On solutions of the singular minimal surface equation

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Dedicated to Anthony Tromba on his seventy-fifth birthday

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Abstract

Results of Bernstein type are proven for supersolutions of the singular minimal surface equation when $\alpha < 0$. In particular the non-existence of “entire” minimal graphs in hyperbolic space is shown. In addition we construct a foliation of $\mathbb{R}^n \times \mathbb{R}^+$ consisting of minimizing surfaces, and solve a Dirichlet problem for the singular minimal surface equation.

Keywords Bernstein-type theorems · Entire supersolutions · Minimal surfaces · Dirichlet problem

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1 Introduction

In this paper, we are interested in the *singular minimal surface equation* (in short: s. m. s. e.)

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{\alpha}{u\sqrt{1 + |Du|^2}} \quad \text{in } \mathbb{R}^n, \quad (*)$$

where $\alpha \in \mathbb{R}$ is some real number and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes a differentiable function. In particular we here address the question of existence and non-existence of *entire*, i. e. on all of $\mathbb{R}^n$ defined (super-)solutions in cases where $\alpha < 0$. As we show in the discussion to follow, the global behaviour of solutions of $(*)$ crucially depends on the sign of α , in fact it is completely different in the three cases $\alpha < 0$, $\alpha = 0$, $\alpha > 0$. In addition we also solve the Dirichlet problem for the s. m. s. e. $(*)$ on interior mean convex regions $\Omega \subset \mathbb{R}^n$.

Obviously, if $\alpha = 0$ then equation $(*)$ reduces to the classical minimal surface equation, which has only trivial entire solutions (affine functions) for $2 \leq n \leq 7$ and nonlinear solutions for all $n \geq 8$, see the fundamental paper by Bombieri-De Giorgi-Giusti [4].

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Now if $\alpha > 0$ there exist entire solutions of the s. m. s. e. in all dimensions $n \geq 2$, e. g. Lipschitz (weak)-solutions $c_n^\alpha(x) = \sqrt{\frac{\alpha}{n-1}} \|x\|$ and also smooth rotational symmetric solutions, cp. the papers by Keiper [17], Dierkes [6] and Lopez [19]. However, if $\alpha + n$ is suitably small (i. e. $\alpha + n < 4 + \sqrt{\frac{2}{n+\alpha}}$) no stable entire solution can exist (see [7,8,10]), nor are there entire solutions of (*) which are of *sublinear* growth (independent of n), see the work of Dierkes and Tennstädt [12]. Recall that “stability” requires that the second-variation $\delta^2 E_\alpha$ is nonnegative, where E_α denotes the “ α -energy” or “ α -area”-functional

$$E_\alpha(u) = \int u^\alpha \sqrt{1 + |Du|^2} \, dx$$

and that (*) is the Euler equation for E_α .

Alternatively equation (*) can also be regarded as an equation of mean curvature type, the mean curvature H with respect to the upper normal being given by

$$H(u, Du) = \frac{\alpha}{u\sqrt{1 + |Du|^2}}.$$

There are several Bernstein-type results for equations of mean curvature type, starting with the celebrated result by Bernstein [3] himself, according to which no graph with mean curvature $\geq H_0 > 0$ can exist on a ball of radius $> \frac{1}{H_0}$, for improvements see Finn [13,14], Heinz [16] and Usami [20]. However, the class comprised and considered in these papers excludes cases like in (*), where the mean curvature depends on both u and Du . We prove the following

Theorem 1.1 *Let $\alpha < 0$. Then any C^2 -function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ which is an entire supersolution of (*), i. e.*

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) \leq \frac{\alpha}{u\sqrt{1 + |Du|^2}} \quad \text{on } \mathbb{R}^n \quad (1.1)$$

is nowhere positive. In particular there are no nonnegative entire C^2 -solutions $u : \mathbb{R}^n \rightarrow \mathbb{R}_0^+ = \{t \geq 0\}$ of inequality (1.1).

Clearly, any constant function $u \equiv c < 0$ satisfies inequality (1.1). On the other hand applying the same reasoning to both u and $(-u)$, we obtain the following Bernstein-type result

Corollary 1.2 *Let $\alpha < 0$, then there is no entire C^2 -solution $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of the s. m. s. e. (*).*

Remark 1.3

- (i) Non-existence of positive, entire C^2 -solutions of the s. m. s. e. (*) in the two-dimensional case has recently been proved by Lopez [19] in a forthcoming paper.
- (ii) For $\alpha = -n$ equation (*) arises in the study of minimal hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}(-1)$ modelled by the upper half space

$$\mathbb{R}^n \times \mathbb{R}^+ := \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; x_{n+1} > 0\}$$

with the hyperbolic metric

$$ds^2 = \frac{1}{x_{n+1}^2} (dx^2 + dx_{n+1}^2)$$

of curvature -1 .

Thus we have

Corollary 1.4 *There is no entire C^2 -minimal graph in $\mathbb{H}^{n+1}(-1)$ for all $n \geq 2$.*

Note that here we have used the expression “entire” in a strictly euclidean sense, meaning that u is defined on all of $\mathbb{R}^n$. Indeed, the existence of complete minimal graphs in $\mathbb{H}^{n+1}(-1)$ with asymptotic boundary at “infinity” $\partial\Omega \subset \mathbb{R}^n \times \{0\}$, that is, solutions $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of (*) with $\alpha = -n$ and $\Omega \subset \mathbb{R}^n$ a mean convex domain, $u = 0$ on $\partial\Omega$, was shown in the papers by Anderson [1] and Lin [18], see also our Theorem 1.6.

