Quaternionic Monge–Ampère operator for unbounded plurisubharmonic functions

Dongrui Wan¹

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Abstract

In this paper, we generalize the definition of the quaternionic Monge–Ampère operator to some unbounded plurisubharmonic functions, and we prove that the quaternionic Monge–Ampère operator is continuous on the monotonically decreasing sequences of plurisubharmonic functions. After introducing the generalized Lelong number of a positive current, Demailly’s comparison theorems are showed. Moreover, we prove that the quaternionic Lelong–Jensen-type formula also holds for the unbounded plurisubharmonic function.

Keywords Monge–Ampère operator · Quaternionic plurisubharmonic function · Positive current

Mathematics Subject Classification 32U15 · 31C10 · 2U40

1 Introduction

Quaternionic analysis, which becomes an active and important direction of research in mathematics, has many applications in physics. Quaternionic manifolds play important roles in the supersymmetric theory in physics. The Baston operator in the quaternionic complexes appears in quantization of $N = 2$ supergravity black holes. Recently, quaternionic version of Calabi–Yau conjecture, which has applications in superstring theory, has attracted many analysts to study on this problem. And they have obtained some results [4–6]. Relating to this problem, it is interesting to study the pluripotential theory, in particular to study the quaternionic Monge–Ampère operator, on quaternionic manifolds [3, 5, 6, 14, 23, 25].

Let Ω be an open set in n -dimensional quaternionic space $\mathbb{H}^n$. The quaternionic Monge–Ampère operator is defined as the Moore determinant of the quaternionic Hessian of u :

✉ Dongrui Wan
wandongrui@szu.edu.cn

¹ College of Mathematics and Statistics, Shenzhen University, Room 415 Science and Technology Building, Shenzhen 518060, People’s Republic of China

$$\det(u) = \det \left[\frac{\partial^2 u}{\partial \bar{q}_j \partial q_k}(q) \right].$$

Alesker proved in [1] a quaternionic version of Chern–Levine–Nirenberg estimate and extended the definition of quaternionic Monge–Ampère operator to continuous quaternionic plurisubharmonic functions. Since it is inconvenient to use the Moore determinant, the studying of the quaternionic Monge–Ampère operator is much more difficult than that of the complex Monge–Ampère operator.

To define the quaternionic Monge–Ampère operator on general quaternionic manifolds, Alesker introduced in [3] an operator in terms of the Baston operator Δ , which is the first operator of the quaternionic complex on quaternionic manifolds. The n th power of this operator is exactly the quaternionic Monge–Ampère operator when the manifold is flat. On the flat space $\mathbb{H}^n$, the Baston operator Δ is the first operator of 0-Cauchy–Fueter complex:

$$0 \rightarrow C^\infty(\Omega, \mathbb{C}) \xrightarrow{\Delta} C^\infty(\Omega, \wedge^2 \mathbb{C}^{2n}) \xrightarrow{D} C^\infty(\Omega, \mathbb{C}^2 \otimes \wedge^3 \mathbb{C}^{2n}) \rightarrow \dots \quad (1.1)$$

Wang [22] wrote down explicitly each operator of the k -Cauchy–Fueter complex in terms of real variables.

Motivated by this, the author and Wang introduced in [19] two first-order differential operators d_0 and d_1 acting on the quaternionic version of differential forms. And the second operator D in (1.1) can be written as $D := \begin{pmatrix} d_0 \\ d_1 \end{pmatrix}$. The behavior of d_0 , d_1 and $\Delta = d_0 d_1$ is

very similar to ∂ , $\bar{\partial}$ and $\partial\bar{\partial}$ in several complex variables. The quaternionic Monge–Ampère operator can be defined as $(\Delta u)^n$ and has a simple explicit expression, which is much more convenient to use than the previous definition by using Moore determinant.

By introducing the quaternionic version of differential forms, the author and Wang defined in [19] the notions of closed positive forms and closed positive currents in the quaternionic case. We proved that Δu is a closed positive 2-current for any quaternionic plurisubharmonic function u , and showed that when functions $u_1, \dots, u_k$ are locally bounded, $\Delta u_1 \wedge \dots \wedge \Delta u_k$ is a well defined closed positive current and is continuous on decreasing sequences. Then we generalized Bedford–Taylor theory [7] in complex analysis to the quaternionic case, i.e., extend the definition of the quaternionic Monge–Ampère operator $(\Delta u)^n$ to locally bounded quaternionic plurisubharmonic function u and prove that the operator is continuous on monotonically decreasing sequences of plurisubharmonic functions.

Quaternionic positive currents are discussed before in [3, 12] and closed currents are discussed in [6]. Our definition of closedness matches positivity well. Therefore we can extend several results of closed positive currents in several complex variables to the quaternionic case [15, 16, 20]. The results about the quaternionic Monge–Ampère operator and quaternionic closed positive currents proved by Wang and the author [15, 16, 18–20] play key roles in this paper.

The purpose of this paper is to generalize the definition of the quaternionic Monge–Ampère operator to some unbounded plurisubharmonic functions and to prove that the quaternionic Monge–Ampère operator is continuous on the monotonically decreasing sequences of plurisubharmonic functions. In this paper, we use Demailly’s methods for the complex Monge–Ampère operator (cf. [8–11]).

We define the *unbounded locus* $L(u)$ to be the set of points $q \in \Omega$ such that u is unbounded in every neighborhood of q .

Theorem 1.1 *Let $u_1, \dots, u_k$ be plurisubharmonic functions in Ω and T be a closed positive $(2n - 2p)$ -current such that*

$$\mathcal{H}_{4p-4m+1}(L(u_{t_1}) \cap \dots \cap L(u_{t_m}) \cap \text{Supp } T) = 0 \quad (1.2)$$

for all choices of indices $t_1 < \dots < t_m$ in $\{1, \dots, k\}$ and $k \leq p$. Then the currents $u_1 \Delta u_2 \wedge \dots \wedge \Delta u_k \wedge T$ and $\Delta u_1 \wedge \dots \wedge \Delta u_k \wedge T$ are well defined and have locally finite mass in Ω .

Theorem 1.2 *Let $u^1, \dots, u^k$ and T satisfy condition (1.2). Let $\{u_j^1\}_{j \in \mathbb{N}}, \dots, \{u_j^k\}_{j \in \mathbb{N}}$ be decreasing sequences of plurisubharmonic functions in Ω such that $\lim_{j \rightarrow \infty} u_j^i = u^i$, pointwise in Ω , $i = 1, \dots, k$. Then as $j \rightarrow \infty$,*

$$u_j^1 \Delta u_j^2 \wedge \dots \wedge \Delta u_j^k \wedge T \longrightarrow \Delta u^1 \Delta u^2 \wedge \dots \wedge \Delta u^k \wedge T \quad \text{weakly}, \quad (a)$$

$$\Delta u_j^1 \wedge \dots \wedge \Delta u_j^k \wedge T \longrightarrow \Delta u^1 \wedge \dots \wedge \Delta u^k \wedge T \quad \text{weakly}. \quad (b)$$

Let Ω be a quaternionic strictly pseudoconvex domain, i.e., Ω has a strictly plurisubharmonic exhaustion function. For a closed positive $(2n - 2p)$ -current T , we define the *generalized Lelong number* $v(T, \varphi)$ of T with respect to the weight φ (the generalized Lelong number in the complex case was introduced by Demailly [11]). Define

$$v(T, \varphi) := \int_{\{\varphi = -\infty\}} T \wedge (\Delta \varphi)^p = \lim_{r \rightarrow -\infty} v(T, \varphi, r), \quad (1.3)$$

where

$$v(T, \varphi, r) = \int_{B_\varphi(r)} T \wedge (\Delta \varphi)^p, \quad (1.4)$$

and

$$B_\varphi(r) := \{q \in \Omega; \varphi(q) < r\}. \quad (1.5)$$

Here $\varphi : \Omega \rightarrow [-\infty, +\infty)$ is a continuous plurisubharmonic function (in general φ may assume $-\infty$ as a value at some discrete points in Ω), satisfying that, there exists a real number R such that $B_\varphi(R) \Subset \Omega$. In particular, if one let $\varphi = -\|q - a\|^{-2}$, which is the fundamental solution of quaternionic Monge–Ampère operator, the generalized Lelong number $v(T, \varphi)$ is just the Lelong number $v_a(T)$ of T at point a that we introduced in [19].

As applications of the main theorems above, we obtain two Demailly’s comparison theorems for the generalized Lelong number. Moreover, we prove that the Lelong–Jensen-type formula we established in [19, Theorem 5.1] also holds for the unbounded function. The Lelong–Jensen-type formula can be regarded as a higher-dimensional version of Poisson integral formula.

