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Cusp Kähler–Ricci flow on compact Kähler manifolds

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Abstract
In this paper, we study the limiting flow of conical Kähler–Ricci flows when the cone angles tend to 0. We prove the existence and uniqueness of this limiting flow with cusp singularity on compact Kähler manifold M which carries a smooth hypersurface D such that the twisted canonical bundle $K_M + D$ is ample. Furthermore, we prove that this limiting flow converge to a unique cusp Kähler–Einstein metric.

Keywords Cusp Kähler–Ricci flow · Conical Kähler–Ricci flow · Cusp Kähler–Einstein metric

Mathematics Subject Classification 53C55 · 32W20

1 Introduction

In this paper, we study the limiting flow of (twisted) conical Kähler–Ricci flows when the cone angles tend to 0. Our motivation for considering this limiting flow is to study the existence of singular Kähler–Einstein metric when the cone angle is 0. In [39], Tian anticipated that the complete Tian–Yau Kähler–Einstein metric on the complement of a divisor should be the limit of conical Kähler–Einstein metrics when the cone angles tend to 0.

Let M be a compact Kähler manifold with complex dimension n and $D \subset M$ be a smooth hypersurface. Here, by supposing that the twisted canonical bundle $K_M + D$ is ample, we prove the long-time existence, uniqueness and convergence of the limiting flow of twisted conical Kähler–Ricci flows when the cone angles tend to 0. Since this limiting flow admits cusp singularity along D, we call it cusp Kähler–Ricci flow. As an application, we show the existence of cusp Kähler–Einstein metric [22,40] by using cusp Kähler–Ricci flow.

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The conical Kähler–Ricci flow was introduced to attack the existence problem of conical Kähler–Einstein metric. This equation was first proposed in Jeffres–Mazzeo–Rubinstein's paper (Section 2.5 in [19]). Song–Wang made some conjectures on the relation between the convergence of conical Kähler–Ricci flow and the greatest Ricci lower bound of M (conjecture 5.2 in [37]). The long-time existence, regularity and limit behavior of conical Kähler–Ricci flow have been widely studied, see the works of Liu–Zhang [27,28], Chen-Wang [7,8], Wang [45], Shen [34,35], Edwards [11], Nomura [33], Liu–Zhang [26] and Zhang [47].

By saying a closed positive $(1, 1)$-current ω is conical Kähler metric with cone angle $2\pi\beta$ ($0 < \beta \leq 1$) along D, we mean that D is locally given by $\{z^n = 0\}$ and ω is asymptotically equivalent to model conical metric

$$\sqrt{-1}\sum_{j=1}^{n-1} dz^j \wedge d\bar{z}^j + \frac{\sqrt{-1}dz^n \wedge d\bar{z}^n}{|z^n|^{2(1-\beta)}}. \tag{1.1}$$

And by saying a closed positive $(1, 1)$-current ω is cusp Kähler metric along D, we mean that D is locally given by $\{z^n = 0\}$ and ω is asymptotically equivalent to model cusp metric

$$\sqrt{-1}\sum_{j=1}^{n-1} dz^j \wedge d\bar{z}^j + \frac{\sqrt{-1}dz^n \wedge d\bar{z}^n}{|z^n|^2 \log^2 |z^n|^2}. \tag{1.2}$$

For more about cusp Kähler metrics, please see Auvray's works [1,2].

Let ω_0 be a smooth Kähler metric on M and satisfy $c_1(K_M) + c_1(D) = [\omega_0]$. We denote $D = \{s = 0\}$, where s is a holomorphic section of the line bundle L_D associated to D. In [28], we proved the long-time existence, uniqueness, regularity and convergence of conical Kähler–Ricci flow with weak initial data $\omega_{\varphi_0} \in \mathcal{E}_p(M, \omega_0)$ when $p > 1$, where

$$\mathcal{E}_p(M, \omega_0) = \left\{\varphi \in \mathcal{E}(M, \omega_0) \mid \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\omega_0^n} \in L^p(M, \omega_0^n)\right\},$$
$$\mathcal{E}(M, \omega_0) = \left\{\varphi \in PSH(M, \omega_0) \mid \int_M (\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_M \omega_0^n\right\}.$$

Thanks to Kołodziej's theorem (Theorem 2.4.2 in [24]), potentials in the class $\mathcal{E}_p(M, \omega_0)$ with $p > 1$ are continuous. Furthermore, by Kołodziej's L^p-estimate (Theorem 2.1 in [23]) and Dinew's uniqueness theorem (Theorem 1.2 in [10], see also Theorem B in [14]), we know that the potentials in $\mathcal{E}_p(M, \omega_0)$ with $p > 1$ are Hölder continuous with respect to ω_0 on M.

Let ρ be a smooth closed $(1, 1)$-form and $\hat{\omega}_\beta = \omega_0 + \sqrt{-1}\tau\partial\bar{\partial}|s|_h^{2\beta}$, where h is a smooth hermitian metric on L_D and τ is a small constant. When $c_1(M) = \mu[\omega_0]+(1-\beta)c_1(D)+[\rho]$ ($\mu \in \mathbb{R}$), by our arguments in [28], there exists a unique long-time solution of twisted conical Kähler–Ricci flow

$$\begin{aligned} \frac{\partial\omega_\beta(t)}{\partial t} &= -Ric(\omega_\beta(t)) + \mu\omega_\beta(t) + (1-\beta)[D] + \rho. \\ \omega_\beta(t)|_{t=0} &= \omega_{\varphi_0} \end{aligned} \tag{1.3}$$

Definition 1.1 We call $\omega_\beta(t)$ a long-time solution to twisted conical Kähler–Ricci flow (1.3) if it satisfies the following conditions.

(1) For any $[\delta, T]$ ($\delta, T > 0$), there exists constant C such that

$$C^{-1}\hat{\omega}_\beta \leq \omega_\beta(t) \leq C\hat{\omega}_\beta \quad \text{on} \quad [\delta, T] \times (M\backslash D);$$

(2) on $(0,\infty)\times(M\backslash D)$, $\omega_\beta(t)$ satisfies smooth twisted Kähler–Ricci flow;
(3) on $(0,\infty)\times M$, $\omega_\beta(t)$ satisfies Eq. (1.3) in the sense of currents;
(4) there exists metric potential $\varphi_\beta(t)\in C^0([0,\infty)\times M)\cap C^\infty((0,\infty)\times(M\backslash D))$ such that $\omega_\beta(t)=\omega_0+\sqrt{-1}\partial\bar\partial\varphi_\beta(t)$ and $\lim\limits_{t\to 0^+}\|\varphi_\beta(t)-\varphi_0\|_{L^\infty(M)}=0$;
(5) on $[\delta,T]$, there exist constant $\alpha\in(0,1)$ and C^* such that the above metric potential $\varphi_\beta(t)$ is C^α on M with respect to ω_0 and $\|\frac{\partial\varphi_\beta(t)}{\partial t}\|_{L^\infty(M\backslash D)}\leqslant C^*$.

From Guenancia's result (Lemma 3.1 in [16]),

$$\omega_\beta=\omega_0-\sqrt{-1}\partial\bar\partial\log\left(\frac{1-|s|_h^{2\beta}}{\beta}\right)^2:=\omega_0+\sqrt{-1}\partial\bar\partial\psi_\beta \tag{1.4}$$

is a conical Kähler metric with cone angle $2\pi\beta$ along D. Hence, $\omega_\beta\in\mathcal{E}_p(M,\omega_0)$ for $p\in(1,\frac{1}{1-\beta})$. By direct calculations, it is obvious that $\omega_\beta\geq\frac{1}{2}\omega_0$ for choosing suitable hermitian metric h. Since ψ_β converge to ψ_0 in C^∞_{loc}-sense outside D and globally in L^1-sense on M as $\beta\to 0$, ω_β converge to cusp Kähler metric

$$\omega_{cusp}=\omega_0-\sqrt{-1}\partial\bar\partial\log\log^2|s|_h^2:=\omega_0+\sqrt{-1}\partial\bar\partial\psi_0 \tag{1.5}$$

in C^∞_{loc}-sense outside D and globally in the sense of currents. Here we remark that $\frac{\omega^n_{\psi_0}}{\omega_0^n}\in L^1(M)$ but $\frac{\omega^n_{\psi_0}}{\omega_0^n}\notin L^p(M)$ for $p>1$. We pick $\theta\in c_1(D)$ a smooth real closed $(1,1)$-form such that $[D]=\theta+\sqrt{-1}\partial\bar\partial\log|s|_h^2$, then we consider twisted conical Kähler–Ricci flow

$$\begin{aligned}\frac{\partial\omega_\beta(t)}{\partial t}&=-Ric(\omega_\beta(t))-\omega_\beta(t)+(1-\beta)[D]+\beta\theta.\\ \omega_\beta(t)|_{t=0}&=\omega_\beta\end{aligned} \tag{1.6}$$

Since $c_1(K_M)+c_1(D)=[\omega_0]$, flow (1.6) preserves the Kähler class, that is, $[\omega_\beta(t)]=[\omega_0]$. We write (1.6) as parabolic complex Monge–Ampère equation on potentials,

$$\begin{aligned}\frac{\partial\varphi_\beta(t)}{\partial t}&=\log\frac{(\omega_0+\sqrt{-1}\partial\bar\partial\varphi_\beta(t))^n}{\omega_0^n}-\varphi_\beta(t)+h_0+(1-\beta)\log|s|_h^2\\ \varphi_\beta(0)&=\psi_\beta\end{aligned} \tag{1.7}$$

on $(0,\infty)\times(M\backslash D)$, where $h_0\in C^\infty(M)$ satisfies $-Ric(\omega_0)+\theta-\omega_0=\sqrt{-1}\partial\bar\partial h_0$. By proving uniform estimates (independent of β) for twisted conical Kähler–Ricci flows (1.6), we obtain the limiting flow which is called cusp Kähler–Ricci flow

$$\begin{aligned}\frac{\partial\omega(t)}{\partial t}&=-Ric(\omega(t))-\omega(t)+[D].\\ \omega(t)|_{t=0}&=\omega_{cusp}\end{aligned} \tag{1.8}$$

Definition 1.2 We call $\omega(t)$ a long-time solution to cusp Kähler–Ricci flow (1.8) if it satisfies the following conditions.