There is a “parametric” version of Theorem 1.1. To give a clear statement, consider a C^2 -submanifold $M \subset \mathbb{R}^n \times \mathbb{R}^+$, $\mathbb{R}^+ = \{t > 0\}$ and let $\mathcal{E}_\alpha$ denote the parametric counterpart of E_α , i. e.

$$\mathcal{E}_\alpha(M) := \int_M x_{n+1}^\alpha d\mathcal{H}_n,$$

where $\mathcal{H}_n$ stands for the n -dimensional Hausdorff-measure. One readily verifies that the first variation $\delta\mathcal{E}_\alpha(M, X)$ vanishes for all compactly supported vectorfields X , if and only if the mean curvature with respect to the unit normal $\nu = (\nu_1, \dots, \nu_{n+1})$ is given by

$$H(x) = \alpha x_{n+1}^{-1} \nu_{n+1}$$

(cp.[10, chap.3.2]). In this case M is called “ α -stationary”. Observe that the vertical planes

$$E := \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+; a \cdot x = \text{const}\}$$

where $a \in \mathbb{R}^n$ are α -stationary hypersurfaces for arbitrary $\alpha \in \mathbb{R}$. By a similar argument as in Theorem 1.1 we can prove

Theorem 1.5 *Let $\alpha < 0$, and put $\mathbb{R}_0^+ = \{t \geq 0\}$. There are no α -stationary C^2 -submanifolds M in $\mathbb{R}^n \times \mathbb{R}^+$ with $\partial M \cap (\mathbb{R}^n \times \mathbb{R}_0^+) = \emptyset$.*

It is well known and easily verified, that every upper halfsphere

$$u(x) = \sqrt{R^2 - \|x - x_0\|^2}, \quad x_0 \in \mathbb{R}^n$$

solves the minimal surface equation in $\mathbb{H}^{n+1}(-1)$ on balls $B_R(x_0)$, i. e. we have

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{-n}{u\sqrt{1 + |Du|^2}} \quad \text{in } B_R(x_0),$$

and $u = 0$ on $\partial B_R(x_0)$.

Furthermore it was shown in [18] that each halfsphere is locally (hyperbolic-)area minimizing in $\mathbb{H}^{n+1}(-1)$. A crucial point in the present paper is the construction of classical $C^2(B_R) \cap C^{0, \frac{1}{2}}(\overline{B_R})$ solutions of the s. m. s. e. (*) for arbitrary $n \geq 2$ and any $\alpha < 0$, which generalizes the upper hemisphere solution described above. Moreover, we show that these solutions give rise to a whole foliation of $\mathbb{R}^n \times \mathbb{R}^+$ by singular minimal surfaces. Using a variant of a method due to Bombieri-De Giorgi-Giusti [4] and the present author [6], we then show that each element of the foliation is in fact $\mathcal{E}_\alpha$ -minimizing in a very general sense, see Theorem 2.1. Combining these solutions with E. Hopf’s boundary point lemma, we can prove Theorem 1.1 in a classical way.

As a further consequence of the minimal foliation constructed in Theorem 2.1 and the general theory of quasilinear elliptic equations (see for example Gilbarg–Trudinger [15]), we find the following

Theorem 1.6 Suppose $\Omega \subset \mathbb{R}^n$ is a bounded C^2 -domain which is inward mean convex, i. e. the inward mean curvature of $\partial\Omega$ is nonnegative and let $\alpha < 0$. Then the Dirichlet problem

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1+|Du|^2}} \quad \text{in } \Omega, \quad (1.2)$$

$$u = 0 \text{ on } \partial\Omega$$

has a unique positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

When $\alpha = -n$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of the Dirichlet problem (1.2), then—as already mentioned above— $\operatorname{graph}(u)$ is a complete minimal hypersurface in hyperbolic space $\mathcal{H}^{n+1}(-1)$ with asymptotic boundary $\partial\Omega \times \{0\}$ at infinity. For this case problem (1.2) was solved by Anderson [1] and—with a different argument—by Lin [18]. Invoking Theorem 2.1 we may apply a similar argument as given in [18] to prove Theorem 1.6, for details see Sect. 3.

The Dirichlet problem associated with the s. m. s. e. when $\alpha > 0$ and $\Omega \subset \mathbb{R}^n$ is mean convex has been considered and solved in Dierkes [5,9] and Dierkes–Huisken [11].

2 Functions of least weighted gradient and boundaries of least α -area

In the following $\mathbb{R}^+ = \{t > 0\}$ stands for the positive real numbers and we work in the open half space $\mathbb{R}^n \times \mathbb{R}^+ = \{x = (\hat{x}, x_{n+1}); \hat{x} \in \mathbb{R}^n, x_{n+1} > 0\}$. Also $\alpha < 0$ denotes a negative real number. If $A \subset \mathbb{R}^n \times \mathbb{R}^+$ is an open set and $f \in BV_{\operatorname{loc}}(A)$, we say that f has “least α -weighted gradient” (or simply “least α -gradient”) in A , if

$$\int_K x_{n+1}^\alpha |Df| \leq \int_K x_{n+1}^\alpha |D(f+g)|$$

for every function $g \in BV_{\operatorname{loc}}(A)$ with compact support $K \subset A$.

