, x_4)$, and the tension field of the Riemannian submersion is $\tau(\phi) = d\phi(\frac{2}{r}\frac{\partial}{\partial r})$ and the mean curvature vector field of the fibers is $\mu = -\frac{1}{r}\frac{\partial}{\partial r}$.

Example 2 (see [5] for details) Let $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ denote the upper-half plane and C be a positive constant. Then, the Riemannian submersion defined by the projection of the warped product

$$\begin{aligned} \pi : (\mathbb{R}_+^2 \times \mathbb{R}, dx^2 + dy^2 + Cy^4 dz^2) &\rightarrow (\mathbb{R}_+^2, dx^2 + dy^2) \\ \pi(x, y, z) &= (x, y), \end{aligned}$$

is a proper biharmonic Riemannian submersion with the tension field $\tau(\pi) = \frac{2}{y}d\pi(\frac{\partial}{\partial y})$ and the mean curvature vector field of fibers $\mu = -\frac{2}{y}\frac{\partial}{\partial y}$, which is basic.

Note that there are many Riemannian submersions (e.g., those given by the projections of a twisted product onto its first factor studied in Corollary 3.2) whose mean curvature vector fields are not basic.

In this paper, we study biharmonicity of a general Riemannian submersion. We first derive the bitension field of a general Riemannian submersion, and we then use it to obtain biharmonic equations for Riemannian submersions with one-dimensional fibers generalizing a result in [13] and Riemannian submersions with basic mean curvature vector fields of fibers. These are used to construct examples of proper biharmonic Riemannian submersions with one-dimensional fibers and to characterize warped products whose projections onto the first factor are biharmonic Riemannian submersions.

2 Biharmonic equations for Riemannian submersions

Biharmonic equations for Riemannian submersions can be obtained as a special case of biharmonic equations for conformal submersion obtained in [1] and [7]. However, the equation obtained this way seems to be hard to apply. For the convenience of applications, we first derive the following form of the bitension field of a Riemannian submersion and then use it to obtain biharmonic equation for Riemannian submersions with one-dimensional fibers or with basic mean curvature vector field of fibers.

Theorem 2.1 *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a Riemannian submersion between Riemannian manifolds. Then, the bitension field of ϕ is given by*

$$\begin{aligned} \tau_2(\phi) = & -(m-n)d\phi\left(\sum_{i=1}^n\left\{\nabla_{e_i}^M(\nabla_{e_i}^M\mu)^{\mathcal{H}}-\nabla_{(\nabla_{e_i}^Me_i)^{\mathcal{H}}}\mu+[\mu,(\nabla_{e_i}^Me_i)^{\mathcal{V}}]\right\}\right. \\ & +\sum_{s=n+1}^m\{[[\mu,e_s],e_s]+[\mu,(\nabla_{e_s}^Me_s)^{\mathcal{V}}]\}-(m-n)\nabla_{\mu}^M\mu\Big) \\ & -(m-n)\operatorname{Ricci}^N(d\phi(\mu)), \end{aligned} \quad (2)$$

where $\{e_i, e_s\}_{i=1,\dots,n,s=1,\dots,m-n}$ is a local orthonormal frame on the total space M with e_i horizontal and e_s vertical and $\mu = \frac{1}{m-n} \sum_{s=1}^{m-n} (\nabla_{e_s}^M e_s)^{\mathcal{H}}$ is the mean curvature vector field of the fibers of the Riemannian submersion.

Proof It is well known (see e.g., [2], Lemma 4.5.1) that for a Riemannian submersion $\phi : (M^m, g) \longrightarrow (N^n, h)$ and any horizontal vector fields X, Y and vertical vector fields V, W , we have

$$\begin{cases} (\nabla d\phi)(X, Y) = 0, \\ (\nabla d\phi)(V, W) = -d\phi(\nabla_V^M W), \\ (\nabla d\phi)(X, V) = -d\phi(\nabla_X^M V). \end{cases} \quad (3)$$

One can easily check that the tension field of the Riemannian submersion is given by

$$\tau(\phi) = -(m-n)d\phi(\mu). \quad (4)$$

By the definition of the second fundamental form of the map ϕ and (3), we have

$$\begin{aligned} \nabla_{e_i}^{\phi} d\phi(\mu) &= d\phi(\nabla_{e_i}^M \mu) = d\phi((\nabla_{e_i}^M \mu)^{\mathcal{H}}), \\ \nabla_{e_i}^{\phi} \nabla_{e_i}^{\phi} d\phi(\mu) &= d\phi(\nabla_{e_i}^M (\nabla_{e_i}^M \mu)^{\mathcal{H}}), \end{aligned} \quad (5)$$

$$-\nabla_{\nabla_{e_i}^M e_i}^{\phi} d\phi(\mu) = -d\phi(\nabla_{(\nabla_{e_i}^M e_i)^{\mathcal{H}}}^M \mu) + d\phi([\mu, (\nabla_{e_i}^M e_i)^{\mathcal{V}}]). \quad (6)$$