The paper is organized as follows. The results about quaternionic Monge–Ampère operator and quaternionic closed positive currents we need are collected in Sect. 2. Theorems 1.1 and 1.2 are proved in Sect. 3. In Sect. 4, we introduce the generalized Lelong number and its relationship with quaternionic boundary measure. Moreover, two Demailly’s comparison theorems for the generalized Lelong number and the quaternionic version of Lelong–Jensen-type formula are showed.

2 Preliminaries on quaternionic Monge–Ampère operator

In this section, we are going to recall some basic definitions and conclusions of quaternionic Monge–Ampère operator and quaternionic closed positive currents following [19,20].

A real valued function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ is called *quaternionic plurisubharmonic* (*PSH*, for short) if it is upper semicontinuous and its restriction to any right quaternionic line is subharmonic (in the usual sense). Any quaternionic *PSH* function is subharmonic (cf. [1,2,6] for more information about *PSH* functions).

We use the well-known embedding of the quaternionic algebra $\mathbb{H}$ into $\text{End}(\mathbb{C}^2)$ defined by

$$x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mapsto \begin{pmatrix} x_0 + \mathbf{i}x_1 & -x_2 - \mathbf{i}x_3 \\ x_2 - \mathbf{i}x_3 & x_0 - \mathbf{i}x_1 \end{pmatrix}.$$

Actually, we will use the conjugate embedding

$$\begin{aligned} \tau : \mathbb{H}^n &\cong \mathbb{R}^{4n} \hookrightarrow \mathbb{C}^{2n \times 2}, \\ (q_0, \dots, q_{n-1}) &\mapsto \mathbf{z} = (z^{j\alpha}) \in \mathbb{C}^{2n \times 2}, \end{aligned}$$

$j = 0, 1, \dots, 2n-1$, $\alpha = 0, 1$, with

$$\begin{pmatrix} z^{(2l)0} & z^{(2l)1} \\ z^{(2l+1)0} & z^{(2l+1)1} \end{pmatrix} := \begin{pmatrix} x_{4l} - \mathbf{i}x_{4l+1} & -x_{4l+2} + \mathbf{i}x_{4l+3} \\ x_{4l+2} + \mathbf{i}x_{4l+3} & x_{4l} + \mathbf{i}x_{4l+1} \end{pmatrix},$$

for $q_l = x_{4l} + \mathbf{i}x_{4l+1} + \mathbf{j}x_{4l+2} + \mathbf{k}x_{4l+3}$, $l = 0, 1, \dots, n-1$. Pulling back to the quaternionic space $\mathbb{H}^n \cong \mathbb{R}^{4n}$ by the embedding above, we define on $\mathbb{R}^{4n}$ first-order differential operators $\nabla_{j\alpha}$ as following:

$$\begin{pmatrix} \nabla_{(2l)0} & \nabla_{(2l)1} \\ \nabla_{(2l+1)0} & \nabla_{(2l+1)1} \end{pmatrix} := \begin{pmatrix} \partial_{x_{4l}} + \mathbf{i}\partial_{x_{4l+1}} & -\partial_{x_{4l+2}} - \mathbf{i}\partial_{x_{4l+3}} \\ \partial_{x_{4l+2}} - \mathbf{i}\partial_{x_{4l+3}} & \partial_{x_{4l}} - \mathbf{i}\partial_{x_{4l+1}} \end{pmatrix}. \quad (2.1)$$

$z^{k\beta}$'s can be viewed as independent variables and $\nabla_{j\alpha}$'s are derivatives with respect to these variables. The operators $\nabla_{j\alpha}$'s play very important roles in the investigating of regular functions in several quaternionic variables [13,21–24].

Let $\wedge^{2k}\mathbb{C}^{2n}$ be the complex exterior algebra generated by $\mathbb{C}^{2n}$, $0 \leq k \leq n$. Fix a basis $\{\omega^0, \omega^1, \dots, \omega^{2n-1}\}$ of $\mathbb{C}^{2n}$. Let Ω be a domain in $\mathbb{R}^{4n}$. Define $d_0, d_1 : C_0^\infty(\Omega, \wedge^p \mathbb{C}^{2n}) \rightarrow C_0^\infty(\Omega, \wedge^{p+1} \mathbb{C}^{2n})$ by

$$\begin{aligned} d_0 F &= \sum_{k,I} \nabla_{k0} f_I \omega^k \wedge \omega^I, \\ d_1 F &= \sum_{k,I} \nabla_{k1} f_I \omega^k \wedge \omega^I, \\ \Delta F &= d_0 d_1 F, \end{aligned} \quad (2.2)$$

for $F = \sum_I f_I \omega^I \in C_0^\infty(\Omega, \wedge^p \mathbb{C}^{2n})$, where the multi-index $I = (i_1, \dots, i_p)$ and $\omega^I := \omega^{i_1} \wedge \dots \wedge \omega^{i_p}$. Although d_0, d_1 are not exterior differential, their behavior is similar to the exterior differential.

Lemma 2.1 (Proposition 2.2 in [19])

- (1) $d_0 d_1 = -d_1 d_0$.
- (2) $d_0^2 = d_1^2 = 0$, thus $d_0 \Delta = d_1 \Delta = 0$.
- (3) For $F \in C_0^\infty(\Omega, \wedge^p \mathbb{C}^{2n})$, $G \in C_0^\infty(\Omega, \wedge^q \mathbb{C}^{2n})$, we have

$$d_\alpha(F \wedge G) = d_\alpha F \wedge G + (-1)^p F \wedge d_\alpha G, \quad \alpha = 0, 1.$$

We say F is *closed* if $d_0 F = d_1 F = 0$. For $u_1, \dots, u_n \in C^2$, $\Delta u_1 \wedge \dots \wedge \Delta u_k$ is closed, $k = 1, \dots, n$. Moreover, it follows easily from Lemma 2.1 that $\Delta u_1 \wedge \dots \wedge \Delta u_n$ satisfies the following remarkable identities:

$$\begin{aligned} \Delta u_1 \wedge \Delta u_2 \wedge \dots \wedge \Delta u_n &= d_0(d_1 u_1 \wedge \Delta u_2 \wedge \dots \wedge \Delta u_n) \\ &= -d_1(d_0 u_1 \wedge \Delta u_2 \wedge \dots \wedge \Delta u_n) = d_0 d_1(u_1 \Delta u_2 \wedge \dots \wedge \Delta u_n) \\ &= \Delta(u_1 \Delta u_2 \wedge \dots \wedge \Delta u_n). \end{aligned} \quad (2.3)$$

To write down the explicit expression, we define for a function $u \in C^2$,

$$\Delta_{ij} u := \frac{1}{2}(\nabla_{i0} \nabla_{j1} u - \nabla_{i1} \nabla_{j0} u), \quad \Delta u = \sum_{i,j=0}^{2n-1} \Delta_{ij} u \omega^i \wedge \omega^j.$$

$$\begin{aligned} \Delta u_1 \wedge \dots \wedge \Delta u_n &= \sum_{i_1, j_1, \dots} \Delta_{i_1 j_1} u_1 \dots \Delta_{i_n j_n} u_n \omega^{i_1} \wedge \omega^{j_1} \wedge \dots \wedge \omega^{i_n} \wedge \omega^{j_n} \\ &= \sum_{i_1, j_1, \dots} \delta_{01 \dots (2n-1)}^{i_1 j_1 \dots i_n j_n} \Delta_{i_1 j_1} u_1 \dots \Delta_{i_n j_n} u_n \Omega_{2n}, \end{aligned}$$

for $u_1, \dots, u_n \in C^2$, where Ω_{2n} is defined as

$$\Omega_{2n} := \omega^0 \wedge \omega^1 \wedge \dots \wedge \omega^{2n-2} \wedge \omega^{2n-1}, \quad (2.4)$$

and $\delta_{01 \dots (2n-1)}^{i_1 j_1 \dots i_n j_n} :=$ the sign of the permutation from $(i_1, j_1, \dots, i_n, j_n)$ to $(0, 1, \dots, 2n-1)$, if $\{i_1, j_1, \dots, i_n, j_n\} = \{0, 1, \dots, 2n-1\}$; otherwise, $\delta_{01 \dots (2n-1)}^{i_1 j_1 \dots i_n j_n} = 0$.