Moreover, a set $C \subset A$ has boundary of “least α -area” with respect to A , if the characteristic function $\varphi_C \in BV_{\operatorname{loc}}(A)$ and has least α -gradient in A .

We have the following

Theorem 2.1 Let $n \geq 2$ and $\alpha < 0$. There exists a foliation of $\mathbb{R}^n \times \mathbb{R}^+$ determined by concave rotational symmetric functions $v_\lambda = v_\lambda(\hat{x}) : B_\lambda(0) \rightarrow \mathbb{R}^+$, $\lambda > 0$ arbitrary, $v_\lambda(0) = \lambda \cdot v_1(0)$ and $Dv_\lambda(0) = 0$. Furthermore, for each $\lambda > 0$ the functions $v_\lambda \in C^\omega(B_\lambda(0)) \cap C^0(\overline{B}_\lambda(0))$, solve the Dirichlet problem

$$\operatorname{div}\left(\frac{Dv_\lambda}{\sqrt{1+|Dv_\lambda|^2}}\right) = \frac{\alpha}{v_\lambda\sqrt{1+|Dv_\lambda|^2}} \quad \text{in } B_\lambda(0),$$

$$v_\lambda = 0 \text{ on } \partial B_\lambda(0)$$

and satisfy $|Dv_\lambda(\hat{x})| \rightarrow \infty$ as $\hat{x} \rightarrow \partial B_\lambda(0)$.

If $\alpha < -1$ the functions v_λ belong to $C^{0, \frac{1}{2}}(\overline{B}_\lambda)$. Finally the subgraphs

$$V_\lambda := \{(\hat{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+; \hat{x} \in B_\lambda(0), 0 < x_{n+1} < v_\lambda(\hat{x})\}, \quad \text{for } \lambda > 0,$$

have boundaries of least α -area in $\mathbb{R}^n \times \mathbb{R}^+$. In particular the functions v_λ minimize locally the (nonparametric) α -area

$$E_\alpha(u) = \int u^\alpha \sqrt{1+|Du|^2} \quad \text{in } BV(B_\lambda(0)).$$

The proof of Theorem 2.1 is as follows: We show that any function f of least α -gradient has (sub-) level sets of least α -area (Lemma 2.3). Moreover, any solution $f \in H_{1,\text{loc}}^1(A)$ of the equation

$$\int_A x_{n+1}^\alpha |Df|^{-1} Df D\varphi dx = 0, \quad \forall \varphi \in C_c^\infty(A)$$

where $Df = (D_1 f, \dots, D_n f)$ has least α -gradient (Lemma 2.2). Here a locally integrable function f is of class $H_{1,\text{loc}}^1(A)$, if its weak derivative is again locally integrable i.e. in $L_{1,\text{loc}}(A)$. Finally a function $f(x) = F(u, v)$ of the two variables

$$u := |\hat{x}| = \{x_1^2 + \dots + x_n^2\}^{\frac{1}{2}}, \quad v := (x_{n+1}^2)^{\frac{1}{2}} = x_{n+1}$$

is constructed, which is of least α -gradient and has level sets with the properties described in Theorem 2.1.

The following Lemmata are analogues to the results in [6], Lemmas 1 and 3 for the case of positive α , so we skip over the proofs referring to the arguments in the paper loc. cit.

Lemma 2.2 *Let $A \subset \mathbb{R}^n \times \mathbb{R}^+$ be an open set and $\mathcal{N}$ be some closed set with $\mathcal{H}_n(\mathcal{N}) = 0$. Suppose $f \in H_{1,\text{loc}}^1(A)$ satisfies*

$$\mathcal{H}_{n+1}(\{x \in A; |Df(x)| = 0\}) = 0,$$

and

$$\int_A x_{n+1}^\alpha |Df|^{-1} Df \cdot D\varphi dx = 0 \quad (2.1)$$

for every $\varphi \in C_c^1(A - \mathcal{N})$. Then f has least α -gradient in A .

For the proof we refer to Lemma 1 in [6]. Observe that here $\mathbb{R}^+$ stands for the positive real numbers, while $\mathbb{R}^+ := \{t \geq 0\}$ in [6].

Lemma 2.3 *Let $A \subset \mathbb{R}^n \times \mathbb{R}^+$ and $f \in BV_{\text{loc}}(A)$ be a function of least α gradient in A . Then every nonempty level set*

$$E_\lambda := \{x \in A; f(x) \geq \lambda\}$$

has an oriented boundary of least α area in A .

The proof follows from the coarea formula and some approximation device analogously to Lemma 3 in [6].