For vertical vector field e_s on M , since μ is horizontal, we have

$$\nabla_{e_s}^\phi d\phi(\mu) = -d\phi([\mu, e_s]). \quad (7)$$

Using (7) and (3), we have

$$\begin{aligned} \nabla_{e_s}^\phi \nabla_{e_s}^\phi d\phi(\mu) &= -\nabla_{e_s}^\phi d\phi([\mu, e_s]) = -\nabla_{e_s}^\phi d\phi([\mu, e_s]^\mathcal{H}) \\ &= d\phi([\mu, e_s]^\mathcal{H}, e_s) = d\phi([\mu, e_s], e_s), \end{aligned} \quad (8)$$

where the last equality was obtained by using the fact that $d\phi([\mu, e_s]^\mathcal{V}, e_s) = 0$.

Using (3) again, we obtain

$$-\nabla_{\nabla_{e_s}^\mu e_s}^\phi d\phi(\mu) = -d\phi(\nabla_{(\nabla_{e_s}^\mu e_s)^\mathcal{H}}^\mu \mu) + d\phi([\mu, (\nabla_{e_s}^\mu e_s)^\mathcal{V}]). \quad (9)$$

Finally, the curvature term in the bitension field formula for a Riemannian submersion becomes

$$\begin{aligned} -\sum_{i=1}^m R^N(d\phi(e_i), \tau(\phi))d\phi(e_i) &= (m-n) \sum_{i=1}^n R^N(d\phi(e_i), d\phi(\mu))d\phi(e_i) \\ &= -(m-n)\text{Ricci}^N(d\phi(\mu)). \end{aligned} \quad (10)$$

Substituting (5), (6), (8), (9) and (10) into the bitension field formula (1), we obtain (2) from which the theorem follows. $\square$

Corollary 2.2 *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a Riemannian submersion with basic mean curvature vector field of the fibers. Then, it is biharmonic if and only if*

$$d\phi\left(\sum_{i=1}^n \{\nabla_{e_i}^M (\nabla_{e_i}^M \mu)^\mathcal{H} - \nabla_{(\nabla_{e_i}^M e_i)^\mathcal{H}}^M \mu\} - (m-n)\nabla_\mu^M \mu\right) + \text{Ricci}^N(d\phi(\mu)) = 0,$$

where $\{e_i, e_s\}_{i=1, \dots, n, s=1, \dots, m-n}$ is a local orthonormal frame on the total space M with e_i horizontal and e_s vertical and $\mu = \frac{1}{m-n} \sum_{s=1}^{m-n} (\nabla_{e_s}^M e_s)^\mathcal{H}$ is the mean curvature of the fibers of the Riemannian submersion.

Proof It is well known (see e.g., [8]) that for a basic vector field X and a vertical vector field V of a Riemannian submersion, $[X, V]$ is vertical and hence $d\phi([X, V]) = 0$. From this and Theorem 2.1, we obtain the corollary. $\square$

Notice that for any basic vector fields X, Y of a Riemannian submersion $\phi : (M^m, g) \longrightarrow (N^n, h)$, i.e., horizontal vector fields that are ϕ -related to vector fields $\bar{X}, \bar{Y}$ which means $d\phi(X) = \bar{X}$ and $d\phi(Y) = \bar{Y}$. In this case, it is well known (see e.g., [8]) that $\nabla_X^M Y$ is basic and ϕ -related to $\nabla_{\bar{X}}^N \bar{Y}$ so

$$d\phi(\nabla_X^M Y) = \nabla_{d\phi(X)}^N d\phi(Y). \quad (11)$$

Since we can always choose a local orthonormal frame $\{e_i, e_s\}$ adapted to a Riemannian submersion so that $\{e_i : i = 1, 2, \dots, n\}$ are basic vector fields, we can use Corollary 2.2 and (11) to have the following

Corollary 2.3 *Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be a Riemannian submersion with basic mean curvature vector field of the fibers. Then, it is biharmonic if and only if*

$$\text{Tr}_h(\nabla^N)^2 \tau(\phi) + \nabla_{\tau(\phi)}^N \tau(\phi) + \text{Ricci}^N(\tau(\phi)) = 0. \quad (12)$$

Remark 1 One can check that the notion of “the tension field of a Riemannian submersion being basic” defined in [9] is equivalent to the mean curvature vector field of the fibers of the Riemannian submersion being basic. Thus, our Corollary 2.3 recovers Theorem 4.1 in [9].