Note that $\Delta u_1 \wedge \dots \wedge \Delta u_n$ is symmetric with respect to the permutation of $u_1, \dots, u_n$. In particular, when $u_1 = \dots = u_n = u$, $\Delta u_1 \wedge \dots \wedge \Delta u_n$ coincides with $(\Delta u)^n := \wedge^n \Delta u = \Delta_n u \Omega_{2n}$. Denote by $\Delta_n(u_1, \dots, u_n)$ the coefficient of the form $\Delta u_1 \wedge \dots \wedge \Delta u_n$, i.e., $\Delta u_1 \wedge \dots \wedge \Delta u_n = \Delta_n(u_1, \dots, u_n) \Omega_{2n}$. Then $\Delta_n(u_1, \dots, u_n)$ coincides with the mixed Monge–Ampère operator $\det(u_1, \dots, u_n)$ while $\Delta_n u$ coincides with the quaternionic Monge–Ampère operator $\det(u)$. This result was proved by Alesker (Proposition 7.1 in [3]). And in Appendix A in [19] we gave an elementary and simpler proof of the identity:

$$\Delta u_1 \wedge \dots \wedge \Delta u_n = n! \det(u_1, \dots, u_n) \Omega_{2n}$$

for C^2 function u .

We defined in [19] the notions of quaternionic positive forms and strongly positive forms. They are counterparts of forms in several complex variables. Please see [19] for detailed definitions. Denoted by $\mathcal{D}^p(\Omega)$ the set of all $C_0^\infty(\Omega)$ functions valued in $\wedge^p \mathbb{C}^{2n}$. The elements of $\mathcal{D}^p(\Omega)$ are often called the *test p -forms*. An element of the dual space $(\mathcal{D}^{2n-p}(\Omega))'$ is called a *p -current*. Denoted by $\mathcal{D}_0^p(\Omega)$ the set of all $C_0(\Omega)$ functions valued in $\wedge^p \mathbb{C}^{2n}$. The elements of the dual space $(\mathcal{D}_0^{2n-p}(\Omega))'$ are called *p -currents of order zero*. The $2n$ -currents are just the distributions on Ω , whereas the $2n$ -currents of order zero are Radon measures on Ω .

If T is a $2k$ -current on Ω , ψ is a $2l$ -form on Ω with coefficients in $C^\infty(\Omega)$, and $k+l \leq n$, then the formula

$$(T \wedge \psi)(\varphi) = T(\psi \wedge \varphi) \quad \text{for } \varphi \in \mathcal{D}^{2n-2k-2l}(\Omega) \quad (2.5)$$

defines a $(2k+2l)$ -current.

A $2k$ -current T is said to be *positive* if we have $T(\eta) \geq 0$ for any strongly positive form $\eta \in \mathcal{D}^{2n-2k}(\Omega)$. Although a $2n$ -form is not an authentic differential form and we cannot integrate it, we can define

$$\int_{\Omega} F := \int_{\Omega} f dV, \quad (2.6)$$

if we write $F = f \Omega_{2n} \in L^1(\Omega, \wedge^{2n} \mathbb{C}^{2n})$, where dV is the Lebesgue measure and Ω_{2n} is given by (2.4). In particular, if F is positive $2n$ -form, then $\int_{\Omega} F \geq 0$. For a $2n$ -current $F = \mu \Omega_{2n}$ with coefficient to be measure μ , define

$$\int_{\Omega} F := \int_{\Omega} \mu. \quad (2.7)$$

Any positive $2k$ -current T on Ω has measure coefficients (i.e., is of order zero) (cf. Proposition 3.4 in [19] for details). For a positive $2k$ -current T and a strongly positive test form φ , we can write $T \wedge \varphi = \mu \Omega_{2n}$ for some Radon measure μ . We have $T(\varphi) = \int_{\Omega} T \wedge \varphi$.

Lemma 2.2 (Stokes-type formula, Lemma 3.2 in [19]) *Assume that $T = \sum_i T_i \hat{\omega}^i$ is a smooth $(2n-1)$ -form in Ω , where $\hat{\omega}^i = \omega^0 \wedge \cdots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \cdots \wedge \omega^{2n-1}$. Then for smooth function h , we have*

$$\int_{\Omega} h d_{\alpha} T = - \int_{\Omega} d_{\alpha} h \wedge T + \sum_{i=0}^{2n-1} (-1)^{i-1} \int_{\partial \Omega} h T_i n_{i\alpha} dS,$$

where $n_{i\alpha}$, $i = 0, 1, \dots, 2n-1$, $\alpha = 0, 1$, is defined by the matrix:

$$\begin{pmatrix} n_{(2l)0} & n_{(2l)1} \\ n_{(2l+1)0} & n_{(2l+1)1} \end{pmatrix} := \begin{pmatrix} n_{4l} + i n_{4l+1} & -n_{4l+2} - i n_{4l+3} \\ n_{4l+2} - i n_{4l+3} & n_{4l} - i n_{4l+1} \end{pmatrix}, \quad (2.8)$$

$l = 0, 1, \dots, n-1$. Here $\mathbf{n} = (n_0, n_1, \dots, n_{4n-1})$ is the unit outer normal vector to $\partial \Omega$ and dS denotes the surface measure of $\partial \Omega$. In particular, if $h = 0$ on $\partial \Omega$, we have

$$\int_{\Omega} h d_{\alpha} T = - \int_{\Omega} d_{\alpha} h \wedge T, \quad \alpha = 0, 1. \quad (2.9)$$

Lemma 2.3 (Proposition 3.6 in [19]) *Let T be a $(2n-1)$ -current such that $\text{supp } T \subset \Omega$. If $d_{\alpha} T$ is a positive $2n$ -current, then $\int_{\Omega} d_{\alpha} T = 0$.*

Bedford–Taylor theory [7] in complex analysis can be generalized to the quaternionic case.

Lemma 2.4 (Proposition 3.9 in [19]) *Let u be a locally bounded PSH function and let T be a closed positive $2k$ -current. Then $\Delta u \wedge T$ defined by*

$$\Delta u \wedge T := \Delta(uT), \quad (2.10)$$

i.e., $(\Delta u \wedge T)(\eta) := uT(\Delta \eta)$, for any test form η , is also a closed positive current. Inductively,

$$\Delta u_1 \wedge \cdots \wedge \Delta u_p \wedge T := \Delta(u_1 \Delta u_2 \wedge \cdots \wedge \Delta u_p \wedge T) \quad (2.11)$$

is a closed positive current, when $u_1, \dots, u_p \in PSH \cap L_{loc}^{\infty}(\Omega)$.

In particular, for $u_1, \dots, u_n \in PSH \cap L_{loc}^{\infty}(\Omega)$, $\Delta u_1 \wedge \cdots \wedge \Delta u_n = \mu \Omega_{2n}$ for a well defined positive Radon measure μ .

For any strongly positive test $(2n - 2p)$ -form ψ on Ω , (2.11) can be rewritten as

$$\int_{\Omega} \Delta u_1 \wedge \cdots \wedge \Delta u_p \wedge \psi = \int_{\Omega} u_1 \Delta u_2 \wedge \cdots \wedge \Delta u_p \wedge \Delta \psi, \quad (2.12)$$

where $u_1, \dots, u_p \in PSH \cap L_{loc}^{\infty}(\Omega)$. Since positive currents have measure coefficients, (2.12) also holds for strongly positive $\psi \in \mathcal{D}_0^{2n-2p}(\Omega)$ vanishing on the boundary.

Alesker gave a quaternionic version of Chern–Levine–Nirenberg estimate in Proposition 6.3 in [3]. And we gave an elementary and simpler proof in [19]. Set

$$\beta_n := \sum_{l=0}^{n-1} \omega^{2l} \wedge \omega^{2l+1} = \frac{1}{8} \Delta(\|q\|^2). \quad (2.13)$$

β_n is a positive 2-form and $\beta_n^n = \wedge^n \beta_n = n! \Omega_{2n}$ is a strongly positive $2n$ -form. Let T be a positive $(2n - 2p)$ -current and K be an arbitrary compact subset. Define

$$\|T\|_K := \int_K T \wedge \beta_n^p. \quad (2.14)$$

In particular, if T is a positive $2n$ -current, $\|T\|_K$ coincides with $\int_K T$ defined by (2.7).

Lemma 2.5 (Chern–Levine–Nirenberg-type estimate, Proposition 3.10 in [19]) *Let Ω be a domain in $\mathbb{H}^n$. Let K, L be compact subsets of Ω such that L is contained in the interior of K . Then there exists a constant C depending only on K, L, Ω such that for any $u_1, \dots, u_n \in PSH \cap C^2(\Omega)$, one has*

$$\|\Delta u_1 \wedge \cdots \wedge \Delta u_k\|_L \leq C \|u_1\|_{L^{\infty}(K)} \cdots \|u_k\|_{L^{\infty}(K)}. \quad (2.15)$$

The following convergence results are powerful tools in developing pluripotential theory for the quaternionic Monge–Ampère operator.