(1) For any $[\delta,T]$ $(\delta,T>0)$, there exists constant C such that

$$C^{-1}\omega_{cusp}\leq\omega(t)\leq C\omega_{cusp}\quad\text{on}\quad[\delta,T]\times(M\backslash D);$$

(2) on $(0,\infty)\times(M\backslash D)$, $\omega(t)$ satisfies smooth Kähler–Ricci flow;
(3) on $(0,\infty)\times M$, $\omega(t)$ satisfies Eq. (1.8) in the sense of currents;

(4) there exists $\varphi(t) \in C^0([0,\infty)\times(M\backslash D)) \cap C^\infty((0,\infty)\times(M\backslash D))$ such that

$$\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t) \text{ and } \lim_{t\to 0^+} \|\varphi(t) - \psi_0\|_{L^1(M)} = 0;$$

(5) on $(0, T]$, $\|\varphi(t) - \psi_0\|_{L^\infty(M\backslash D)} \leqslant C$;
(6) on $[\delta, T]$, there exist constant C such that $\|\frac{\partial\varphi(t)}{\partial t}\|_{L^\infty(M\backslash D)} \leqslant C$.

There are some important results on Kähler–Ricci flows (as well as its twisted versions with smooth twisting forms) from weak initial data, such as Chen–Ding [5], Chen–Tian–Zhang [6], Guedj–Zeriahi [15], Di Nezza–Lu [9], Song-Tian [36], Székelyhidi–Tosatti [38] and Zhang [48] etc. In [29,30], Lott–Zhang proved some significant results on various Kähler–Ricci flows with singularities. In particular in [29], in $M\backslash D$, they thoroughly studied the existence and convergence of Kähler–Ricci flow whose initial metric is finite volume Kähler metric with "superstandard spatial asymptotics" (Definition 8.10 in [29]) and gave some interesting examples. Their flow keeps "superstandard spatial asymptotics" and this type of metrics contain cusp Kähler metrics. Here we study the limiting behavior of conical Kähler–Ricci flows when the cone angles tend to 0 on M. The limiting flow admits non-smooth twisting form globally, cusp singularity along D and weak initial data, it is a solution of cusp Kähler–Ricci flow (1.8) and can be seen as Lott–Zhang's case when we restrict it in $M\backslash D$. In this limiting process, we need prove uniform estimates (independent of β) for a sequence of conical flows and consider the asymptotic behavior of the weak solution to the limiting flow when t tend to 0^+. There are some other interesting results on singular Ricci flows, see Ji–Mazzeo–Sesum [20], Kleiner–Lott [21], Mazzeo–Rubinstein–Sesum [32], Topping [43] and Topping–Yin [44].

In [28], we studied conical Kähler–Ricci flows which are twisted by non-smooth twisting forms and start from weak initial data with L^p-density for $p > 1$. Here, by limiting conical flows (1.6), we prove that the limiting flow with weak initial data ω_{cusp} is a long-time solution to cusp Kähler–Ricci flow (1.8). This initial metric only admits L^1-density. For obtaining this limiting flow, in addition to getting uniform estimates (independent of β) of flows (1.6), it is important to prove that $\varphi(t)$ converge to ψ_0 globally in L^1-sense and locally in L^∞-sense outside D as $t \to 0^+$. In this process, we need to construct auxiliary function, and we also need a key observation (Proposition 2.8 and 2.9) that both ψ_β and $\varphi_\beta(t)$ are monotone decreasing as $\beta \searrow 0$. Then we obtain a uniqueness result of cusp Kähler–Ricci flow. In fact, we obtain the following theorem.

Theorem 1.3 *Let M be a compact Kähler manifold and ω_0 be a smooth Kähler metric. Assume that $D \subset M$ is a smooth hypersurface which satisfies $c_1(K_M) + c_1(D) = [\omega_0]$. Then the twisted conical Kähler–Ricci flows* (1.6) *converge to a unique long-time solution $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ of cusp Kähler–Ricci flow* (1.8) *in C^∞_{loc}-sense in $(0,\infty)\times(M\backslash D)$ and globally in the sense of currents.*

Remark 1.4 The uniqueness in Theorem 1.3 need to be understood in this sense: if $\phi(t) \in C^0([0,\infty)\times(M\backslash D)) \cap C^\infty((0,\infty)\times(M\backslash D))$ is a solution to equation

$$\begin{aligned} \frac{\partial\phi(t)}{\partial t} &= \log\frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t))^n}{\omega_0^n} - \phi(t) + h_0 + \log|s|_h^2 \\ \phi(0) &= \psi_0 \end{aligned} \tag{1.9}$$

on $(0,\infty)\times(M\backslash D)$ and satisfies (1), (4), (5) and (6) in Definition 1.2, then $\phi(t)$ lies below $\varphi(t)$ which is obtained by limiting twisted conical Kähler–Ricci flows (1.6) in Theorem 1.3.

When $n = 1$, this uniqueness property is called "maximally stretched" in Topping's (Remark 1.9 in [42]) and Giesen-Topping's (Theorem 1.2 in [13]) works.

Remark 1.5 Since $K_M + D$ is ample, $K_M + (1-\beta)D$ is also ample for sufficiently small β. Guenancia [16] proved that cusp Kähler–Einstein metric is the limit of conical Kähler–Einstein metrics with background metrics $\omega_0 - \beta\theta$ as $\beta \to 0$. The cohomology classes are changing in this process. But in the flow case, we cannot obtain above uniqueness of the limiting flow if we choose the approximating flows that are conical Kähler–Ricci flows with background metrics $\omega_0 - \beta\theta$. In fact, if we choose the approximating flows that are conical Kähler–Ricci flows

$$\begin{aligned} \frac{\partial \tilde{\omega}_\beta(t)}{\partial t} &= -Ric(\tilde{\omega}_\beta(t)) - \tilde{\omega}_\beta(t) + (1-\beta)[D] \\ \tilde{\omega}_\beta(t)|_{t=0} &= \omega_0 - \beta\theta + \sqrt{-1}\partial\bar{\partial}\psi_\beta \end{aligned} \tag{1.10}$$

with background metrics $\omega_0 - \beta\theta$, that is, $\tilde{\omega}_\beta(t) = \omega_0 - \beta\theta + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\beta(t)$, we can also get a long-time solution $\tilde{\omega}(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}(t)$ to Eq. (1.8). But we do not know whether $\tilde{\varphi}(t)$ is unique or maximal. We can only prove $\tilde{\varphi}_\beta(t) + \beta \log |s|_h^2 \nearrow \tilde{\varphi}(t)$ outside D as $\beta \searrow 0$. However, by the uniqueness result in Theorem 1.3, $\tilde{\varphi}(t)$ must lie below $\varphi(t)$. Therefore, we set the background metric to ω_0 in this paper.

At last, we prove the convergence of cusp Kähler–Ricci flow (1.8).

Theorem 1.6 *Cusp Kähler–Ricci flow* (1.8) *converges to a Kähler–Einstein metric with cusp singularity along D in C^∞_{loc}-topology outside hypersurface D and globally in the sense of currents.*

Kobayashi [22] and Tian–Yau [40] asserted that if the twisted canonical bundle $K_M + D$ is ample, then there is a unique (up to constant multiple) complete cusp Kähler–Einstein metric with negative Ricci curvature in $M\backslash D$. The above convergence result recovers the existence of this cusp Kähler–Einstein metric.

The paper is organized as follows. In Sect. 2, we prove the long-time existence and uniqueness of cusp Kähler–Ricci flow (1.8) by limiting twisted conical Kähler–Ricci flows (1.6) and constructing auxiliary function. In Sect. 3, we prove the convergence theorem.

2 The long-time existence of cusp Kähler–Ricci flow

In this section, we prove the long-time existence of cusp Kähler–Ricci flow by limiting twisted conical Kähler–Ricci flows (1.6), and we also prove the uniqueness theorem. For further consideration in the following arguments, we shall pay attention to the estimates which are independent of β.