Theorem 2.4 *The Riemannian submersion*

$$\phi : (M^m \times N^n, g_M + e^{2\lambda} g_N) \longrightarrow (M^m, g_M), \quad \phi(x, y) = x$$

defined by the projection of a warped product onto its first factor is biharmonic if and only if

$$\operatorname{grad} \Delta \lambda + 2\operatorname{Ricci}^M(\operatorname{grad} \lambda) + \frac{n}{2} \operatorname{grad} |\operatorname{grad} \lambda|^2 = 0, \quad (13)$$

where grad and Δ are the gradient and the Laplace operators on (M^m, g_M) .

Proof First of all, one can easily see that if λ is constant, then the Riemannian submersion ϕ is harmonic since it has minimal fibers. In this case, we also have (13). Now suppose that λ is not constant. Let $P = (M \times N, g_M + e^{2\lambda} g_N)$ be the warped product of (M^m, g_M) and (N^n, g_N) with the warping function $f = e^\lambda$. We denote by ∇^P , ∇^M and ∇^N the connections of the warped product $(P, g = g_M + e^{2\lambda} g_N)$, the base (M^m, g_M) and the fiber (N^n, g_N) spaces, respectively. Then, we have (see, e.g., [4])

$$\begin{cases} \nabla_X^P Y = \nabla_X^M Y, & \nabla_X^P V = \nabla_V^P X = X(\lambda)V \\ \nabla_V^P W = -g(V, W)\operatorname{grad} \lambda + \nabla_V^N W, \end{cases} \quad (14)$$

where for all $X, Y \in \mathfrak{X}(M)$ and $V, W \in \mathfrak{X}(N)$.

Let $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}\}$ be a local orthonormal frame on the warped product $P = (M \times N, g_M + e^{2\lambda} g_N)$ such that $\{e_1, \dots, e_m\}$ are horizontal and $\{e_{m+1}, \dots, e_{m+n}\}$ are vertical. Clearly this orthonormal frame is adapted to the Riemannian submersion ϕ . Using (14), we obtain the tension field of the Riemannian submersion

$$\tau(\phi) = -n d\phi(\mu) = - \sum_{s=m+1}^{m+n} d\phi(\nabla_{e_s}^P e_s) = n d\phi(\operatorname{grad} \lambda) = n \operatorname{grad} \lambda, \quad (15)$$

where the last equality was obtained by using the fact that ϕ is a projection and hence $d\phi(X) = X$ for any horizontal vector field X .

Clearly, in this case, the mean curvature vector field $\mu = -\operatorname{grad} \lambda$ of the fibers of the Riemannian submersion is basic. So, by Corollary 2.3 and (15), the Riemannian submersion ϕ is biharmonic if and only if

$$\operatorname{Tr}_{g_M} (\nabla^M)^2 \operatorname{grad} \lambda + n \nabla_{\operatorname{grad} \lambda}^M \operatorname{grad} \lambda + \operatorname{Ricci}^M(\operatorname{grad} \lambda) = 0. \quad (16)$$

We can check (see e.g., [1]) that

$$\operatorname{Tr}_{g_M} (\nabla^M)^2 (\operatorname{grad} \lambda) = \operatorname{grad} \Delta \lambda + \operatorname{Ricci}^M(\operatorname{grad} \lambda). \quad (17)$$

On the other hand, it is well known (see e.g., [10]) that

$$\nabla_{\operatorname{grad} \lambda}^M \operatorname{grad} \lambda = \frac{1}{2} \operatorname{grad} (|\operatorname{grad} \lambda|^2). \quad (18)$$

Substituting (17) and (18) into (16), we obtain Eq. (13), which completes the proof of the theorem. $\square$