Lemma 2.6 (1) (Theorem 3.1 in [19])

Let $v^1, \dots, v^k \in PSH \cap L_{loc}^{\infty}(\Omega)$. Let $\{v_j^1\}_{j \in \mathbb{N}}, \dots, \{v_j^k\}_{j \in \mathbb{N}}$ be decreasing sequences of PSH functions in Ω such that $\lim_{j \rightarrow \infty} v_j^t = v^t$ pointwise in Ω for each t . Then the currents $\Delta v_j^1 \wedge \cdots \wedge \Delta v_j^k$ converge weakly to $\Delta v^1 \wedge \cdots \wedge \Delta v^k$ as $j \rightarrow \infty$.

(2) (Proposition 4.1 in [20])

Let $\{u_j\}_{j \in \mathbb{N}}$ be a sequence in $PSH \cap L_{loc}^{\infty}(\Omega)$ that increases to $u \in PSH \cap L_{loc}^{\infty}(\Omega)$ almost everywhere in Ω (with respect to Lebesgue measure). Then the currents $(\Delta u_j)^n$ converge weakly to $(\Delta u)^n$ as $j \rightarrow \infty$.

3 Definition of wedge product $\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T$ for unbounded PSH functions

First we show that the Chern–Levine–Nirenberg-type estimate in Lemma 2.5 also holds for the closed current $\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T$ with closed positive current T .

Proposition 3.1 *Let Ω be a domain in $\mathbb{H}^n$. Let K, L be compact subsets of Ω such that L is contained in the interior of K . Then there exists a constant C depending only on K, L such that for any $u_1, \dots, u_k \in PSH \cap C^2(\Omega)$ and closed positive $(2n - 2p)$ -current T with $p \geq k$, one has*

$$\|\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T\|_L \leq C \prod_{i=1}^k \|u_i\|_{L^{\infty}(K)} \|T\|_K, \quad (3.1)$$

where $\|\cdot\|_L$ is defined by (2.14).

Proof By Lemma 2.4, $\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T$ is already a closed positive current. By the induction definition (2.11), it suffices to prove that

$$\|\Delta u_1 \wedge T\|_L \leq C \|u_1\|_{L^\infty(K)} \|T\|_K.$$

Since L is compact, there is a covering of L by a family of balls $B'_j \Subset B_j \subset K$. Let $\chi \geq 0$ be a smooth function equals to 1 on $\overline{B'_j}$ with support in B_j . Then

$$\begin{aligned} \|\Delta u_1 \wedge T\|_{L \cap \overline{B'_j}} &= \int_{\overline{B'_j}} \Delta u_1 \wedge T \wedge \beta_n^{p-1} \leq \int_{B_j} \chi \Delta u_1 \wedge T \wedge \beta_n^{p-1} \\ &= (T \wedge \beta_n^{p-1})(\chi \Delta u_1) = (T \wedge \beta_n^{p-1})(u_1 \Delta \chi) = \int_{B_j} u_1 \Delta \chi \wedge T \wedge \beta_n^{p-1} \\ &\leq \frac{1}{\varepsilon} \|u_1\|_{L^\infty(K)} \|\Delta \chi\| \int_{B_j} T \wedge \beta_n^p, \end{aligned}$$

where the second identity follows from Proposition 3.8 in [19] and the last inequality follows from Lemma 3.1 below. $\square$

Lemma 3.1 (Lemma 3.3 in [19]) *For each $\eta \in \wedge^{2k} \mathbb{C}^{2n*}$, $\beta_n^k \pm \varepsilon \eta$ is a positive $2k$ -form when ε is sufficiently small.*

Corollary 3.1 *Let $u_1, \dots, u_s, u_1^{(1)}, \dots, u_l^{(1)}, u_1^{(2)}, \dots, u_l^{(2)} \in PSH \cap L^\infty(\Omega)$, $s + l \leq p$, and let T be a closed positive $(2n - 2p)$ -current. Then the mixed product $\Theta = \Delta u_1 \wedge \cdots \wedge \Delta u_s \wedge \gamma(u_1^{(1)}, u_1^{(2)}) \wedge \cdots \wedge \gamma(u_l^{(1)}, u_l^{(2)}) \wedge T$ is also well defined, where*

$$\gamma(u_t^{(1)}, u_t^{(2)}) := d_0 u_t^{(1)} \wedge d_1 u_t^{(2)} + d_0 u_t^{(2)} \wedge d_1 u_t^{(1)}, \quad t = 1, \dots, l. \quad (3.2)$$

And for $L \subset K^o$, we have

$$\|\Theta\|_L \leq C \|T\|_K \prod_{m=1}^s \|u_m\|_{L^\infty(K)} \prod_{t=1}^l \|u_t^{(1)}\|_{L^\infty(K)} \|u_t^{(2)}\|_{L^\infty(K)}. \quad (3.3)$$

Proof For each t , by Lemma 2.1(3) we have the polarization identity

$$\begin{aligned} &2 \left(d_0 u_t^{(1)} \wedge d_1 u_t^{(2)} + d_0 u_t^{(2)} \wedge d_1 u_t^{(1)} \right) \\ &= d_0 d_1 (u_t^{(1)} + u_t^{(2)})^2 - d_0 d_1 (u_t^{(1)})^2 - d_0 d_1 (u_t^{(2)})^2 - 2u_t^{(1)} d_0 d_1 u_t^{(2)} - 2u_t^{(2)} d_0 d_1 u_t^{(1)}. \end{aligned}$$

And it follows that

$$\begin{aligned} 2\gamma(u_t^{(1)}, u_t^{(2)}) \wedge \Theta' &= \Delta(u_t^{(1)} + u_t^{(2)})^2 \wedge \Theta' - \Delta(u_t^{(1)})^2 \wedge \Theta' - \Delta(u_t^{(2)})^2 \wedge \Theta' \\ &\quad - 2u_t^{(1)} \Delta u_t^{(2)} \wedge \Theta' - 2u_t^{(2)} \Delta u_t^{(1)} \wedge \Theta', \end{aligned} \quad (3.4)$$

where $\Theta' = \Delta u_1 \wedge \cdots \wedge \Delta u_s \wedge T$. Then (3.3) follows by applying estimates (3.1). Noting that all functions $u_t^{(\alpha)}$, $\alpha = 0, 1$, are bounded, we can let $u_t^{(\alpha)} \geq 0$ by adding a positive constant. So $(u_t^{(\alpha)})^2$ is also plurisubharmonic. Therefore, (3.1) can be applied to $(u_t^{(\alpha)})^2$ and $(u_t^{(1)} + u_t^{(2)})^2$. It is similar to handle with several $\gamma(u_t^{(1)}, u_t^{(2)})$'s. $\square$

When all functions are locally bounded, the mixed product $\gamma(u, u) \wedge \Delta w^1 \wedge \cdots \wedge \Delta w^k$ is well defined as a positive $(2k + 2)$ -current. Moreover, this positive current is also continuous on decreasing sequences (see Proposition 3.1 in [15] for more discussion about the mixed product).

Now we are going to show that $\Delta v_1 \wedge \cdots \wedge \Delta v_k \wedge T$ is continuous on decreasing sequences when $v_1, \dots, v_k$ are locally bounded *PSH* functions and T is a closed positive current. See this conclusion in the complex case in Demailly [11] and Bedford and Taylor [7].

Lemma 3.2 (Lemma 1.9 in [11]) *Let f_j be a decreasing sequence of upper semicontinuous functions converging to f pointwise on some separable locally compact space X and μ_j a sequence of positive measures converging weakly to μ on X . Then every weak limit ν of $f_j \mu_j$ satisfies $\nu \leq f \mu$.*

Proposition 3.2 *Let $v^1, \dots, v^k \in PSH \cap L_{loc}^\infty(\Omega)$ and let T be a closed positive $(2n - 2p)$ -current with $p \geq k$. Let $\{v_j^1\}_{j \in \mathbb{N}}, \dots, \{v_j^k\}_{j \in \mathbb{N}}$ be decreasing sequences of *PSH* functions in Ω such that $\lim_{j \rightarrow \infty} v_j^t = v^t$, pointwise in Ω , for $t = 1, \dots, k$. Then*

$$v_j^1 \Delta v_j^2 \wedge \cdots \wedge \Delta v_j^k \wedge T \longrightarrow v^1 \Delta v^2 \wedge \cdots \wedge \Delta v^k \wedge T \quad \text{weakly}, \quad (a)$$

$$\Delta v_j^1 \wedge \cdots \wedge \Delta v_j^k \wedge T \longrightarrow \Delta v^1 \wedge \cdots \wedge \Delta v^k \wedge T \quad \text{weakly}. \quad (b)$$

Proof Since the results are purely local, we can assume that Ω is an open ball $B = B(a, r)$ and the functions v_j^t are smooth in B and coincide outside a compact subset of B as in the proof of Lemma 2.6(1) (Theorem 3.1 in [19]; see also Bedford and Taylor [7]).