3 Proofs

It follows from Sect. 2 that we have to find a function f of least α -gradient in $\mathbb{R}^n \times \mathbb{R}^+$ which has level sets given by functions $v_\lambda : B_\lambda(0) \rightarrow \mathbb{R}^+$ with the properties described in Theorem 2.1. Since we are looking for rotational symmetric solutions, it is convenient to introduce the variables

$$\begin{cases} u := (x_1^2 + \dots + x_n^2)^{\frac{1}{2}} = |\hat{x}|, \text{ and} \\ v := (x_{n+1}^2)^{\frac{1}{2}} = x_{n+1}, \end{cases} \quad (3.1)$$

where $x = (\hat{x}, x_{n+1}) = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+$, $\mathbb{R}^+ = \{t > 0\}$ and consider functions $f(x) := F(u(x), v(x))$ where $F : \mathbb{R}_0^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $\mathbb{R}_0^+ := \{t \geq 0\}$. Let the open set $\tilde{\Omega} \subset \mathbb{R}^+ \times \mathbb{R}^+$ correspond to $\Omega \subset \mathbb{R}^n \times \mathbb{R}^+$ under the transformation (3.1), then $f = f(x) \in C^2(\Omega)$, $|\nabla f| \neq 0$, satisfies the Euler equation

$$\sum_{i=1}^{n+1} D_i \left\{ x_{n+1}^\alpha \frac{D_i f}{|Df|} \right\} = 0 \text{ in } \Omega, \quad (3.2)$$

if and only if $F = F(u, v)$ fulfils

$$\frac{\partial}{\partial u} \left\{ \frac{u^{n-1} v^\alpha F_u}{|\nabla F|} \right\} + \frac{\partial}{\partial v} \left\{ \frac{u^{n-1} v^\alpha F_v}{|\nabla F|} \right\} = 0 \text{ in } \tilde{\Omega}, \quad (3.3)$$

where we have put

$$F_u = \frac{\partial F}{\partial u}, F_v = \frac{\partial F}{\partial v} \text{ and } |\nabla F|^2 = F_u^2 + F_v^2.$$

(3.3) in turn is equivalent to

$$F_v^2 F_{uu} - 2F_u F_v F_{uv} + F_u^2 F_{vv} + (F_u^2 + F_v^2) \left[p \frac{F_u}{u} + \alpha \frac{F_v}{v} \right] = 0 \text{ in } \tilde{\Omega} \quad (3.4)$$

with $p := n - 1$.

The unknown function $F = F(u, v)$ will now be constructed from its level sets $\{F = \text{const}\}$. Along a regular level curve, we have

$$dF = F_u du + F_v dv = 0,$$

which yields when applied to (3.4) the equation

$$u''v' - v''u' + \{(u')^2 + (v')^2\} \left[\alpha \frac{u'}{v} - p \frac{v'}{u} \right] = 0, \quad (3.5)$$

where $(u(t), v(t)) \in \mathbb{R}^+ \times \mathbb{R}^+$, $t \in \mathbb{R}$, is a twice differentiable parametrization of the level curve.

We introduce the coordinates

$$\begin{cases} \varphi := \arctg\left(\frac{v}{u}\right), & \text{the "polar angle" and} \\ \theta := \arctg\left(\frac{v'}{u'}\right), & \text{the "tangent angle"} \end{cases} \quad (3.6)$$

which are both invariant under homotheties

$$(u(t), v(t)) \rightarrow \lambda(u(t), v(t)) \text{ for } \lambda > 0.$$

From (3.6) we infer the relations

$$\varphi' = \frac{v'u - u'v}{u^2 + v^2} \quad \text{and} \quad \theta' = \frac{v''u' - v'u''}{(u')^2 + (v')^2},$$

whereas from

$$\begin{aligned} u &= \sqrt{u^2 + v^2} \cos \varphi, & v &= \sqrt{u^2 + v^2} \sin \varphi & \text{and} \\ u' &= \sqrt{(u')^2 + (v')^2} \cos \theta, & v' &= \sqrt{(u')^2 + (v')^2} \sin \theta \end{aligned}$$

we have

$$\varphi' = \left\{ \frac{(u')^2 + (v')^2}{u^2 + v^2} \right\}^{\frac{1}{2}} \sin(\theta - \varphi). \quad (3.7)$$

On the other hand (3.5) applied to θ' gives

$$\theta' = \alpha \frac{u'}{v} - p \frac{v'}{u} = \left\{ \frac{(u')^2 + (v')^2}{u^2 + v^2} \right\}^{\frac{1}{2}} \left\{ \alpha \frac{\cos \theta}{\sin \varphi} - p \frac{\sin \theta}{\cos \varphi} \right\}$$

which together with (3.7) yields the formula

$$2[\alpha \cos \theta \cos \varphi - p \sin \theta \sin \varphi] \varphi' - [\sin 2\varphi \sin(\theta - \varphi)] \theta' = 0.$$

Whence the level curves of the function F are described by the ordinary planar system

$$\begin{aligned} \frac{d\varphi}{dt} &= \sin(2\varphi) \sin(\theta - \varphi) =: \Phi(\varphi, \theta) \\ \frac{d\theta}{dt} &= 2\alpha \cos \theta \cos \varphi - 2p \sin \theta \sin \varphi =: \Psi(\varphi, \theta) \end{aligned} \quad (3.8)$$

which we analyse according to Poincaré-Bendixson theory in the rectangle $(\varphi, \theta) \in [0, \frac{\pi}{2}] \times [-\frac{\pi}{2}, 0]$.