Remark 2 (i) It is interesting to note that in their study of biharmonic maps from warped product manifolds, Balmus–Montaldo–Oniciuc [3] introduced the notion of *biharmonic warping function* defined to be a warping function f so that the identity map $\text{Id} : M \times_f N \equiv (M \times N, g_M + f^2 g_N) \longrightarrow (M \times N, g_M + g_N)$, $\text{Id}(x, y) = (x, y)$ is a proper biharmonic map for some manifold N of dimension greater than zero. They also proved that the projection

$$\phi : (M^m \times N^n, g_M + f^2 g_N) \longrightarrow (M^m, g_M), \quad \phi(x, y) = x$$

is proper biharmonic if and only if

$$\text{Tr}_g \nabla^2 \text{grad} \ln f + \text{Ricci}^M(\text{grad} \ln f) + \frac{n}{2} \text{grad} |\text{grad} \ln f|^2 = 0. \quad (19)$$

(ii) Note also that the biharmonic equation for the projection of a warped product in the form of (12) was obtained in [11] as in this case the mean curvature vector field of the Riemannian submersion is basic. So our Theorem 2.4 gives an improvement of the equation obtained in [11].

Corollary 2.5 *If (M^m, g_M) is an Einstein manifold with $\text{Ricci}^M = a g_M$, then the Riemannian submersion*

$$\phi : (M^m \times N^n, g_M + e^{2\lambda} g_N) \longrightarrow (M^m, g_M), \quad \phi(x, y) = x$$

defined by the projection of a warped product onto its first factor is biharmonic if and only if

$$\Delta \lambda + 2a\lambda + \frac{n}{2} |\text{grad} \lambda|^2 = C, \quad (20)$$

where C is a constant and Δ is the Laplace operator on (M^m, g_M) .

Proof If (M^m, g_M) is an Einstein manifold with $\text{Ricci}^M = a g_M$, then one can easily check that $\text{Ricci}^M(\text{grad} \lambda) = a \text{grad} \lambda$. Substituting this into (13), we conclude that the Riemannian submersion

$$\phi : (M^m \times N^n, g_M + e^{2\lambda} g_N) \longrightarrow (M^m, g_M), \quad \phi(x, y) = x$$

defined by the projection of a warped product onto its first factor is biharmonic if and only if

$$\text{grad} \left(\Delta \lambda + 2a\lambda + \frac{n}{2} |\text{grad} \lambda|^2 \right) = 0,$$

from which we obtain the corollary. $\square$

Corollary 2.6 *The Riemannian submersion*

$$\phi : (M^m \times N^n, g_M + e^{2\lambda} g_N) \longrightarrow (M^m, g_M), \quad \phi(x, y) = x$$

defined by the projection of a warped product onto a compact manifold with negative Ricci curvature is biharmonic if and only if it is harmonic.

Proof By Theorem 2.4, the Riemannian submersion

$$\phi : (M^m \times N^n, g_M + e^{2\lambda} g_N) \longrightarrow (M^m, g_M), \quad \phi(x, y) = x$$

defined by the projection of a warped product onto its first factor is biharmonic if and only if

$$\text{grad} \Delta \lambda + 2\text{Ricci}^M(\text{grad} \lambda) + \frac{n}{2} \text{grad} |\text{grad} \lambda|^2 = 0. \quad (21)$$

On the other hand, it is well known (see, e.g., [10]) that for any function λ on M , we have the following Bochner formula

$$\frac{1}{2} \Delta |\text{grad} \lambda|^2 = |\nabla \text{d} \lambda|^2 + \langle \text{grad} \lambda, \text{grad} \Delta \lambda \rangle + \text{Ricci}^M(\text{grad} \lambda, \text{grad} \lambda).$$

Substituting the term $\text{grad } \Delta \lambda$ from (21) into the Bochner formula, we have

$$\frac{1}{2} \Delta |\text{grad } \lambda|^2 = |\nabla \lambda|^2 - \frac{n}{2} \langle \text{grad } \lambda, \text{grad } |\text{grad } \lambda|^2 \rangle - \text{Ricci}^M(\text{grad } \lambda, \text{grad } \lambda). \quad (22)$$

If λ is constant, then the Riemannian submersion is harmonic and the corollary follows. Now suppose λ is not a constant, then, since M is compact the function $|\nabla \lambda|^2$ attains its maximum at some point $p \in M$. Evaluating both sides of Eq. (22) at the point p , we see that the right-hand side is positive by the fact that $(\text{grad } |\text{grad } \lambda|^2)(p) = 0$ and the assumption of negative Ricci curvature, while the left-hand side is non-positive since the Laplacian being the trace of the Hessian of $|\text{grad } \lambda|^2$, which is semi-negative definite at p . The contradiction shows that λ has to be a constant on M . This completes the proof of the corollary. $\square$