Note that (a) $\Rightarrow$ (b) by the induction definition (2.11), and the case of $k = 1$ for (a) is obvious. Suppose that (a) has been proved for $k - 1$. So we have

$$S_j = \Delta v_j^2 \wedge \cdots \wedge \Delta v_j^k \wedge T \longrightarrow S = \Delta v^2 \wedge \cdots \wedge \Delta v^k \wedge T \quad \text{weakly}.$$

To prove (a) for k , it suffices to show that the weak limit Θ of $v_j^1 S_j$ is equal to $v^1 S$. Let η be a strongly positive test $(2p - 2k + 2)$ -form. Then $S_j \wedge \eta$ converges weakly to $S \wedge \eta$. Denote $S_j \wedge \eta = \mu_j \Omega_{2n}$ and $S \wedge \eta = \mu \Omega_{2n}$ with μ_j, μ positive measures. Apply Lemma 3.2 to v_j^1, v^1, μ_j, μ to get $\Theta \wedge \eta \leq v^1 S \wedge \eta$. Thus the weak limit $\Theta \leq v^1 S$. To prove the equality, it suffices to show that

$$\int_B v^1 \Delta v^2 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \leq \liminf_{j \rightarrow \infty} \int_B v_j^1 \Delta v_j^2 \wedge \cdots \wedge \Delta v_j^k \wedge T \wedge \beta_n,$$

for $\beta_n = \frac{1}{8} \Delta \rho, \rho(q) = \|q - a\|^2 - r^2$.

Since the functions v_j^t are bounded on the ball B , we can also assume that $-M < v_j^t < -1$ after adding a constant. Then for large A the function $\max(v_j^t, A\rho)$ will coincide with $A\rho$ near the boundary and remain the same near the center. Therefore we can assume that all v_j^t are smooth after regularization and vanish on the boundary. Now since $v^1 \leq v_j^1$, by Lemma 2.1

we have

$$\begin{aligned}
 \int_B v^1 \Delta v^2 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n &\leq \int_B v_j^1 \Delta v^2 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &= \int_B v_j^1 d_0 d_1 v^2 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &= - \int_B d_0 v_j^1 \wedge d_1 v^2 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &= \int_B d_1 v^2 \wedge d_0 v_j^1 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &= - \int_B v^2 d_1 d_0 v_j^1 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &= \int_B v^2 \Delta v_j^1 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &\leq \int_B v_j^2 \Delta v_j^1 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \\
 &= \dots = \int_B v_j^1 \Delta v_j^2 \wedge \Delta v^3 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n.
 \end{aligned} \tag{3.5}$$

Here we use Stokes-type formula (2.9) for $h = v_j^1$ in the second identity and for $h = v^2$ in the fourth identity. Repeating this argument to obtain

$$\int_B v^1 \Delta v^2 \wedge \cdots \wedge \Delta v^k \wedge T \wedge \beta_n \leq \int_B v_j^1 \Delta v_j^2 \wedge \cdots \wedge \Delta v_j^k \wedge T \wedge \beta_n.$$

Finally it suffices to let $j \rightarrow \infty$. $\square$

The proof of Theorem 1.1 is also an induction on k , similar to the inductive process in the proof of Lemma 2.4. All that we need is the following Proposition 3.3, which is an improved version of Chern–Levine–Nirenberg-type estimate. To prove this proposition, we need the following lemma (cf. Lemma 2.7 in [11] for the proof in the complex case).

Lemma 3.3 *Let $F \subset \mathbb{H}^n$ be a closed set such that $\mathcal{H}_{4s+1}(F) = 0$ for some integer $0 \leq s < n$. Then for almost all choices of unitary coordinates $(q_1, \dots, q_n) = (q', q'')$ with $q' = (q_1, \dots, q_s)$, $q'' = (q_{s+1}, \dots, q_n)$ and almost all radii of balls $B'' = B(0, r'') \subset \mathbb{H}^{n-s}$, the set $\{0\} \times \partial B''$ does not intersect F .*

Proof The unitary group $U_{\mathbb{H}}(n)$ has real dimension $2n^2 + n$ by calculating the dimension of its Lie algebra {quaternionic $(n \times n)$ –matrix A ; $A + A^* = 0$ }. There is a proper submersion

$$\Phi : U_{\mathbb{H}}(n) \times (\mathbb{H}^{n-s} \setminus \{0\}) \rightarrow \mathbb{H}^n \setminus \{0\}, \quad (g, q'') \mapsto g(0, q''),$$

whose fibers have real dimension $N = 2n^2 + n - 4s$. It follows that the inverse image $\Phi^{-1}(F)$ has zero Hausdorff measure $\mathcal{H}_{N+4s+1} = \mathcal{H}_{2n^2+n+1}$. The set of pairs $(g, r'') \in U_{\mathbb{H}}(n) \times \mathbb{R}_+^*$ such that $g(\{0\} \times \partial B'')$ intersects F is precisely the image of $\Phi^{-1}(F)$ in $U_{\mathbb{H}}(n) \times \mathbb{R}_+^*$ by the Lipschitz map $(g, q'') \mapsto (g, |q''|)$. Hence this set has zero $\mathcal{H}_{2n^2+n+1}$ -measure. $\square$

We define the *unbounded locus* $L(u)$ to be the set of points $q \in \Omega$ such that u is unbounded in every neighborhood of q .

Proposition 3.3 *Let $u_1, \dots, u_k$ be PSH functions in Ω and let T be a closed positive $(2n - 2p)$ -current, $k \leq p$. Let $U_1, \dots, U_k \subset \Omega$ be closed sets such that*

$$\mathcal{H}_{4p-4m+1}(U_{t_1} \cap \dots \cap U_{t_m} \cap \text{Supp } T) = 0 \quad (3.6)$$

for all choices of indices $t_1, \dots, t_m$ in $\{1, \dots, k\}$. Then for all compact sets K, L of Ω with $L \subset K^\circ$, there exist neighborhoods V_t of $K \cap U_t$, $t = 1, \dots, k$, and a constant $C = C(K, L, U_1, \dots, U_k)$ such that the conditions $u_t \leq 0$ on K and $L(u_t) \subset U_t$ imply

$$\|u_1 \Delta u_2 \wedge \dots \wedge \Delta u_k \wedge T\|_L \leq C \|u_1\|_{L^\infty(K \setminus V_1)} \dots \|u_k\|_{L^\infty(K \setminus V_k)} \|T\|_K \quad (a_k)$$

$$\|\Delta u_1 \wedge \dots \wedge \Delta u_k \wedge T\|_L \leq C \|u_1\|_{L^\infty(K \setminus V_1)} \dots \|u_k\|_{L^\infty(K \setminus V_k)} \|T\|_K. \quad (b_k)$$

Proof It is sufficient to show that every point $x_0 \in K^\circ$ has a neighborhood L such that (a), (b) hold. After a mollification process $u_t = \lim_{\varepsilon \rightarrow 0} u_t * \chi_\varepsilon$ for $t = k, k-1, \dots, 1$ in this order, we may assume that $u_1, \dots, u_k$ are smooth. We proceed by induction on k in two steps:

Step 1 $(b_{k-1}) \Rightarrow b_k$,

Step 2 (a_{k-1}) and $(b_k) \Rightarrow (a_k)$.

First (b_0) is the trivial statement $\|T\|_L \leq \|T\|_K$. Let $x_0 = 0 \in K^\circ$ and L be a neighborhood of 0. To prove Step 1, we need to show that

$$\int_L \Delta u_1 \wedge \dots \wedge \Delta u_k \wedge T \wedge \beta_n^s \leq C \|u_1\|_{L^\infty(K \setminus V_1)} \dots \|u_k\|_{L^\infty(K \setminus V_k)} \|T\|_K, \quad (3.8)$$

with $s = p - k$. By Corollary 2.2 in [19], if we change the unitary basis in $\mathbb{H}^n$, Δu_t is invariant. So we only need to show the estimate (3.8) for $\Delta u_1 \wedge \dots \wedge \Delta u_k \wedge T \wedge \omega^0 \wedge \omega^1 \wedge \dots \wedge \omega^{2s-1}$.

Suppose first $0 \in U_1 \cap \dots \cap U_k$ and set $F = U_1 \cap \dots \cap U_k \cap \text{Supp } T$. Since $\mathcal{H}_{4p-4k+1}(F) = 0$, by Lemma 3.3 above, there are coordinates $q' = (q_1, \dots, q_s)$, $q'' = (q_{s+1}, \dots, q_n)$ and a ball $\overline{B''}$ such that $F \cap (\{0\} \times \partial B'') = \emptyset$ and $\{0\} \times \overline{B''} \subset K^\circ$. Since K is compact, we can find neighborhoods W_t of $K \cap U_t$ and a ball $B' = B(0, r') \subset \mathbb{H}^s$ such that $\overline{B'} \times \overline{B''} \subset K^\circ$ and

$$\overline{W_1} \cap \dots \cap \overline{W_k} \cap \text{Supp } T \cap \left(\overline{B'} \times \left(\overline{B''} \setminus (1 - \delta) \overline{B''} \right) \right) = \emptyset \quad (3.9)$$

for $\delta > 0$ small.