Since $\Phi(\varphi, \theta) = \sin(2\varphi) \sin(\theta - \varphi) \neq 0$ for all $(\varphi, \theta) \in (0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, 0)$ system (3.8) has no singular points in the open rectangle $(0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, 0)$. Indeed, the only singular points in the closed interval $[0, \frac{\pi}{2}] \times [-\frac{\pi}{2}, 0]$ are the points $(\frac{\pi}{2}, 0)$ and $(0, -\frac{\pi}{2})$. We distinguish between the following cases:

First case. The singular point $(\frac{\pi}{2}, 0)$ is a *saddle point* and the linearized system has matrix

$$\begin{pmatrix} 2 & 0 \\ -2\alpha & -2p \end{pmatrix}.$$

The direction of the stable manifolds is $\pm \infty$, while the two unstable manifolds have directions

$$t_1 := \frac{-\alpha}{1+p} = \frac{-\alpha}{n} > 0.$$

Second case. The singular point $(\varphi_0, \theta_0) = (0, -\frac{\pi}{2})$ is a stable node and the linearized system (3.8) has the matrix

$$\begin{pmatrix} -2 & 0 \\ 2p & 2\alpha \end{pmatrix}.$$

If $-1 < \alpha < 0$ or $\alpha < -1$ the principal direction is $\pm \infty$ or $t_2 := \frac{-p}{1+\alpha} > 0$, respectively; while the exceptional directions are

$$t_3 := \frac{-p}{1+\alpha} < 0 \quad (\text{for } -1 < \alpha < 0) \text{ and } \pm \infty \quad (\text{if } \alpha < -1) \text{ respectively.}$$

Finally, if $\alpha = -1$ the node is *improper* and all stable manifolds have direction $\pm \infty$.

Lemma 3.1 *Let $\alpha < 0$ and $p = n - 1 \geq 1$. There exists a trajectory $\gamma(t) = (\varphi(t), \theta(t))$, $t \in (-\infty, \infty)$ which solves the system (3.8) and has the following properties:*

- (i) $\gamma(-\infty) = (\frac{\pi}{2}, 0)$, $\gamma(\infty) = (0, -\frac{\pi}{2})$
- (ii) $\gamma(t) \in (0, \frac{\pi}{2}) \times (-\frac{\pi}{2}, 0)$ for all $t \in (-\infty, \infty)$.
- (iii) $\frac{d\varphi}{dt} < 0$ and $\frac{d\theta}{dt} < 0$ for all $t \in (-\infty, \infty)$.

In particular the integral curve γ has a nonparametric representation $\theta = \Theta(\varphi)$ for some differentiable function $\Theta : (0, \frac{\pi}{2}) \rightarrow (-\frac{\pi}{2}, 0)$ with $\Theta(0) = -\frac{\pi}{2}$, $\Theta(\frac{\pi}{2}) = 0$ and $\frac{d\Theta}{d\varphi} > 0$.

Proof of Lemma 3.1 Consider the rectangle

$$R := \left\{ (\varphi, \theta); 0 < \varphi < \frac{\pi}{2} \text{ and } -\frac{\pi}{2} < \theta < 0 \right\}$$

with the sides

$$\begin{aligned} \ell_1 &:= \left\{ 0 < \varphi < \frac{\pi}{2}, \quad \theta = 0 \right\}, \\ \ell_2 &:= \left\{ \varphi = \frac{\pi}{2}, \quad -\frac{\pi}{2} < \theta < 0 \right\}, \\ \ell_3 &:= \left\{ 0 < \varphi < \frac{\pi}{2}, \quad \theta = -\frac{\pi}{2} \right\}, \text{ and} \\ \ell_4 &:= \left\{ \varphi = 0, \quad -\frac{\pi}{2} < \theta < 0 \right\}. \end{aligned}$$

By (3.8) we have on ℓ_1, ℓ_2, ℓ_3 and ℓ_4 , respectively

$$\begin{aligned} \Phi|_{\ell_1} &= \sin(2\varphi) \sin(-\varphi) < 0, \quad \Psi|_{\ell_1} = 2\alpha \cos \varphi < 0, \\ \Phi|_{\ell_2} &= 0, \quad \Psi|_{\ell_2} = -2p \sin \theta > 0, \\ \Phi|_{\ell_3} &= -\sin(2\varphi) \cos \varphi < 0, \quad \Psi|_{\ell_3} = 2p \sin \varphi > 0, \quad \text{and} \\ \Phi|_{\ell_4} &= 0, \quad \Psi|_{\ell_4} = 2\alpha \cos \theta < 0. \end{aligned}$$

Therefore, the right-hand side $(\Phi(\varphi, \theta), \Psi(\varphi, \theta))$ of the system (3.8) at a regular boundary point of the rectangle $\partial R = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ is directed into the closure $\bar{R}$ of the quadrilateral R , in fact in the interior along the sides ℓ_1 and ℓ_3 . It follows from the preceding discussion of system (3.8) that one of the unstable manifolds, $\gamma(t) := (\varphi(t), \theta(t))$, starting at the saddle point $(\frac{\pi}{2}, 0)$ with direction $t_1 = -\frac{\alpha}{n} > 0$ into the interior of R cannot leave $\bar{R}$ and hence approaches the stable node $(0, -\frac{\pi}{2})$ with slope $t_2 = -\frac{p}{1+\alpha}$ or $+\infty$. This implies (i) and (ii) of the Lemma. Since $\Phi(\varphi, \theta) < 0$ on all of R also $\frac{d\varphi}{dt} < 0$ is obvious. For $t > -\infty$ “close to $-\infty$ ” we conclude from the discussion above, that $\theta'(t) < 0$. We claim that $\theta'(t) < 0$ for all $t \in \mathbb{R}$. Indeed, suppose on the contrary that there is (by analyticity) a smallest $t_0 > -\infty$ with $\theta'(t_0) = 0$ and $\theta''(t_0) \geq 0$. Then (3.8) yields the contradiction $\theta''(t_0) = -2\alpha \cos \theta \sin \varphi \cdot \varphi'(t_0) - 2p \sin \theta \cos \varphi \cdot \varphi'(t_0) < 0$, since $\alpha < 0$, $\varphi'(t_0) < 0$ and $0 < \varphi(t_0) < \frac{\pi}{2}$, $-\frac{\pi}{2} < \theta(t_0) < 0$.