Example 3 For any constants $a_i > 0$, b_i , $i = 1, 2, \dots, n$, let $\kappa_i(x_i) = -\frac{2}{x_i + b_i}$, $\kappa_i(x_i) = a_i \tan\left(\frac{a_i}{2}(x_i + b_i)\right)$, or $\kappa_i(x_i) = \frac{a_i(1 + e^{a_i(x_i + b_i)})}{1 - e^{a_i(x_i + b_i)}}$, and let $\lambda(x_1, x_2, \dots, x_n) = -\sum_{i=1}^n \int \kappa_i(x_i) dx_i$. Then, the Riemannian submersion

$$\pi : (\mathbb{R}^n \times \mathbb{R}, g_0 + e^{2\lambda(x)} dt^2) \longrightarrow (\mathbb{R}^n, g_0), \quad \pi(x, t) = x$$

given by the projection of a warped product onto its first factor is a proper biharmonic map. In particular, the projections

$$\pi : ((0, \infty)^n \times \mathbb{R}, g_0 + (x_1 x_2 \cdots x_n)^4 dt^2) \longrightarrow ((0, \infty)^n, g_0), \quad \pi(x, t) = x,$$

and

$$\pi : ((0, \infty)^n \times \mathbb{R}, g_0 + (\sinh x_1 \sinh x_2 \cdots \sinh x_n)^2 dt^2) \longrightarrow ((0, \infty)^n, g_0), \quad \pi(x, t) = x,$$

are both biharmonic Riemannian submersions.

In fact, since the base manifold is an open submanifold of a Euclidean space which is an Einstein space with $a = 0$, we use this and Eq. (20) with $n = 1$ to conclude that the Riemannian submersion π is biharmonic if and only if

$$\Delta \lambda + \frac{1}{2} |\text{grad } \lambda|^2 = C, \quad (23)$$

where the Laplacian and the gradient operators are defined by the Euclidean metric. If we look for the special solutions of the form $\lambda = -\sum_{i=1}^n \int \kappa_i(x_i) dx_i$, then a straightforward computation shows that Eq. (23) is equivalent to

$$-\sum_{i=1}^n \left(\kappa'_i - \frac{\kappa_i^2}{2} \right) = C. \quad (24)$$

Since $\kappa_i = \kappa_i(x_i)$, one can easily check that (24) is equivalent to

$$\kappa'_i - \frac{\kappa_i^2}{2} = C_i, \quad i = 1, 2, \dots, n, \quad (25)$$

which can be solved by the following cases.

Case I For $C_i = 0$, we solve (25) to have solutions

$$\kappa_i = -\frac{2}{x_i + b_i}, \quad i = 1, 2, \dots, n,$$

for some constant b_i .

Case II For $C_i = \frac{a_i^2}{2} > 0$ with constant $a_i > 0$, then it is easily checked that (25) can be solved to have

$$\kappa_i = a_i \tan\left(\frac{a_i}{2}(x_i + b_i)\right), \quad i = 1, 2, \dots, n,$$

for some constant b_i .

Case III For $C_i = -\frac{a_i^2}{2} < 0$, we solve (25) to obtain

$$\kappa_i = \frac{a_i(1 + e^{a_i(x_i + b_i)})}{(1 - e^{a_i(x_i + b_i)})}, \quad i = 1, 2, \dots, n,$$

for some constant b_i .

From these, we obtain the example.

3 Biharmonic Riemannian submersions with one-dimensional fibers

In this section, we will follow the idea from [13] to describe biharmonicity of a Riemannian submersion with one-dimensional fibers from a generic $(n + 1)$ -manifold by using the integrability data of a special orthonormal frame adapted to a Riemannian submersion. We also use the biharmonic equation derived to study biharmonicity of a Riemannian submersion defined by the projection of a twisted product onto its first factor.

Let $\phi : (M^{n+1}, g) \rightarrow (N^n, h)$ be a Riemannian submersion. A local orthonormal frame is said to be **adapted to the Riemannian submersion** ϕ if the vector fields in the frame that are tangent to the horizontal distribution are basic (i.e., they are ϕ -related to a local orthonormal frame in the base space). Such a frame always exists (cf. e.g., [2]). Let $\{e_1, \dots, e_n, e_{n+1}\}$ be an orthonormal frame adapted to ϕ with e_{n+1} being vertical. Then, it is well known (see [8]) that $[e_i, e_{n+1}]$ ($i = 1, 2, \dots, n$) are vertical and $[e_i, e_j]$ ($i, j = 1, 2, \dots, n$) are ϕ -related to $[\varepsilon_i, \varepsilon_j]$, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is an orthonormal frame in the base manifold. If we assume that