When $0 \notin U_t$ for some t , we set instead W_t to be a small neighborhood of 0 such that $\overline{W_t} \subset (\overline{B'} \times (1 - \delta) \overline{B''}) \setminus U_t$, then (3.9) is satisfied automatically. Let $\chi_t \geq 0$ be a function with compact support in W_t , equal to 1 near $K \cap U_t$ if $0 \in U_t$ (resp. equal to 1 near 0 if $0 \notin U_t$). Let $\chi(q') \geq 0$ be a function equal to 1 on $\frac{1}{2} B'$ with compact support in B' .

Note that $\tilde{\alpha} := \chi(q') (\Delta(\|q'\|^2))^s = \chi(q') 8^s s! \omega^0 \wedge \omega^1 \wedge \dots \wedge \omega^{2s-1}$ is closed, where $q' = (q_1, \dots, q_s)$. This is because,

$$d_\alpha \tilde{\alpha} = 8^s s! \sum_{k=0}^{2n-1} \nabla_{k\alpha}(\chi(q')) \omega^k \wedge \omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^{2s-1} = 0, \quad (3.10)$$

for $\alpha = 0, 1$, by considering the type and $\nabla_{k\alpha}(\chi(q')) \equiv 0$ for $k = 2s, \dots, 2n-1$. Since T and $\tilde{\alpha}$ are closed, we find that

$$\Delta(\chi_1 u_1) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \tilde{\alpha} = d_0[d_1(\chi_1 u_1) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \tilde{\alpha}]. \quad (3.11)$$

Then it follows from Lemma 2.3 and (3.9) that

$$\begin{aligned} & \int_{B' \times B''} \Delta(\chi_1 u_1) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \tilde{\alpha} \\ &= \int_{B' \times B''} d_0[d_1(\chi_1 u_1) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \tilde{\alpha}] = 0. \end{aligned} \quad (3.12)$$

The last identity holds because $d_1(\chi_1 u_1) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \tilde{\alpha}$ vanishes near the boundary of $B' \times B''$ by (3.9).

If we expand all factors $\Delta(\chi_t u_t)$, we find a term

$$\chi_1 \cdots \chi_k \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T = \Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T \geq 0 \quad (3.13)$$

on a neighborhood of 0 where $\chi_1 = \cdots = \chi_k = 1$. Other terms involve $\gamma(\chi_t, u_t) + u_t \Delta \chi_t$ for at least one index t , where $\gamma(\chi_t, u_t) = d_0 \chi_t \wedge d_1 u_t + d_0 u_t \wedge d_1 \chi_t$ is the mixed term. However all derivatives of χ_t vanish on some neighborhood V'_t of $K \cap U_t$. And note that u_t is bounded on $(\overline{B'} \times \overline{B''}) \setminus V'_t$ by the fact that $L(u_t) \subset U_t$.

Let K' be a compact neighborhood of $\overline{B'} \times \overline{B''}$ with $K' \subset K^o$, and let V_t be a neighborhood of $K \cap U_t$ with $\overline{V}_t \subset V'_t$. To control these forms, for $L' := (\overline{B'} \times \overline{B''}) \setminus V'_t \subset (K' \setminus V_t)^o$, we apply the usual Chern–Levine–Nirenberg-type estimate in Corollary 3.1 to the product of the positive current $\Theta = \Delta u_1 \wedge \cdots \wedge \widehat{\Delta u_t} \wedge \cdots \wedge \Delta u_k \wedge T$ with the mixed term $\gamma(\chi_t, u_t)$ to get

$$\|\gamma(\chi_t, u_t) \wedge \Theta\|_{\overline{B'} \times \overline{B''}} = \|\gamma(\chi_t, u_t) \wedge \Theta\|_{L'} \leq C_1 \|u_t\|_{L^\infty(K' \setminus V_t)} \|\Theta\|_{K' \setminus V_t}.$$

Then apply the induction hypothesis (b_{k-1}) to the current Θ to get

$$\|\Theta\|_{K' \setminus V_t} \leq \|\Theta\|_{K'} \leq C_2 \|u_1\|_{K \setminus V_1} \cdots \|\widehat{u_t}\| \cdots \|u_k\|_{K \setminus V_k} \|T\|_{K'}.$$

Thus the mixed terms are uniformly bounded. For the same reason, the terms involving $u_t \Delta \chi_t$ are also uniformly bounded. For the terms involving several copies of $\gamma(\chi_t, u_t)$ and $u_t \Delta \chi_t$, one can get the desired bounds by repeating the above process. Finally, it follows from (3.12) and (3.13) that $\Delta u_1 \wedge \cdots \wedge \Delta u_k \wedge T \wedge \omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^{2s-2} \wedge \omega^{2s-1}$ satisfies estimate (b_k) on a neighborhood of 0. Thus Step 1 is finished.

Now we proceed by induction to prove Step 2. Note that $d_\alpha(\chi(q')(\Delta(\|q'\|^2))^s) = 0$ in (3.10). By applying Lemma 2.3 twice, we get

$$\begin{aligned} & \int_{\overline{B'} \times \overline{B''}} \chi_1 u_1 \Delta(\chi_2 u_2) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \chi(q')(\Delta(\|q'\|^2))^s \wedge \Delta(|q_{s+1}|^2) \\ &= \int_{\overline{B'} \times \overline{B''}} |q_{s+1}|^2 \Delta(\chi_1 u_1) \wedge \Delta(\chi_2 u_2) \wedge \cdots \wedge \Delta(\chi_k u_k) \wedge T \wedge \chi(q')(\Delta(\|q'\|^2))^s. \end{aligned} \quad (3.14)$$

By Corollary 3.1 and (b_k) , we already know that all terms in the right hand side integral admit the desired bound. And this also means $(b_1) \Rightarrow (a_1)$.

Except for $\chi_1 \cdots \chi_k u_1 \Delta u_2 \wedge \cdots \wedge \Delta u_k \wedge T$, all terms in the left hand side of (3.14) involve derivatives of χ_t for at least one index t , $t \geq 2$. Note that all terms in the right hand

side integral admit the desired bound. Once one can show that the terms in the left hand side involving derivatives of χ_t for at least one index t , $t \geq 2$, admit the desired bound, then it follows the desired bound for the integral of $\chi_1 \dots \chi_k u_1 \Delta u_2 \wedge \dots \wedge \Delta u_k \wedge T$, which equals $u_1 \Delta u_2 \wedge \dots \wedge \Delta u_k \wedge T$ on a neighborhood of 0. Since the support of derivatives of χ_t , $t \geq 2$, is disjoint from U_t , the integral involving these derivatives is actually on some neighborhood of 0 which is disjoint from U_t . Therefore we only have to obtain a bound for $\|u_1 \Delta u_2 \wedge \dots \wedge \Delta u_k \wedge T \wedge \alpha'\|_L$ when $L := \overline{B}(0, r)$ is disjoint from U_t for some $t \geq 2$, say $L \cap U_2 = \emptyset$, and α' is a constant positive $(2p - 2k + 2)$ -form.

Then for some $\varepsilon > 0$, and some neighborhood V_2 of $K \cap U_2$, we have $\overline{B}(x_0, r + \varepsilon) \subset K^o \setminus \overline{V}_2$. Replace u_2 by a *PSH* function $\tilde{u}_2$, which equals to u_2 in L and to $\frac{M}{\varepsilon r}(\|q\|^2 - r^2) - M$ in $\overline{B}(0, r + \varepsilon) \setminus B(0, r + \frac{\varepsilon}{2})$, with $M = \|u_2\|_{L^\infty(K \setminus V_2)}$. Let $\chi \geq 0$ be a smooth function equal to 1 in $B(0, r)$ with support in $B(0, r + \frac{\varepsilon}{2})$. Then $\chi \tilde{u}_2$ equal to u_2 in L and equal to 0 out of $B(0, r + \frac{\varepsilon}{2})$, and $\|\chi \tilde{u}_2\|_{B(0, r + \frac{\varepsilon}{2})} \leq O(1)\|u_2\|_{L^\infty(K \setminus V_2)}$. By Lemma 2.3, we get

$$\begin{aligned} & \int_{B(0, r + \varepsilon)} u_1 \Delta(\chi \tilde{u}_2) \wedge \Delta u_3 \wedge \dots \wedge \Delta u_k \wedge T \wedge \alpha' \\ &= \int_{B(0, r + \varepsilon/2)} \chi \tilde{u}_2 \Delta(u_1) \wedge \Delta u_3 \wedge \dots \wedge \Delta u_k \wedge T \wedge \alpha' \\ &\leq O(1)\|u_1\|_{L^\infty(K \setminus V_1)} \dots \|u_k\|_{L^\infty(K \setminus V_k)} \|T\|_K, \end{aligned} \quad (3.15)$$

where the last estimate follows from (b_{k-1}) applied to $\Delta u_1 \wedge \Delta u_3 \wedge \dots \wedge \Delta u_k \wedge T$. Note that

$$\Delta(\chi \tilde{u}_2) = \chi \Delta \tilde{u}_2 + (\text{smooth terms involving derivatives of } \chi)$$

coincides with Δu_2 in L , and (a_{k-1}) implies the required estimate for the other terms in the left hand side of (3.15). Then we get finally the desired bound. $\square$