Concluding we can write $t = T(\varphi)$ to denote the inverse function of $\varphi = \varphi(t)$ and $\theta = \theta(t) = \theta(T(\varphi)) =: \Theta(\varphi)$, where $\Theta : [0, \frac{\pi}{2}] \rightarrow [-\frac{\pi}{2}, 0]$ is monotonically increasing with $\Theta(0) = -\frac{\pi}{2}$ and $\Theta(\frac{\pi}{2}) = 0$. $\square$

Proof of Theorem 2.1 Recalling the transformation (3.6)

$$\varphi = \operatorname{arctg} \frac{v}{u} \quad \text{and} \quad \theta = \operatorname{arctg} \frac{v'}{u'}$$

we see that $\gamma(t) = (\varphi(t), \theta(t))$ induces an analytic curve Γ given by $(u(t), v(t))$ in the (u, v) -plane, which by virtue of Lemma 3.1 lies completely in the quadrant

$$T = \{(u, v); u \geq 0, v \geq 0\}.$$

Since φ, θ are invariant under homotheties, we obtain a “field” of curves Γ_λ , $\lambda > 0$, which covers T simply (quartermcircles if $\alpha = -n$).

Let Γ_1 be the curve issuing vertically from the point $(1, 0)$. By Lemma 3.1 we may introduce φ as a parameter on Γ_1 which leads to the representation for Γ_1

$$u = u_1(\varphi), \quad v = v_1(\varphi), \quad \varphi \in \left[0, \frac{\pi}{2}\right], \quad (u_1(\varphi) = \sin \varphi, \quad v_1(\varphi) = \cos \varphi \text{ if } \alpha = -n).$$

Alternatively, since $\theta \in (0, -\frac{\pi}{2})$ for $\varphi \in (0, \frac{\pi}{2})$ the curve Γ_1 may be written as graph $v = v_1(u)$, $0 \leq u \leq 1$ for some differentiable function v_1 with $v_1(0) > 0$, $Dv_1(0) = 0$ and $Dv_1(u) \rightarrow -\infty$ for $u \rightarrow 1$. In addition, since $\frac{d\theta}{d\varphi} < 0$, the representation v_1 must be a concave function.

Now any function $F = F(u, v)$ with $|\nabla F| \neq 0$, defined on the open set $\mathring{T} = \{u > 0, v > 0\}$ which has the curves Γ_λ as level curves must satisfy Eq. (3.3). We choose the homogeneous function

$$F(u, v) := (u^2 + v^2) \cdot H\left(\operatorname{arctg} \frac{v}{u}\right), \quad \text{where } (u, v) \in \mathring{T},$$

normalized by the requirement

$$F(u_1(\varphi), v_1(\varphi)) = 1 \quad \text{for all } \varphi \in \left(0, \frac{\pi}{2}\right),$$

whence

$$H(\varphi) = (u_1^2(\varphi) + v_1^2(\varphi))^{-1}, \quad \text{for } \varphi \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad H(0) = 1.$$

Clearly, $F = F(u, v)$ extends continuously onto the closure T and is—by construction—an analytic solution of (3.4) in the interior $\mathring{T}$. Moreover, we have

$$|\nabla F(u, v)| = (u^2 + v^2)^{\frac{1}{2}} (4H^2 + H'^2)^{\frac{1}{2}} \neq 0 \quad \text{and} \quad F(\Gamma_\lambda) = \lambda^2, \lambda > 0.$$

On putting

$$f(x) := F(u(x), v(x))$$

we obtain a classical solution of Eq. (3.2) in the open set $\mathbb{R}^n \times \mathbb{R}^+ \cap \{u \neq 0\}$ which has locally bounded gradient on $\mathbb{R}^n \times \mathbb{R}^+$. Hence we may apply Lemma 2.1 with $A := \mathbb{R}^n \times \mathbb{R}^+$ and $\mathcal{N} := \{x \in \mathbb{R}^n \times \mathbb{R}_0^+; u = 0\}$, to conclude that $f = f(x)$ has least α -gradient in $\mathbb{R}^n \times \mathbb{R}^+$. Recall that the level curves Γ_λ , $\lambda > 0$ constitute a foliation of the quadrant $T = \{u, v \geq 0\}$ and that—by virtue of Lemma 3.1—each Γ_λ gives rise to a function $v_\lambda = v_\lambda(\hat{x})$ defined on balls $\overline{B_\lambda(0)} \subset \mathbb{R}^n$ such that

$$v_\lambda \in C^1(B_\lambda(0)) \cap C^\infty(B_\lambda(0) - \{0\}) \cap C^0(\overline{B_\lambda}) \quad \text{and} \quad v_\lambda = 0 \quad \text{on} \quad \partial B_\lambda(0).$$