From our arguments (Sections 2 and 3 in [28]), we know that there exists a unique long-time solution $\varphi_\beta(t) \in C^0([0,\infty) \times M) \bigcap C^\infty((0,\infty) \times (M\backslash D))$ to Eq. (1.7). Let $\phi_\beta(t) = \varphi_\beta(t) - \psi_\beta$, we write the Eq. (1.7) as

$$\begin{aligned} \frac{\partial \phi_\beta(t)}{\partial t} &= \log \frac{(\omega_\beta + \sqrt{-1}\partial\bar{\partial}\phi_\beta(t))^n}{\omega_\beta^n} - \phi_\beta(t) + h_\beta \\ \phi_\beta(0) &= 0 \end{aligned} \tag{2.1}$$

in $(0,\infty) \times (M\backslash D)$, where $h_\beta = -\psi_\beta + h_0 + \log \frac{|s|_h^{2(1-\beta)}\omega_\beta^n}{\omega_0^n}$ is uniformly bounded by constant C independent of β.

Lemma 2.1 *There exists constant C independent of β and t such that*

$$\|\phi_\beta(t)\|_{L^\infty(M)} \leqslant C. \tag{2.2}$$

Proof Fix $T > 0$. For any $\varepsilon > 0$, we let $\chi_{\beta,\varepsilon}(t) = \phi_\beta(t) + \varepsilon \log |s|_h^2$. Since $\chi_{\beta,\varepsilon}(t)$ is smooth in $M\backslash D$, bounded from above and goes to $-\infty$ near D, it achieves its maximum in $M\backslash D$. Let (t_0, x_0) be the maximum point of $\chi_{\beta,\varepsilon}(t)$ on $[0, T] \times M$ with $x_0 \in M\backslash D$. If $t_0 = 0$, then we have

$$\phi_\beta(t) \leqslant -\varepsilon \log |s|_h^2. \tag{2.3}$$

If $t_0 \neq 0$. At (t_0, x_0), we have

$$\begin{aligned} 0 \leqslant \frac{\partial \chi_{\beta,\varepsilon}(t)}{\partial t} &= \log \frac{(\omega_\beta + \sqrt{-1}\partial\bar{\partial}\phi_\beta(t))^n}{\omega_\beta^n} - \phi_\beta(t) + h_\beta \\ &= \log \frac{(\omega_\beta + \sqrt{-1}\partial\bar{\partial}\chi_{\beta,\varepsilon}(t) + \varepsilon\theta)^n}{\omega_\beta^n} - \phi_\beta(t) + h_\beta \\ &\leq n \log 2 - \phi_\beta(t) + C. \end{aligned}$$

Hence, $\phi_\beta(t_0, x_0) \leqslant C$ and

$$\phi_\beta(t) \leqslant C - \varepsilon \log |s|_h^2, \tag{2.4}$$

where constant C independent of β, t and ε. Let $\varepsilon \to 0$, we have $\phi_\beta(t) \leqslant C$ in $M\backslash D$. Since $\phi_\beta(t)$ is continuous, $\phi_\beta(t) \leqslant C$ on M.

For the lower bound, we can reproduce the same arguments with $\tilde{\chi}_{\beta,\varepsilon}(t) = \phi_\beta(t) - \varepsilon \log |s|_h^2$, and get $\phi_\beta(t) \geq C$ on M. □

We now prove the uniform equivalence of volume forms along complex Monge–Ampère Eq. (2.1). We first recall the following lemma.

Lemma 2.2 *If ω_1 and ω_2 are positive $(1, 1)$-forms, then*

$$n \left(\frac{\omega_1^n}{\omega_2^n}\right)^{\frac{1}{n}} \leqslant tr_{\omega_2}\omega_1 \leqslant n \left(\frac{\omega_1^n}{\omega_2^n}\right) (tr_{\omega_1}\omega_2)^{n-1}. \tag{2.5}$$

The proof of Lemma 2.2 follows from eigenvalue considerations (section 2 [46]).

Lemma 2.3 *For any $T > 0$, there exists constant C independent of β such that for any $t \in (0, T]$,*

$$\frac{t^n}{C} \leq \frac{(\omega_\beta + \sqrt{-1}\partial\bar{\partial}\phi_\beta(t))^n}{\omega_\beta^n} \leq e^{\frac{C}{t}} \quad in \;\; M\backslash D. \tag{2.6}$$

Proof For any $t > 0$, we assume that $t \in [\delta, T]$ with $\delta > 0$. Let $\Delta_{\beta,t}$ be the Laplacian operator associated to $\omega_\beta(t) = \omega_\beta + \sqrt{-1}\partial\bar{\partial}\phi_\beta(t)$. Straightforward calculations show that

$$\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) \dot{\phi}_\beta(t) = -\dot{\phi}_\beta(t). \tag{2.7}$$

Let $H^+_{\beta,\varepsilon}(t) = (t-\delta)\dot{\phi}_\beta(t) - \phi_\beta(t) + \varepsilon \log |s|_h^2$. Since $H^+_{\beta,\varepsilon}(t)$ is smooth in $M\backslash D$, bounded from above and goes to $-\infty$ near D, it achieves its maximum in $M\backslash D$. Let (t_0, x_0) be the maximum point of $H^+_{\beta,\varepsilon}(t)$ on $[\delta, T] \times M$ with $x_0 \in M\backslash D$. If $t_0 = \delta$, then

$$(t - \delta)\dot{\phi}_\beta(t) \leq C - \varepsilon \log |s|_h^2, \tag{2.8}$$

where constant C independent of β, δ, t and ε. If $t_0 \neq \delta$, then at (t_0, x_0), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) H^{+}_{\beta,\varepsilon}(t) = -(t-\delta)\dot{\phi}_\beta(t) + n + tr_{\omega_\beta(t)}(-\omega_\beta + \varepsilon\theta)$$
$$\leqslant -(t-\delta)\dot{\phi}_\beta(t) + n \tag{2.9}$$

for sufficiently small ε. By the maximum principle, we have

$$(t-\delta)\dot{\phi}_\beta(t) \leq C - \varepsilon \log |s|^2_h, \tag{2.10}$$

where constant C independent of β, δ, t and ε. Let $\varepsilon \to 0$ and then $\delta \to 0$, we have

$$\dot{\phi}_\beta(t) \leq \frac{C}{t} \quad on \ \ (0, T] \times (M \backslash D), \tag{2.11}$$

where constant C independent of β and t.

Let $H^{-}_{\beta,\varepsilon}(t) = \dot{\phi}_\beta(t) + 2\phi_\beta(t) - n\log(t-\delta) - \varepsilon \log |s|^2_h$. Then $H^{-}_{\beta,\varepsilon}(t)$ tend to $+\infty$ either $t \to \delta^+$ or $x \to D$. By computing, we also have

$$\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) H^{-}_{\beta,\varepsilon}(t) \geq \dot{\phi}_\beta(t) - 2n - \frac{n}{t-\delta} + tr_{\omega_\beta(t)}\omega_\beta. \tag{2.12}$$

Assume that (t_0, x_0) is the minimum point of $H^{-}_{\beta,\varepsilon}(t)$ on $[\delta, T] \times M$ with $t_0 > \delta$ and $x_0 \in M \backslash D$. Thanks to Lemmas 2.1 and 2.2, there exists constant C_1 and C_2 such that

$$0 \geqslant \left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) H^{-}_{\beta,\varepsilon}(t)|_{(t_0,x_0)} \geq \left(C_1 \left(\frac{\omega^n_\beta}{\omega^n_\beta(t)}\right)^{\frac{1}{n}} + \log \frac{\omega^n_\beta(t)}{\omega^n_\beta} - \frac{C_2}{t-\delta}\right)|_{(t_0,x_0)}$$
$$\geq \left(\frac{C_1}{2}\left(\frac{\omega^n_\beta}{\omega^n_\beta(t)}\right)^{\frac{1}{n}} - \frac{C_2}{t-\delta}\right)|_{(t_0,x_0)}, \tag{2.13}$$

where constant C_1 depends only on n, C_2 depends only on n, ω_0 and T. In inequality (2.13), we assume $\frac{\omega^n_\beta}{\omega^n_\beta(t)} > 1$ and $\frac{C_1}{2}(\frac{\omega^n_\beta}{\omega^n_\beta(t)})^{\frac{1}{n}} + \log \frac{\omega^n_\beta(t)}{\omega^n_\beta} \geq 0$ at (t_0, x_0). Other cases are easy to get the lower bound (2.16) for $\dot{\phi}_\beta$. By the maximum principle, we have

$$\omega^n_\beta(t_0, x_0) \geq C_4(t_0 - \delta)^n \omega^n_\beta(x_0), \tag{2.14}$$

where C_4 independent of β, ε and δ. Then it easily follows that

$$\dot{\phi}_\beta(t) \geq -C + n\log(t-\delta) + \varepsilon \log |s|^2_h, \tag{2.15}$$

where constant C independent of β, ε and δ. Let $\varepsilon \to 0$ and then $\delta \to 0$, we have

$$\dot{\phi}_\beta(t) \geq -C + n\log t \quad \text{on} \ \ (0, T] \times (M \backslash D), \tag{2.16}$$

where constant C independent of β. By (2.11) and (2.16), we obtain (2.6). □

We first recall Guenancia's results about the curvature of ω_β (Theorem 3.2 [16]).

Lemma 2.4 *There exists a constant C depending only on M such that for all $\beta \in (0, \frac{1}{2}]$, the holomorphic bisectional curvature of ω_β is bounded by C.*

Next, we prove the uniform equivalence of metrics along twisted conical Kähler–Ricci flows (1.6) by Chern–Lu inequality.