Proof of Theorem 1.2 As in the proof of Proposition 3.3, we take $x_0 = 0$. We proceed by induction on k and also on the number m of functions u^t , $t = 1, \dots, k$, which are unbounded near the given point x_0 . By Lemma 3.2 and the arguments already used in the proof of Proposition 3.2, it suffices to prove that

$$\begin{aligned} & \int_{B' \times B''} \chi_1 \dots \chi_k u^1 \Delta u^2 \wedge \dots \wedge \Delta u^k \wedge T \wedge \alpha \\ &\leq \liminf_{j \rightarrow \infty} \int_{B' \times B''} \chi_1 \dots \chi_k u_j^1 \Delta u_j^2 \wedge \dots \wedge \Delta u_j^k \wedge T \wedge \alpha, \end{aligned} \quad (3.16)$$

where $\alpha = \chi(q')(\Delta(\|q'\|^2))^s$, and B' , B'' and the functions χ_t , $t = 1, \dots, k$, are chosen as in the proof of Step 1 in Proposition 3.3. There product $\chi_1 \dots \chi_k$ has compact support in $B' \times B''$ and $\chi_1 = \dots = \chi_k = 1$ in a neighborhood of x_0 . If u^t is bounded near x_0 , we take $W_t'' \Subset W_t' \Subset W_t$ to be small balls of center x_0 on which u^t is bounded and we modify the sequence u_j^t on the ring $W_t \setminus W_t''$ so as to make it constant and equal to a smooth function $A\|q - x_0\|^2 + B$ on the smaller ring $W_t \setminus W_t'$. In this case, we take χ_t equal to 1 near $\overline{W}_t'$ and $\text{Supp} \chi_t \subset W_t$.

For every $l = 1, \dots, k$, as the process in (3.5), we replace $\chi_1 u_j^1 \Delta(\chi_l u^l)$ by $\chi_l u^l \Delta(\chi_1 u_j^1)$ and use the inequality $u^l \leq u_j^l$ to get

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \int_{B' \times B''} \chi_1 u_j^1 \Delta(\chi_2 u_j^2) \wedge \dots \wedge \Delta(\chi_{l-1} u_j^{l-1}) \wedge \\ & \quad \Delta(\chi_l u^l) \wedge \Delta(\chi_{l+1} u^{l+1}) \wedge \dots \wedge \Delta(\chi_k u^k) \wedge T \wedge \alpha \\ & \leq \liminf_{j \rightarrow \infty} \int_{B' \times B''} \chi_1 u_j^1 \Delta(\chi_2 u_j^2) \wedge \dots \wedge \Delta(\chi_{l-1} u_j^{l-1}) \wedge \\ & \quad \Delta(\chi_l u_j^l) \wedge \Delta(\chi_{l+1} u^{l+1}) \wedge \dots \wedge \Delta(\chi_k u^k) \wedge T \wedge \alpha. \end{aligned} \quad (3.17)$$

If we expand all factors $\Delta(\chi_l u^l)$ in (3.17), we find an inequality involving only the term $\chi_l \Delta u^l$, that is,

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \int_{B' \times B''} \chi_1 \dots \chi_k u_j^1 \Delta u_j^2 \wedge \dots \wedge \Delta u_j^{l-1} \wedge \Delta u^l \wedge \Delta u^{l+1} \wedge \dots \wedge \Delta u^k \wedge T \wedge \alpha \\ & \leq \liminf_{j \rightarrow \infty} \int_{B' \times B''} \chi_1 \dots \chi_k u_j^1 \Delta u_j^2 \wedge \dots \wedge \Delta u_j^{l-1} \wedge \Delta u_j^l \wedge \Delta u^{l+1} \wedge \dots \wedge \Delta u^k \wedge T \wedge \alpha, \end{aligned}$$

which gives (3.16). The other terms involve $d_0 \chi_l \wedge d_1 u^l + d_0 u^l \wedge d_1 \chi_l + u^l \Delta \chi_l$ for at least one index l . It suffices to show that these terms do not give any contribution in the inequalities.

Note that the support of all derivatives of χ_t is disjoint from the unbounded locus of u^l when u^l is unbounded, and contained in $W_t \setminus \overline{W'_t}$ when u^l is bounded. In the first case, the number m of unbounded functions is therefore replaced by $m - 1$. Whereas in the second case $u_j^l = u^l$ is constant and smooth on the support of the derivatives of χ_t , so k can be replaced by $k - 1$. By induction on $k + m$ and the polarization identity (3.4), the limit of the terms involving derivatives of χ_t is equal on both sides to the corresponding terms obtained by suppressing all indices j . $\square$

Let $k = p = n$ and T be a constant in Theorems 1.1 and 1.2. Then we can get the following conclusion for the positive measure $\Delta u^1 \wedge \dots \wedge \Delta u^n$.

Corollary 3.2 *Let $u^1, \dots, u^n$ be PSH functions in Ω such that*

$$\mathcal{H}_{4n-4m+1}(L(u^{t_1}) \cap \dots \cap L(u^{t_m})) = 0 \quad (3.18)$$

for all choices of indices $t_1 < \dots < t_m$ in $\{1, \dots, n\}$. Then the quaternionic Monge–Ampère measure $\Delta u^1 \wedge \dots \wedge \Delta u^n$ is well defined and has locally finite mass in Ω . Moreover, let $\{u_j^1\}_{j \in \mathbb{N}}, \dots, \{u_j^n\}_{j \in \mathbb{N}}$ be decreasing sequences of PSH functions in Ω such that $\lim_{j \rightarrow \infty} u_j^i = u^i$, pointwise in Ω , $i = 1, \dots, n$. Then $\Delta u_j^1 \wedge \dots \wedge \Delta u_j^n$ converges weakly to $\Delta u^1 \wedge \dots \wedge \Delta u^n$ as $j \rightarrow \infty$.

4 Generalized Lelong number

For a closed positive $(2n - 2p)$ -current T , $a \in \Omega$, the Lelong number of T at point a was defined in [19] as the limit

$$v_a(T) := \lim_{r \rightarrow 0+} \frac{\sigma_T(a, r)}{r^{4p}}, \quad (4.1)$$

where

$$\sigma_T(a, r) := \int_{B(a, r)} T \wedge \beta_n^p \quad (4.2)$$

for small r . We showed that $\frac{\sigma_T(a,r)}{r^{4p}}$ is an increasing function of r .

Let $\varphi = -\|q - a\|^{-2}$, which is the fundamental solution of quaternionic Monge–Ampère operator, the generalized Lelong number $\nu(T, \varphi)$ given by (1.3) is just the Lelong number $\nu_a(T)$ of T at point a . This is because:

$$\begin{aligned} \nu(T, -\|q - a\|^{-2}) &= \lim_{r \rightarrow -\infty} \nu(T, -\|q - a\|^{-2}, r) = \lim_{r \rightarrow 0} \nu(T, -\|q - a\|^{-2}, -r^{-2}) \\ &= \lim_{r \rightarrow 0} \int_{\{-\|q - a\|^{-2} < -r^{-2}\}} T \wedge (\Delta(-\|q - a\|^{-2}))^p \\ &= \lim_{r \rightarrow 0} \int_{B(a,r)} T \wedge (\Delta(-\|q - a\|^{-2}))^p = \nu_a(T). \end{aligned}$$

The Lelong–Jensen-type formula we established in [19, Theorem 5.1] also holds for the unbounded case, that is, the requirement of local boundedness of V in Lelong–Jensen-type formula is not necessary. Recall that the quaternionic boundary measure associated with φ is the nonnegative Borel measure $\mu_{\varphi,r}$ defined by:

$$\mu_{\varphi,r} = \Delta_n(\varphi_r) - \chi_{\Omega \setminus B_\varphi(r)} \Delta_n \varphi,$$

where $\varphi_r := \max\{\varphi, r\}$ and $\Delta_n \varphi$ is the coefficient of the $2n$ -current $(\Delta \varphi)^n$. The measure $\mu_{\varphi,r}$ supports on $S_\varphi(r) := \{q \in \Omega; \varphi(q) = r\}$.