In particular the subgraph

$$V_\lambda := \{(\hat{x}, x_{n+1}) \in B_\lambda(0) \times \mathbb{R}^+; x_{n+1} < v_\lambda(\hat{x})\}$$

has an oriented boundary of least α -area in $\mathbb{R}^n \times \mathbb{R}^+$. As in [2] one can then show that any function v_λ locally minimizes the nonparametric α -area

$$E_\alpha(u) := \int u^\alpha \sqrt{1 + |Du|^2} \quad \text{in } BV_{\text{loc}}(B_\lambda(0)).$$

Therefore, the s. m. s. e. (*) is satisfied classically in $B_\lambda - \{0\}$ and weakly in $B_\lambda(0)$; by elliptic regularity theory it follows that $v_\lambda \in C^\omega(B_\lambda(0))$ and solves s. m. s. e. (*) in the usual sense on the ball $B_\lambda(0)$.

To conclude with the proof of Theorem 2.1 we finally show $\frac{1}{2}$ -höldercontinuity of v_λ on $\overline{B_\lambda(0)}$ if $\alpha < -1$. By the discussion preceding Lemma 3.1 the point $(\varphi, \theta) = (0, -\frac{\pi}{2})$ is a stable node with principal direction $t_2 = -\frac{p}{1+\alpha} > 0$ and exceptional direction $\pm\infty$. Hence the separatrix $\gamma(t)$ in Lemma 3.1 must have direction t_2 at the point $(0, -\frac{\pi}{2})$. Consequently we have the asymptotic expansion

$$\theta = -\frac{\pi}{2} + t_2\varphi + o(\varphi) \text{ as } \varphi \rightarrow 0^+,$$

while on the other hand

$$\begin{aligned} \frac{v_1}{u_1} &= \operatorname{tg} \varphi, \quad \frac{v'_1}{u'_1} = \operatorname{tg} \theta \quad \text{i. e.} \\ \frac{u'_1}{u_1} &= \frac{1}{\cos^2 \varphi} \cdot \frac{1}{\operatorname{tg} \theta - \operatorname{tg} \varphi} = \frac{\cos \theta}{\cos \varphi} \cdot \frac{1}{\sin(\theta - \varphi)}. \end{aligned}$$

Asymptotically we find

$$\begin{aligned} \frac{u'_1}{u_1} &\sim -\cos \theta \sim -\cos\left(t_2\varphi - \frac{\pi}{2}\right) = -\sin(t_2\varphi) = -t_2\varphi + o(\varphi) \text{ as } \varphi \rightarrow 0^+ \\ \text{whence } u_1(\varphi) &= 1 - \frac{t_2}{2}\varphi^2 + o(\varphi^2) \text{ as } \varphi \rightarrow 0^+. \end{aligned}$$

But $v_1 = u_1 \cdot \operatorname{tg} \varphi$, so $v_1(\varphi) = \varphi + o(\varphi)$ as $\varphi \rightarrow 0^+$ from which we conclude that $v_1 \in C^{0, \frac{1}{2}}(\overline{B_1(0)})$ and also $v_\lambda \in C^{0, \frac{1}{2}}(\overline{B_\lambda})$.

Theorem 2.1 follows. $\square$

Proof of Theorem 1.1 Suppose by contradiction that a solution u of inequality (1.1) exists for which $u(x_0) > 0$ at some point $x_0 \in \mathbb{R}^n$. Without loss of generality we can assume that $x_0 = 0$ and consider the symmetric minimizers $v_\lambda : \overline{B_\lambda(0)} \rightarrow \mathbb{R}^+ \cup \{0\}$ with $|Dv_\lambda(x)| \rightarrow \infty$ as $x \rightarrow \partial B_\lambda(0)$, $v_\lambda(x) = 0$ for all $x \in \partial B_\lambda$ and $v_\lambda(0) = \lambda \cdot v_1(0) > 0$, $\lambda > 0$. If $\lambda > 0$ is sufficiently small the graph of v_λ lies below the graph of u in the ball $B_\lambda(0)$. Now let λ increase monotonically up to a certain value λ_0 until points of contact (y_0, u_0) , $u_0 := u(y_0)$, first appear. Denote by

$$\mathcal{T} := \{x \in \overline{B_{\lambda_0}(0)}; u(x) = v_{\lambda_0}(x)\}$$

the closed “touching set” of v_{λ_0} and u (note that $v_{\lambda_0} \leq u$ on $\overline{B_{\lambda_0}(0)}$). Since $|Du|$ is bounded in $\overline{B_{\lambda_0}(0)}$ and $|Dv_{\lambda_0}(x)| \rightarrow \infty$ as $x \rightarrow \partial B_{\lambda_0}(0)$, $\mathcal{T}$ must be contained in the interior of $B_{\lambda_0}(0)$. Denote by B_ε an open ball with the properties

- (a) $\overline{B_\varepsilon} \subset B_{\lambda_0}(0)$ and
- (b) $B_\varepsilon \cap \mathcal{T} = \emptyset$, $\partial B_\varepsilon \cap \mathcal{T} \neq \emptyset$.