Lemma 2.5 *For any $T > 0$, there exists constant C independent of β such that for any $t \in (0, T]$ and $\beta \in (0, \frac{1}{2}]$,*

$$e^{-\frac{C}{t}}\omega_\beta \leq \omega_\beta(t) \leq e^{\frac{C}{t}}\omega_\beta \quad in \ \ M\backslash D. \tag{2.17}$$

Proof By Chern–Lu inequality [4,31] (see also Proposition 7.1 in [19]), in $M\backslash D$, we have

$$\Delta_{\beta,t} \log tr_{\omega_\beta(t)}\omega_\beta \geqslant \frac{\big(Ric(\omega_\beta(t)), \omega_\beta\big)_{\omega_\beta(t)}}{tr_{\omega_\beta(t)}\omega_\beta} - Ctr_{\omega_\beta(t)}\omega_\beta, \tag{2.18}$$

where $(\ ,\ )_{\omega_\beta(t)}$ is the inner product with respect to $\omega_\beta(t)$ and constant C depends on the upper bound for the holomorphic bisectional curvature of ω_β. In $M\backslash D$, we also have

$$\frac{\partial}{\partial t}\log tr_{\omega_\beta(t)}\omega_\beta = \frac{\big(Ric(\omega_\beta(t)) + \omega_\beta(t) - \beta\theta, \omega_\beta\big)_{\omega_\beta(t)}}{tr_{\omega_\beta(t)}\omega_\beta}. \tag{2.19}$$

By using (2.18) and (2.19), we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right)\log tr_{\omega_\beta(t)}\omega_\beta \leq Ctr_{\omega_\beta(t)}\omega_\beta + 1, \tag{2.20}$$

where constant C independent of β.

Let $H_{\beta,\varepsilon}(t) = (t-\delta)\log tr_{\omega_\beta(t)}\omega_\beta - A\phi_\beta(t) + \varepsilon\log|s|_h^2$, A be a sufficiently large constant and (t_0, x_0) be the maximum point of $H_{\beta,\varepsilon}(t)$ on $[\delta, T] \times (M\backslash D)$. We know that $x_0 \in M\backslash D$ and we need only consider $t_0 > \delta$. By direct calculations,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right)H_{\beta,\varepsilon}(t) &\leq \log tr_{\omega_\beta(t)}\omega_\beta + Ctr_{\omega_\beta(t)}\omega_\beta - A\dot{\phi}_\beta(t) - Atr_{\omega_\beta(t)}\omega_\beta + \varepsilon tr_{\omega_\beta(t)}\theta + C\\ &\leq -\frac{A}{2}tr_{\omega_\beta(t)}\omega_\beta + \log tr_{\omega_\beta(t)}\omega_\beta - A\log\frac{\omega_\beta^n(t)}{\omega_\beta^n} + C,\end{aligned}$$

where constant C independent of β and δ.

Without loss of generality, we assume that $-\frac{A}{4}tr_{\omega_\beta(t)}\omega_\beta + \log tr_{\omega_\beta(t)}\omega_\beta \leq 0$ at (t_0, x_0). Then at (t_0, x_0), by Lemma 2.3, we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right)H_{\beta,\varepsilon}(t) \leq -\frac{A}{4}tr_{\omega_\beta(t)}\omega_\beta - An\log(t-\delta) + C. \tag{2.21}$$

By the maximum principle, at (t_0, x_0),

$$tr_{\omega_\beta(t)}\omega_\beta \leq C\log\frac{1}{t-\delta} + C, \tag{2.22}$$

which implies that

$$(t-\delta)\log tr_{\omega_\beta(t)}\omega_\beta \leq (t_0-\delta)\log\left(C\log\frac{1}{t_0-\delta} + C\right) + C - \varepsilon\log|s|_h^2. \tag{2.23}$$

Let $\varepsilon \to 0$ and then $\delta \to 0$, on $(0, T] \times (M\backslash D)$,

$$tr_{\omega_\beta(t)}\omega_\beta \leq e^{\frac{C}{t}}. \tag{2.24}$$

By using Lemmas 2.2 and 2.3, we have

$$tr_{\omega_\beta}\omega_\beta(t) \leq e^{\frac{C}{t}}, \tag{2.25}$$

where C independent of β. From (2.24) and (2.25), we prove the lemma. □

By the argument as that in Lemma 3.1 of [28], we get the following local Calabi's C^3-estimates and curvature estimates.

Lemma 2.6 *For any $T > 0$ and $B_r(p) \subset\subset M \backslash D$, there exist constants C, C' and C'' depend only on n, T, ω_0 and $dist_{\omega_0}(B_r(p), D)$ such that*

$$S_{\omega_\beta(t)} \leq \frac{C'}{r^2} e^{\frac{C}{t}},$$
$$|Rm_{\omega_\beta(t)}|^2_{\omega_\beta(t)} \leq \frac{C''}{r^4} e^{\frac{C}{t}}$$

on $(0, T] \times B_{\frac{r}{2}}(p)$.

Since $\phi_\beta(t) = \varphi_\beta(t) - \psi_\beta$ and $\psi_\beta \in C^0(M) \cap C^\infty(M \backslash D)$, establishing local uniform estimates for $\phi_\beta(t)$ in $M \backslash D$ is equivalent to establish the estimates for $\varphi_\beta(t)$. By using the standard parabolic Schauder regularity theory (Theorem 4.9 in [25]), we obtain the following proposition.

Proposition 2.7 *For any $0 < \delta < T < \infty$, $k \in \mathbb{N}^+$ and $B_r(p) \subset\subset M \backslash D$, there exists constant $C_{\delta,T,k,p,r}$ depends only on n, δ, k, T, ω_0 and $dist_{\omega_0}(B_r(p), D)$ such that for $\beta \in (0, \frac{1}{2}]$,*

$$\|\varphi_\beta(t)\|_{C^k([\delta,T]\times B_r(p))} \leq C_{\delta,T,k,p,r}. \tag{2.26}$$

Through a further observation to ψ_β and Eq. (1.7), we prove the monotonicity of ψ_β and $\varphi_\beta(t)$ with respect to β.

Proposition 2.8 *For any $x \in M$, $\psi_\beta(x)$ is monotone decreasing as $\beta \searrow 0$.*

Proof By direct computations, for any $x \in M \backslash D$, we have

$$\frac{d\psi_\beta}{d\beta} = 2\frac{\beta|s|_h^{2\beta}\log|s|_h^2 + 1 - |s|_h^{2\beta}}{\beta(1 - |s|_h^{2\beta})}. \tag{2.27}$$

Denote $f_\beta(a) = \beta a^\beta \log a + 1 - a^\beta$ for $\beta > 0$ and $a \in [0, 1]$. By computing, we get

$$f'_\beta(a) = \beta^2 a^{\beta-1}\log a \leq 0. \tag{2.28}$$

Hence $f_\beta(a) \geq f_\beta(1) = 0$ and we have $\frac{d\psi_\beta}{d\beta} \geq 0$. □

Proposition 2.9 *For any $(t, x) \in (0, \infty) \times M$, $\varphi_\beta(t, x)$ is monotone decreasing as $\beta \searrow 0$.*

Proof By the arguments in section 3 of [28], we obtain Eq. (1.7) by approximating equations

$$\begin{aligned}\frac{\partial \varphi_{\beta,\varepsilon}(t)}{\partial t} &= \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\beta,\varepsilon}(t))^n}{\omega_0^n} - \varphi_{\beta,\varepsilon}(t) + h_0 + \log(\varepsilon^2 + |s|_h^2)^{1-\beta} \\ \varphi_{\beta,\varepsilon}(0) &= \psi_\beta\end{aligned} \tag{2.29}$$

For $\beta_1 < \beta_2$, let $\psi_{1,2}(t) = \varphi_{\beta_1,\varepsilon}(t) - \varphi_{\beta_2,\varepsilon}(t)$. On $[\eta, T] \times M$ with $\eta > 0$ and $T < \infty$,

$$\begin{aligned}&\frac{\partial}{\partial t}(e^{t-\eta}\psi_{1,2}(t)) \\ &\leq e^{t-\eta}\log\frac{\left(e^{t-\eta}\omega_0 + \sqrt{-1}\partial\bar{\partial}e^{t-\eta}\varphi_{\beta_2,\varepsilon}(t) + \sqrt{-1}\partial\bar{\partial}e^{t-\eta}\psi_{1,2}(t)\right)^n}{(e^{t-\eta}\omega_0 + \sqrt{-1}\partial\bar{\partial}e^{t-\eta}\varphi_{\beta_2,\varepsilon}(t))^n}.\end{aligned} \tag{2.30}$$

Let $\tilde{\psi}_{1,2}(t) = e^{t-\eta}\psi_{1,2}(t) - \delta(t-\eta)$ with $\delta > 0$ and (t_0, x_0) be the maximum point of $\tilde{\psi}_{1,2}(t)$ on $[\eta, T] \times M$. If $t_0 > \eta$, by the maximum principle, at this point,

$$0 \leq \frac{\partial}{\partial t}\tilde{\psi}_{1,2}(t) = \frac{\partial}{\partial t}\left(e^{t-\eta}\psi_{1,2}(t)\right) - \delta \leq -\delta \tag{2.31}$$

which is impossible, hence $t_0 = \eta$. So for any $(t, x) \in [\eta, T] \times M$,

$$\psi_{1,2}(t, x) \leq e^{-t+\eta} \sup_M \psi_{1,2}(\eta, x) + T\delta. \tag{2.32}$$