$$[\varepsilon_i, \varepsilon_j] = F_{ij}^k \varepsilon_k, \quad (26)$$

where and in the sequel, $F_{ij}^k \in C^\infty(N)$ and the Einstein convention is used, then we have

$$\begin{cases} [e_i, e_{n+1}] = \kappa_i e_{n+1}, \\ [e_i, e_j] = f_{ij}^k e_k - 2\sigma_{ij} e_{n+1}, \quad i, j = 1, 2, \dots, n, \end{cases} \quad (27)$$

where $f_{ij}^k = F_{ij}^k \circ \phi$, κ_i , and $\sigma_{ij} \in C^\infty(M)$ for all $i, j = 1, 2, \dots, n$. We will call $\{f_{ij}^k, \kappa_i, \sigma_{ij}\}$ **the integrability data** of the adapted frame of the Riemannian submersion ϕ . It follows from (27) that

$$f_{ij}^k = -f_{ji}^k \text{ and } \sigma_{ij} = -\sigma_{ji}, \quad \forall i, j, k = 1, 2, \dots, n. \quad (28)$$

Theorem 3.1 *Let $\phi : (M^{n+1}, g) \rightarrow (N^n, h)$ be a Riemannian submersion with the adapted frame $\{e_1, \dots, e_{n+1}\}$ and the integrability data $\{f_{ij}^k, \kappa_i, \sigma_{ij}\}$. Then, the Riemannian submersion ϕ is biharmonic if and only if*

$$\begin{aligned} \Delta \kappa_k + \sum_{i,j=1}^n \left(2e_i(\kappa_j) P_{ij}^k + \kappa_j [e_i(P_{ij}^k) + P_{ij}^l P_{il}^k - \kappa_i P_{ij}^k - P_{ii}^l P_{lj}^k] \right) \\ + \text{Ricci}^N(d\phi(\mu), d\phi(e_k)) = 0, \quad k = 1, 2, \dots, n, \end{aligned} \quad (29)$$

where $P_{ij}^k = \frac{1}{2}(-f_{ik}^j - f_{jk}^i + f_{ij}^k)$ for all $i, j, k = 1, 2, \dots, n$ and Δ denotes the Laplacian of (M^{n+1}, g) .

Proof Let $P_{ij}^k = \frac{1}{2}(-f_{ik}^j - f_{jk}^i + f_{ij}^k)$ for all $i, j, k = 1, 2, \dots, n$. Then, one can easily check that

$$P_{ij}^k = P_{ji}^k + f_{ij}^k, \quad P_{ii}^k = -f_{ik}^i. \quad (30)$$

A straightforward computation using (27) and Koszul formula gives

$$\nabla_{e_i}^M e_j = P_{ij}^k e_k - \sigma_{ij} e_{n+1} \quad \text{for any } i, j = 1, 2, \dots, n, \quad (31)$$

$$\nabla_{e_{n+1}}^M e_{n+1} = \sum_{i=1}^n \kappa_i e_i. \quad (32)$$

The mean curvature vector field of the fibers of the Riemannian submersion ϕ is given by

$$\mu = (\nabla_{e_{n+1}}^M e_{n+1})^{\mathcal{H}} = \sum_{i=1}^n \kappa_i e_i. \quad (33)$$