Theorem 4.1 (Lelong–Jensen-type formula) *Let Ω be a quaternionic strictly pseudoconvex domain. Let $\varphi : \Omega \rightarrow [-\infty, +\infty)$ be a continuous PSH function, satisfying that, there exists a real number R such that $B_\varphi(R) \Subset \Omega$. Let V be a PSH function in Ω , then*

$$\begin{aligned} \mu_{\varphi,r}(V) - \int_{B_\varphi(r)} V(\Delta \varphi)^n &= \int_{B_\varphi(r)} (r - \varphi) \Delta V \wedge (\Delta \varphi)^{n-1} \\ &= \int_{-\infty}^r dt \int_{B_\varphi(t)} \Delta V \wedge (\Delta \varphi)^{n-1} \\ &= \int_{-\infty}^r \nu(\Delta V, \varphi, t) dt, \end{aligned} \quad (4.3)$$

where $B_\varphi(r)$ is given by (1.5).

Proof For arbitrary $V \in PSH(\Omega)$, let V_h be a decreasing sequence of smooth functions such that $V = \lim_{h \rightarrow 0} V_h$ with $V_h \in PSH(\Omega')$, $\Omega' \Subset \Omega$. Then $\Delta V_h \wedge (\Delta \varphi)^{n-1}$ converges weakly to $\Delta V \wedge (\Delta \varphi)^{n-1}$ by Corollary 3.2. So by Fatou lemma we have

$$\liminf_{h \rightarrow 0} \int_{B_\varphi(r)} (r - \varphi) \Delta V_h \wedge (\Delta \varphi)^{n-1} \geq \int_{B_\varphi(r)} (r - \varphi) \Delta V \wedge (\Delta \varphi)^{n-1}$$

with equality if φ is finite. $\int_{B_\varphi(r)} V_h(\Delta \varphi)^n \rightarrow \int_{B_\varphi(r)} V(\Delta \varphi)^n$ and $\mu_{\varphi,r}(V_h) \rightarrow \mu_{\varphi,r}(V)$ by the monotone convergence theorem. Apply (4.3) to V_h with φ continuous and let $h \rightarrow 0$ to get

$$\mu_{\varphi,r}(V) - \int_{B_\varphi(r)} V(\Delta \varphi)^n \geq \int_{B_\varphi(r)} (r - \varphi) \Delta V \wedge (\Delta \varphi)^{n-1},$$

with equality if φ is finite. On the other hand, for V smooth and l sufficiently large, in the proof of Theorem 5.1 in [19] we have

$$\mu_{\varphi,r}(V) - \int_{B_\varphi(r)} V(\Delta \varphi_{-l})^n = \int_{-l}^r dt \int_{B_\varphi(t)} \Delta V \wedge (\Delta \varphi_{-l})^{n-1}. \quad (4.4)$$

Now apply (4.4) to V_h and let $h \rightarrow 0$ then we get (4.4) for arbitrary V . Letting $l \rightarrow +\infty$ on both side of (4.4), and noting that

$$\int_{B_\varphi(r)} -V(\Delta\varphi)^n \leq \liminf_{l \rightarrow +\infty} \int_{B_\varphi(r)} -V(\Delta\varphi_{-l})^n$$

by Fatou lemma, we finally get the converse inequality

$$\mu_{\varphi,r}(V) - \int_{B_\varphi(r)} V(\Delta\varphi)^n \leq \int_{-\infty}^r dt \int_{B_\varphi(t)} \Delta V \wedge (\Delta\varphi)^{n-1}.$$

□

The following is some relationship between the generalized Lelong number and the quaternionic boundary measure.

Proposition 4.1 *Suppose that Ω is a quaternionic strictly pseudoconvex domain, φ is a continuous PSH function satisfying that, there exists a real number R such that $B_\varphi(R) \Subset \Omega$. Let $V \in PSH(\Omega)$. Assume that $(\Delta\varphi)^n = 0$ on $\Omega \setminus S_\varphi(-\infty)$. Then $r \mapsto \mu_{\varphi,r}(V)$ is a convex increasing function of r and the generalized Lelong number $v(\Delta V, \varphi)$ is given by:*

$$v(\Delta V, \varphi) = \lim_{r \rightarrow -\infty} \frac{\mu_{\varphi,r}(V)}{r}.$$

Proof For $r < r_0$, by (4.3) and (1.4) we have

$$\mu_{\varphi,r}(V) = \mu_{\varphi,r_0}(V) + \int_{r_0}^r v(\Delta V, \varphi, t) dt.$$

It follows from $v(\Delta V, \varphi, t)$ being increasing and nonnegative that $r \mapsto \mu_{\varphi,r}(V)$ is increasing and

$$\begin{aligned} \lim_{r \rightarrow -\infty} \frac{\mu_{\varphi,r}(V)}{r} &= \lim_{r \rightarrow -\infty} \frac{\mu_{\varphi,r_0}(V) + \int_{r_0}^r v(\Delta V, \varphi, t) dt}{r} \\ &= \lim_{r \rightarrow -\infty} v(\Delta V, \varphi, r) = v(\Delta V, \varphi). \end{aligned}$$

□

For $a \in \Omega$, the quaternionic pluri-Green function with pole at a is defined as:

$$g_\Omega(q, a) = \sup \left\{ u(q) : u \in PSH(\Omega, [-\infty, 0]); u(q) + \frac{1}{\|q - a\|^2} \leq O(1) \text{ as } q \rightarrow a \right\}.$$

The quaternionic pluri-Green function $g_\Omega(q, a)$ is maximal and continuous in $\Omega \setminus \{a\}$. This can be proved by using the method as in the case of real k -Green function (cf. Appendix A. in [17]).

In Lelong–Jensen-type formula (4.3), let φ be $g = g_\Omega(\cdot, a)$, we have

$$V(a) = \frac{1}{C_n} \mu_{g,r}(V) - \frac{1}{C_n} \int_{B_g(r)} (r - g) \Delta V \wedge (\Delta g)^{n-1},$$

where C_n is a positive constant such that $(\Delta g)^n = C_n \delta_a$. In particular, if V is pluriharmonic in $B_g(r)$, then $\Delta V = 0$. So $V(a) = \frac{1}{C_n} \mu_{g,r}(V)$. In the special case $n = 1$,

$$V(a) = \frac{1}{C_n} \int_{S_g(r)} V \sum_{ij} \delta_{01}^{ij} \nabla_{j1} g \cdot n_{i0} dS,$$

where $S_g(r) = \{q \in \Omega : g(q) = r\}$ and n_{i0} is given by (2.8). This is the Poisson integral formula and $\sum_{ij} \delta_{01}^{ij} \nabla_{j1} g n_{i0} dS$ is the harmonic measure. Therefore, the Lelong–Jensen-type formula can be regarded as a higher-dimensional version of Poisson integral formula.

We can define quaternionic limit boundary measure associated to φ and prove the comparison theorem for the quaternionic limit boundary measure as in the case of real k -Hessian operator (see Section 4 in [17] for detailed discussion).

By repeating the methods for complex PSH functions and positive currents (cf. [8, 11]), one can get the following two Demailly’s comparison theorems for the generalized Lelong number.

Corollary 4.1 (First comparison theorem) *Let Ω be a quaternionic strictly pseudoconvex domain and let T be a closed positive (p, p) -current. Let φ, ψ be continuous PSH functions such that each one satisfies the hypotheses of Theorem 1.1 and that*

$$l = \limsup \frac{\psi(q)}{\varphi(q)} < +\infty \text{ as } q \in \text{Supp } T \text{ and } \varphi(q) \rightarrow -\infty.$$

Then $v(T, \psi) \leq l^p v(T, \varphi)$, and the equality holds if $l = \lim \frac{\psi}{\varphi}$.

Corollary 4.2 (Second comparison theorem) *Let Ω be a quaternionic strictly pseudoconvex domain and let $u_1, \dots, u_k$ and $v_1, \dots, v_k$ be PSH functions such that each k -tuple satisfies the hypotheses of Theorem 1.1 with respect to T , where T is a closed positive (p, p) -current in Ω . Suppose that $u_j = -\infty$ on $\text{Supp } T \cap \varphi^{-1}(-\infty)$ and that*

$$l_j := \limsup \frac{v_j(q)}{u_j(q)} < +\infty, \text{ where } z \in \text{Supp } T \setminus u_j^{-1}(-\infty), \varphi(q) \rightarrow -\infty.$$

Then

$$v(\Delta v_1 \wedge \dots \wedge \Delta v_k \wedge T, \varphi) \leq l_1 \dots l_k v(\Delta u_1 \wedge \dots \wedge \Delta u_k \wedge T, \varphi).$$

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