Since $\lim_{t\to 0^+} \| \varphi_{\beta,\varepsilon}(t) - \psi_\beta \|_{L^\infty(M)} = 0$, let $\eta \to 0$, we get

$$\psi_{1,2}(t, x) \leq e^{-t} \sup_M(\psi_{\beta_1} - \psi_{\beta_2}) + T\delta \leq T\delta. \tag{2.33}$$

Here we use Proposition 2.8 in the last inequality. Let $\delta \to 0$ and then $\varepsilon \to 0$, we conclude that $\varphi_{\beta_1}(t, x) \leq \varphi_{\beta_2}(t, x)$. □

For any $[\delta, T] \times K \subset\subset (0, \infty) \times M\backslash D$ and $k \geq 0$, $\|\varphi_\beta(t)\|_{C^k([\delta,T]\times K)}$ is uniformly bounded by Proposition 2.7. Let δ approximate to 0, T approximate to ∞ and K approximate to $M\backslash D$, by diagonal rule, we get a sequence $\{\beta_i\}$, such that $\varphi_{\beta_i}(t)$ converge in C^∞_{loc}-topology in $(0, \infty) \times (M\backslash D)$ to a function $\varphi(t)$ that is smooth on $C^\infty((0, \infty) \times (M\backslash D))$ and satisfies equation

$$\frac{\partial \varphi(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\omega_0^n} - \varphi(t) + h_0 + \log |s|_h^2 \tag{2.34}$$

in $(0, \infty) \times (M\backslash D)$. Since $\varphi_\beta(t)$ is monotone decreasing as $\beta \to 0$, $\varphi_\beta(t)$ converge in C^∞_{loc}-topology in $(0, \infty) \times (M\backslash D)$ to $\varphi(t)$. For any $T > 0$,

$$e^{-\frac{C}{t}}\omega_{cusp} \leq \omega(t) \leq e^{\frac{C}{t}}\omega_{cusp} \;\; on\ (0, T] \times (M\backslash D), \tag{2.35}$$

where $\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$, constants C depend only on n, ω_0 and T.

Next, by using the monotonicity of $\varphi_\beta(t)$ with respect to β and constructing auxiliary function, we prove the L^1-convergence of $\varphi(t)$ as $t \to 0^+$ as well as $\varphi(t)$ converge to ψ_0 in L^∞-norm as $t \to 0^+$ on any compact subset $K \subset\subset M\backslash D$.

Lemma 2.10 *There exists a unique $v_\beta \in PSH(M, \omega_0) \bigcap L^\infty(M)$ to equation*

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}v_\beta)^n = e^{v_\beta - h_0}\frac{\omega_0^n}{|s|_h^{2(1-\beta)}}. \tag{2.36}$$

Furthermore, $v_\beta \in C^{2,\alpha,\beta}(M)$ and $\| v_\beta - \psi_\beta \|_{L^\infty(M)}$ can be uniformly bounded by constant C independent of β.

Proof By Kołodziej's theorem (Theorem 2.4.2 in [23], see also Theorem 4.1 in [12]), there exists a unique continuous solution v_β to Eq. (2.36). Then by Guenancia–P*ă*un's regularity estimates (Theorem *B* in [17], see also Theorem 1.4 in [26]), $v_\beta \in C^{2,\alpha,\beta}(M)$ (readers can refer to page 5731 in [3] for more details about the space $C^{2,\alpha,\beta}(M)$). Next, we prove $\| v_\beta - \psi_\beta \|_{L^\infty(M)}$ can be uniformly bounded. Let $u_\beta = v_\beta - \psi_\beta$, we write Eq. (2.36) as

$$(\omega_\beta + \sqrt{-1}\partial\bar{\partial}u_\beta)^n = e^{u_\beta + h_\beta}\omega_\beta^n, \tag{2.37}$$

where $h_\beta = \psi_\beta - h_0 + \log \frac{\omega_0^n}{|s|_h^{2(1-\beta)}\omega_\beta^n}$ is uniformly bounded independent of β. Define $\chi_{\beta,\varepsilon} = u_\beta + \varepsilon \log |s|_h^2$. Then $\sqrt{-1}\partial\bar{\partial}\chi_{\beta,\varepsilon} = \sqrt{-1}\partial\bar{\partial}u_\beta - \varepsilon\theta$ in $M\backslash D$. Since $\chi_{\beta,\varepsilon}$ is smooth in $M\backslash D$, bounded from above and goes to $-\infty$ near D, it achieves its maximum in $M\backslash D$. Let x_0 be the maximum point of $\chi_{\beta,\varepsilon}$ on M with $x_0 \in M\backslash D$. By the maximum principle,

$$e^{u_\beta+h_\beta}\omega_\beta^n(x_0) = (\omega_\beta + \sqrt{-1}\partial\bar\partial u_\beta)^n(x_0) = (\omega_\beta + \sqrt{-1}\partial\bar\partial\chi_{\beta,\varepsilon} + \varepsilon\theta)^n(x_0) \le 2^n\omega_\beta^n(x_0).$$

Hence, $u_\beta \le C - \varepsilon\log|s|_h^2$, where constant C independent of β and ε. Let $\varepsilon \to 0$, we get the uniform upper bound of u_β. By the similar arguments, we can obtain the uniform lower bound of u_β. □

Proposition 2.11 *$\varphi(t) \in C^0([0,\infty)\times(M\backslash D))$ and*

$$\lim_{t\to 0^+}\|\varphi(t) - \psi_0\|_{L^1(M)} = 0. \tag{2.38}$$

Proof By the monotonicity of $\varphi_\beta(t)$ with respect to β, for any $(t,z)\in(0,T]\times(M\backslash D)$, we have

$$\begin{aligned}\varphi(t,z) - \psi_0(z) &\le \varphi_\beta(t,z) - \psi_0(z)\\ &\le |\varphi_\beta(t,z) - \psi_\beta(z)| + |\psi_\beta(z) - \psi_0(z)|.\end{aligned} \tag{2.39}$$

Since ψ_β converge to ψ_0 in C^∞_{loc}-sense outside D as $\beta \to 0$, we can insure that for any $\epsilon > 0$ and $K \subset\subset M\backslash D$, there exists N such that for $\beta_1 < \frac{1}{N}$,

$$\|\psi_{\beta_1}(z) - \psi_0(z)\|_{L^\infty(K)} < \frac{\epsilon}{2}. \tag{2.40}$$

Fix such β_1. Since by the definition of the flow (1.6)

$$\lim_{t\to 0^+}\|\varphi_\beta(t,z) - \psi_\beta\|_{L^\infty(M)} = 0, \tag{2.41}$$

there exists $0 < \delta_1 < T$ such that

$$\sup_{[0,\delta_1]\times M}|\varphi_{\beta_1}(t,z) - \psi_{\beta_1}| < \frac{\epsilon}{2}. \tag{2.42}$$

Combining the above inequalities together, for any $t \in (0,\delta_1]$ and $z \in K$

$$\sup_{[0,\delta_1]\times K}(\varphi(t,z) - \psi_0(z)) < \epsilon. \tag{2.43}$$

We define function

$$H_\beta(t) = (1 - te^{-t})\psi_\beta + te^{-t}v_\beta + h(t)e^{-t}, \tag{2.44}$$

where v_β and $u_\beta = v_\beta - \psi_\beta$ are obtained in Lemma 2.10, and

$$h(t) = (1 - e^t - t)\|u_\beta\|_{L^\infty(M)} + n(t\log t - t)e^t - n\int_0^t e^s s\log s ds.$$

Straightforward calculations show that

$$\begin{aligned}\frac{\partial}{\partial t}H_\beta(t) + H_\beta(t) &= \psi_\beta + e^{-t}u_\beta - e^{-t}\|u_\beta\|_{L^\infty(M)} - \|u_\beta\|_{L^\infty(M)} + n\log t - nt\\ &\le \psi_\beta + u_\beta + n\log t - nt\\ &= v_\beta + n\log t - nt.\end{aligned}$$

Therefore, we have

$$e^{\frac{\partial}{\partial t}H_\beta(t)+H_\beta(t)}\omega_0^n \le t^n e^{-nt}e^{v_\beta}\omega_0^n.$$

Note that $te^{-t} < 1$, we have

$$\begin{aligned}\omega_0 + \sqrt{-1}\partial\overline{\partial} H_\beta(t) &= (1 - te^{-t})(\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_\beta) + te^{-t}(\omega_0 + \sqrt{-1}\partial\overline{\partial} v_\beta)\\ &\geq te^{-t}(\omega_0 + \sqrt{-1}\partial\overline{\partial} v_\beta).\end{aligned}$$

Combining the above inequalities,

$$\begin{aligned}(\omega_0 + \sqrt{-1}\partial\overline{\partial} H_\beta(t))^n &\geq t^n e^{-nt}(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_\beta)^n\\ &\geq e^{-h_0 + \frac{\partial}{\partial t}H_\beta(t) + H_\beta(t)} \frac{\omega_0^n}{|s|_h^{2(1-\beta)}},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{\partial}{\partial t}H_\beta(t) &\leq \log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial} H_\beta(t))^n}{\omega_0^n} - H_\beta(t) + h_0 + \log |s|_h^{2(1-\beta)}.\\ H_\beta(0) &= \psi_\beta \end{aligned} \tag{2.45}$$

Next, we prove $H_\beta(t) \leq \varphi_\beta(t)$ by using Jeffres' trick [18]. For any $0 < t_1 < T < \infty$ and $a > 0$.