A straightforward computation using (27), (31), (32) and (33) yields

$$\begin{aligned} \sum_{i=1}^n \{ \nabla_{e_i}^M (\nabla_{e_i}^M \mu)^{\mathcal{H}} \}^{\mathcal{H}} &= \left\{ \sum_{i,j=1}^n \nabla_{e_i}^M \left(e_i(\kappa_j) + \sum_{l=1}^n \kappa_l P_{il}^j \right) e_j \right\}^{\mathcal{H}} \\ &= \sum_{i,k=1}^n e_i \left(e_i(\kappa_k) + \sum_{j=1}^n \kappa_j P_{ij}^k \right) e_k + \sum_{i,j=1}^n \left(e_i(\kappa_j) + \sum_{l=1}^n \kappa_l P_{il}^j \right) P_{ij}^k e_k \\ &= \sum_{k=1}^n \left(\sum_{i=1}^n e_i e_i(\kappa_k) + 2 \sum_{i,j=1}^n e_i(\kappa_j) P_{ij}^k + \sum_{i,j=1}^n \kappa_j (e_i P_{ij}^k) + \sum_{i,j=1}^n \kappa_j P_{ij}^l P_{il}^k \right) e_k, \\ &\quad \left\{ -\nabla_{(\nabla_{e_i}^M e_i)^{\mathcal{H}}}^M \mu \right\}^{\mathcal{H}} = - \sum_{i,k,j=1}^n (P_{ii}^j (e_j \kappa_k) + \kappa_j P_{ii}^l P_{lj}^k) e_k, \\ [\mu, (\nabla_{e_i}^M e_i)^{\mathcal{V}}] &= 0, \quad \text{since } (\nabla_{e_i}^M e_i)^{\mathcal{V}} = 0, \\ [[\mu, e_{n+1}], e_{n+1}] &= \sum_{k=1}^n e_{n+1} (e_{n+1} \kappa_k) e_k, \\ [\mu, (\nabla_{e_{n+1}}^M e_{n+1})^{\mathcal{V}}] &= 0, \quad \text{since } (\nabla_{e_{n+1}}^M e_{n+1})^{\mathcal{V}} = 0, \\ \{ -\nabla_{\mu}^M \mu \}^{\mathcal{H}} &= - \sum_{i,k=1}^n \kappa_i \left(e_i(\kappa_k) + \sum_{j=1}^n \kappa_j P_{ij}^k \right) e_k, \quad \text{and} \\ \text{Ricci}^N(d\phi(\mu)) &= \sum_{k=1}^n \text{Ricci}^N(d\phi(\mu), d\phi(e_k)) d\phi(e_k). \end{aligned}$$

Using these and Theorem 2.1, we conclude that the Riemannian submersion ϕ is biharmonic if and only if

$$\begin{aligned} & \sum_{k=1}^n \left(\sum_{i=1}^n e_i e_i (\kappa_k) + 2 \sum_{i,j=1}^n e_i (\kappa_j) P_{ij}^k + \sum_{i,j=1}^n \kappa_j (e_i P_{ij}^k) + \sum_{i,j=1}^n \kappa_j P_{ij}^l P_{il}^k \right. \\ & \quad \left. - \sum_{i,j=1}^n (P_{ii}^j (e_j \kappa_k) + \kappa_j P_{ii}^l P_{lj}^k) + e_{n+1} (e_{n+1} \kappa_k) - \sum_{i=1}^n \kappa_i e_i (\kappa_k) - \sum_{i,j=1}^n \kappa_i \kappa_j P_{ij}^k \right. \\ & \quad \left. + \text{Ricci}^N(d\phi(\mu), d\phi(e_k)) \right) d\phi(e_k) = 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \sum_{i=1}^{n+1} e_i e_i (\kappa_k) - \sum_{i=1}^n P_{ii}^j (e_j \kappa_k) - \sum_{i=1}^n \kappa_i e_i (\kappa_k) \\ & \quad + \sum_{i,j=1}^n \left(2e_i (\kappa_j) P_{ij}^k + \kappa_j (e_i P_{ij}^k) + \kappa_j P_{ij}^l P_{il}^k - \kappa_i \kappa_j P_{ij}^k - \kappa_j P_{ii}^l P_{lj}^k \right) \\ & \quad + \text{Ricci}^N(d\phi(\mu), d\phi(e_k)) = 0, \quad k = 1, 2, \dots, n \end{aligned} \quad (34)$$

since $\{d\phi(e_k) \mid k = 1, 2, \dots, n\}$ is an orthonormal frame on $\phi^{-1}TN$. Finally, a straightforward computation of the Laplacian $\Delta \kappa_k$ shows that Eq. (34) is equivalent to (29) from which we obtain the theorem. $\square$

Remark 3 One can check that when $n = 2$, the integrability data are given by

$$\begin{aligned} f_{12}^1 &= f_1, \quad f_{12}^2 = f_2, \\ P_{11}^2 &= -f_1, \quad P_{21}^2 = -f_2, \quad P_{12}^1 = f_1, \quad P_{22}^1 = f_2, \quad \text{all other } P_{ij}^k = 0. \end{aligned}$$