By relation (1.1) we have on $\overline{B_\varepsilon}$

$$\Delta u - \frac{D_i u D_j u D_{ij} u}{1 + |Du|^2} - \frac{\alpha}{u} \leq 0,$$

and by Theorem 2.1,

$$\Delta v_{\lambda_0} - \frac{D_i v_{\lambda_0} D_j v_{\lambda_0} D_{ij} v_{\lambda_0}}{(1 + |Dv_{\lambda_0}|^2)} - \frac{\alpha}{v_{\lambda_0}} = 0.$$

By a standard device invoking the mean value theorem, we obtain for the difference $w := v_{\lambda_0} - u$ the elliptic inequality

$$Lw = a_{ij}(x)D_{ij}w + b_i D_i w + cw \geq 0 \text{ in } B_\varepsilon,$$

with suitable coefficients a_{ij} , b_i and $c := \alpha(uv_{\lambda_0})^{-1} < 0$. Concluding we have $w < 0$ on B_ε and $w(z_0) = 0$ for any point $z_0 \in \partial B_\varepsilon \cap \mathcal{T}$. Also, since each $z_0 \in \mathcal{T}$ is a point of tangency for both surfaces u and v_{λ_0} , we must have $Dw(z_0) = 0$. This however contradicts E. Hopf's boundary point Lemma, according to which the outer normal derivative of w at a boundary maximum must be positive. Theorem 1.1 is completely proved. $\square$

Proof of Theorem 1.5 The proof is similar, using that the submanifold M has no boundary in $\mathbb{R}^n \times \mathbb{R}_0^+$. Indeed, suppose on the contrary the existence of a stationary submanifold M with empty boundary, then we could find some number $\lambda_0 > 0$ such that the solution v_{λ_0} touches M at some interior point, again an obvious contradiction to Hopf's boundary point lemma. $\square$

We now turn to the

Proof of Theorem 1.6 Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and put

$$Q(u) := \Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_{ij} u - \frac{\alpha}{u},$$

then by the *maximum principle* (see for example Gilbarg–Trudinger [15], chapters 3 and 10.1) we have the following comparison and uniqueness result:

$$\begin{aligned} &\text{Suppose } Q(u) \geq Q(v) \text{ in } \Omega \text{ and } u \leq v \text{ on } \partial\Omega \text{ then } u \leq v \text{ in } \Omega; \\ &\text{if in addition } u(x_0) = v(x_0) \text{ at some point } x_0 \in \Omega \text{ then } u \equiv v \text{ in } \Omega. \end{aligned} \quad (\text{M})$$

Hence the solutions $v_\lambda \in C^2(B_\lambda(0)) \cap C^0(\overline{B_\lambda(0)})$, $\lambda > 0$, obtained in Theorem 2.1 may be used as barriers to conclude the a priori L_∞ -bound

$$0 < c(K, \Omega, \alpha, n) \leq u(x) \leq C(\Omega, \alpha, n) \quad (3.9)$$

for all $x \in K$, any compact set $K \subset \Omega$, any solution u of (1.2) and constants $c \leq C$ depending only on the quantities indicated.

For $\varepsilon > 0$ consider the approximate problems

$$\begin{cases} Q(u^\varepsilon) = 0 \text{ in } \Omega, & u^\varepsilon > 0 \text{ in } \Omega, \text{ and} \\ u^\varepsilon = \varepsilon \text{ on } \partial\Omega. \end{cases} \quad (3.10)$$

By the a priori bound (3.9) and classical Schauder theory (see for example [15] chapter 10–15, in particular Corollary 14.7, where—in this special case—the mean convexity condition of $\partial\Omega$ is required) we obtain a unique $C^2(\Omega) \cap C^0(\overline{\Omega})$ solution u^ε of (3.10) satisfying the Hölder norm estimate

$$\|u^\varepsilon\|_{2,\beta,K} \leq C(K, \alpha, n, \Omega) \quad (3.11)$$

for all $\varepsilon \in (0, 1)$, all compact subsets $K \subset \Omega$ and some $\beta > 0$.

The maximum principle (M) implies that u^ε are monotonic in ε , whence

$$u(x) := \lim_{\varepsilon \searrow 0} u^\varepsilon(x) \quad (3.12)$$

exists pointwise in Ω . By (3.9), (3.10) and (3.11) u is a positive solution of

$$Q(u) = 0 \text{ in } \Omega.$$

Finally, let $0 < \varepsilon_1 \leq \varepsilon_2 < 1$, then by (M) also $u_{\varepsilon_1} \leq u_{\varepsilon_2}$ in $\overline{\Omega}$. On the other hand we have the inequality

$$Q(u^{\varepsilon_1} + (\varepsilon_2 - \varepsilon_1)) = \alpha(\varepsilon_2 - \varepsilon_1)(u^{\varepsilon_1}(u^{\varepsilon_1} + \varepsilon_2 - \varepsilon_1))^{-1} \leq 0$$

so we infer -again by the maximum principle (M)- that also

$$u^{\varepsilon_2} \leq u^{\varepsilon_1} + (\varepsilon_2 - \varepsilon_1) \text{ in } \overline{\Omega},$$

whence the convergence in (3.12) is uniform. In particular $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ and $u = 0$ on $\partial\Omega$. Uniqueness follows similarly. $\square$

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