Denote $\Psi(t) = H_\beta(t) + a|s|_h^{2q} - \varphi_\beta(t)$ and $\hat{\Delta} = \int_0^1 g^{i\bar{j}}_{sH_\beta(t)+(1-s)\varphi_\beta(t)} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \mathrm{d}s$, where $0 < q < 1$ is determined later. $\Psi(t)$ evolves along the following equation

$$\frac{\partial \Psi(t)}{\partial t} \leqslant \hat{\Delta}\Psi(t) - a\hat{\Delta}|s|_h^{2q} - \Psi(t) + a|s|_h^{2q}.$$

Since

$$\begin{aligned}\omega_0 + \sqrt{-1}\partial\overline{\partial} H_\beta(t) &\geq (1 - te^{-t})(\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_\beta) \geq \frac{1}{4}\omega_0,\\ \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_\beta(t) &\geq e^{-\frac{C(T)}{t_1}}\omega_\beta \geq \frac{1}{2}e^{-\frac{C(T)}{t_1}}\omega_0,\\ \sqrt{-1}\partial\overline{\partial}|s|_h^{2q} &= q^2|s|_h^{2q}\sqrt{-1}\partial \log |s|_h^2 \wedge \overline{\partial} \log |s|_h^2 + q|s|_h^{2q}\sqrt{-1}\partial\overline{\partial} \log |s|_h^2,\end{aligned}$$

we obtain the following inequalities

$$\frac{1}{2}\min\left(\frac{1}{2}, e^{-\frac{C(T)}{t_1}}\right)\omega_0 \leq \frac{1}{2}s\omega_\beta + (1-s)e^{-\frac{C(T)}{t_1}}\omega_\beta \leq s\omega_{H_\beta(t)} + (1-s)\omega_{\varphi_\beta(t)}.$$

Hence, we have

$$\begin{aligned}\hat{\Delta}|s|_h^{2q} &\geq q|s|_h^{2q}\int_0^1 g_{sH_\beta(t)+(1-s)\varphi_\beta(t)}\left(\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log |s|_h^2\right)\mathrm{d}s\\ &= -q|s|_h^{2q}\int_0^1 g^{i\bar{j}}_{sH_\beta(t)+(1-s)\varphi_\beta(t)}\theta_{i\bar{j}}\mathrm{d}s\\ &\geq -\frac{1}{2}C(t_1, T)q|s|_h^{2q} g_\beta^{i\bar{j}} g_{0,i\bar{j}} \geq -C(t_1, T)\end{aligned}$$

in $M\backslash D$, where constant $C(t_1, T) = 2K\max(2, e^{\frac{C(T)}{t_1}})$ independent of a and $\theta \leqslant K\omega_0$. Then we have

$$\frac{\partial \Psi(t)}{\partial t} \leq \hat{\Delta}\Psi(t) - \Psi(t) + aC(t_1, T).$$

Let $\tilde{\Psi} = e^{(t-t_1)}\Psi - aC(t_1, T)e^{(t-t_1)} - \varepsilon(t - t_1)$. By choosing suitable $0 < q < 1$, we can assume that the space maximum of $\tilde{\Psi}$ on $[t_1, T] \times M$ is attained away from D. Let (t_0, x_0) be the maximum point. If $t_0 > t_1$, by the maximum principle, at (t_0, x_0), we have

$$0 \leq \left(\frac{\partial}{\partial t} - \hat{\Delta}\right)\tilde{\Psi}(t) \leq -\varepsilon,$$

which is impossible, hence $t_0 = t_1$. Then for $(t, x) \in [t_1, T] \times M$, we obtain

$$H_\beta(t) - \varphi_\beta(t) \leq \|H_\beta(t_1, x) - \varphi_\beta(t_1, x)\|_{L^\infty(M)} + aC(t_1, T) + \varepsilon T$$

Since $\lim\limits_{t\to 0^+} \|H_\beta(t, z) - \psi_\beta\|_{L^\infty(M)} = 0$ and (2.41), let $a \to 0$ and then $t_1 \to 0^+$,

$$H_\beta(t) - \varphi_\beta(t) \leq \varepsilon T.$$

It shows that $H_\beta(t) \leq \varphi_\beta(t)$ after we let $\varepsilon \to 0$. For any $(t, z) \in (0, T] \times (M\backslash D)$

$$\begin{aligned}\varphi_\beta(t, z) - \psi_0(z) &\geq te^{-t}u_\beta + h(t)e^{-t} + \psi_\beta - \psi_0 \\ &\geq -Ct - C(1 - e^{-t}) + h_1(t)e^{-t},\end{aligned} \tag{2.46}$$

where $h_1(t) = n(t \log t - t)e^t - n\int_0^t e^s s \log s \, \mathrm{d}s$, constant C is independent of β thanks to Lemma 2.10. Letting $\beta \to 0$, we have

$$\varphi(t, z) - \psi_0(z) \geq -Ct - C(1 - e^{-t}) + h_1(t)e^{-t}. \tag{2.47}$$

There exists δ_2 such that for any $t \in [0, \delta_2]$ and $z \in M\backslash D$,

$$\varphi(t, z) - \psi_0(z) > -\frac{\epsilon}{2}. \tag{2.48}$$

Let $\delta = \min(\delta_1, \delta_2)$, then for any $t \in (0, \delta]$ and $z \in K$,

$$-\epsilon < \varphi(t, z) - \psi_0(z) < \epsilon. \tag{2.49}$$

This, together with Proposition 2.7, insures that $\varphi(t) \in C^0([0, \infty) \times (M\backslash D))$. Since ψ_β converge to ψ_0 in L^1-sense on M, for sufficiently small β_2, we have

$$\int_M |\psi_{\beta_2}(z) - \psi_0(z)|\, \omega_0^n < \frac{\epsilon}{2}. \tag{2.50}$$

By (2.39), (2.41) and (2.48), there exists δ such that for any $t \in (0, \delta)$,

$$\int_M |\varphi(t) - \psi_0(z)|\, \omega_0^n < \epsilon, \tag{2.51}$$

which implies (2.38). □

Theorem 2.12 *$\omega(t) = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ is a long-time solution to cusp Kähler–Ricci flow (1.8).*

Proof We should only prove that $\omega(t)$ satisfies Eq. (1.8) in the sense of currents on $(0, \infty) \times M$.

Let $\eta = \eta(t, x)$ be a smooth $(n-1, n-1)$-form with compact support in $(0, \infty) \times M$. Without loss of generality, we assume that its compact support is included in $(\delta, T) \times M$ $(0 < \delta < T < \infty)$. On $[\delta, T] \times (M\backslash D)$, $\log \frac{\omega_\beta^n(t)|s|_h^{2(1-\beta)}}{\omega_0^n} - \psi_\beta$, $\log \frac{\omega^n(t)|s|_h^2}{\omega_0^n} - \psi_0$, $\varphi_\beta(t) - \psi_\beta$ and $\varphi(t) - \psi_0$ are uniformly bounded. On $[\delta, T]$, we have

$$\int_M \frac{\partial \omega_\beta(t)}{\partial t} \wedge \eta = \int_M \sqrt{-1}\partial\bar{\partial}\frac{\partial \varphi_\beta(t)}{\partial t} \wedge \eta$$

$$
\begin{aligned}
&= \int_M \left(\log \frac{\omega_\beta^n(t)|s|_h^{2(1-\beta)}}{\omega_0^n} - \psi_\beta - (\varphi_\beta(t) - \psi_\beta) + h_0 \right) \sqrt{-1}\partial\bar{\partial}\eta \\
&\xrightarrow{\beta\to 0} \int_M \left(\log \frac{\omega^n(t)|s|_h^2}{\omega_0^n} - \psi_0 - (\varphi(t) - \psi_0) + h_0 \right) \sqrt{-1}\partial\bar{\partial}\eta \\
&= \int_M (-Ric(\omega(t)) - \omega(t) + [D]) \wedge \eta.
\end{aligned}
$$

In the above limit process, we make use of the uniform convergence theorem. At the same time, there also holds

$$
\begin{aligned}
\int_M \omega_\beta(t) \wedge \frac{\partial\eta}{\partial t} &= \int_M \omega_0 \wedge \frac{\partial\eta}{\partial t} + \int_M \varphi_\beta(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial\eta}{\partial t} \\
&\xrightarrow{\beta\to 0} \int_M \omega_0 \wedge \frac{\partial\eta}{\partial t} + \int_M \varphi(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial\eta}{\partial t} \\
&= \int_M \omega(t) \wedge \frac{\partial\eta}{\partial t}. \qquad (2.52)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_M \omega_\beta(t) \wedge \eta &= \int_M \varphi_\beta(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial\eta}{\partial t} + \int_M \omega_0 \wedge \frac{\partial\eta}{\partial t} + \int_M \frac{\partial\varphi_\beta(t)}{\partial t}\sqrt{-1}\partial\bar{\partial}\eta \\
&\xrightarrow{\beta\to 0} \int_M \varphi(t)\sqrt{-1}\partial\bar{\partial}\frac{\partial\eta}{\partial t} + \int_M \omega_0 \wedge \frac{\partial\eta}{\partial t} + \int_M \frac{\partial\varphi}{\partial t}\sqrt{-1}\partial\bar{\partial}\eta \\
&= \frac{\partial}{\partial t}\int_M \omega(t) \wedge \eta. \qquad (2.53)
\end{aligned}
$$