Substituting these and $\text{Ricci}^N(d\phi(\mu), d\phi(e_k)) = \kappa_k K^N$ into Eq. (29), we obtain that a Riemannian submersion $\phi : (M^3, g) \rightarrow (N^2, h)$ is biharmonic if and only if

$$\begin{aligned} & \Delta \kappa_1 + (e_1 \kappa_2) f_1 + (e_2 \kappa_2) f_2 + e_1 (\kappa_2 f_1) + e_2 (\kappa_2 f_2) \\ & \quad - \kappa_1 \kappa_2 f_1 - (\kappa_2)^2 f_2 - \kappa_1 (-K^N + f_1^2 + f_2^2) = 0, \end{aligned} \quad (35)$$

$$\begin{aligned} & \Delta \kappa_2 - (e_1 \kappa_1) f_1 - (e_2 \kappa_1) f_2 - e_1 (\kappa_1 f_1) - e_2 (\kappa_1 f_2) \\ & \quad + \kappa_1 \kappa_2 f_2 + (\kappa_1)^2 f_1 - \kappa_2 (-K^N + f_1^2 + f_2^2) = 0. \end{aligned} \quad (36)$$

This is exactly Proposition 2.1 in [13] of which our Theorem 3.1 gives a generalization.

Corollary 3.2 Let g_0 be the standard Euclidean metric on $\mathbb{R}^n$. Then, the Riemannian submersion defined by the projection of the twisted product

$$\phi : (\mathbb{R}^n \times \mathbb{R}, g_0 + e^{2\lambda(x,t)} dt^2) \rightarrow (\mathbb{R}^n, g_0), \quad \phi(x, t) = x$$

is biharmonic if and only if its tension field viewed as a map

$$\tau(\phi) : (\mathbb{R}^n \times \mathbb{R}, g_0 + e^{2\lambda(x,t)} dt^2) \rightarrow \mathbb{R}^n, \quad \tau(\phi)(x, t) = (\kappa_1(x, t), \kappa_2(x, t), \dots, \kappa_n(x, t)) \quad (37)$$

is harmonic, where $\tau(\phi) = \sum_{i=1}^n \kappa_i(x, t) d\phi(e_i)$ is the tension field of the Riemannian submersion with respect to an adapted frame.

Proof In this case, we can check that the orthonormal frame $\{e_i = \frac{\partial}{\partial x_i} \mid i = 1, \dots, n, e_{n+1} = e^{-\lambda} \frac{\partial}{\partial t}\}$ is adapted to the Riemannian submersion ϕ with $d\phi(e_i) = \varepsilon_i, i = 1, 2, \dots, n$ and e_{n+1} being vertical, where $\varepsilon_i = \frac{\partial}{\partial x_i}, i = 1, 2, \dots, n$ form an orthonormal frame on the base space $(\mathbb{R}^n, g_0)$. An easy computation gives

$$[e_i, e_j] = 0, \quad i, j = 1, 2, \dots, n, \quad [e_i, e_{n+1}] = \kappa_i e_{n+1},$$

where $\kappa_i = -\lambda_{x_i}$. It is easily checked that the integrability data in this case are given by

$$f_{ij}^k = \sigma_{ij} = 0, \quad \forall i, j, k = 1, 2, \dots, n.$$

It follows that

$$P_{ij}^k = \sigma_{ij} = 0, \quad \forall i, j, k = 1, 2, \dots, n. \quad (38)$$

Using these and Theorem 3.1, we conclude that the Riemannian submersion defined by the twisted product onto its first factor is biharmonic if and only if

$$\Delta \kappa_i = 0, \quad \forall i = 1, 2, \dots, n, \quad (39)$$

where κ_i is the component of the tension field $\tau(\phi) = -\sum_{i=1}^n \kappa_i e_i$ obtained from (4) and (32).

This means exactly that the tension field of the Riemannian submersion viewed as a map

$$\tau(\phi) : (\mathbb{R}^n \times \mathbb{R}, g_0 + e^{2\lambda(x,t)} dt^2) \longrightarrow \mathbb{R}^n, \quad \tau(\phi)(x, t) = (\kappa_1, \kappa_2, \dots, \kappa_n) \quad (40)$$

is harmonic. $\square$

Remark 4 (i) We would like to point out that the mean curvature vector field of the Riemannian submersion studied in Corollary 3.2 is $\mu = -\sum_{i=1}^n \frac{\partial \lambda(x,t)}{\partial x_i} e_i$, which is not a basic vector field.

(ii) One can also check that

$$\Delta_M \kappa_i = \sum_{j=1}^n \frac{\partial^2 \kappa_i}{\partial x_j^2} + e^{-2\lambda} \frac{\partial^2 \kappa_i}{\partial t^2} - \sum_{j=1}^n \kappa_j \frac{\partial \kappa_i}{\partial x_j} - e^{-2\lambda} \frac{\partial \lambda}{\partial t} \frac{\partial \kappa_i}{\partial t}. \quad (41)$$

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