Combining equality

$$
\frac{\partial}{\partial t}\int_M \omega_\beta(t) \wedge \eta = \int_M \frac{\partial\omega_\beta(t)}{\partial t} \wedge \eta + \int_M \omega_\beta(t) \wedge \frac{\partial\eta}{\partial t}
$$

with equalities (2.52)–(2.53), on $[\delta, T]$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_M \omega(t) \wedge \eta &= \int_M (-Ric(\omega(t)) - \omega(t) + [D]) \wedge \eta \\
&\quad + \int_M \omega(t) \wedge \frac{\partial\eta}{\partial t}. \qquad (2.54)
\end{aligned}
$$

Since $Supp\ \eta \subset (\delta, T) \times M$, we have

$$
\int_0^{+\infty} \frac{d}{dt}\int_M \omega(t) \wedge \eta \, \mathrm{d}t = 0. \qquad (2.55)
$$

Integrating form 0 to ∞ on both sides of Eq. (2.54),

$$
\begin{aligned}
\int_{(0,\infty)\times M} \frac{\partial\omega(t)}{\partial t} \wedge \eta \, \mathrm{d}t &= -\int_{(0,\infty)\times M} \omega(t) \wedge \frac{\partial\eta}{\partial t} \, \mathrm{d}t = -\int_0^\infty \int_M \omega(t) \wedge \frac{\partial\eta}{\partial t} \, \mathrm{d}t \\
&= \int_0^\infty \int_M (-Ric(\omega(t)) - \omega(t) + [D]) \wedge \eta \, \mathrm{d}t \\
&= \int_{(0,\infty)\times M} (-Ric(\omega(t)) - \omega(t) + [D]) \wedge \eta \, \mathrm{d}t.
\end{aligned}
$$

By the arbitrariness of η, we prove that $\omega(t)$ satisfies cusp Kähler–Ricci flow (1.8) in the sense of currents on $(0, \infty) \times M$. □

Now we prove the uniqueness theorem.

Theorem 2.13 *Let $\tilde{\varphi}(t) \in C^0\left([0, \infty) \times (M \backslash D)\right) \bigcap C^\infty\big((0, \infty) \times (M \backslash D)\big)$ be a long-time solutions to parabolic Monge–Ampère equation*

$$\frac{\partial \varphi(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\omega_0^n} - \varphi(t) + h_0 + \log |s|_h^2 \tag{2.56}$$

in $(0, \infty) \times (M \backslash D)$. If $\tilde{\varphi}$ satisfies

- *For any $0 < \delta < T < \infty$, there exists uniform constant C such that*
$$C^{-1}\omega_{cusp} \leq \omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}(t) \leq C\omega_{cusp} \quad on \;\; [\delta, T] \times (M \backslash D);$$
- *on $(0, T]$, $\|\tilde{\varphi}(t) - \psi_0\|_{L^\infty(M \backslash D)} \leqslant C$;*
- *on $[\delta, T]$, there exist constant C^* such that $\|\frac{\partial \tilde{\varphi}(t)}{\partial t}\|_{L^\infty(M \backslash D)} \leqslant C^*$;*
- $\lim\limits_{t \to 0^+} \|\tilde{\varphi}(t) - \psi_0\|_{L^1(M)} = 0.$

Then $\tilde{\varphi}(t) \leq \varphi(t)$.

Proof For any $0 < t_1 < T < \infty$ and $a > 0$. Denote $\Psi(t) = \tilde{\varphi}(t) + a \log |s|_h^2 - \varphi_\beta(t)$ and $\hat{\Delta} = \int_0^1 g^{i\bar{j}}_{s\tilde{\varphi}(t) + (1-s)\varphi_\beta(t)} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \mathrm{d}s$. We note that $\tilde{\varphi}(t)$ is bounded from above because it is a ω_0-psh function. $\Psi(t)$ evolves along the following equation

$$\frac{\partial \Psi(t)}{\partial t} = \hat{\Delta}\Psi(t) - a\hat{\Delta}\log |s|_h^2 - \Psi(t) + (a + \beta)\log |s|_h^2.$$

Since $-\sqrt{-1}\partial\bar{\partial}\log |s|_h^2 = \theta$, we obtain

$$-\hat{\Delta}\log |s|_h^2 = \int_0^1 g^{i\bar{j}}_{s\tilde{\varphi}(t) + (1-s)\varphi_\beta(t)} \theta_{i\bar{j}} \mathrm{d}s \leq C(t_1, T)$$

in $M \backslash D$. Then we obtain

$$\frac{\partial \Psi(t)}{\partial t} \leq \hat{\Delta}\Psi(t) - \Psi(t) + aC(t_1, T).$$

Then by the arguments as that in Proposition 2.11, on $[t_1, T] \times (M \backslash D)$,

$$\tilde{\varphi}(t) - \varphi_\beta(t) \leq e^{-(t - t_1)} \sup_M \left(\tilde{\varphi}(t_1) - \varphi_\beta(t_1)\right).$$

Since $\tilde{\varphi}(t_1)$ converge to ψ_0 in L^1-sense and $\varphi_\beta(t)$ converge to ψ_β in L^∞-sense as $t_1 \to 0^+$, by Hartogs Lemma, we have

$$\tilde{\varphi}(t) - \varphi_\beta(t) \leq e^{-t} \sup_M (\psi_0 - \psi_\beta) \leq 0,$$

after we let $t_1 \to 0$. Hence $\tilde{\varphi}(t) \leq \varphi(t)$ in $(0, \infty) \times (M \backslash D)$. □

Remark 2.14 If M is a compact Kähler manifold with smooth hypersurface D. We can also consider unnormalized cusp Kähler–Ricci flow

$$\frac{\partial \hat{\omega}(t)}{\partial t} = -Ric(\hat{\omega}(t)) + [D].$$

$$\hat{\omega}(t)|_{t=0} = \omega_{cusp} \tag{2.57}$$

If we define $\omega(t) = e^{-t}\hat{\omega}(e^t - 1)$, then flow (2.57) is actually the same as normalized cusp Kähler–Ricci flow (1.8) only modulo a scaling. Let

$$T_0 = \sup\{\, t \mid [\omega_0] - t(c_1(M) - c_1(D)) > 0\}. \tag{2.58}$$

Combining the arguments of Tian–Zhang [41] and Liu–Zhang [28] with the arguments in this paper, there exists a unique solution to flow (2.57) on $[0, T_0)$ in some weak sense which is similar as Definition 1.2.

3 The convergence of cusp Kähler–Ricci flow

In this section, we prove the convergence theorem of cusp Kähler–Ricci flow (1.8).

Proof of Theorem 1.6: Differentiating Eq. (2.56) in time t, we have

$$\left(\frac{d}{\mathrm{d}t} - \Delta_t\right)\frac{\partial\varphi}{\partial t} = -\frac{\partial\varphi}{\partial t} \tag{3.1}$$

on $[\delta, T] \times (M\backslash D)$ with $\delta > 0$. For any $\varepsilon > 0$,

$$\begin{aligned}\left(\frac{d}{\mathrm{d}t} - \Delta_t\right)\left(\frac{\partial\varphi}{\partial t} + \varepsilon\log|s|_h^2\right) &= -\frac{\partial\varphi}{\partial t} + \varepsilon tr_{\omega(t)}\theta \\ &\le -\left(\frac{\partial\varphi}{\partial t} + \varepsilon\log|s|_h^2\right) + \varepsilon C(\delta, T),\end{aligned} \tag{3.2}$$

where constant $C(\delta, T)$ independent of ε. For any $\eta > 0$, let $H = e^{t-\delta}(\frac{\partial\varphi}{\partial t} + \varepsilon\log|s|_h^2) - \varepsilon e^{t-\delta}C(\delta, T) - \eta(t-\delta)$. Since $\frac{\partial\varphi}{\partial t}$ is bounded on $[\delta, T]\times(M\backslash D)$, the maximum point (t_0, x_0) of H satisfies $x_0 \in M\backslash D$. If $t_0 > \delta$, by the maximum principle, we get a contradiction. Hence, $t_0 = \delta$. Then we have

$$\frac{\partial\varphi}{\partial t} \le C(\delta)e^{-t} - \varepsilon\log|s|_h^2 + \varepsilon C(\delta, T) + \eta T. \tag{3.3}$$

Let $\varepsilon \to 0$, $\eta \to 0$ and then $T \to \infty$, we obtain

$$\frac{\partial\varphi}{\partial t} \le C(\delta)e^{-t} \quad in \ [\delta, \infty) \times (M\backslash D). \tag{3.4}$$

By the same arguments, we can get the lower bound of $\frac{\partial\varphi}{\partial t}$. In fact, we obtain

$$|\frac{\partial\varphi}{\partial t}| \le C(\delta)e^{-t} \quad in \ [\delta, \infty) \times (M\backslash D). \tag{3.5}$$

For $\delta < t < s$,

$$|\varphi(t) - \varphi(s)| \le C(\delta)(e^{-t} - e^{-s}) \quad in \ [\delta, \infty) \times (M\backslash D). \tag{3.6}$$

Therefore, $\varphi(t)$ converge exponentially fast in L^∞-sense to φ_∞ in $M\backslash D$. Then making use of the arguments in the proofs of Lemma 2.5, Lemma 2.6 and Proposition 2.7 for flow (1.8), we can prove that $\varphi(t)$ are locally C^k-bounded (independent of t) for any $k \in \mathbb{N}^+$ outside D.

Since $\varphi(t)$ converge in L^∞-sense to φ_∞, $\varphi(t)$ must converge to φ_∞ in C^∞_{loc}-sense in $M\backslash D$. At the same time, For any smooth $(n-1,n-1)$-form η,

$$\int_M \frac{\partial \omega(t)}{\partial t} \wedge \eta = \int_M \frac{\partial \varphi(t)}{\partial t} \sqrt{-1}\partial\bar{\partial}\eta \xrightarrow{t\to\infty} 0 \tag{3.7}$$

while

$$\begin{aligned}
\int_M \frac{\partial \omega(t)}{\partial t} \wedge \eta &= \int_M \sqrt{-1}\partial\bar{\partial}(\log \frac{|s|_h^2(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\omega_0^n} - \varphi(t) + h_0) \wedge \eta \\
&= \int_M \left(\log \frac{|s|_h^2(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\omega_0^n} - \psi_0 - (\varphi(t)-\psi_0) + h_0\right) \sqrt{-1}\partial\bar{\partial}\eta \\
&\xrightarrow{t\to\infty} \int_M \left(\log \frac{|s|_h^2(\omega_0+\sqrt{-1}\partial\bar{\partial}\varphi_\infty)^n}{\omega_0^n} - \psi_0 - (\varphi_\infty-\psi_0) + h_0\right) \sqrt{-1}\partial\bar{\partial}\eta \\
&= \int_M (-Ric(\omega_\infty) - \omega_\infty + [D]) \wedge \eta.
\end{aligned}$$

which implies the convergence in the sense of currents. □

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References

1. Auvray, H.: The space of Poincaré type Kähler metrics on the complement of a divisor. Journal für die reine und angewandte Mathematik **722**, 1–64 (2017)
2. Auvray, H.: Metrics of Poincaré type with constant scalar curvature: a topological constraint. J. Lond. Math. Soc. **87**, 607–621 (2017)
3. Brendle, S.: Ricci flat Kähler metrics with edge singularities. Int. Math. Res. Not. **24**, 5727–5766 (2013)
4. Chern, S.S.: On holomorphic mappings of Hermitian manifolds of the same dimension. Proc. Symp. Pure Math. **11**, 157–170 (1968)
5. Chen, X.X., Ding, W.Y.: Ricci flow on surfaces with degenerate initial metrics. J. Partial Differ. Equ. **20**, 193–202 (2007)
6. Chen, X.X., Tian, G., Zhang, Z.: On the weak Kähler–Ricci flow. Trans. Am. Math. Soc. **363**, 2849–2863 (2011)
7. Chen, X.X., Wang, Y.Q.: Bessel functions. Heat kernel and the conical Kähler–Ricci flow. J. Funct. Anal. **269**, 551–632 (2015)
8. Chen, X.X., Wang, Y.Q.: On the long-time behaviour of the conical Kähler–Ricci flows. Journal für die reine und angewandte Mathematik (2014)
9. Di Nezza, E., Lu, C.H.: Uniqueness and short time regularity of the weak Kähler–Ricci flow. Adv. Math. **305**, 953–993 (2017)
10. Dinew, S.: Uniqueness in $\mathcal{E}(X,\omega_0)$. J. Funct. Anal. **256**, 2113–2122 (2009)
11. Edwards, G.: A scalar curvature bound along the conical Kähler–Ricci flow. J. Geom. Anal. **28**, 225–252 (2018)
12. Eyssidieux, P., Guedj, V., Zeriahi, A.: Singular Kähler–Einstein metrics. J. Am. Math. Soc. **22**, 607–63 (2009)
13. Giesen, G., Topping, P.: Existence of Ricci flows of incomplete surfaces. Commun. Partial Differ. Equ. **36**, 1860–1880 (2011)
14. Guedj, V., Zeriahi, A.: The weighted Monge–Ampère energy of quasiplurisubharmonic functions. J. Funct. Anal. **250**, 442–482 (2007)

15. Guedj, V., Zeriahi, A.: Regularizing properties of the twisted Kähler-Ricci flow. Journal für die reine und angewandte Mathematik **729**, 275–304 (2017)
16. Guenancia, H.: Kähler–Einstein metrics: from cones to cusps. arXiv:1504.01947
17. Guenancia, H., Păun, M.: Conic singularities metrics with prescribed Ricci curvature: the case of general cone angles along normal crossing divisors. J. Differ. Geom. **103**, 15–57 (2016)
18. Jeffres, T.: Uniqueness of Kähler–Einstein cone metrics. Publ. Mat. **44**, 437–448 (2000)
19. Jeffres, T., Mazzeo, R., Rubinstein, Y.: Kähler–Einstein metrics with edge singularities. Ann. Math. **183**, 95–176 (2016)
20. Ji, L., Mazzeo, R., Sesum, N.: Ricci flow on surfaces with cusps. Math. Ann. **345**, 819–834 (2009)
21. Kleiner, B., Lott, J.: Singular Ricci flows I. Acta Math. **219**, 65–134 (2017)
22. Kobayashi, R.: Kähler–Einstein metric on an open algebraic manifolds. Osaka J. Math. **21**, 399–418 (1984)
23. Kołodziej, S.: The complex Monge–Ampère equation. Acta Math. **180**, 69–117 (1998)
24. Kołodziej, S.: Hölder continuity of solutions to the complex Monge–Ampère equation with the right-hand side in L^p: the case of compact Kähler manifolds. Math. Ann. **342**, 379–386 (2008)
25. Lieberman, G.: Second Order Parabolic Differential Equations. World Scientific, Singapore (1996)
26. Liu, J.W., Zhang, C.J.: The conical complex Monge–Ampère equations on Kähler manifolds. Calc. Var. Partial Differ. Equ. **57**, 44 (2018)
27. Liu, J.W., Zhang, X.: The conical Kähler–Ricci flow on Fano manifolds. Adv. Math. **307**, 1324–1371 (2017)
28. Liu, J.W., Zhang, X.: The conical Kähler–Ricci flow with weak initial data on Fano manifold. Int. Math. Res. Not. **17**, 5343–5384 (2017)
29. Lott, J., Zhang, Z.: Ricci flow on quasi-projective manifolds. Duke Math. J. **156**, 87–123 (2011)
30. Lott, J., Zhang, Z.: Ricci flow on quasiprojective manifolds II. J. Eur. Math. Soc. **18**, 1813–1854 (2016)
31. Lu, Y.C.: Holomorphic mappings of complex manifolds. J. Differ. Geom. **2**, 299–312 (1968)
32. Mazzeo, R., Rubinstein, Y., Sesum, N.: Ricci flow on surfaces with conic singularities. Anal. PDE **8**, 839–882 (2015)
33. Nomura, R.: Blow-up behavior of the scalar curvature along the conical Kähler–Ricci flow with finite time singularities. arXiv:1607.03004
34. Shen, L.M.: Unnormalize conical Kähler–Ricci flow. arXiv:1411.7284
35. Shen, L.M.: $C^{2,\alpha}$-estimate for conical Kähler–Ricci flow. arXiv:1412.2420
36. Song, J., Tian, G.: The Kähler–Ricci flow through singularities. Invent. Math. **201**, 519–595 (2017)
37. Song, J., Wang, X.W.: The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality. Geom. Topol. **20**, 49–102 (2016)
38. Śzekelyhidi, G., Tosatti, V.: Regularity of weak solutions of a complex Monge–Ampère equation. Anal. PDE **4**, 369–378 (2011)
39. Tian, G.: Kähler–Einstein metrics on algebraic manifolds, transcendental methods in algebraic geometry (Cetraro 1994). Lect. Not. Math. **1646**, 143–185 (1994)
40. Tian, G., Yau, S.T.: Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. Adv. Ser. Math. Phys. **11**, 574–628 (1987)
41. Tian, G., Zhang, Z.: On the Kähler–Ricci flow on projective manifolds of general type. Chin. Ann. Math. Ser. B **27**, 179–192 (2006)
42. Topping, P.: Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics. J. Eur. Math. Soc. **12**, 1429–1451 (2010)
43. Topping, P.: Uniqueness and nonuniqueness for Ricci flow on surfaces: reverse cusp singularities. Int. Math. Res. Not. **12**, 2356–2376 (2012)
44. Topping, P., Yin, H.: Rate of curvature decay for the contracting cusp Ricci flow. arXiv:1606.07877
45. Wang, Y.Q.: Smooth approximations of the conical Kähler–Ricci flows. Math. Ann. **365**, 835–856 (2016)
46. Yau, S.T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation I. Commun. Pure Appl. Math. **31**, 339–411 (1978)
47. Zhang, Y.S.: A note on conical Kähler–Ricci flow on minimal elliptic Kähler surfaces. Acta Math. Sci. **38**, 169–176 (2018)
48. Zhang, Z.: Kähler–Ricci flow with degenerate initial class. Trans. Am. Math. Soc. **366**, 3389–3403 (2014)